ON THE LIMIT OF FAMILIES OF ALGEBRAIC SUBVARIETIES WITH UNBOUNDED VOLUME

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Dedicated to José Manuel Aroca on the occasion of his 60th birthday

Abstract. We prove that the limit of a sequence of generic semi-algebraic sets given by a finite number of formulas always exists and is a semi-algebraic set that can be explicitly given as a Boolean expression involving the primitives of the additive forms of the formulas.

1. Introduction

Bishop [2] proved that the limit set of a sequence of complex purely k-dimensional algebraic subvarieties whose real volumes are uniformly bounded is again a purely k-dimensional algebraic subvariety. On the other hand, there are many reasons why one should be interested in analyzing the limit sets of algebraic subvarieties with unbounded volume. One reason is the existence of families of algebraic curves of increasing degree that are integrals of families of polynomials differential equations on the plane with bounded degree, a badly understood phenomenon related to the sixteenth Hilbert Problem (see [4], for instance). Another reason is that, despite the existence of topologically complicated limit sets of curves with unbounded volume (see [6], for instance), much can be said about the limit sets of algebraic subvarieties which lie in a family of subvarieties with finite complexity (see [5] for a definition of this concept).

In this paper we consider the limit sets of one-parameter families of algebraic subvarieties, indexed by a natural number n, defined by a finite number of equations, each equation defined by a *formula*. Informally, a formula is a polynomial expression in which n appears in exponents only. Associated to each formula there is a *height*, which is the maximum number of nested n-th powers that appear in it. Here is the formal definition:

Definition 1. *Formulas* and their *heights* are defined recursively as follows:

(1) Every polynomial $F \in \mathbb{C}[X_1, \dots, X_m]$ is a formula of height zero.
(2) If F_1 and F_2 are formulas, then $F_1 + F_2$ and $F_1 F_2$ are formulas of height $\max(h_1, h_2)$, where h_i is the height of F_i.
(3) If F is a formula of height h, then F^n is a formula of height $h + 1$.

A formula of height zero is also called a *primitive* formula; it is simply a complex polynomial.

At times we shall need to evaluate a formula F at a point $z \in \mathbb{C}^m$ and for a particular n. In this case, we shall write $F(z; n)$.

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The height is a measure of the complexity of the formula: it measures how the degree increases with n. A formula of height h has degree proportional to n^h. More precisely, the degree of a formula of height h is $\Theta(n^h)$, using Landau's asymptotic notation as modified by Knuth [3].

An example of a formula of height 3 is

$$xy^2(((x^2 - y + 1)^n - 1)^n + x)^n + (xy)^n + (y^n - 1)^2 + 1.$$

Note that the degree is $2n^3 + 3 = \Theta(n^3)$.

The same polynomial family may be given by different formulas. For instance,

$$(x^n + y)^2 = (x^n)^2 + 2x^n y + y^2.$$

For our purposes, a convenient way to handle this issue is to express formulas in additive form. A formula is in *additive form* when it can be expressed as

$$Q_1 A_1^n + Q_2 A_2^n + \cdots + Q_l A_l^n - P,$$

where $Q_1, \ldots, Q_l$, and P are primitive formulas and $A_1, \ldots, A_l$ are arbitrary subformulas (necessarily of smaller height than the original formula). As we shall see later, additive forms help us to use induction on the height when working with formulas.

Lemma 1. *Every formula can be written in additive form.*

Proof. The proof is by induction on the number of operations required to obtain the formula according to Definition 1. If F is a primitive formula, then we can take $l = 0$ and $P = -F$. If $F = A^n$, then F is already in additive form because we can take $l = 1$, $Q_1 = 1$, $A_1 = A$, and $P = 0$. If $F = A + B$, then by induction A and B can be expressed in additive form, whose combination gives an additive form for F. If $F = AB$, then again by induction A and B can be expressed in additive form. By performing the multiplication AB on their additive forms, we get an additive form for F. □

As an example of the procedure described in the proof above, $(x^n + y)^2$ can be written in additive form as $(x^2)^n + (2y)x^n + y^2$. Note that the expression $(x^n)^2 + 2x^n y + y^2$ given earlier for $(x^n + y)^2$ is *not* in additive form.

Definition 2. The *limit* (as $n \to \infty$) of a sequence (Ω_n) of subsets of $\mathbb{C}^m$ is the set $\lim \Omega_n$ of points that are limits of sequences of points lying in a subsequence of (Ω_n). More precisely,

$$\lim \Omega_n = \{\, z \in \mathbb{C}^m : \exists (z_n), z_n \to z, \exists (k_n), k_n \to \infty,\ z_n \in \Omega_{k_n} \text{ for sufficiently large } n \,\}.$$

Thus, according to this definition, the family of real curves $x^{2n} + y^{2n} = 1$ converges to the border of the unit square given by $x^2 \le 1$, $y^2 \le 1$. Actually, the definition of limit applies to the curves $x^n + y^n = 1$ (note that we now allow both even and odd exponents). These curves converge to the union of the border of the unit square with the two rays given by $x = -y$, $x^2 \ge 1$ (the curves actually alternate between these two limit sets, but our definition of limit covers this). Considered as a family of complex curves, $x^n + y^n = 1$ has as limit set the subset of $\mathbb{C}^2$ given by $\partial([|x| < 1] \cap [|y| < 1]) \cup [|x| = |y| > 1]$, as it is easy to verify.

We shall consider two situations: limit sets in $\mathbb{R}^m$ of families of algebraic subvarieties given by a finite number of formulas and limit sets in $\mathbb{C}^m$ of families of complex algebraic subvarieties.

In the real case it turns out that it is easier to describe the limits of semi-algebraic subsets, instead of algebraic subsets. Semi-algebraic subsets will also play a role in the complex case. An *algebraic subvariety* is the set of points that satisfy a polynomial equation $f(z) = 0$. For simplicity, we shall write this set as $[f = 0]$. A *semi-algebraic set* in $\mathbb{R}^m$ is one given by a Boolean expression on subsets of the form $[f > 0]$ or $[f \geq 0]$. We shall also deal with *basic closed semi-algebraic subsets*, which are the solutions of a system of polynomial inequalities: $[f_1 \geq 0, \dots, f_k \geq 0]$, and with *basic open semi-algebraic subsets*, which are given by strict inequalities: $[f_1 > 0, \dots, f_k > 0]$.

One main difficulty in the theory of semi-algebraic sets is that the closure of a basic open semi-algebraic set is not necessarily the corresponding basic closed semi-algebraic set obtained by relaxing the strict inequalities. That is, the closure of $[f_1 > 0, \dots, f_k > 0]$ is not always $[f_1 \geq 0, \dots, f_k \geq 0]$. Nor is the interior of a closed semi-algebraic set equal to the corresponding basic open semi-algebraic set obtained by restricting the inequalities. That is, the interior of $[f_1 \geq 0, \dots, f_k \geq 0]$ is not always $[f_1 > 0, \dots, f_k > 0]$. However, these statements are true *generically*, in two senses: (i) they are true if we perturb the polynomials slightly, and (ii) relaxing or restricting the inequalities only adds or removes lower dimensional components. We say that a basic closed semi-algebraic set is *generic* when it coincides with the closure of the corresponding basic open semi-algebraic set obtained by restricting the inequalities. In other words, a basic closed semi-algebraic set given by $[f_1 \geq 0, \dots, f_k \geq 0]$ is generic when $[f_1 \geq 0, \dots, f_k \geq 0] = \text{closure}[f_1 > 0, \dots, f_k > 0]$. A *generic algebraic set* is, by definition, the boundary of a generic semi-algebraic subset. For a full discussion of real algebraic and semi-algebraic sets, see the book by Benedetti and Risler [1].

Our main result is the following:

Theorem 1. *The limit of a sequence of generic semi-algebraic sets given by a finite number of formulas always exists and is a semi-algebraic set that can be explicitly given as a Boolean expression involving the primitives of the additive forms of the formulas.*

The corresponding algebraic version is also valid:

Theorem 2. *The limit of a sequence of generic algebraic sets given by a finite number of formulas always exists and is an algebraic set that can be explicitly given as a Boolean expression involving the primitives of the additive forms of the formulas.*

In the complex case, the limit set of a family of algebraic sets given by a finite number of formulas has also an underlying semi-algebraic structure in the sense that it projects, by means of a rational map, onto a proper real semi-algebraic subset defined by expressions involving the absolute values of the primitives of the formulas. More precisely, we have the following result:

Theorem 3. *The limit of a sequence of generic algebraic subsets given by a finite number of formulas with complex coefficients always exists; it is a subset with a complex structure obtained by means of a rational pull-back on semi-algebraic subsets defined explicitly in terms of Boolean expressions involving the absolute values of the primitives of the formulas.*

As an example of the situation in the complex case, we consider the following generalization of the $x^n + y^n = 1$ example given earlier. Let A_1, A_2, and P be complex polynomials. Then

$$\lim[A_1^n + A_2^n = P] = \partial([|A_1| < 1] \cap [|A_2| < 1] \cap [P \neq 0]) \cup [|A_1| = |A_2| > 1]).$$

This limit can be also understood as the pull-back by the polynomial map

$$(A_1, A_2)\colon \mathbb{C}^2 \to \mathbb{C}^2$$

of the Reinhardt preimage of the semi-algebraic subset of $\mathbb{R}^2$ given by the second member of the equation above, where the axes of $\mathbb{R}^2$ are taken as $|A_1|$ and $|A_2|$.

2. The real case

We start with the simplest cases and continue to more complicated cases until we reach general formulas in additive form. To simplify the exposition, we assume that *all semi-algebraic sets are generic* and we consider only formulas in which *all n-th powers are even.*

The simplest non-trivial formula of height 1 is $A^{2n} - P$, where A and P are real polynomials. We want to describe the limit of the algebraic subsets $[A^{2n} = P]$. As mentioned before, it is simpler to describe the limit of the semi-algebraic sets $\Omega_n = [A^{2n} \leq P]$. The strategy in the following lemma and in all subsequent lemmas in this section will be to give a candidate Ω for $\Omega_\infty = \lim \Omega_n$ and to show that $\Omega_\infty \subseteq \Omega$ and $\Omega \subseteq \Omega_\infty$, thus establishing that $\Omega_\infty = \Omega$.

All lemmas in this section say that the limit of a formula can be expressed as a Boolean combination of formulas of smaller height. Thus, they will provide a basis for proving Theorem 1 by induction on the height of the formula.

Lemma 2. *Let A and P be polynomials. Then* $\lim[A^{2n} \leq P] = [A^2 \leq 1, P \geq 0]$.

Proof. Let $\Omega_n = [A^{2n} \leq P]$, $\Omega_\infty = \lim \Omega_n$, and $\Omega = [A^2 \leq 1, P \geq 0]$. We shall show that $\Omega_\infty = \Omega$.

Take $z \in \Omega_\infty$. Then, by definition, there are sequences $z_n \to z$ and $k_n \to \infty$ with $z_n \in \Omega_{k_n}$, that is, $A(z_n)^{2k_n} \leq P(z_n)$. Since $A(z_n)^{2k_n} \geq 0$, we get $P(z_n) \geq 0$ and hence $P(z) = \lim P(z_n) \geq 0$. Moreover, the sequence $(P(z_n))$ is bounded and so $P(z_n) \leq L$ for some $L > 0$. This implies that $A(z_n)^2 \leq P(z_n)^{1/k_n} \leq L^{1/k_n}$. Therefore, $A(z)^2 = \lim A(z_n)^2 \leq \lim L^{1/k_n} = 1$. Hence, $z \in \Omega$.

Reciprocally, take $z \in \Omega$. Since Ω is generic, we have that $z = \lim z_n$, with $z_n \in [A^2 < 1, P > 0]$. From $A(z_n)^2 < 1$ we get that $A(z_n)^{2k} \to 0$ as $k \to \infty$. Since $P(z_n) > 0$, there is a k_n such that $A(z_n)^{2k_n} < P(z_n)$, that is, $z_n \in \Omega_{k_n}$. By increasing k_n beyond n if necessary to get $k_n \to \infty$, we conclude that $z \in \Omega_\infty$. □

The genericity hypothesis is essential to the lemma as stated. Although the proof shows that $\Omega_\infty \subseteq \Omega$ even when Ω is not generic, the reverse inclusion is not always true when Ω is not generic. The following example gives a taste of how things are more complicated in the general case. Let $A = y(y-1)^2 + 1$ and $P = x^2(x-1)$. Note that $[P \geq 0]$ is not the closure of $[P > 0]$ because $[P \geq 0]$ contains the line $[x = 0]$, which is not in the closure of $[P > 0]$ since P is negative around $x = 0$. Similarly, $[A^2 \leq 1]$ is not the closure of $[A^2 < 1]$ because of the line $[y = 1]$. As a consequence, $[A = 1, P \geq 0]$ is only partially contained in $\lim[A^n \leq P]$; only $[A = 1, P \geq 1]$ is part of the limit set. This example is typical of what happens in general: $\lim[A^{2n} \leq P]$ is equal to $[A^2 \leq 1, P \geq 0]$, except that $P \geq 1$ when $A = 1^+$, and $A = 1$ when $P = 0^-$.

The next lemma generalizes Lemma 2 and the $x^n + y^n = 1$ example given in §1:

Lemma 3. *Let $A_1, \dots, A_k$ and P be polynomials. Then*

$$\lim[A_1^{2n} + \cdots + A_k^{2n} \le P] = \bigcap_{i=1}^{k} \lim[A_i^{2n} \le P] = [A_1^2 \le 1, \dots, A_k^2 \le 1, P \ge 0].$$

Proof. Take $z \in \lim[A_1^{2n} + \cdots + A_k^{2n} \le P]$. Then there are sequences $z_n \to z$ and $k_n \to \infty$ such that $A_i(z_n)^{2k_n} \le A_1(z_n)^{2k_n} + \cdots + A_k(z_n)^{2k_n} \le P(z_n)$. So

$$\lim[A_1^{2n} + \cdots + A_k^{2n} \le P] \subseteq \bigcap_{i=1}^{k} \lim[A_i^{2n} \le P] = \bigcap_{i=1}^{k} [A_i^2 \le 1, P \ge 0],$$

by Lemma 2. Hence $z \in [A_1^2 \le 1, \dots, A_k^2 \le 1, P \ge 0]$.

Reciprocally, take $z \in [A_1^2 \le 1, \dots, A_k^2 \le 1, P \ge 0]$. Then $z = \lim z_n$ with $A_i(z_n) < 1$ and $P(z_n) > 0$. Since $A_i(z_n)^{2r} \to 0$ as $r \to \infty$, we have $A_i(z_n)^{2k_n} < P(z_n)/k$ for sufficiently large k_n. As in Lemma 2, we can ensure that $k_n \to \infty$ and conclude that $z \in \lim[A_1^{2n} + \cdots + A_k^{2n} \le P]$. □

The next lemma generalizes Lemma 2 for formulas of larger height.

Lemma 4. *Let A be a formula and P be a primitive formula. Then*

$$\lim[A^{2n} \le P] = \lim[A^2 \le 1] \cap [P \ge 0].$$

Proof. Take $z \in \lim[A^{2n} \le P]$. Then there are sequences $z_n \to z$ and $k_n \to \infty$ such that $A(z_n; k_n)^{2k_n} \le P(z_n)$. Clearly, $P(z) = \lim P(z_n) \ge 0$. As in Lemma 2, the sequence $(P(z_n))$ is bounded and we have $A(z_n; k_n)^{2k_n} \le L$ for some $L > 0$. This implies $A(z_n; k_n)^2 \le L^{1/k_n}$. Clearly, $\lim[A^2 \le L^{1/n}] = \lim[A^2 \le 1]$, because $L^{1/n} \to 1$. Hence, $z \in \lim[A^2 \le 1] \cap [P \ge 0]$. Reciprocally, take $z \in \lim[A^2 \le 1] \cap [P \ge 0]$. Assume for the moment that $P(z) > 0$. Since $[P \ge 0]$ is generic, there are sequences $z_n \to z$ and $k_n \to \infty$ such that $A(z_n; k_n)^2 \le 1$ and $P(z_n) > 0$. Since $(A(z_n; k_n)^{2k_n}/P(z_n))$ is bounded we have $A(z_n; k_n)^{2k_n} \le LP(z_n)$, for some $L > 0$. Since $\lim[A^{2n} \le LP] = \lim[A^{2n} \le P]$, we conclude that $z \in \lim[A^{2n} \le P]$. Finally, if $P(z) = 0$, then $z = \lim z_n$ with $P(z_n) > 0$, again because $[P \ge 0]$ is generic. Since $\lim[A^{2n} \le P]$ is closed, we conclude that $z \in \lim[A^{2n} \le P]$. □

The next lemma handles the reverse inequality.

Lemma 5. *Let A be a formula and P be a primitive formula. Then*

$$\lim[A^{2n} \ge P] = [P \le 0] \cup (\lim[A^2 \ge 1] \cap [P \ge 0]).$$

Proof. Take $z \in \lim[A^{2n} \ge P]$. Then there are sequences $z_n \to z$ and $k_n \to \infty$ such that $A(z_n; k_n)^{2k_n} \ge P(z_n)$. So, either $P(z) \le 0$, or $P(z) > 0$ and $A(z_n; k_n)^2 \ge P(z_n)^{1/k_n}$. Since $\lim[P^{-1/n}A^2 \ge 1] = \lim[A^2 \ge 1]$, we obtain $z \in [P \le 0] \cup (\lim[A^2 \ge 1] \cap [P \ge 0])$. Reciprocally, take $z \in [P \le 0] \cup (\lim[A^2 \ge 1] \cap [P \ge 0])$. Since $[P \ge 0]$ is generic, there are sequences $z_n \to z$ and $k_n \to \infty$ such that $A(z_n; k_n)^2 \ge 1$ and $P(z_n) > 0$. As in Lemma 4, we may assume that $P(z) > 0$, and then the sequence $(A(z_n; k_n)^{2k_n}/P(z_n))$ is bounded below by $L > 0$, i.e., $A(z_n; k_n)^{2k_n} \ge LP(z_n)$. Since $\lim[A^{2n} \ge LP] = \lim[A^{2n} \ge P]$, we conclude that $z \in \lim[A^{2n} \ge P]$. □

Lemma 6. *Let A be a formula and P and Q be primitive formulas. Then*

$$\lim[QA^{2n} \le P] = ([Q > 0] \cap \lim[A^{2n} \le P]) \cup ([Q < 0] \cap \lim[A^{2n} \ge -P]) \cup [Q = 0, P \ge 0].$$

Proof. If $Q(z) > 0$ and $z \in \lim[QA^{2n} \leq P]$, then there are sequences $z_n \to z$ and $k_n \to \infty$ such that $Q(z_n)A(z_n;k_n)^{2k_n} \leq P(z_n) \leq L$, with $L > 0$. Since $\lim[Q^{1/n}A^2 \leq L^{1/n}] = \lim[A^2 \leq 1]$ and $P(z) \geq 0$, we obtain that $z \in [Q > 0] \cap \lim[A^{2n} \leq P]$ by Lemma 4. If $Q(z) < 0$, then there are sequences $z_n \to z$ and $k_n \to \infty$ such that $A(z_n;k_n)^{2k_n} \geq -P(z_n)/Q(z_n)$. By Lemma 5, either $-P(z)/Q(z) \leq 0$ or $z \in \lim[A^2 \geq 1, -P/Q \geq 0]$, or equivalently $P(z) \leq 0$ or $\lim[A^2 \geq 1, P(z) \geq 0]$, i.e., $z \in \lim[A^{2n} \geq -P]$. □

By setting $Q = A$ in the limit above when A is a primitive formula, we get an expression for $\lim[A^{2n+1} \leq P]$, and from this an expression for $\lim[A^n \leq P]$, which should convince the reader that restricting to even powers simplifies the exposition.

Lemma 7. *Let A and B be formulas and P be a primitive formula. Then*

$$\lim[A^{2n} \leq P+B^{2n}] = (\lim[B^2 < 1] \cap \lim[A^{2n} \leq P]) \cup (\lim[B^2 \geq 1] \cap \lim[A^2 \leq B^2]).$$

Proof. Take $z \in \lim[A^{2n} \leq P + B^{2n}]$. Then there are sequences $z_n \to z$ and $k_n \to \infty$ such that $A(z_n;k_n)^{2k_n} \leq P(z_n) + B(z_n;k_n)^{2k_n}$. If $B(z_n;k_n) < 1$, then $B(z_n;k_n)^{2k_n} \to 0$ and we have $P(z) \geq 0$ and $A(z_n;k_n)^{2k_n} \leq L$, where L is a constant. Thus $A(z_n;k_n)^2 \leq L^{1/k_n}$ and so $z \in [P \geq 0] \cap \lim[A^2 \leq 1] = \lim[A^{2n} \leq P]$ by Lemma 4. So we get $z \in \lim[B^2 < 1] \cap \lim[A^{2n} \leq P]$. If $\lim B(z_n;k_n)^2 \geq 1$, then for n large $P(z_n) \leq KB(z_n;k_n)^{2k_n}$ for some constant $K > 0$. Thus $A(z_n;k_n)^{2k_n} \leq (K+1)B(z_n;k_n)^{2k_n}$, so $z \in \lim[A^{2n} \leq B^{2n}]$. Reciprocally, if $z \in \lim[B^2 < 1] \cap \lim[A^{2n} \leq P]$, then there are sequences $z_n \to z$ and $k_n \to \infty$ such that $A(z_n;k_n)^{2k_n} \leq P(z_n) \leq P(z_n) + B(z_n;k_n)^{2k_n}$. If $z \in \lim[B^2 \geq 1] \cap \lim[A^{2n} \leq B^{2n}]$, then we have two possibilities: either $P(z) > 0$ and then $A(z_n;k_n)^{2k_n} \leq B(z_n;k_n)^{2k_n} \leq B(z_n;k_n)^{2k_n} + P(z_n)$, or $P(z) \leq 0$ and then for n large $B(z_n;k_n)^{2k_n} > -2P(z_n)$. Since $\lim[A^{2n} \leq B^{2n}] = \lim[2A^{2n} \leq B^{2n}]$ we can write $2A(z_n;k_n)^{2k_n} \leq B(z_n;k_n)^{2k_n} = 2B(z_n;k_n)^{2k_n} - B(z_n;k_n)^{2k_n} < 2B(z_n;k_n)^{2k_n} + 2P(z_n)$, i.e., $A(z_n;k_n)^{2k_n} \leq B(z_n;k_n)^{2k_n} + P(z_n)$. □

The next lemma shows that Lemma 7 is an important tool for the general case:

Lemma 8. *Let $A_1, \ldots, A_k, B_1, \ldots, B_l$ be formulas and P be a primitive formula. Then*

$$\lim[A_1^{2n} + \cdots + A_k^{2n} \leq P + B_1^{2n} + \cdots + B_l^{2n}] = \\ = \bigcap_{i=1}^{k} \lim[A_i^{2n} \leq P + B_1^{2n} + \cdots + B_l^{2n}] = \bigcap_{i=1}^{k} \bigcup_{j=1}^{l} \lim[A_i^{2n} \leq P + B_j^{2n}].$$

Proof. Define $P_1 := P + B_1^{2n} + \cdots + B_l^{2n}$. We first show that

$$\lim[A_1^{2n} + \cdots + A_k^{2n} \leq P_1] = \bigcap_{i=1}^{k} \lim[A_i^{2n} \leq P_1].$$

Indeed, $\lim[A_1^{2n} + \cdots + A_k^{2n} \leq P_1] \subseteq \bigcap_{i=1}^{k} \lim[A_i^{2n} \leq P_1]$, because $A_i^{2n} \leq A_1^{2n} + \cdots + A_k^{2n} \leq P_1$. Reciprocally, $\bigcap_{i=1}^{k} \lim[A_i^{2n} \leq P_1] \subseteq \lim[A_1^{2n} + \cdots + A_k^{2n} \leq P_1]$. because $\lim[A_i^{2n} \leq P_1] = \lim[A_i^{2n} \leq (1/k)P_1]$, as in Lemma 3.

We now proceed to show that

$$\lim[A^{2n} \leq P + B_1^{2n} + \cdots + B_l^{2n}] = \bigcup_{j=1}^{l} \lim[A^{2n} \leq P + B_j^{2n}].$$

On the one hand, it is clear that $\lim[A^{2n} \leq P + B_j^{2n}] \subseteq \lim[A^{2n} \leq P + B_1^{2n} + \cdots + B_l^{2n}]$. On the other hand, if $z \in \lim[A^{2n} \leq P + B_1^{2n} + \cdots + B_l^{2n}]$, then we have a relation $B_{i_1}(z)^2 \leq \cdots \leq B_{i_l}(z)^2$, where $i_1, \ldots, i_l = 1, \ldots, l$. Then $A(z_n; k_n)^{2k_n} \leq P(z_n) + B_1(z_n; k_n)^{2k_n} + \cdots + B_l(z_n; k_n)^{2k_n} \leq P(z_n) + lB_{i_l}(z_n; k_n)^{2k_n}$, i.e., $z \in \lim[A^{2n} \leq P + lB_{i_l}^{2n}] = \lim[A^{2n} \leq P + B_{i_l}^{2n}]$. □

The next lemma is the stepping stone to the proof of Theorem 1. Its proof is similar to that of Lemma 6, and we leave it to the reader.

Lemma 9. *Let $A_1, \ldots, A_k, B_1, \ldots, B_l$ be formulas and $P, Q_1, \ldots, Q_k, R_1, \ldots, R_l$ be primitive formulas. Then*

$$\begin{aligned}&\lim[Q_1 A_1^{2n} + \cdots + Q_k A_k^{2n} \leq P + R_1 B_1^{2n} + \cdots + R_l B_l^{2n}] \\ &= \lim[A_1^{2n} + \cdots + A_k^{2n} \leq P + B_1^{2n} + \cdots + B_l^{2n}]\end{aligned}$$

provided that $Q_1, \ldots, Q_k, R_1, \ldots, R_l$ are positive. □

We are now ready to prove Theorem 1.

Proof of Theorem 1. By Lemma 1, every formula can be expressed in additive form and the question is reduced to determining

$$\lim[Q_1 A_1^{2n} + \cdots + Q_k A_k^{2n} \leq P + R_1 B_1^{2n} + \cdots + R_l B_l^{2n}],$$

where $Q_1, \ldots, Q_k, R_1, \ldots, R_l$ are positive, since the complete limit can be written as a finite union of expressions as above. By Lemma 9 it is enough to find

$$\lim[A_1^{2n} + \cdots + A_k^{2n} \leq P + B_1^{2n} + \cdots + B_l^{2n}],$$

which by Lemma 8 is

$$\bigcap_{i=1}^{k} \bigcup_{j=1}^{l} \lim[A_i^{2n} \leq P + B_j^{2n}].$$

Thus, it is enough to find the limit of formulas of the type $[A^{2n} \leq P + B^{2n}]$. Proceeding by induction on the height h of $A^n - B^n - P$, we have by Lemma 7 that for $h = 0$ $\lim[A^{2n} \leq P + B^{2n}] = [B^2 < 1] \cap [A^2 \leq 1] \cap [P \geq 0] \cup [B^2 \geq 1] \cap [A^2 \leq B^2]$ and so this limit can be given by a Boolean expression involving the primitives of the formula. Again, by Lemma 7 if $h > 0$, then $\lim[A^{2n} \leq P + B^{2n}] = \lim[B^2 \leq 1] \cap \lim[[A^2 \leq 1] \cap [P \geq 0] \cup \lim[B^2 \geq 1] \cap \lim[A^2 \leq B^2]$ is expressed in terms of limits of formulas of height smaller than h. Thus by induction hypothesis we conclude that $\lim[A^{2n} \leq P + B^{2n}]$ exists, has a semi-algebraic structure, and can be given in terms of a Boolean expression involving the primitives of the formula. □

3. The complex case

Consider now a formula of height $h \geq 1$ written in additive form: $Q_1 A_1^n + \cdots + Q_l A_l^n - P$, where $Q_1, \ldots, Q_l$, and P are complex polynomials in m variables and $A_1, \ldots, A_l$ are formulas of height $\leq h - 1$. We wish to describe $\lim[Q_1 A_1^n + \cdots + Q_l A_l^n = P]$. As in the real case, we start with the simplest situation, $\lim[A^n = P]$.

Lemma 10. *Let A and P be complex polynomials. Suppose that $P \not\equiv 0$ and that A and P are independent in the sense that $P \nmid dP \wedge dA$ in the region where $|A| < 1$. Then* $\lim[A^n = P] = \partial([|A| < 1] \cup [P \neq 0])$.

Proof. Let $z \in \lim[A^n = P]$. Then there are sequences $z_n \to z$ and $k_n \to \infty$ such that $A(z_n)^{k_n} = P(z_n)$. There are two possibilities: $|A(z)| < 1$, then $|A(z_n)| < 1$ for large n and $P(z) = \lim P(z_n) = 0$, i.e., $z \in [|A| < 1, P = 0]$; and $|A(z)| = 1$, then $z \in [|A| = 1] = \overline{[|A| = 1] \cup [P \neq 0]}$. Since $\partial([|A| < 1] \cap [P \neq 0]) = \overline{[|A| < 1]} \cap [P = 0] \cup [|A| = 1] \cap \overline{[P \neq 0]}$, we obtain that $z \in \partial([|A| < 1] \cap [P \neq 0])$.

Conversely, we wish to prove that $\overline{[|A| < 1]} \cap [P = 0] \cup [|A| = 1] \cap \overline{[P \neq 0]} \subseteq \lim[A^n = P]$. Since $\lim[A^n = P]$ is closed, it is enough to show that

$$[|A| < 1] \cap [P = 0] \cup [|A| = 1] \cap [P \neq 0] \subseteq \lim[A^n = P].$$

First take $z \in [|A| < 1] \cap [P = 0]$. Then $|A(z)| < 1$ and $P(z) = 0$. In the plane (A, P) the graph G_{k_n} of the map $P = A^{k_n}$ approaches any point $(A, 0)$ with $|A| < 1$ as $k_n \to \infty$. Thus, given $\varepsilon > 0$ there is N such that for each $n \geq N$ the point $(A(z), \xi_n) \in G_{k_n}$ satisfies $|\xi_n| < \varepsilon$. Since $S := A^{-1}(A(z)) \cap P^{-1}(P(z))$ is an algebraic subvariety of codimension ≥ 2, there is a 1-disc $z \in U_\varepsilon \subseteq A^{-1}(A(z))$, in general position with S, such that $P|_{U_\varepsilon}$ is a covering map of U_ε over a neighborhood of $0 \in \mathbb{C}$. Thus, for k_n large enough, there is $w_n \in U_\varepsilon$ such that $P(w_n) = \xi_n$. Since $A(w_n) = A(z)$ and $(A(z), \xi_n) \in G_{k_n}$ we obtain that $P(w_n) = A(w_n)^{k_n}$. Clearly, $w_n \to z$ and so $z \in \lim[A^n = P]$.

Suppose now that $z \in [|A| = 1] \cap [P \neq 0]$. Then $|A(z)| = 1$ and $P(z) \neq 0$. In the plane (A, P) the horizontal line through the point $(0, P(z))$ intersects the graph G_{k_n} of the map $P = A^{k_n}$ in k_n points over the points $\mathfrak{A}_n = \{P(z)^{1/k_n}\}$ in the A-axis. For each of the points $w \in A^{-1}(\mathfrak{A}_n)$ we have $P(z) = P(w) = A(w)^{k_n}$. Since $|P(z)|^{1/k_n} \to 1$, the graph G_{k_n} approaches the set$|A| = 1$, thus the set $\mathfrak{A}_n$ tends to fill the unitary circle. Therefore for each n we can find $w_n \in A^{-1}(\mathfrak{A}_n)$, $w_n \to z$, such that $P(w_n) = A(w_n)^{k_n}$. □

Lemma 11. $\lim(|A|^n = |P|) = \partial([|A| < 1] \cap [|P| \neq 0]) = \lim[A^n = P]$.

Proof. Same as above. □

Lemma 12. *Suppose P and Q are polynomials, not identically zero, and let A be a formula of positive height h. Assuming that, for n large, $P \nmid dP \wedge dA$ and $Q \nmid dQ \wedge dA$, we have*

$$\lim[QA = P] = \partial(\lim[|A| < 1] \cap [P \neq 0]) \cup \partial([\lim[|A| > 1] \cap [Q \neq 0]).$$

Proof. Take $z \in \lim[QA^n = P]$. Then there are sequences $z_n \to z$ and $k_n \to \infty$ such that $Q(z_n)A(z_n; k_n)^{k_n} = P(z_n)$. We have the following possibilities:

- $\lim |A(z_n; k_n)| < 1$. Then for, n large, $|A(z_n; k_n)| < 1$ and $P(z) = \lim P(z_n) = 0$. Thus $z \in \lim[|A| < 1] \cap [P = 0]$.
- $\lim |A(z_n; k_n)| = 1$. Then $z \in \lim[|A| = 1] = \lim[|A| = 1] \cap \overline{[P \neq 0]} = \lim[|A| = 1] \cap \overline{[Q \neq 0]}$
- $\lim |A(z_n; k_n)| > 1$. Then for n large $|A(z_n; k_n)| > 1$ and $Q(z) = \lim Q(z_n) = \lim P(z_n)A(z_n; k_n)^{-k_n} = 0$.

Reciprocally, if $z \in \lim[|A| < 1] \cap [P = 0]$, then there are sequences $z_n \to z$ and $k_n \to \infty$ such that $P(z) = 0$ and $\lim |A(z_n; k_n)| < 1$. Assume that $Q(z) \neq 0$. Let $\mathfrak{D} = \{w : |A(w; k_n)| \leq 1, n \geq 1\}$. Then $\mathfrak{D} \neq \emptyset$ and since $(A(\,\cdot\,, k_n)$ is bounded in $\mathfrak{D}$, it is a normal family. Then there is a subsequence, say $(A(\,\cdot\,; k_n))$, which converges to a holomorphic function A, i.e., $\lim A(w; k_n) = A(w)$. Since $|A(z)| < 1$, we have $\xi_n = A(z)^{l_n} \to 0$ as $l_n \to \infty$. As by hypothesis $S_n := A^{-1}(A(z); k_n) \cap (P/Q)^{-1}((P/Q)(z))$ is a codimension 2 algebraic subvariety for n large, there is a neighborhood $z \in U$ such that $(P/Q)|_{U\cap A^{-1}(A(z);k_n)}$ projects onto a neighborhood of $0 \in \mathbb{C}$. Thus, there is $w_n \in U \cap A^{-1}(A(z);k_n)$, such that $(P/Q)(w_n) = \xi_n$. Therefore $P(w_n) = Q(w_n)\xi_n = Q(w_n)A(z)^{l_n} = Q(w_n)A(w_n;k_n)^{l_n}$. Clearly, $w_n \to z$ and so $z \in \lim[QA^n = P]$. Similarly, if $z \in \lim[|A| = 1]$ we have that $z \in \lim[QA^n]$. On the other hand, if $z \in \lim[|A| > 1] \cap [Q = 0]$ then $Q(z) = 0$ and $\lim |A(z_n;k_n)| > 1$. If $P = 0$ then $z \in \lim[QA^n = P]$. We assume $P(z) \neq 0$. Define the domain $\widetilde{\mathfrak{D}} = \{w : |A(w;k_n)^{-1}| < 1, n > 1\}$. On $\widetilde{\mathfrak{D}}$ the sequence $(A(\,\cdot\,;k_n)^{-1})$ is normal and converges to a holomorphic function B, i.e., $\lim A(w;k_n)^{-1} = B(w)^{-1}$. Thus $|B(z)| > 1$ and $\eta_n = B(z)^{-l_n} \to 0$ as $l_n \to \infty$. By hypothesis $A^{-1}(B(z);k_n) \cap Q^{-1}(0)$ is a codimension 2 algebraic subvariety for n large. Then, since $P(z) \neq 0$, there is a neighborhood $z \in U$ such that $Q/P|^{-1}_{U\cap A(\,\cdot\,;k_n)}(B(z))$ projects over a neighborhood of $0 \in \mathbb{C}$. Thus there is $w_n \in A^{-1}(B(z);k_n) \cap U$ such that $(Q/P)(w_n) = \eta_n = B(z)^{-l_n} = A(w_n;k_n)^{-l_n}$ or $Q(w_n)A(w_n;k_n)^{l_n} = P(w_n)$. Clearly, $w_n \to z$ and so $z \in \lim[QA^n = P]$. □

Example. Let us compute $\lim[(A^n + P)^n = Q]$.

$$\begin{aligned}
\lim[(A^n+P)^n = Q] &= \partial(\lim[|A^n+P|<1] \cap [Q \neq 0])\\
&= \overline{\lim[|A^n+P|<1]} \cap [Q=0]) \cup (\lim[|A^n|=1] \cap \overline{[Q\neq 0]})\\
\lim[|A^n+P|<1] &= \lim[|A|^n < 1 + |P|]\\
&= [|A|^2 < 1]\\
\lim[|A|^n = 1] &= [|A| = 1]\\
\text{Thus,}\qquad&\\
\lim[(A^n+P)^n = Q] &= [|A|^2 \le 1] \cap [Q=0] \cup \overline{[|A|=1] \cap [Q \neq 0]}\\
&= [|A|^2 \le 1] \cap [Q=0] \cup [|A|=1].
\end{aligned}$$

Thus this limit is the pull back by a rational map of a Reinhardt variety over a semi-algebraic subset of $\mathbb{R}^2$. This example reflects pretty well the general picture described in Theorem 3.

Suppose that $Q_1, \dots, Q_l, P \in \mathbb{C}[x_1, \dots, x_m]$. In what follows we will write

$$\begin{aligned}
\mathcal{Q}\mathcal{A}^n &= [Q_1A_1^n + \cdots + Q_lA_l^n = P]\\
\mathcal{Q}\mathcal{A}^n(\hat{i}) &= [Q_1A_1^n + \cdots + \widehat{Q_iA_i^n} + \cdots + Q_lA_l^n = P]
\end{aligned}$$

Lemma 13. *Suppose that $Q_1, \dots, Q_l, P \in \mathbb{C}[x_1, \dots, x_m]$ and assume that Z_{Q_i}, Z_{Q_j}, Z_{Q_k} intersect in general position if $i \neq j \neq k \neq i$. Let $A_1, \dots, A_l$ be formulas of positive height h. Then*

(1) $\lim[Q_1A_1^n + \cdots + Q_lA_l^n = P] = \bigcup_{i,j=1}^{l} \lim[|A_i| = |A_j| > 1] \cup (Z_{Q_i} \cap Z_{Q_j}) \bigcup_{i=1}^{l} \lim[|A_i| < 1] \cap \lim[Q_1A_1^n + \cdots + \widehat{Q_iA_i^n} + \cdots + Q_lA_l^n = P]$
(2) $\lim[Q_1A_1^n + \cdots + Q_lA_l^n = P] = \bigcup_{i,j=1}^{l} \lim[|A_i| = |A_j| > 1] \cup (Z_{Q_i} \cap Z_{Q_j}) \cup \partial(\lim[|A_1| < 1] \cap \cdots \cap \lim[|A_l| < 1] \cap [P \neq 0])$

Proof. Write $\mathcal{R}_1 := \bigcup_{i=1}^{l} \lim[|A_i| < 1$, $\mathcal{R}_2 := \bigcap_{i=1}^{l} \lim[|A_i| \geq 1]$. Then Lemma 13 follows from the next two lemmas. □

Lemma 14.

$$\begin{aligned}
&\lim[Q_1A_1^n+\cdots+Q_lA_l^n=P]\cap\mathcal{R}_1\\
&= \bigcup_{i=1}^{l}\lim[|A_i|<1]\cap\lim[Q_1A_1^n+\cdots+\widehat{Q_iA_i^n}+\cdots+Q_lA_l^n=P]\\
&= \partial(\lim[|A_1|<1]\cap\cdots\cap\lim[|A_l|<1]\cap[P\neq 0])
\end{aligned}$$

Proof. Let $z\in\lim[Q_1A_1^n+\cdots+Q_lA_l^n=P]\cap\mathcal{R}_1$. Then there are sequences $z_n\to z$ and $k_n\to\infty$ such that $Q_1(z_n)A_1(z_n;k_n)^{k_n}+\cdots+Q_l(z_n)A_l(z_n;k_n)^{k_n}=P(z_n)$. Suppose first that $\lim|A_1(z_n;k_n)|<1$. Then $\varepsilon_n(z_n)=Q_1(z_n)A_1(z_n;k_n)^{k_n}\to 0$ as $n\to\infty$ and if we define

$$f_n(z_n):=P(z_n)-Q_2(z_n)A_2(z_n;k_n)^{k_n}-\cdots-Q_l(z_n)A_l(z_n;k_n)^{k_n},$$

then we have $f_n(z_n)=\varepsilon_n(z_n)$. Let $Z_n=f_n^{-1}(0)$ and $Z=\lim Z_n$. We claim that $z\in Z$. Indeed, if $z\notin Z$ then there are neighborhoods $z\in V$ and $Z\subseteq W$ with $V\cap W=\emptyset$. For n large $z_n\in V$ and $f_n^{-1}(\varepsilon_n)\subseteq W$, a contradiction since $f_n(z_n)=\varepsilon_n$ and $f_n(z_n)=\varepsilon_n$ and $z_n\to z$. Then there is $w_n\in Z_n=f_n^{-1}(0), w_n\to z$, i.e., $f_n(w_n)=0, w_n\to z$. This means that

$$Q_2(w_n)A_{2k_n}(w_n)^{k_n}+\cdots+Q_l(w_n)A_{lk_n}(w_n)^{k_n}=P(w_n)$$

and so $z\in\lim[|A_1|<1]\cap\lim[Q_2A_2^n+\cdots+Q_lA_l^n=P]$. Similarly, if $z\in\mathcal{R}_1$, then

$$z\in\bigcup_{i=1}^{l}\lim[|A_i|<1]\cap\lim[Q_1A_1^n+\cdots+\widehat{Q_iA_i^n}+\cdots+Q_lA_l^n=P].$$

Reciprocally, suppose there are sequences $z_n\to z$ and $w_n\to z$ such that $f_n(w_n)=0$ and $\varepsilon_n(z_n)\to 0$. Then $Z_n=f_n^{-1}(0)\to Z$. Since $w_n\in Z_n$ and $w_n\to z$, then $z\in Z$. Therefore for any δ small positive there is $y_n\in\varepsilon_n^{-1}(\delta)\cap f_n^{-1}(\delta)\neq\emptyset$, i.e., $\varepsilon_n(y_n)=f_n(y_n)$. We will show now that for the points in the region $\mathcal{R}_1$ we have

$$\begin{aligned}
&\partial([|A_1|<1]\cap\cdots\cap[|A_l|<1]\cap[P\neq 0])=\\
&= \bigcup_{i=1}^{l}[|A_i|<1]\cap\lim[Q_1A_1^n+\cdots+\widehat{Q_iA_i^n}+\cdots+Q_lA_l^n=P] \qquad (**)\\
&= \lim[Q_1A_1^n+\cdots+Q_lA_l^n=P].
\end{aligned}$$

We proceed by induction on l. For $l=2$ we have, by Lemma 12,

$$\begin{aligned}
&\partial[[|A_1|<1]\cap[|A_2|<1\cap[P\neq 0]]=\\
&= [|A_1|<1]\cap\partial([|A_2|<1]\cap[P\neq 0])\cup[|A_2|<1]\cap\partial([|A_1|<1]\cap[P\neq 0])\\
&= [|A_1|<1]\cap\lim[Q_2A_2^n=P]\cup[|A_2|<1]\cap\lim[Q_1A_1^n=P]\\
&= \lim[Q_1A_1^n+Q_2A_2^n=P].
\end{aligned}$$

For $l > 2$, we have

$$\begin{aligned}
&\partial([|A_1| < 1)] \cap \cdots \cap (|A_l| < 1) \cap [P \neq 0]) = \\
= \; &\bigcup_{i=1}^{l} (|A_i| < 1) \cap \partial([|A_1| < 1] \cap \cdots \cap \widehat{(|A_i|)} \cap \cdots \cap (|A_l| < 1) \cap [P \neq 0]) \\
= \; &\bigcup_{i=1}^{l} (|A_i| < 1) \cap \lim[Q_1 A_1^n + \cdots + \widehat{Q_i A_i^n} + \cdots + Q_l A_l^n = P] \\
= \; &\lim[Q_1 A_1^n + \cdots + Q_l A_l^n = P] = \lim[Q_1 A_1^n + \cdots + Q_l A_l^n = P],
\end{aligned}$$

where the last two equalities are derived by induction hypothesis on $(**)$. □

Lemma 15.

$$\lim[Q_1 A_1^n + \cdots + Q_l A_l^n = P] \cap \mathcal{R}_2 = \bigcup_{i,j=1}^{l} \lim[|A_i| = |A_j| > 1] \cup (Z_{Q_i} \cap Z_{Q_j})$$

Proof. Suppose now that

$$z \in \lim[|A_1| > 1] \cap \cdots \cap \lim[|A_q| > 1] \cup \lim[|A_{q+1}| = 1] \cap \cdots \cap \lim[|A_l| = 1].$$

Then $q \neq 1$ and

$$Q_1(z_n)(A_1)(z_n; k_n)^{k_n} + \cdots + Q_q(z_n)(A_q)(z_n; k_n)^{k_n} = R(z_n),$$

where

$$R(z_n) := P(z_n) - Q_{q+1}(z_n)(A_{q+1})(z_n; k_n)^{k_n} - \cdots - Q_l(z_n)(A_l)(z_n; k_n)^{k_n}$$

is locally bounded at z. For any $i, j = 1, \ldots, q, \; i \neq j$, we can write the next inequality where, for simplicity, we wrote $i = 1$ and $j = 2$:

$$|Q_1(z_n)||A_1(z_n; k_n)|^{k_n} - |Q_2(z_n)||A_2(z_n; k_n)|^{k_n} \leq \sum_{t=3}^{q} |Q_t(z_n)||A_t(z_n; k_n)|^{k_n} + |R(z_n)|.$$

Thus, dividing both members of this expression by $\prod_{t=3}^{q} |A_t(z_n)|^{k_n}$, we obtain a left member locally bounded at z. Then there is a bounded sequence $\{\lambda_n\}$ such that

$$|Q_1(z_n)||A_1(z_n)|^{k_n} / \prod_{t=3}^{q} |A_t(z_n)|^{k_n} = \lambda_n |Q_2(z_n)||A_2(z_n)|^{k_n} / \prod_{t=3}^{q} |A_t(z_n)|^{k_n},$$

i.e.,

$$|Q_1(z_n)||A_1(z_n)|^{k_n} = \lambda_n |Q_2(z_n)||A_2(z_n)|^{k_n}.$$

Thus, either $z \in Z_{Q_1} \cap Z_{Q_2}$, or $|A_1(z_n)| = (\lambda_n |Q_2(z_n)|/|Q_1(z_n)|)^{1/k_n} A_2(z_n)$. Therefore, $\lim[|A_1(z_n; k_n)|] = \lim[|A_2(z_n; k_n)|]$. Thus, $z \in \bigcup_{i,j=1}^{q} [\lim[|A_i|] = \lim[|A_j|]] \cup (Z_{Q_i} \cap Z_{Q_j})$. This shows that for $l > 1$,

$$\lim[Q_1 A_1^n + \cdots + Q_l A_l^n = P] \cap \mathcal{R}_2 \subseteq [\lim[|A_i|] = \lim[|A_j|]] \cup (Z_{Q_i} \cap Z_{Q_j}) \qquad (*)$$

We now proceed to show the converse to $(*)$. Suppose $z \in [|A_i| = |A_j| > 1] \cup (Z_{Q_i} \cap Z_{Q_j})$. For simplicity take $i = 1, \; j = 2$ and $z \in [|A_1| = |A_2| > 1]$, $|A_i(z)| > 1, \; i = 1, \ldots, k, \; |A_j(z)| \leq 1, \; j = k+1, \ldots, l$. Consider the expression

$$a_n := Q_1(A_1/A_3 \ldots A_k)^n + Q_2(A_2/A_3 \ldots A_k)^n$$

We claim that the curve $a_n = 0$ approaches z as $n \to \infty$. Indeed, from $a_n(w) = 0$ we obtain

$$(A_1/A_2)^n(w) = -(Q_2/Q_1)(w).$$

For any w close to z such that $\arg(-Q_2/Q_1)(w)$ is irrational we have that $(-Q_2/Q_1)(w)^{1/n}$ approaches the circle of center $0 \in \mathbb{C}$ and radius 1 as $n \to \infty$. Therefore $(A_1/A_2)(z)$ is in the closure of the sequence $((-Q_2/Q_1(w))^{1/n})_n$. On the other hand, if

$$b_n := P/(A_3 \cdots A_k)^n - 1/(A_3 \cdots A_k)^n \sum_{j=3}^{l} Q_j A_j^n,$$

then $b_n(z) \to 0$ as $n \to \infty$. Therefore the curve $b_n = 0$ approaches z as $n \to \infty$. Thus there is $z_n \in [a_n = b_n], z_n \to z$, i.e.,

$$(Q_1 A_1^n/(A_3 \cdots A_k)^n + Q_2 A_2^n/(A_3 \cdots A_k)^n)(z_n) = 1/(A_3 \cdots A_k)^n (P - \sum_{j=3}^{l} Q_j A_j^n)(z_n)$$

or $Q_1(z_n)A_1(z_n)^n + \cdots + Q_l(z_n)A_l(z_n)^n = P(z_n)$. □

Lemma 16. *Suppose that $Q_1 A_1^n + \cdots + Q_l A_l^n$ is a formula of positive height h. Then*

$$\begin{aligned} &\lim[|Q_1 A_1^n + \cdots + Q_l A_l^n| < 1] \\ = &\bigcup_{i=1}^{l} \lim[|A_i < 1|] \cap \lim[|Q_1 A_1^n + \cdots + \widehat{Q_i A_i^n} + \cdots + Q_l A_l^n| < 1] \\ &\bigcup_{i,j=1, i \neq j}^{l} [\lim |A_i| = \lim |A_j| > 1] \cup (Z_{Q_i} \cap Z_{Q_j}). \end{aligned}$$

Proof. Let $z \in \lim[|Q_1 A_1^n + \cdots + Q_l A_l^n| \leq 1]$. There is $z_n \to z$ and $k_n \to \infty$ such that $|Q_1(z_n)A_1(z_n)^{k_n} + \cdots + Q_l(z_n)A_l(z_n)^{k_n}| < 1$. Suppose that $\lim |A_1(z_n; k_n)| < 1$, then $\varepsilon_n = |Q_1 A_1(z_n)^{k_n}| \to 0$ and there is $w_n \to z$ such that $|Q_2(w_n)A_2(w_n)^{k_n} + \cdots + Q_l(w_n)A_l(w_n)^{k_n}| < 1$. Therefore, $z \in \bigcup_{i=1}^{l} \lim[|A_i| < 1]$, then $z \in \bigcup_{i=1}^{l} \lim[|A_i < 1|] \cap \lim[|Q_1 A_1^n + \cdots + \widehat{Q_i A_i^n} + \cdots + Q_l A_l^n| < 1$. On the other hand, if $z \in \bigcap_{i=1}^{q} \lim[|A_i| > 1] \bigcap_{j=q+1}^{l} \lim[|A_j < 1|]$, then

$$Q_1(z_n)A_1(z_n; k_n)^{k_n} + \cdots + Q_q(z_n)A_q(z_n; k_n)^{k_n} \leq 1 + S(z_n),$$

where $S(z_n) = |\sum_{j=q+1}^{l} Q_j(z_n)A_j(z_n)^{k_n}|$, is locally bounded at z. Proceeding as in Lemma 13, we obtain that for any $i, j = 1, \ldots, q$ either $z \in (\lim |A_i| = \lim |A_j|)$ or $z \in Z_{Q_i} \cap Z_{Q_j}$. The proof of the converse follows the same line of arguments of Lemma 13. □

Proof of Theorem 2. In order to describe $\lim[Q_1 A_1^n + \cdots + Q_l A_l^n = P]$ we first use induction on l by means of Lemma 13, which reduces the problem to describing $\lim[QA^n = P]$ and $\lim[|A| < 1]$ where $QA^n - P$ has height $h \geq 1$. Then we proceed by induction on h. For $h = 1$ Lemma 12 gives $\lim[QA^n = P] = \partial(\lim[|A| < 1) \cap (P \neq 0)] \cup \partial(\lim[|A| > 1] \cap [Q \neq 0])$, which reduces the problem to height $h - 1$. It only remains to find $\lim[|A| < 1]$ and this follows from Lemma 14.

Thus we have shown that this limit can be expressed by algebraic relations between $|A_1|, \ldots, |A_l|$ and $|P|$. □

4. Algebraic curves as integrals of differential equations

Lemma 17. *Given polynomials A and P, there is a family $(\mathcal{X}_n)$ of polynomial vector fields of fixed degree such that $[A^{2n} = P]$ is an integral curve of $\mathcal{X}_n$.*

Proof. Let $\mathcal{X}_n$ be the field corresponding to the following differential equation:

$$\dot{x} = -2nPA_y + P_yA, \qquad \dot{y} = 2nPA_x - P_xA.$$

Let $f = A^{2n} - P$. Then

$$\dot{x}f_x + \dot{y}f_y = 2n(P_yA_x - P_xA_y)f,$$

as can be easily verified. This shows that $[f = 0]$ is an integral curve of $\mathcal{X}_n$. □

Thus, we have curves of increasing degree that are integral curves of polynomial fields of fixed degree. The next lemma says that in this case the field is essentially unique. The following proof is essentially due to B. Scárdua.

Lemma 18. *Suppose that $[f_n = 0]$ is a family of polynomial curves indexed by their degree. Assume that each curve is an integral curve of two differential equations of bounded degree: $\omega_n = 0$ and $\Omega_n = 0$. Then, for n large enough, $\omega_n = 0$ and $\Omega_n = 0$ define the same foliation.*

Proof. Forget the indices, for simplicity.

The hypotheses imply that

$$\begin{aligned} df \wedge \omega &= f\ell dx \wedge dy \\ df \wedge \Omega &= fLdx \wedge dy, \end{aligned}$$

where ℓ and L are polynomials.

Assume that $\omega \wedge \Omega \neq 0$.

If $df \wedge \Omega \neq 0$, then we can write

$$\omega = \alpha df + \beta\Omega.$$

The coefficients α and β are determined as follows:

$$\omega \wedge \Omega = \alpha df \wedge \Omega \quad \Rightarrow \quad \alpha = \frac{\omega \wedge \Omega}{df \wedge \Omega}$$

$$df \wedge \omega = \beta df \wedge \Omega \quad \Rightarrow \quad \beta = \frac{df \wedge \omega}{df \wedge \Omega}.$$

Therefore

$$\beta = \frac{\ell}{L}, \quad \alpha = \frac{\omega \wedge \Omega}{fLdx \wedge dy}$$

and so

$$\omega = \frac{\omega \wedge \Omega}{dx \wedge dy} \cdot \frac{df}{fL} + \frac{\ell}{L}\Omega,$$

or

$$L\omega = \frac{\omega \wedge \Omega}{dx \wedge dy} \cdot \frac{df}{f} + \ell\Omega.$$

Assume that f is irreducible. Since $L\omega - \ell\Omega$ has bounded degree, we must have that $fdx \wedge dy$ divides $\omega \wedge \Omega$, that is,

$$\omega \wedge \Omega = f\mu dx \wedge dy,$$

for some polynomial μ. Hence, $L\omega = \mu df + \ell\Omega$.

Now $\partial\ell = \partial\omega - 1$ and $\partial L = \partial\Omega - 1$, and so μdf has bounded degree. Since $df_n \to \infty$ we conclude that $\mu_n = 0$ for large n.

If $df \wedge \Omega = 0$, then we take $df \wedge \omega \neq 0$. If both expressions vanish identically, then ω, Ω, and df define the same foliation. □

Moreover, as the next lemma indicates, formulas that are more complicated than $A^{2n} = P$ are not likely to be integral curves of fields of fixed degree.

Lemma 19. *Let A, B, and P be bivariate polynomials such that $A(0,0) = 0 = B(0,0)$ and $(A,B) = 1$. Then the curves in the family $A^n + B^n = P$ are not integral curves of a family of polynomial fields of degree 2.*

Proof. Suppose that A and B have degree k and P has degree j. Let $f = A^n + B^n - P$. Suppose that f is an integral curve of the 1-form

$$\omega = adx + bdy,$$

with a and b polynomials of degree 2. Then

$$df \wedge \omega = fLdx \wedge dy,$$

with L a polynomial of degree 1. This equation is equivalent to

$$(nA^{n-1}A_x + nB^{n-1}B_x - P_x)b - (nA^{n-1}A_y + nB^{n-1}B_y - P_y)a = (A^n + B^n - P)L.$$

For n large, because $A(0,0) = 0 = B(0,0)$, we obtain

$$P_x b - P_y a = PL \qquad (1)$$

$$nA^{n-1}(A_x b - A_y a) + nB^{n-1}(B_x b - B_y a) = (A^n + B^n)L. \qquad (2)$$

Because $(A,B) = 1$, we get

$$\begin{aligned} n(A_x b - A_y a) &= AL \\ n(B_x b - B_y a) &= BL \end{aligned} \qquad (3)$$

(The proof is at the end.)

Suppose that P is homogeneous of degree j, A and B are homogeneous of degree k, and a and b are homogeneous of degree 2, in equations (1) and (3). This is not a restriction because it suffices to compare the homogeneous parts of highest degree in these equations.

Equation (3) can be written as

$$n \begin{pmatrix} -A_y & A_x \\ -B_y & B_x \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = L \begin{pmatrix} A \\ B \end{pmatrix}$$

If $\Delta = A_x B_y - A_y B_x$, then

$$n \begin{pmatrix} a \\ b \end{pmatrix} = \frac{L}{\Delta} \begin{pmatrix} B_x & -A_x \\ B_y & -A_y \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$

Since, by Euler's formula, $kA = A_x x + A_y y$ and $kB = B_x x + B_y y$, we get

$$\begin{aligned} n \begin{pmatrix} a \\ b \end{pmatrix} &= \frac{L}{k\Delta} \begin{pmatrix} B_x & -A_x \\ B_y & -A_y \end{pmatrix} \begin{pmatrix} A_x & A_y \\ B_x & B_y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \frac{L}{k\Delta} \begin{pmatrix} 0 & -\Delta \\ \Delta & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \frac{L}{k} \begin{pmatrix} -y \\ x \end{pmatrix}, \end{aligned}$$

which implies that

$$a = -\frac{L}{kn}y, \qquad b = \frac{L}{kn}x.$$

From (1), we get

$$P_x \frac{L}{kn}x + P_y \frac{L}{kn}y = PL,$$

that is,

$$P_x x + P_y y = nkP,$$

which implies that P is homogeneous of degree nk. Since n is arbitrarily large and P has a fixed degree, this is cannot happen. Therefore, f is not an integral curve of ω.

We still have to prove that (2) implies (3). In fact, let $\alpha = A_x b - A_y a$ and $\beta = B_x b - B_y a$. Then

$$nA^{n-1}\alpha + nB^{n-1}\beta = L(A^n + B^n),$$

that is,

$$A^{n-1}(n\alpha - LA) = B^{n-1}(-n\beta + LB).$$

Since $(A, B) = 1$, this implies that $A^{n-1}|(-n\beta + LB)$ and $B^{n-1}|(n\alpha - LA)$. Hence, there is a polynomial λ such that

$$\begin{aligned} \lambda A^{n-1} &= -n\beta + LB \\ \lambda B^{n-1} &= (n\alpha - LA) \end{aligned}$$

Comparing degrees, we get $\lambda = 0$ for large n. Therefore,

$$(-n\beta + LB) = 0 = (n\alpha - LA),$$

as claimed. □

Define the *length* of a formula as the minimum number of its primitives of degree ≥ 1. So, for instance, the formula

$$(x+1)^{2n} + ((x-y-1)^n + y)^n + y^2 - 1$$

has length 4.

Suppose that $\mathcal{C}$ is a family of curves given by the zeros of a formula of positive height. Let l be the length of the formula and assume that the curves defined by the zeros of its primitives intersect transversely in the complex domain. If $\mathcal{V}$ is a family of vector fields of degree k such that the elements of $\mathcal{C}$ are integral curves of the corresponding elements of $\mathcal{V}$, then $l \leq k^2 + k + 1$, as this last expression is the number of singular points of the elements of $\mathcal{V}$. In particular if $l > k^2 + k + 1$ the elements in $\mathcal{C}$ can not be integral curves of a family of polynomial vector fields of degree $\leq k$.

Theorem 4. *Every generic basic closed one-dimensional semi-algebraic set in the plane is the limit of an family of algebraic curves that are integral curves of a family of polynomial vector fields of fixed degree.*

Proof. Let Ω be a generic basic closed semi-algebraic set. It is known (but hard to prove) that every basic open semi-algebraic set in the plane can actually be given by *two* inequalities [1]. Since Ω is generic, this also applies to Ω and we can write $\Omega = [P \geq 0, Q \geq 0]$. We shall show that $\Omega = \lim[A^{2n} \leq P]$ for

$$A = \frac{Q}{n} - 1.$$

Indeed,

$$[A^2 \leq 1] = [(\frac{Q}{n} - 1)^2 \leq 1] = [0 \leq Q \leq 2n]$$

Hence,

$$\{\, z : A^2(z) \leq 1, \text{ for sufficiently large } n \,\} = [Q \geq 0].$$

and so

$$\{\, z : P(z) \geq 0,\ A^2(z) \leq 1, \text{ for sufficiently large } n \,\} = [P \geq 0, Q \geq 0].$$

Lemma 2 then says that

$$\lim[A^{2n} \leq P] = [P \geq 0, Q \geq 0] = \Omega,$$

if $[A^2 \leq 1, P \geq 0]$ is generic for sufficiently large n.

As mentioned in §4, the curves $[A^{2n} = P]$ are integral curves of a family of polynomial vector fields of fixed degree. (Note that, although A has coefficients that depend on n, the vector fields are still of fixed degree. The general case is described in Lemma 20 below.) □

Lemma 20. *Let*

$$A(z;n) = \sum_{j=-k}^{\ell} a_j(z) n^j$$

be a real polynomial in z *and* n*. Then*

$$\lim_{n\to\infty} [A(z;n)^{2n} \leq P(z)] = \begin{cases} \emptyset, & \text{if } a_j \not\equiv 0 \text{ for some } j \geq 1; \\ [a_0^2 < 1, 0 \leq P] \cup [a_0^2 = 1, e^{2a_{-1}(z)} \leq P(z)], & \text{otherwise.} \end{cases}$$

Proof. First, notice that if $a_j \not\equiv 0$ for some $j \geq 1$, then the limit is empty.

Next, suppose that $z \in \lim[A^{2n} \leq P]$. Then, there is a sequence $z_n \to z$ such that

$$\left(\frac{a_{-k}(z_n)}{n^k} + \frac{a_{-k+1}(z_n)}{n^{k-1}} + \cdots + \frac{a_{-1}(z_n)}{n} + a_0(z_n) \right)^{2n} \leq P(z_n).$$

If $a_0(z) = 0$, then $0 \leq P(z)$.

Finally, assume that $a_0(z) \neq 0$. Then, $a_0(z_n) \neq 0$ for n large enough. Letting

$$B(w;n) = \frac{a_{-1}(w)}{a_0(w)} \cdot \frac{1}{n} + \cdots + \frac{a_{-k}(w)}{a_0(w)} \cdot \frac{1}{n^k},$$

we have

$$a_0(z_n)^{2n}(1 + B(z_n;n))^{2n} \leq P(z_n).$$

For n large enough, we have $|B(z_n;n)| < 1$ and then

$$n \log a_0(z_n)^2 + 2n \log(1 + B(z_n;n)) \leq \log P(z_n).$$

If $a_0(z)^2 = 1$, we have $2a_{-1}(z) \leq \log P(z)$, i.e., $e^{2a_{-1}(z)} \leq P(z)$. If $a_0(z)^2 < 1$, we have $0 \leq P(z)$. □

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