Approximating Implicit Curves on Triangulations with Affine Arithmetic

Afonso Paiva
ICMC-USP

with
Filipe C. Nascimento (ICMC-USP), Luiz Henrique de Figueiredo (IMPA)
and Jorge Stolfi (UNICAMP)

Overview

Problem Setup

Given a planar triangulation $\mathcal{T}$ and $f : \mathbb{R}^2 \to \mathbb{R}$, compute a *robust* adaptive polygonal approximation of the curve given implicitly by f on $\mathcal{T}$: $\mathcal{C} = \{(x, y) \in \mathcal{T} : f(x, y) = 0\}$.

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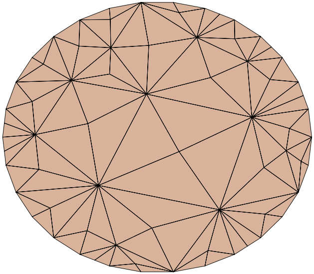

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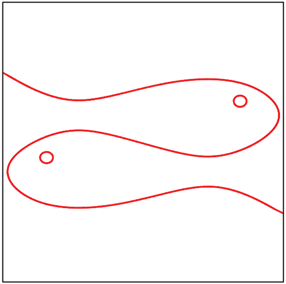

$= ?$

$\mathcal{C} = f^{-1}(0)$

Possible Solution?

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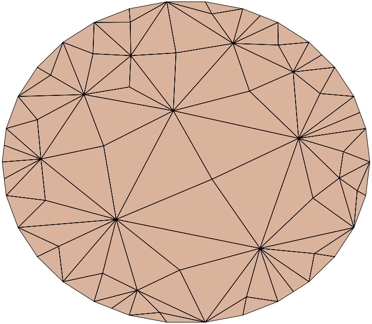

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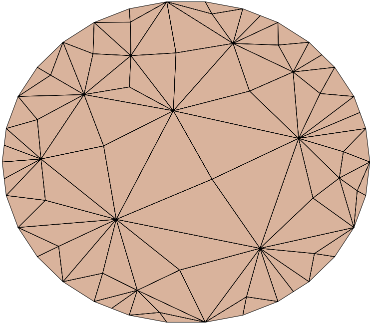

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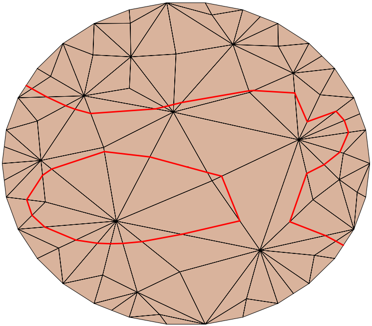

Marching Triangles

$\# \triangle = 101$

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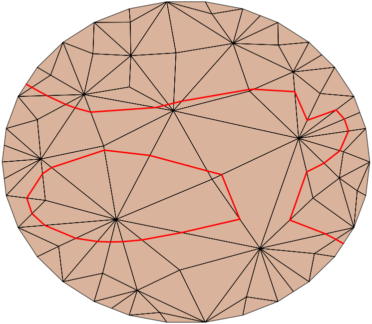

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Possible Solution?

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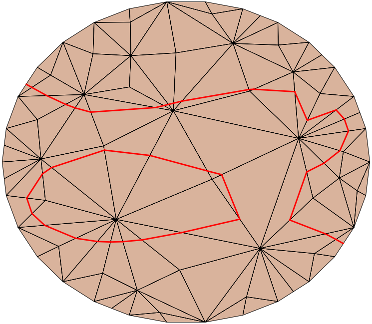

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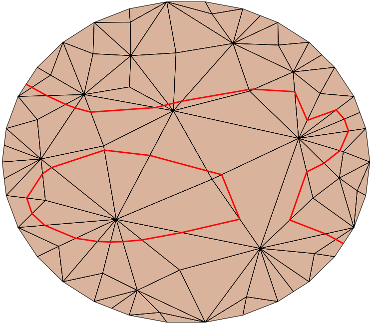

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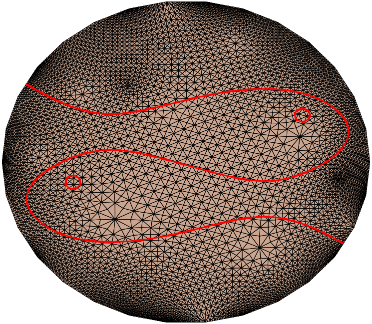

Marching Triangles

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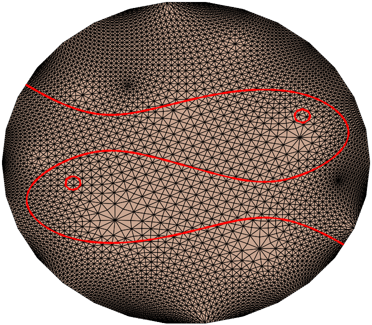

Marching Triangles

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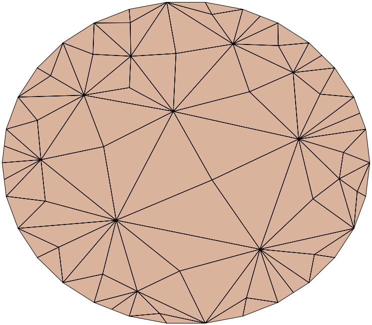
level 0



Our Method

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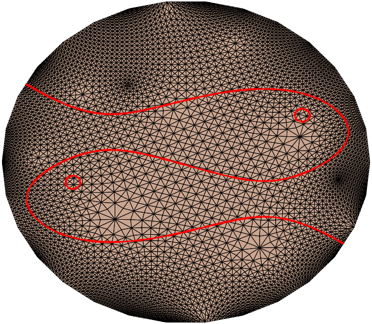

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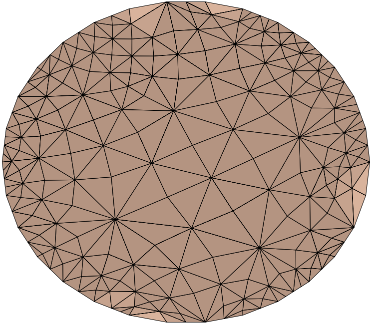

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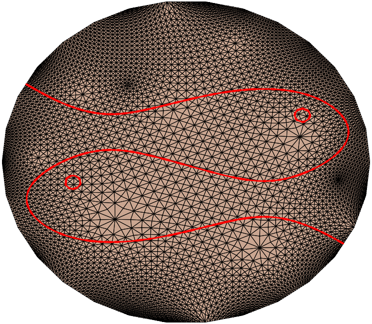

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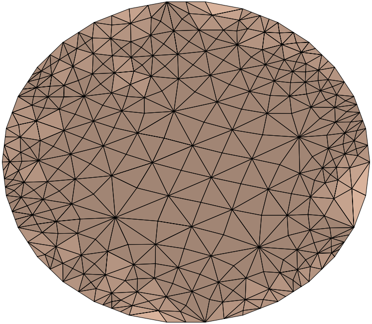
level 2



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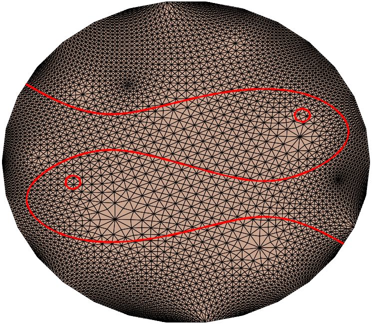

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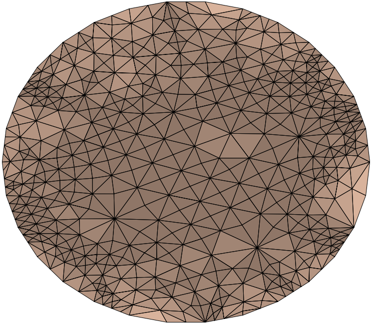

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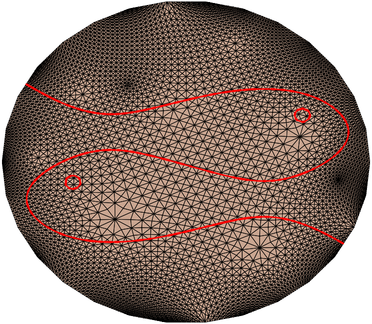

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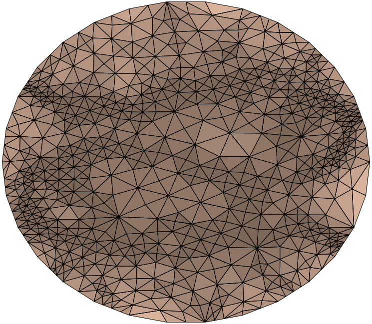

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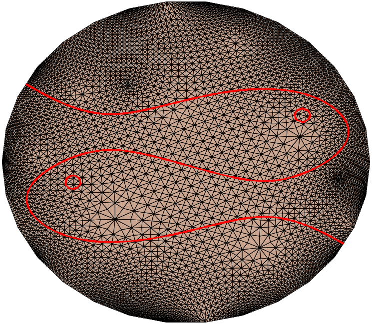

Marching Triangles

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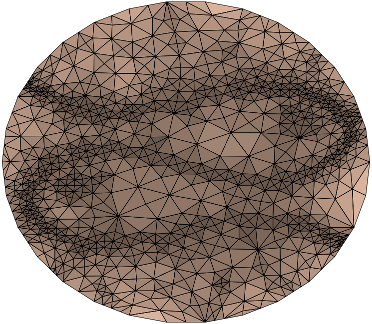

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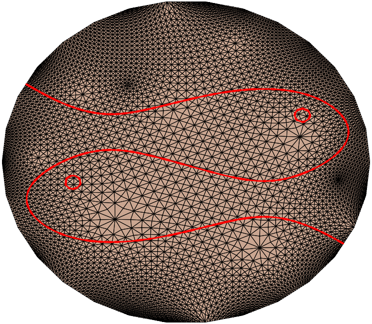

Marching Triangles

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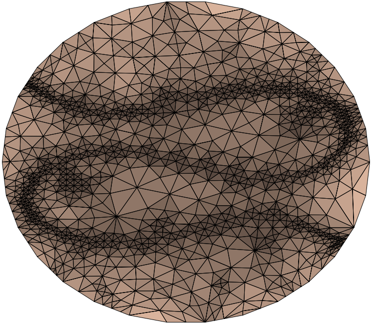
level 6



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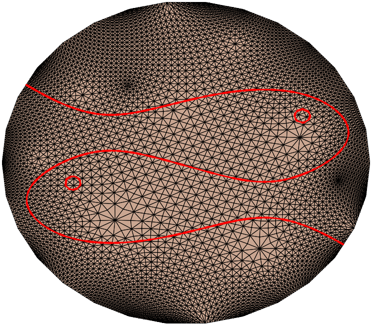

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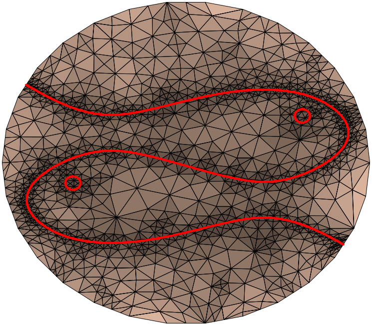

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  - Interval arithmetic (IA) *and* affine arithmetic (AA)

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- Represents a quantity z with an *affine form*:

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  - Good AA libraries in C/C++ available

Intervals in Affine Arithmetic

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- IA form $\Rightarrow$ AA form
  - $z \in [a, b] \Rightarrow \hat{z} = z_0 + z_1\varepsilon_1$ where
    $z_0 = (a + b)/2$
    $z_1 = (b - a)/2$

Bounding Iimplicit Curves with Strips on □

On axis-aligned rectangles:

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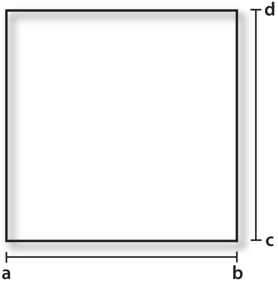
On axis-aligned rectangles: we need to evaluate $f(\square)$ with AA



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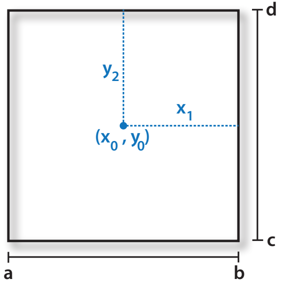

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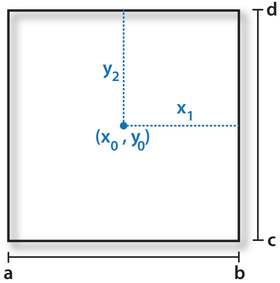

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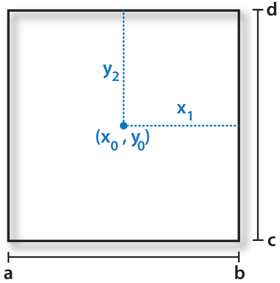

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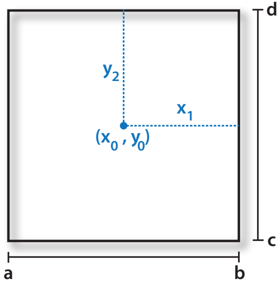

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$\varepsilon_3, \ldots, \varepsilon_n$ are noise symbols related to non-affine operations

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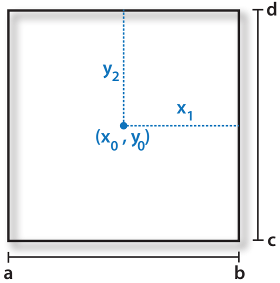

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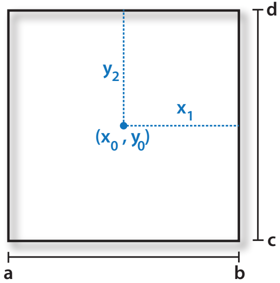

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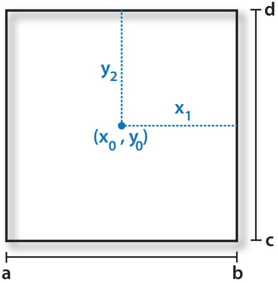
Bounding Iimplicit Curves with Strips on □

On axis-aligned rectangles: we need to evaluate $f(\square)$ with AA



$$\hat{x} = x_0 + x_1\varepsilon_1, \quad x_0 = (a+b)/2, \quad x_1 = (b-a)/2$$

$$\hat{y} = y_0 + y_2\varepsilon_2, \quad y_0 = (c+d)/2, \quad y_2 = (d-c)/2$$

AA form of f

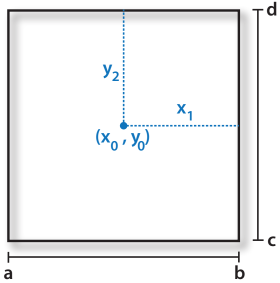
$$\hat{f} = f_0 + f_1\varepsilon_1 + f_2\varepsilon_2 + f_3\varepsilon_3$$

Spatial criteria

$$0 \notin [\hat{f}(\square)] \Rightarrow \texttt{discard}(\square)$$

Bounding Iimplicit Curves with Strips on $\square$

On axis-aligned rectangles: we need to evaluate $f(\square)$ with AA



$$\hat{x} = x_0 + x_1\varepsilon_1, \quad x_0 = (a+b)/2, \quad x_1 = (b-a)/2$$

$$\hat{y} = y_0 + y_2\varepsilon_2, \quad y_0 = (c+d)/2, \quad y_2 = (d-c)/2$$

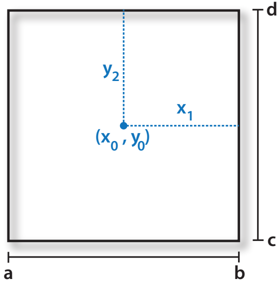
AA form of f

$$\hat{f} = f_0 + f_1\varepsilon_1 + f_2\varepsilon_2 + f_3\varepsilon_3$$

Geometric bounds using the AA form of $\hat{f}$

Bounding Iimplicit Curves with Strips on □

On axis-aligned rectangles: we need to evaluate $f(\square)$ with AA



$$\hat{x} = x_0 + x_1\varepsilon_1, \quad x_0 = (a+b)/2, \quad x_1 = (b-a)/2$$

$$\hat{y} = y_0 + y_2\varepsilon_2, \quad y_0 = (c+d)/2, \quad y_2 = (d-c)/2$$

AA form of f

$$\hat{f} = f_0 + f_1\varepsilon_1 + f_2\varepsilon_2 + f_3\varepsilon_3$$

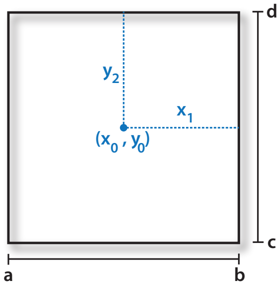
Geometric bounds using the AA form of $\hat{f}$

the graph of $z = f(x, y)$ over $\square$ is sandwiched between the planes:

$$z = f_0 + f_1\varepsilon_1 + f_2\varepsilon_2 \pm f_3$$

Bounding Iimplicit Curves with Strips on □

On axis-aligned rectangles: we need to evaluate $f(\square)$ with AA



$$\varepsilon_1 = \frac{x - x_0}{x_1} \qquad \varepsilon_2 = \frac{y - y_0}{y_2}$$

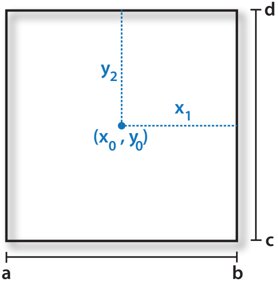
Geometric bounds using the AA form of $\hat{f}$

the graph of $z = f(x, y)$ over □ is sandwiched between the planes:

$$z = f_0 + f_1\varepsilon_1 + f_2\varepsilon_2 \pm f_3$$

Bounding Iimplicit Curves with Strips on $\square$

On axis-aligned rectangles: we need to evaluate $f(\square)$ with AA



$$\varepsilon_1 = \frac{x - x_0}{x_1} \qquad \varepsilon_2 = \frac{y - y_0}{y_2}$$

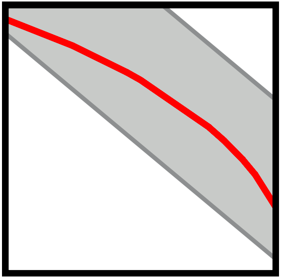
Geometric bounds using the AA form of $\hat{f}$

z in cartesian coordinates:

$$z = f_0 + \frac{f_1}{x_1}(x - x_0) + \frac{f_2}{y_2}(y - y_0) \pm f_3$$

Bounding Iimplicit Curves with Strips on $\Box$

On axis-aligned rectangles: we need to evaluate $f(\Box)$ with AA



$$\varepsilon_1 = \frac{x - x_0}{x_1} \qquad \varepsilon_2 = \frac{y - y_0}{y_2}$$

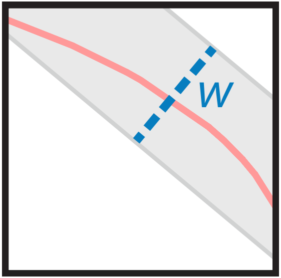
Geometric bounds using the AA form of $\hat{f}$

f is zero inside the strip defined by the two parallel lines:

$$0 = f_0 + \frac{f_1}{x_1}(x - x_0) + \frac{f_2}{y_2}(y - y_0) \pm f_3$$

Bounding Iimplicit Curves with Strips on □

On axis-aligned rectangles: we need to evaluate $f(\square)$ with AA



The width between the lines

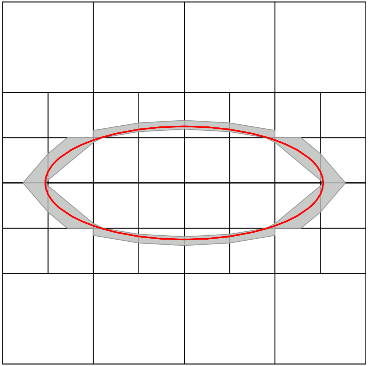
$$w = \frac{2f_3}{\sqrt{\left(\frac{f_1}{x_1}\right)^2 + \left(\frac{f_2}{y_2}\right)^2}}$$

Geometric bounds using the AA form of $\hat{f}$

f is zero inside the strip defined by the two parallel lines:

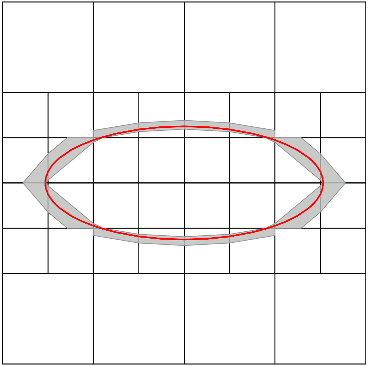
$$0 = f_0 + \frac{f_1}{x_1}(x - x_0) + \frac{f_2}{y_2}(y - y_0) \pm f_3$$

Bounding Implicit Curves with Strips on □



$$\frac{x^2}{6} + y^2 = 1$$

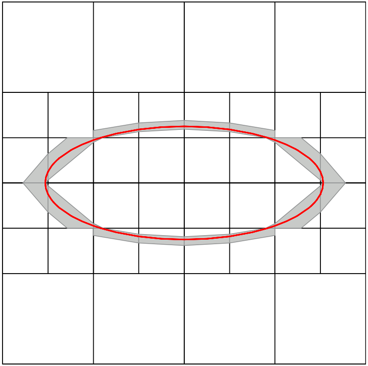
Bounding Implicit Curves with Strips on □



wide strips $\Rightarrow$ high curvature

$$\frac{x^2}{6} + y^2 = 1$$

Bounding Implicit Curves with Strips on $\square$



$$\frac{x^2}{6} + y^2 = 1$$

wide strips $\Rightarrow$ high curvature

Geometric criteria

$w > \mathit{threshold} \Rightarrow$ `subdivide(□)`

Comparing with IA: method proposed by Lopes *et al.* in SIBGRAPI 2001

Bounding Implicit Curves with Strips on □

Comparing with IA: method proposed by Lopes *et al.* in SIBGRAPI 2001

- requires the evaluation ∇f using IA and automatic differentiation

Bounding Implicit Curves with Strips on □

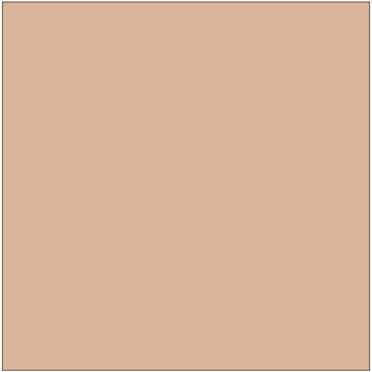
Comparing with IA: method proposed by Lopes *et al.* in SIBGRAPI 2001

- requires the evaluation ∇f using IA and automatic differentiation
- adaptive quadtree

Bounding Implicit Curves with Strips on □

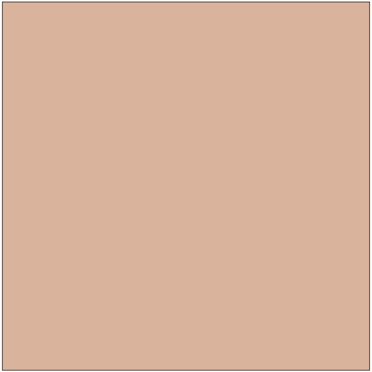
Comparing with IA: method proposed by Lopes *et al.* in SIBGRAPI 2001

- requires the evaluation ∇f using IA and automatic differentiation
- adaptive quadtree



IA

level 0

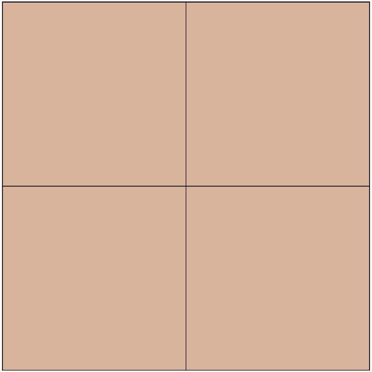


AA

Bounding Implicit Curves with Strips on $\square$

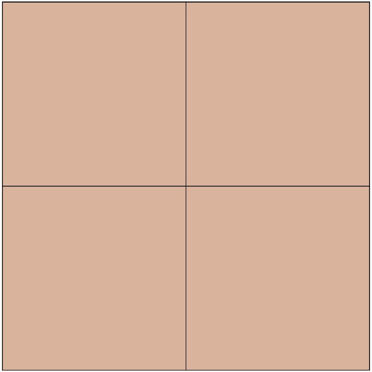
Comparing with IA: method proposed by Lopes *et al.* in SIBGRAPI 2001

- requires the evaluation ∇f using IA and automatic differentiation
- adaptive quadtree



IA

level 1

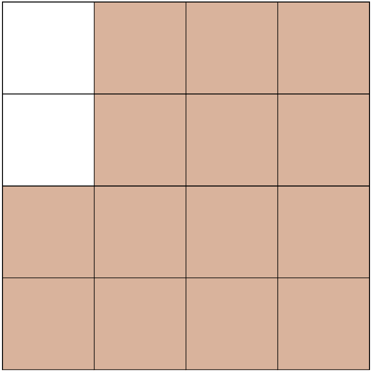


AA

Bounding Implicit Curves with Strips on □

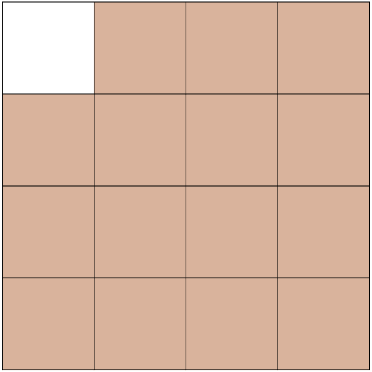
Comparing with IA: method proposed by Lopes *et al.* in SIBGRAPI 2001

- requires the evaluation ∇f using IA and automatic differentiation
- adaptive quadtree



IA

level 2

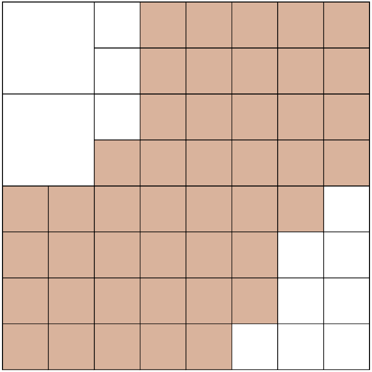


AA

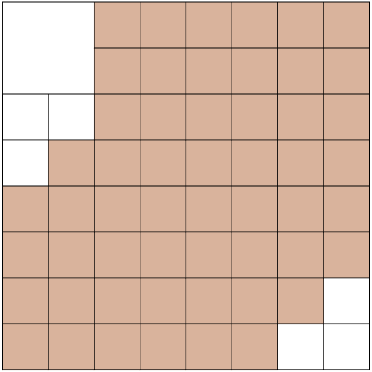
Bounding Implicit Curves with Strips on □

Comparing with IA: method proposed by Lopes *et al.* in SIBGRAPI 2001

- requires the evaluation ∇f using IA and automatic differentiation
- adaptive quadtree



IA



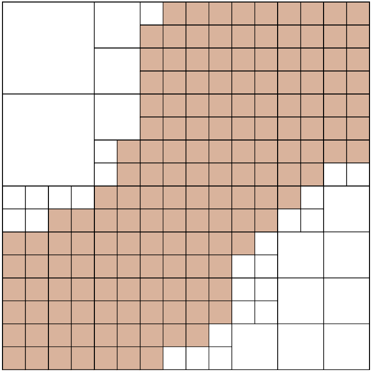
AA

level 3

Bounding Implicit Curves with Strips on □

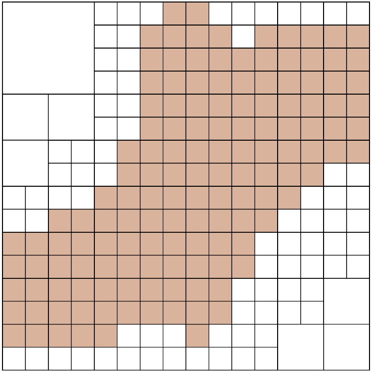
Comparing with IA: method proposed by Lopes *et al.* in SIBGRAPI 2001

- requires the evaluation ∇f using IA and automatic differentiation
- adaptive quadtree



IA

level 4

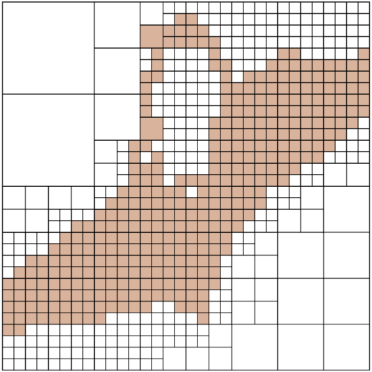


AA

Bounding Implicit Curves with Strips on □

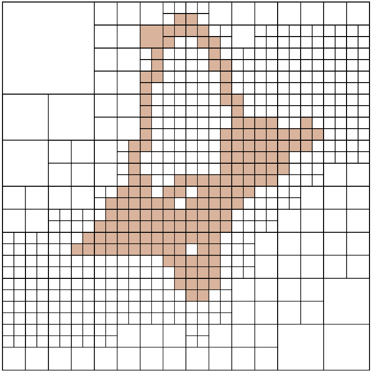
Comparing with IA: method proposed by Lopes *et al.* in SIBGRAPI 2001

- requires the evaluation ∇f using IA and automatic differentiation
- adaptive quadtree



IA

level 5

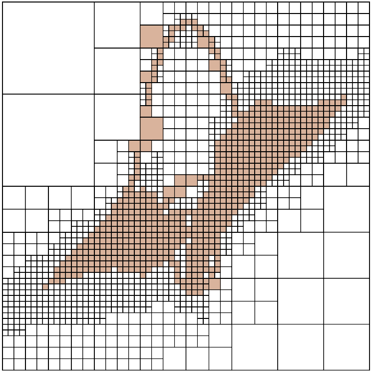


AA

Bounding Implicit Curves with Strips on $\square$

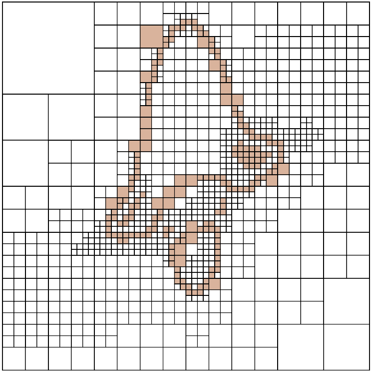
Comparing with IA: method proposed by Lopes *et al.* in SIBGRAPI 2001

- requires the evaluation ∇f using IA and automatic differentiation
- adaptive quadtree



IA

level 6

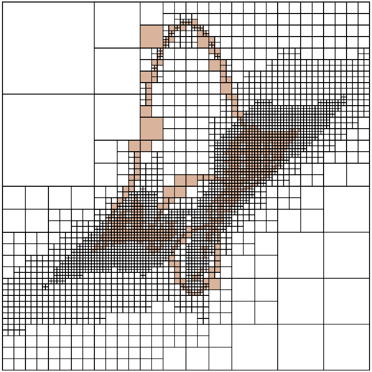


AA

Bounding Implicit Curves with Strips on □

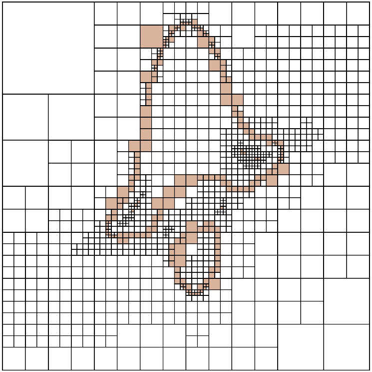
Comparing with IA: method proposed by Lopes *et al.* in SIBGRAPI 2001

- requires the evaluation ∇f using IA and automatic differentiation
- adaptive quadtree



IA

level 7

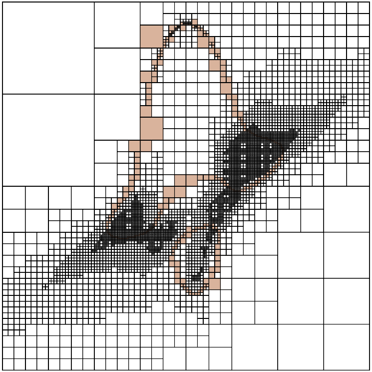


AA

Bounding Implicit Curves with Strips on □

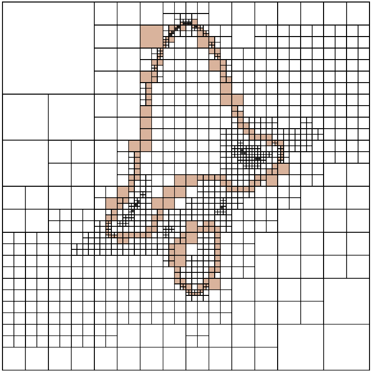
Comparing with IA: method proposed by Lopes *et al.* in SIBGRAPI 2001

- requires the evaluation ∇f using IA and automatic differentiation
- adaptive quadtree



IA

level 8

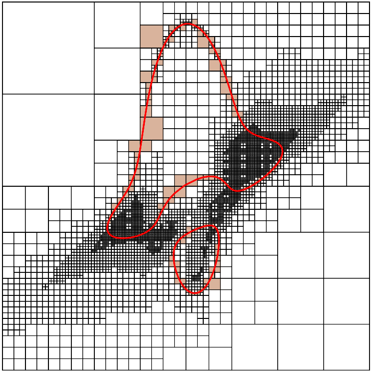


AA

Bounding Implicit Curves with Strips on □

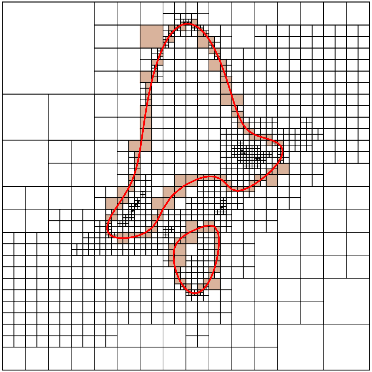
Comparing with IA: method proposed by Lopes *et al.* in SIBGRAPI 2001

- requires the evaluation ∇f using IA and automatic differentiation
- adaptive quadtree



IA

level 8

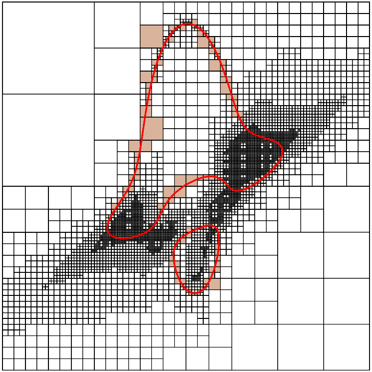


AA

Bounding Implicit Curves with Strips on $\square$

Comparing with IA: method proposed by Lopes *et al.* in SIBGRAPI 2001

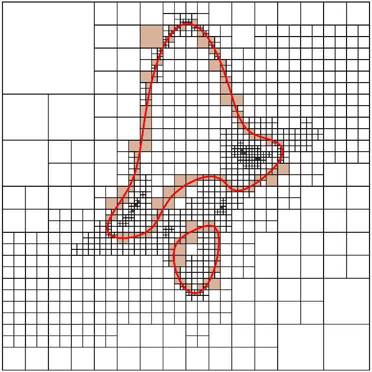
- requires the evaluation ∇f using IA and automatic differentiation
- adaptive quadtree



IA

#cells visited: 6997

level 8



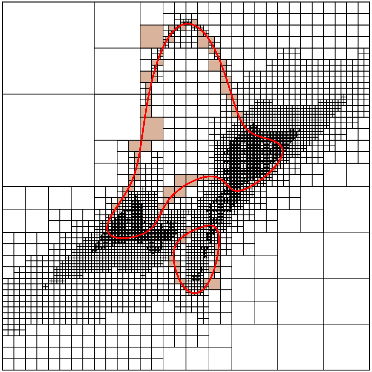
AA

#cells visited: 1697

Bounding Implicit Curves with Strips on □

Comparing with IA: method proposed by Lopes *et al.* in SIBGRAPI 2001

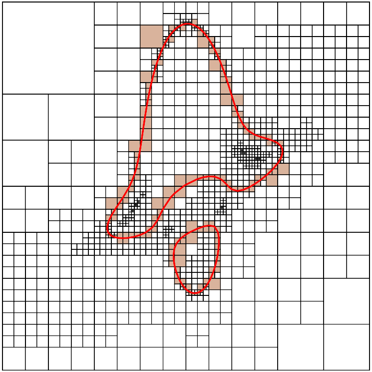
- requires the evaluation ∇f using IA and automatic differentiation
- adaptive quadtree



IA

#leaves: 341

level 8



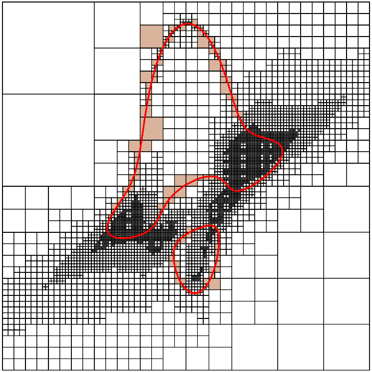
AA

#leaves: 221

Bounding Implicit Curves with Strips on □

Comparing with IA: method proposed by Lopes *et al.* in SIBGRAPI 2001

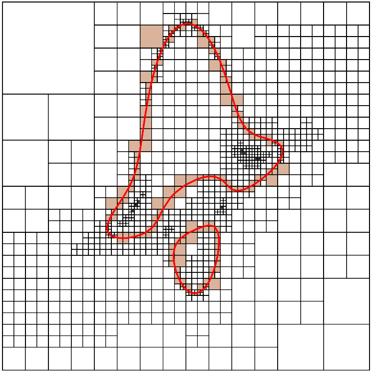
- requires the evaluation ∇f using IA and automatic differentiation
- adaptive quadtree



IA

CPU time: 394 msec

level 8



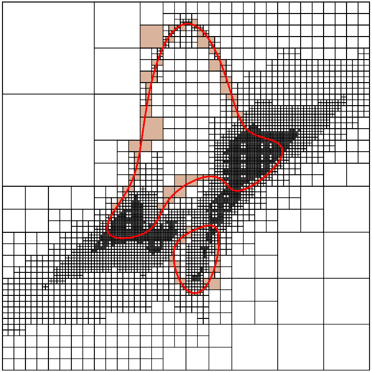
AA

CPU time: 139 msec

Bounding Implicit Curves with Strips on □

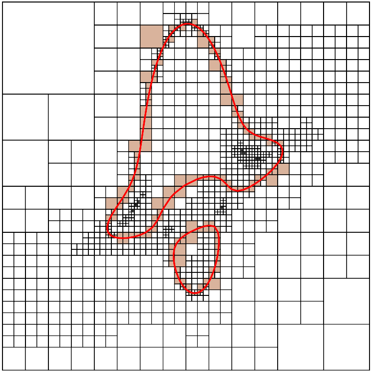
Comparing with IA: method proposed by Lopes *et al.* in SIBGRAPI 2001

- requires the evaluation ∇f using IA and automatic differentiation
- adaptive quadtree



IA
linear convergence

level 8



AA
quadratic convergence

Bounding Implicit Curves with Strips on $\Diamond$

On parallelograms:

Bounding Implicit Curves with Strips on $\Diamond$

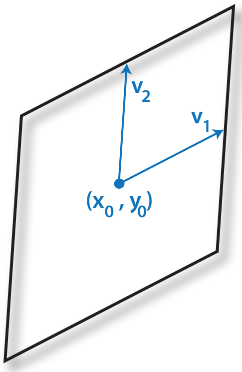
On parallelograms:

evaluate $f(\Diamond)$ with AA $\Rightarrow$ write ε_1 and ε_2 in terms of x and y

Bounding Implicit Curves with Strips on $\lozenge$

On parallelograms:

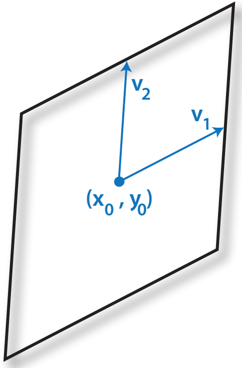
evaluate $f(\lozenge)$ with AA $\Rightarrow$ write ε_1 and ε_2 in terms of x and y



Bounding Implicit Curves with Strips on $\Diamond$

On parallelograms:

evaluate $f(\Diamond)$ with AA $\Rightarrow$ write ε_1 and ε_2 in terms of x and y

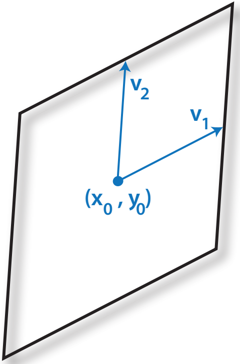


$$v_1 = (x_1, y_1) \qquad v_2 = (x_2, y_2)$$

Bounding Implicit Curves with Strips on $\Diamond$

On parallelograms:

evaluate $f(\Diamond)$ with AA $\Rightarrow$ write ε_1 and ε_2 in terms of x and y



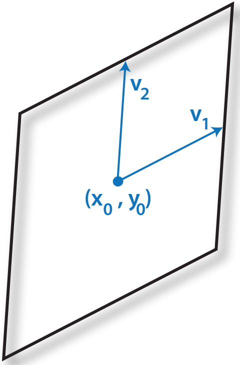
$$v_1 = (x_1, y_1) \qquad v_2 = (x_2, y_2)$$

$$\hat{x} = x_0 + x_1\varepsilon_1 + x_2\varepsilon_2 \qquad \hat{y} = y_0 + y_1\varepsilon_1 + y_2\varepsilon_2$$

Bounding Implicit Curves with Strips on $\Diamond$

On parallelograms:

evaluate $f(\Diamond)$ with AA $\Rightarrow$ write ε_1 and ε_2 in terms of x and y



$$v_1 = (x_1, y_1) \qquad v_2 = (x_2, y_2)$$

$$\hat{x} = x_0 + x_1\varepsilon_1 + x_2\varepsilon_2 \qquad \hat{y} = y_0 + y_1\varepsilon_1 + y_2\varepsilon_2$$

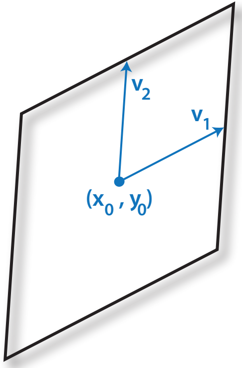
In matrix form

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} \cdot \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix}$$

Bounding Implicit Curves with Strips on $\Diamond$

On parallelograms:

evaluate $f(\Diamond)$ with AA $\Rightarrow$ write ε_1 and ε_2 in terms of x and y



$$v_1 = (x_1, y_1) \qquad v_2 = (x_2, y_2)$$

$$\hat{x} = x_0 + x_1\varepsilon_1 + x_2\varepsilon_2 \qquad \hat{y} = y_0 + y_1\varepsilon_1 + y_2\varepsilon_2$$

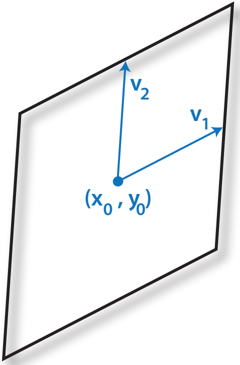
In matrix form

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}^{-1} \cdot \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}$$

Bounding Implicit Curves with Strips on $\Diamond$

On parallelograms:

evaluate $f(\Diamond)$ with AA $\Rightarrow$ write ε_1 and ε_2 in terms of x and y



$$v_1 = (x_1, y_1) \qquad v_2 = (x_2, y_2)$$

$$\hat{x} = x_0 + x_1\varepsilon_1 + x_2\varepsilon_2 \qquad \hat{y} = y_0 + y_1\varepsilon_1 + y_2\varepsilon_2$$

In matrix form

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}^{-1} \cdot \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}$$

the matrix is invertible $\iff$ the parallelogram is not degenerate

Bounding Implicit Curves with Strips on $\triangle$

On triangles:

Bounding Implicit Curves with Strips on $\triangle$

On triangles: replace the evaluation of $f(\triangle) \Rightarrow f(\Diamond)$ with AA

Bounding Implicit Curves with Strips on $\triangle$

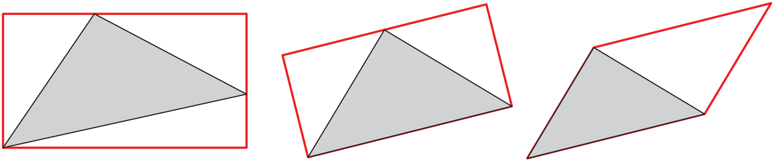
On triangles: replace the evaluation of $f(\triangle) \Rightarrow f(\Diamond)$ with AA

- include a triangle into a parallelogram

Bounding Implicit Curves with Strips on $\triangle$

On triangles: replace the evaluation of $f(\triangle) \Rightarrow f(\Diamond)$ with AA

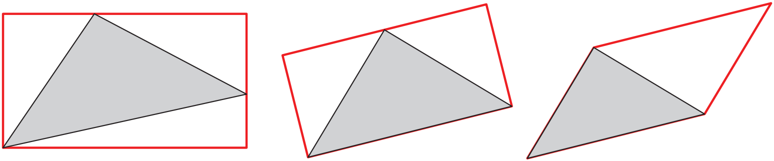
- include a triangle into a parallelogram



Bounding Implicit Curves with Strips on $\triangle$

On triangles: replace the evaluation of $f(\triangle) \Rightarrow f(\Diamond)$ with AA

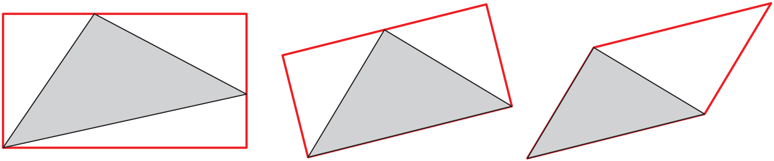
- include a triangle into a parallelogram
  - evaluate f outside of its domain



Bounding Implicit Curves with Strips on $\triangle$

On triangles: replace the evaluation of $f(\triangle) \Rightarrow f(\lozenge)$ with AA

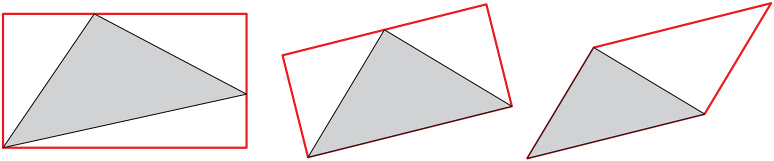
- include a triangle into a parallelogram
  - evaluate f outside of its domain
  - it does not work for surfaces



Bounding Implicit Curves with Strips on $\triangle$

On triangles: replace the evaluation of $f(\triangle) \Rightarrow f(\Diamond)$ with AA

- include a triangle into a parallelogram
  - evaluate f outside of its domain
  - it does not work for surfaces

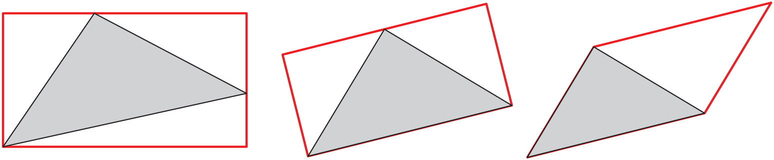


- split a triangle in three parallelograms

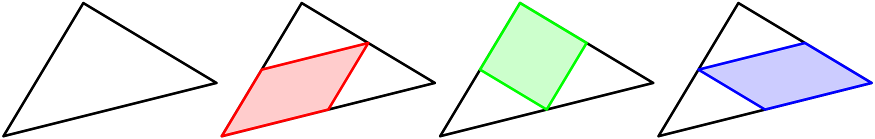
Bounding Implicit Curves with Strips on $\triangle$

On triangles: replace the evaluation of $f(\triangle) \Rightarrow f(\Diamond)$ with AA

- include a triangle into a parallelogram
  - evaluate f outside of its domain
  - it does not work for surfaces



- split a triangle in three parallelograms



Our Adaptive Method

procedure *Explore*($\triangle$)
  $\Diamond_1, \Diamond_2, \Diamond_3 \leftarrow$ *Parallelograms*($\triangle$)
  $\hat{f}_i \leftarrow f(\Diamond_i)$ with AA
  if $0 \in [\hat{f}_i]$ for some i **then**
    $w_i \leftarrow$ width of $\hat{f}$ in $\Diamond_i$
    if $w_i \leq \epsilon_{user}$, for all i **then**
      Approximate($\triangle$)
    else
      $\triangle_i \leftarrow$ *Subdivide*($\triangle$)
      for each i, *Explore*($\triangle_i$)
    end
  end
end

Our Adaptive Method

procedure *Explore*$(\triangle)$

$\Diamond_1, \Diamond_2, \Diamond_3 \leftarrow$ *Parallelograms*$(\triangle)$

$\hat{f}_i \leftarrow f(\Diamond_i)$ with AA

if $0 \in [\hat{f}_i]$ for some i **then**

$w_i \leftarrow$ width of $\hat{f}$ in $\Diamond_i$

if $w_i \leq \epsilon_{user}$, for all i **then**

Approximate$(\triangle)$

else

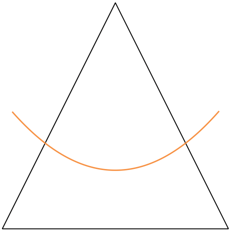
$\triangle_i \leftarrow$ *Subdivide*$(\triangle)$

for each i, *Explore*$(\triangle_i)$

end

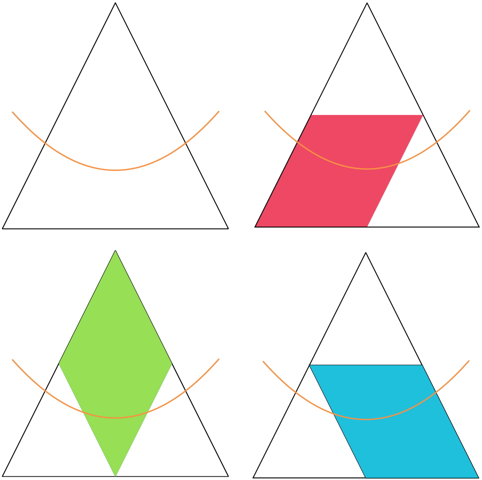
end

end



Our Adaptive Method

procedure $\mathit{Explore}(\triangle)$
  $\lozenge_1, \lozenge_2, \lozenge_3 \leftarrow \mathit{Parallelograms}(\triangle)$
  $\hat{f}_i \leftarrow f(\lozenge_i)$ with AA
  if $0 \in [\hat{f}_i]$ for some i **then**
    $w_i \leftarrow$ width of $\hat{f}$ in $\lozenge_i$
    if $w_i \leq \epsilon_{user}$, for all i **then**
      $\mathit{Approximate}(\triangle)$
    else
      $\triangle_i \leftarrow \mathit{Subdivide}(\triangle)$
      for each i, $\mathit{Explore}(\triangle_i)$
    end
  end
end



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procedure $\mathit{Explore}(\triangle)$
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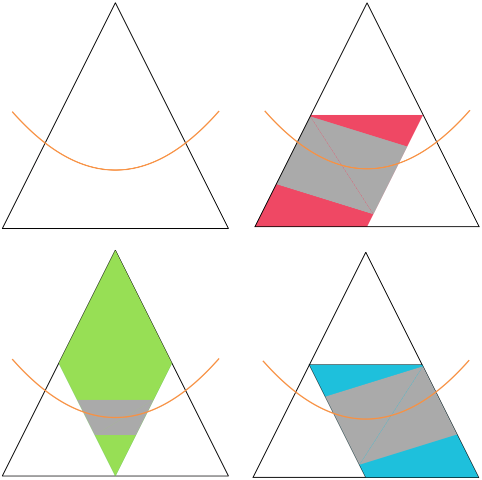
    if $w_i \leq \epsilon_{user}$, for all i **then**
      $\mathit{Approximate}(\triangle)$
    else
      $\triangle_i \leftarrow \mathit{Subdivide}(\triangle)$
      for each i, $\mathit{Explore}(\triangle_i)$
    end
  end
end



Our Adaptive Method

procedure *Explore*$(\triangle)$

$\Diamond_1, \Diamond_2, \Diamond_3 \leftarrow$ *Parallelograms*$(\triangle)$

$\hat{f}_i \leftarrow f(\Diamond_i)$ with AA

if $0 \in [\hat{f}_i]$ for some i **then**

$w_i \leftarrow$ width of $\hat{f}$ in $\Diamond_i$

if $w_i \leq \epsilon_{user}$, for all i **then**

Approximate$(\triangle)$

else

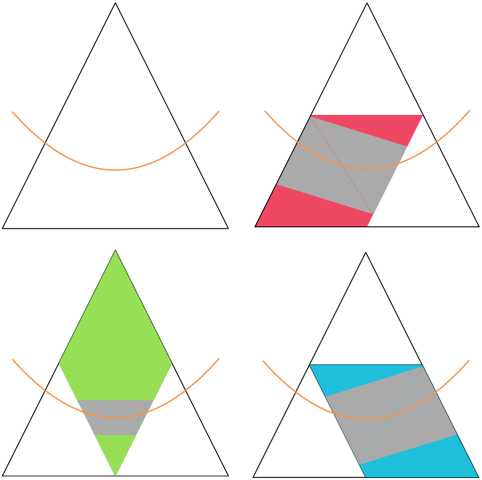
$\triangle_i \leftarrow$ *Subdivide*$(\triangle)$

for each i, *Explore*$(\triangle_i)$

end

end

end



Our Adaptive Method

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$\Diamond_1, \Diamond_2, \Diamond_3 \leftarrow$ *Parallelograms*$(\triangle)$

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if $w_i \leq \epsilon_{user}$, for all i **then**

Approximate$(\triangle)$

else

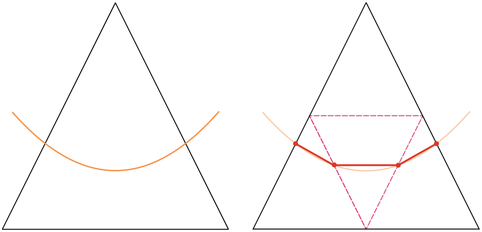
$\triangle_i \leftarrow$ *Subdivide*$(\triangle)$

for each i, *Explore*$(\triangle_i)$

end

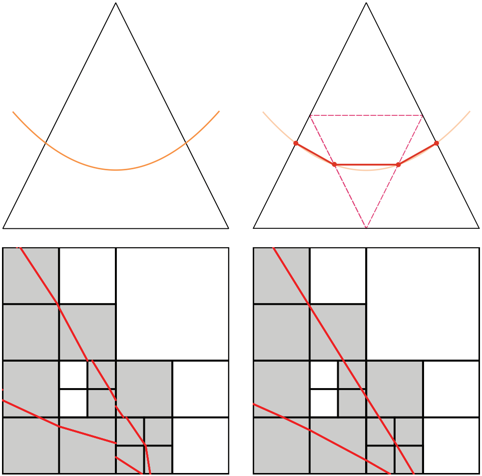
end

end



Our Adaptive Method

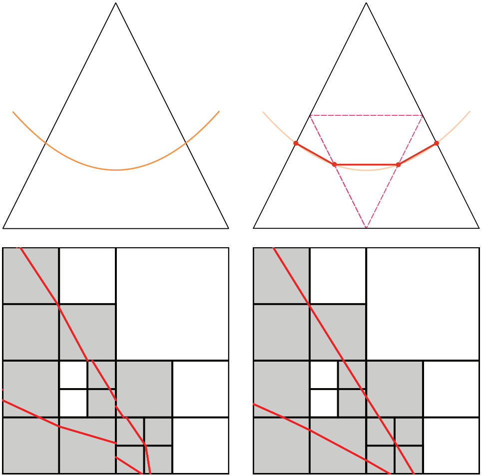
procedure $\mathit{Explore}(\triangle)$
    $\lozenge_1, \lozenge_2, \lozenge_3 \leftarrow \mathit{Parallelograms}(\triangle)$
    $\hat{f}_i \leftarrow f(\lozenge_i)$ with AA
    if $0 \in [\hat{f}_i]$ for some i **then**
        $w_i \leftarrow$ width of $\hat{f}$ in $\lozenge_i$
        if $w_i \leq \epsilon_{user}$, for all i **then**
            $\mathit{Approximate}(\triangle)$
        else
            $\triangle_i \leftarrow \mathit{Subdivide}(\triangle)$
            for each i, $\mathit{Explore}(\triangle_i)$
        end
    end
end



linear interpolation   bissection method

Our Adaptive Method

procedure $\mathit{Explore}(\triangle)$
  $\lozenge_1, \lozenge_2, \lozenge_3 \leftarrow \mathit{Parallelograms}(\triangle)$
  $\hat{f}_i \leftarrow f(\lozenge_i)$ with AA
  if $0 \in [\hat{f}_i]$ for some i **then**
    $w_i \leftarrow$ width of $\hat{f}$ in $\lozenge_i$
    if $w_i \leq \epsilon_{user}$, for all i **then**
      $\mathit{Approximate}(\triangle)$
    else
      $\triangle_i \leftarrow \mathit{Subdivide}(\triangle)$
      for each i, $\mathit{Explore}(\triangle_i)$
    end
  end
end



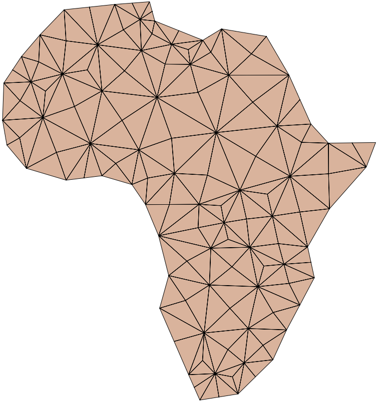
linear interpolation   bissection method

Our Adaptive Method

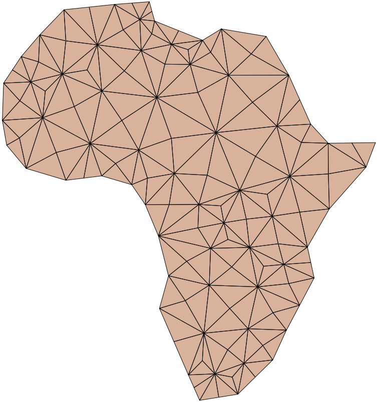
Our method does not care what mesh subdivision method is used

Our Adaptive Method

Our method does not care what mesh subdivision method is used



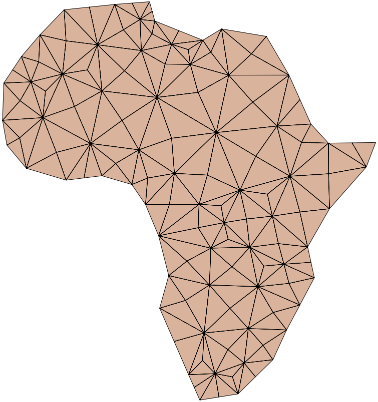
triangle soup



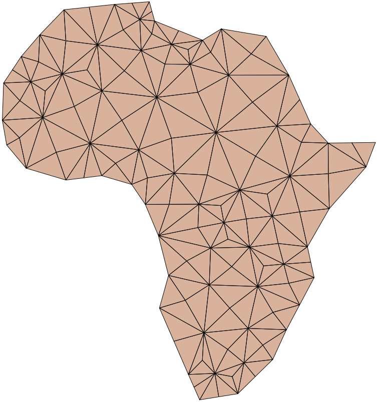
mesh with connecticity

Our Adaptive Method

Our method does not care what mesh subdivision method is used



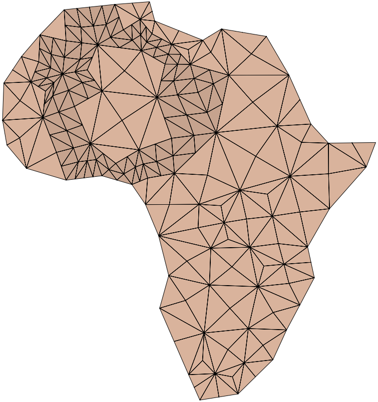
triangle soup
midpoint splitting



mesh with connecticity
$\sqrt{3}$, J_1^a, 4-8 meshes, ...

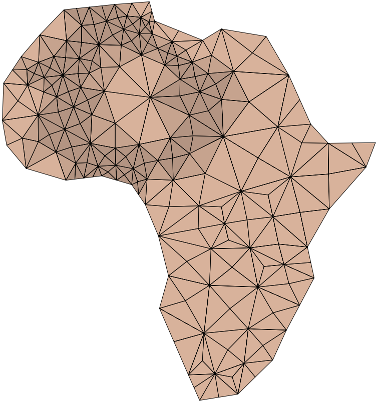
Our Adaptive Method

Our method does not care what mesh subdivision method is used



triangle soup

$\#\triangle = 193$

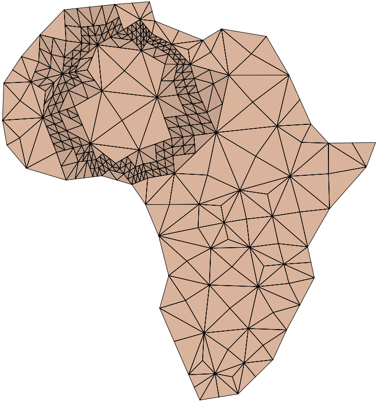


mesh with connecticity

$\#\triangle = 193$

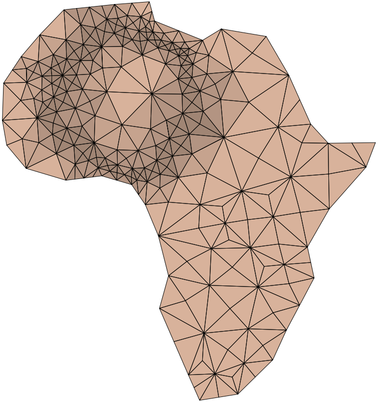
Our Adaptive Method

Our method does not care what mesh subdivision method is used



triangle soup

$\#\triangle = 307$

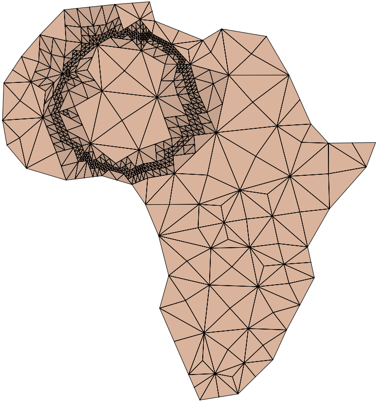


mesh with connecticity

$\#\triangle = 325$

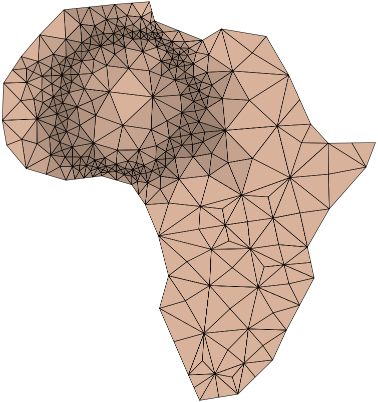
Our Adaptive Method

Our method does not care what mesh subdivision method is used



triangle soup

$\#\triangle = 512$

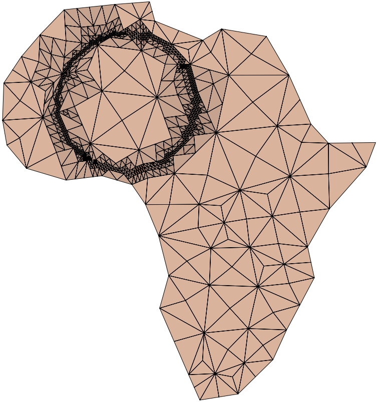


mesh with connecticity

$\#\triangle = 427$

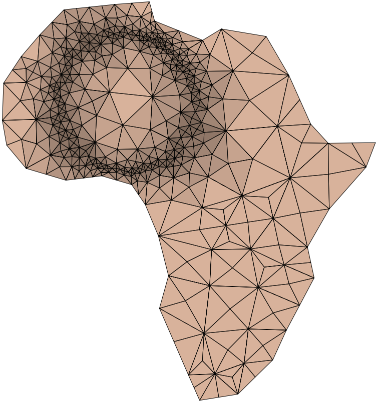
Our Adaptive Method

Our method does not care what mesh subdivision method is used



triangle soup

$\#\triangle = 922$

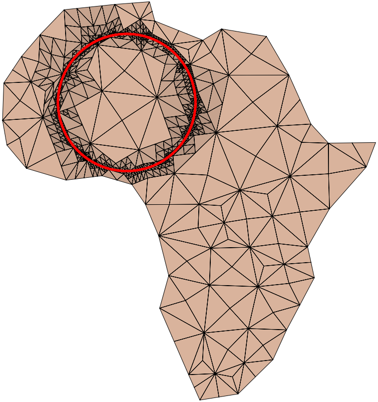


mesh with connecticity

$\#\triangle = 574$

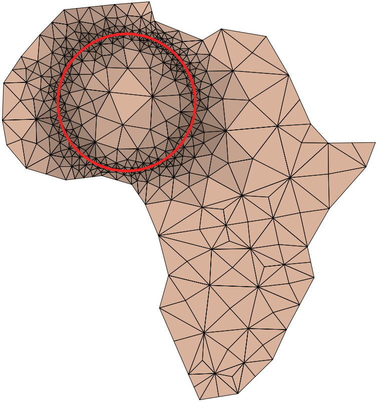
Our Adaptive Method

Our method does not care what mesh subdivision method is used



triangle soup

$\#\triangle = 1384$

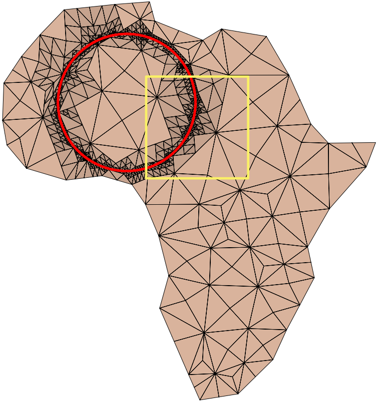


mesh with connecticity

$\#\triangle = 779$

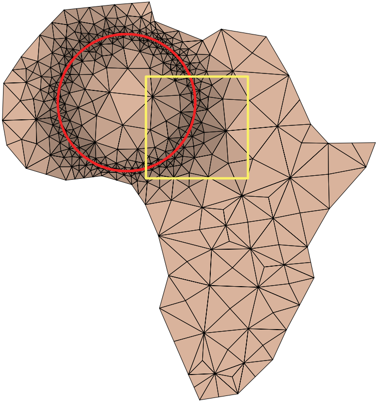
Our Adaptive Method

Our method does not care what mesh subdivision method is used



triangle soup

$\# \triangle = 1384$

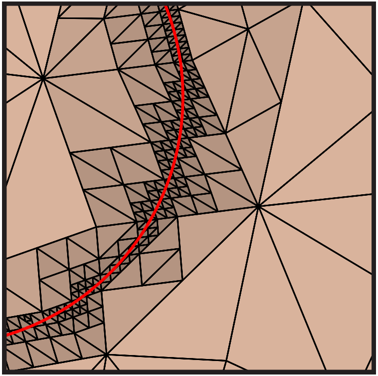


mesh with connecticity

$\# \triangle = 779$

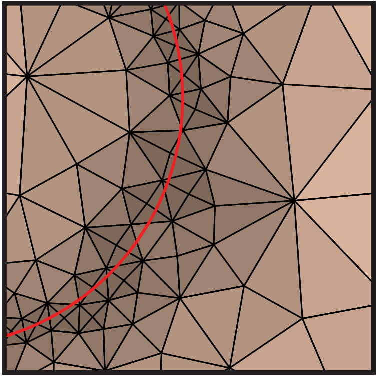
Our Adaptive Method

Our method does not care what mesh subdivision method is used



triangle soup

$\#\triangle = 1384$

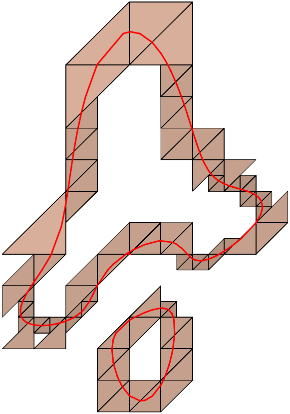


mesh with connecticity

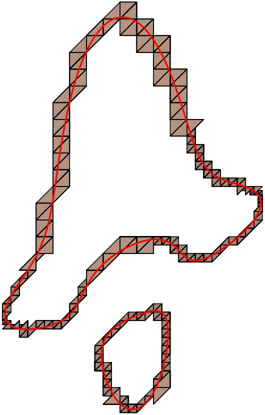
$\#\triangle = 779$

Our Adaptive Method

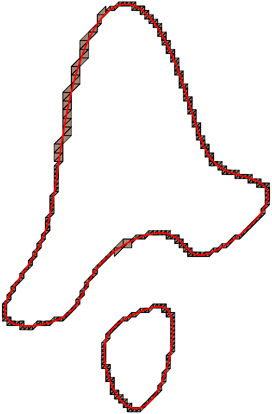
The effect of the geometric criteria on the curve in a triangular quadtree



$\epsilon_{user} = 1$

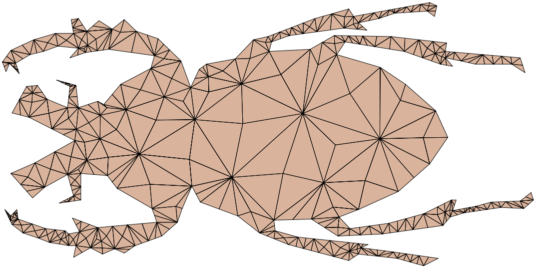


$\epsilon_{user} = 0.1$



$\epsilon_{user} = 0.01$

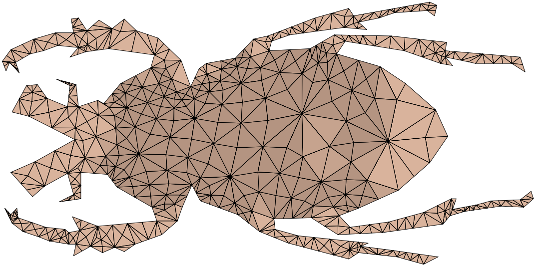
Results



level 0

$$(x+1)^3(1-x)-4y^4=0$$

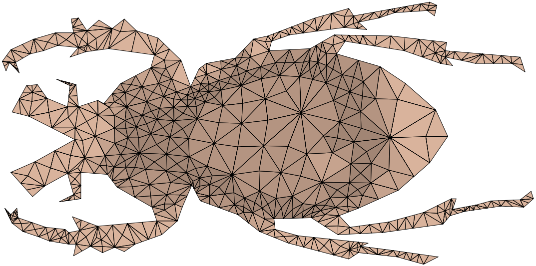
Results



level 1

$$(x+1)^3(1-x) - 4y^4 = 0$$

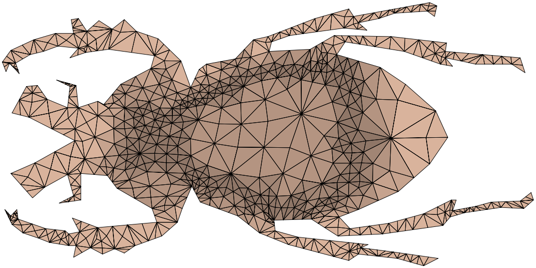
Results



level 2

$$(x+1)^3(1-x)-4y^4=0$$

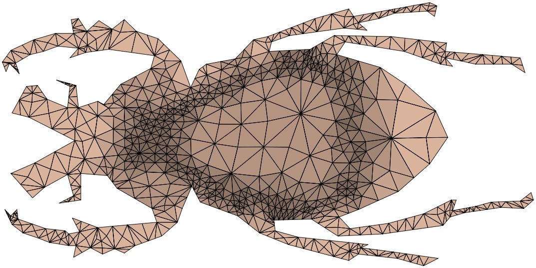
Results



level 3

$$(x+1)^3(1-x) - 4y^4 = 0$$

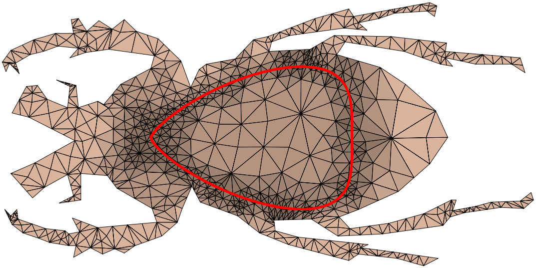
Results



level 4

$$(x+1)^3(1-x) - 4y^4 = 0$$

Results



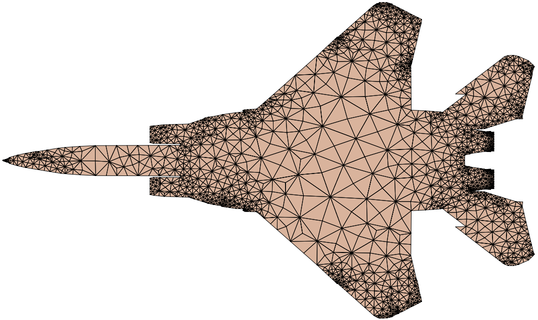
level 4

$\#\triangle_{in} = 940$
$\#\triangle_{out} = 1771$

CPU time = 280 msec

$$(x+1)^3(1-x) - 4y^4 = 0$$

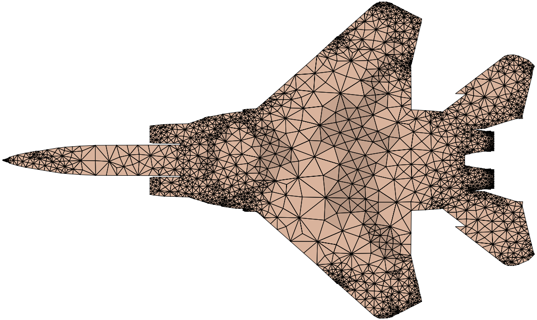
Results



level 0

$$x^3 + x - y^2 = 0$$

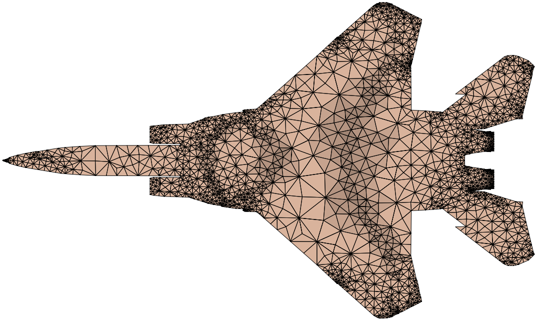
Results



level 1

$$x^3 + x - y^2 = 0$$

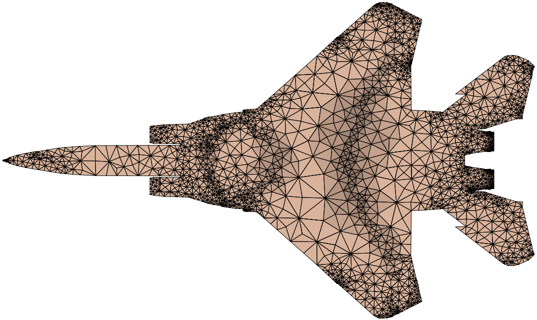
Results



level 2

$$x^3 + x - y^2 = 0$$

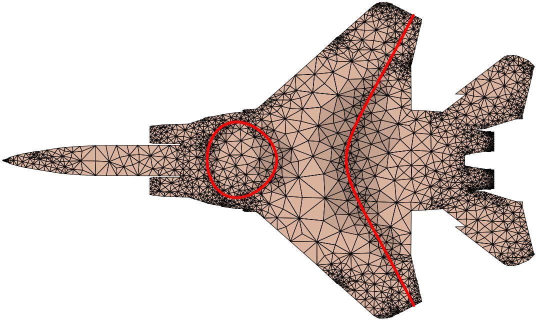
Results



level 3

$$x^3 + x - y^2 = 0$$

Results



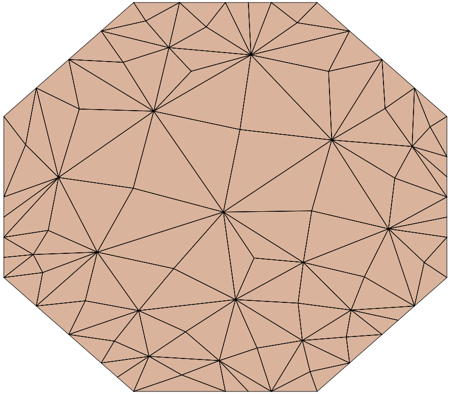
level 3

$\#\triangle_{in} = 3530$
$\#\triangle_{out} = 4142$

CPU time = 333 msec

$x^3 + x - y^2 = 0$

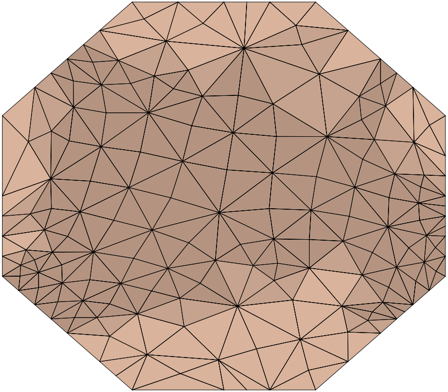
Results



level 0

$$y^2(0.75^2 - x^2) - (x^2 + 1.5y - 0.75^2)^2 = 0$$

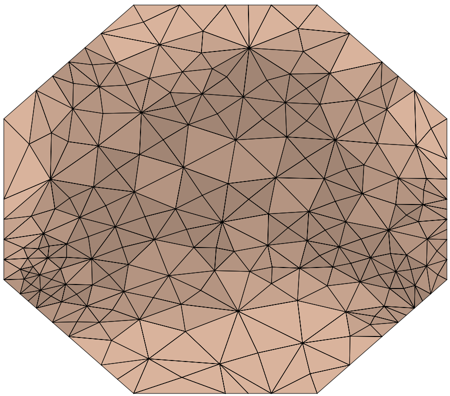
Results



level 1

$$y^2(0.75^2 - x^2) - (x^2 + 1.5y - 0.75^2)^2 = 0$$

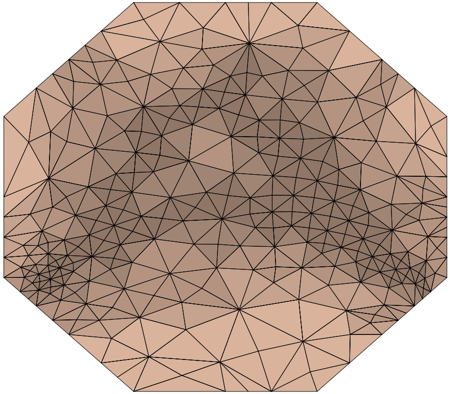
Results



level 2

$$y^2(0.75^2 - x^2) - (x^2 + 1.5y - 0.75^2)^2 = 0$$

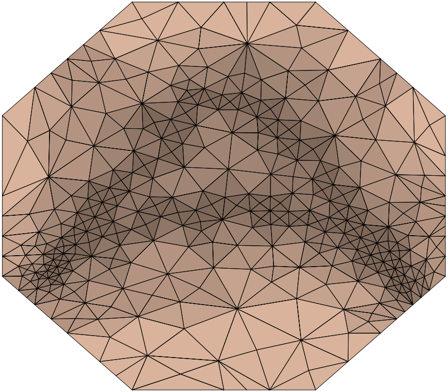
Results



level 3

$$y^2(0.75^2 - x^2) - (x^2 + 1.5y - 0.75^2)^2 = 0$$

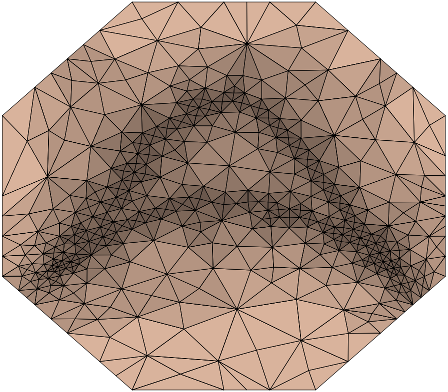
Results



level 4

$$y^2(0.75^2 - x^2) - (x^2 + 1.5y - 0.75^2)^2 = 0$$

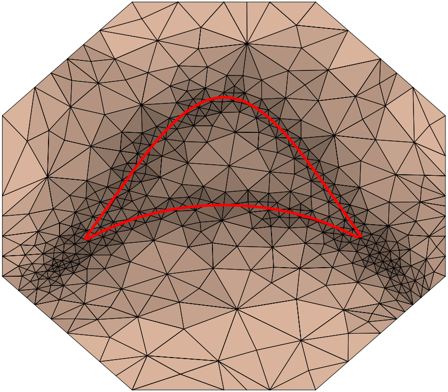
Results



level 5

$$y^2(0.75^2 - x^2) - (x^2 + 1.5y - 0.75^2)^2 = 0$$

Results



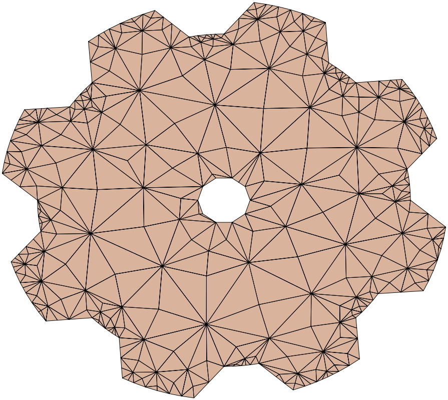
level 5

$\#\triangle_{in} = 126$
$\#\triangle_{out} = 1168$

CPU time = 123 msec

$$y^2(0.75^2 - x^2) - (x^2 + 1.5y - 0.75^2)^2 = 0$$

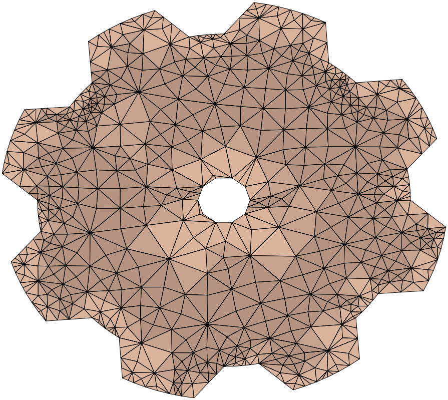
Results



level 0

$$(x^2 + y^2 - 1)^3 - x^2y^3 = 0$$

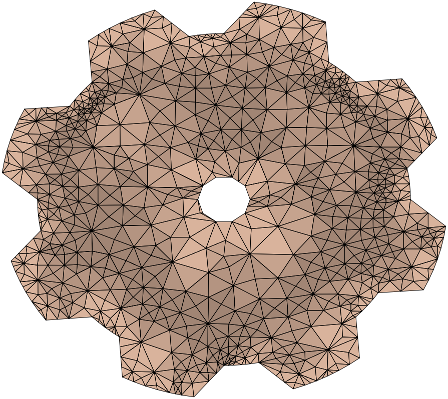
Results



level 1

$$(x^2 + y^2 - 1)^3 - x^2 y^3 = 0$$

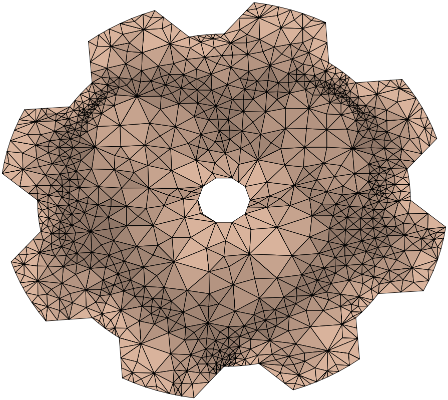
Results



level 2

$$(x^2 + y^2 - 1)^3 - x^2y^3 = 0$$

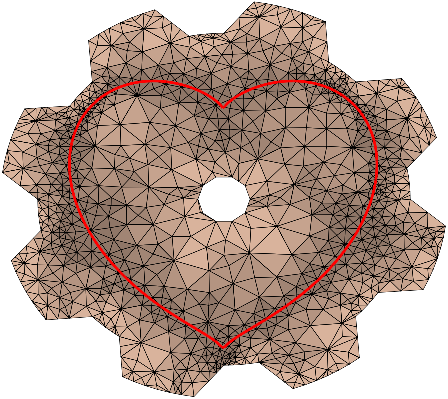
Results



level 3

$$(x^2 + y^2 - 1)^3 - x^2y^3 = 0$$

Results



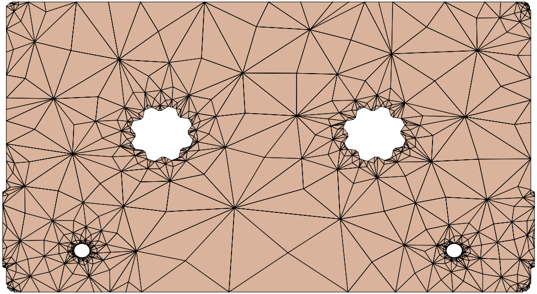
$$(x^2 + y^2 - 1)^3 - x^2y^3 = 0$$

level 3

$\#\triangle_{in} = 1424$
$\#\triangle_{out} = 3298$

CPU time = 547 msec

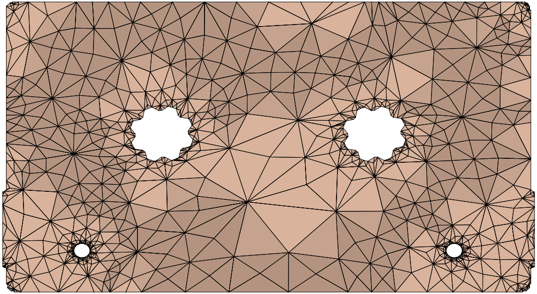
Results



level 0

$$(y - x^2 + 1)^4 + (x^2 + y^2)^4 - 1 = 0$$

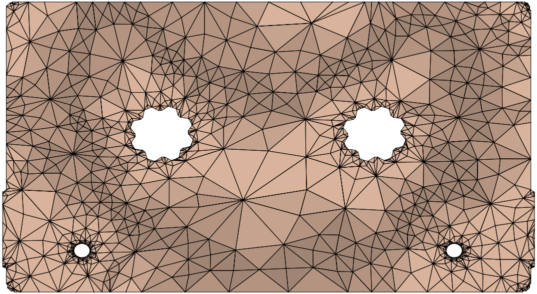
Results



level 1

$$(y - x^2 + 1)^4 + (x^2 + y^2)^4 - 1 = 0$$

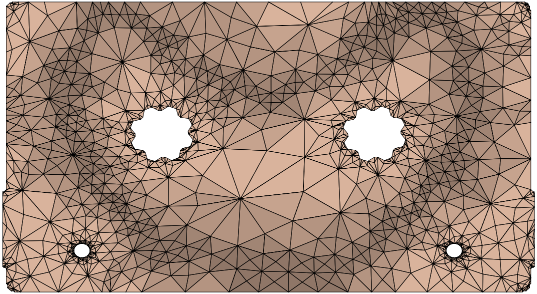
Results



level 2

$$(y - x^2 + 1)^4 + (x^2 + y^2)^4 - 1 = 0$$

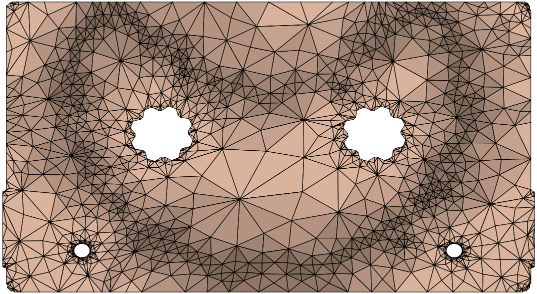
Results



level 3

$$(y - x^2 + 1)^4 + (x^2 + y^2)^4 - 1 = 0$$

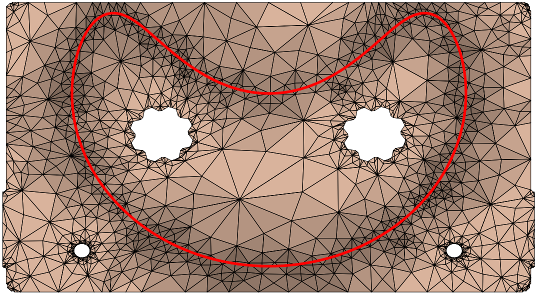
Results



level 4

$$(y - x^2 + 1)^4 + (x^2 + y^2)^4 - 1 = 0$$

Results



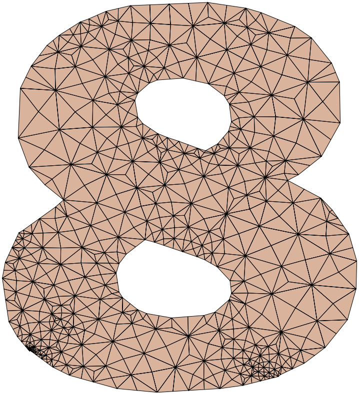
level 4

$\#\triangle_{in} = 1006$
$\#\triangle_{out} = 2134$

CPU time = 391 msec

$$(y - x^2 + 1)^4 + (x^2 + y^2)^4 - 1 = 0$$

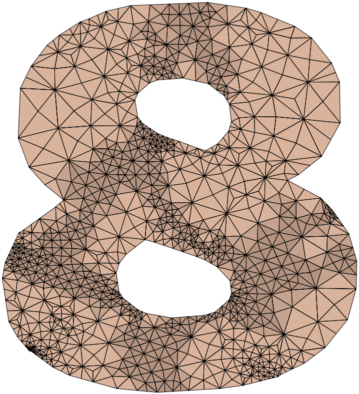
Results



level 0

$$(xy + \cos(x+y))(xy + \sin(x+y)) = 0$$

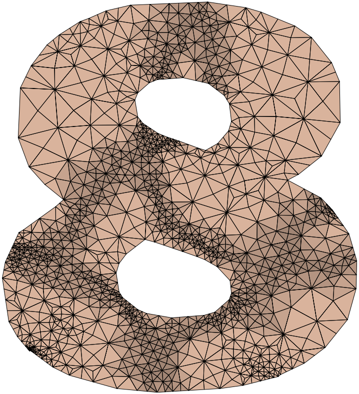
Results



level 1

$$(xy + \cos(x + y))(xy + \sin(x + y)) = 0$$

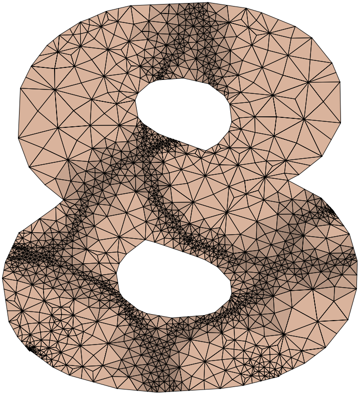
Results



level 2

$$(xy + \cos(x + y))(xy + \sin(x + y)) = 0$$

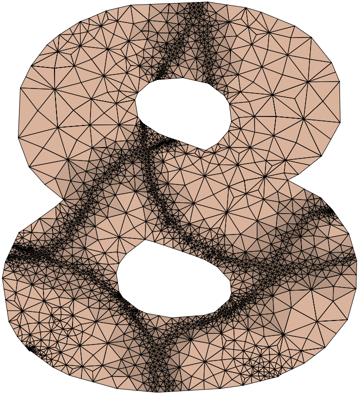
Results



level 3

$$(xy + \cos(x+y))(xy + \sin(x+y)) = 0$$

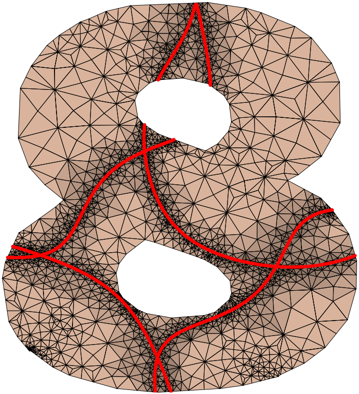
Results



level 4

$$(xy + \cos(x + y))(xy + \sin(x + y)) = 0$$

Results



level 4

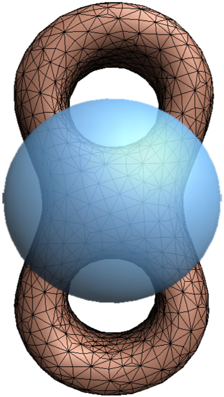
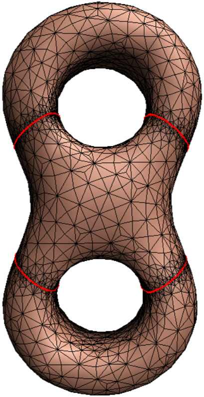
$\#\triangle_{in} = 1032$
$\#\triangle_{out} = 3897$

CPU time $= 454$ msec

$$(xy + \cos(x+y))(xy + \sin(x+y)) = 0$$

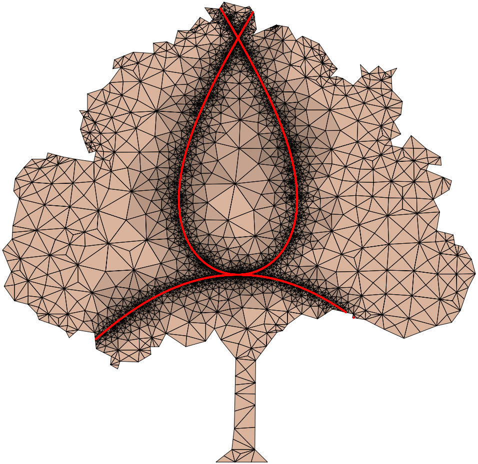
Work in Progress

Implicit curves on surfaces

curve given implicitly by $x^2 + y^2 + z^2 = 1$ on bitorus mesh

Approximating Implicit Curves on Triangulations with Affine Arithmetic



Thanks!