Oktobermat

Images of Julia sets that you can trust

Luiz Henrique de Figueiredo

impa 60 years 1952 - 2012

with

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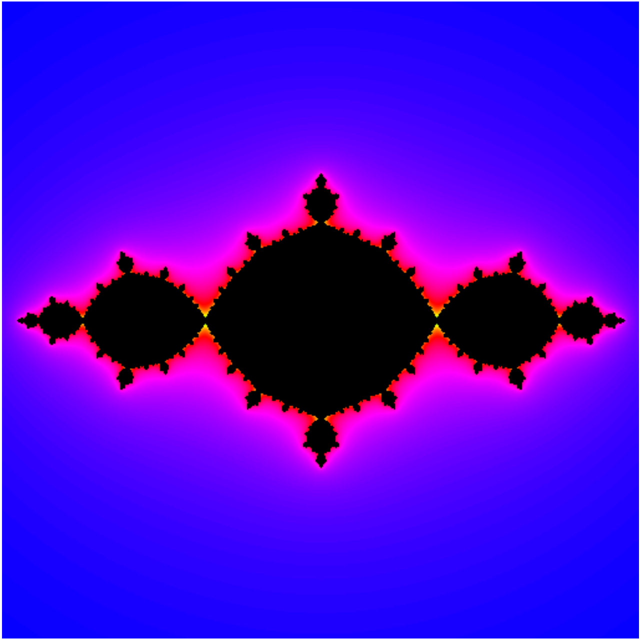
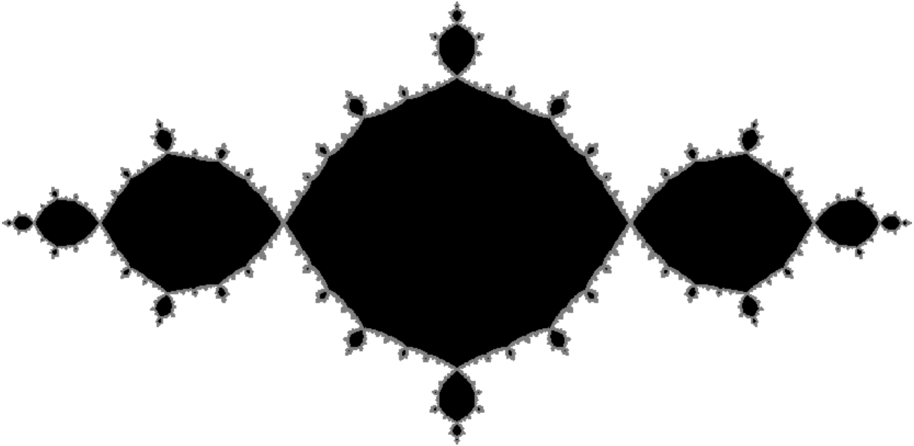
Can we trust this beautiful image?



Curtis McMullen

Julia sets

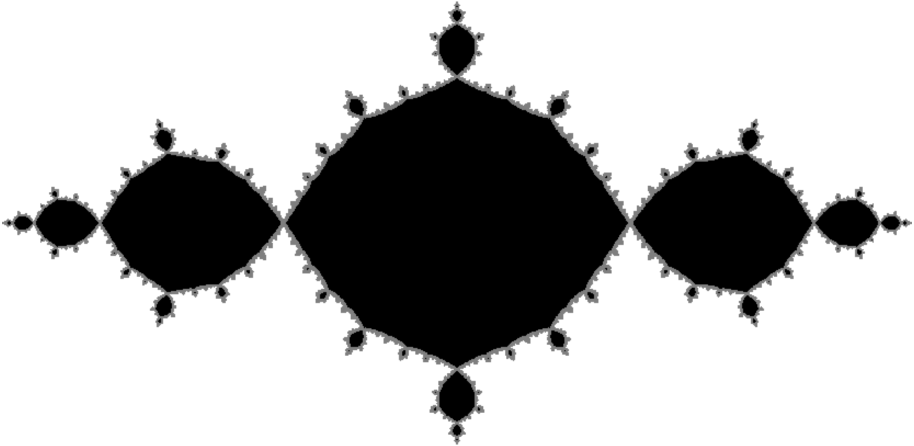


Study the dynamics of $f(z) = z^2 + c$ for $c \in \mathbb{C}$ fixed

$$z_1 = f(z_0), \quad z_2 = f(z_1), \quad \ldots, \quad z_n = f(z_{n-1}) = f^n(z_0)$$

What happens with the orbit of $z_0 \in \mathbb{C}$ under f ?

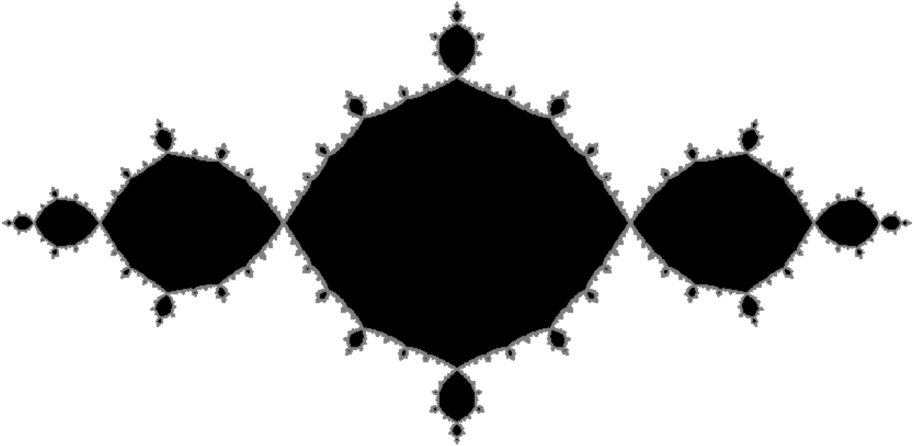
Julia sets



☐ unbounded orbits

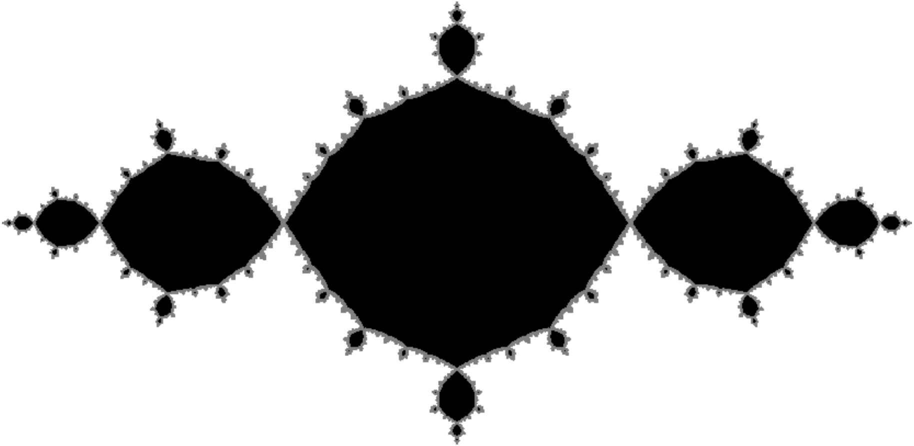
■ bounded orbits

Julia sets



□	unbounded orbits	attraction basin of ∞	$A(\infty)$
■	bounded orbits		

Julia sets



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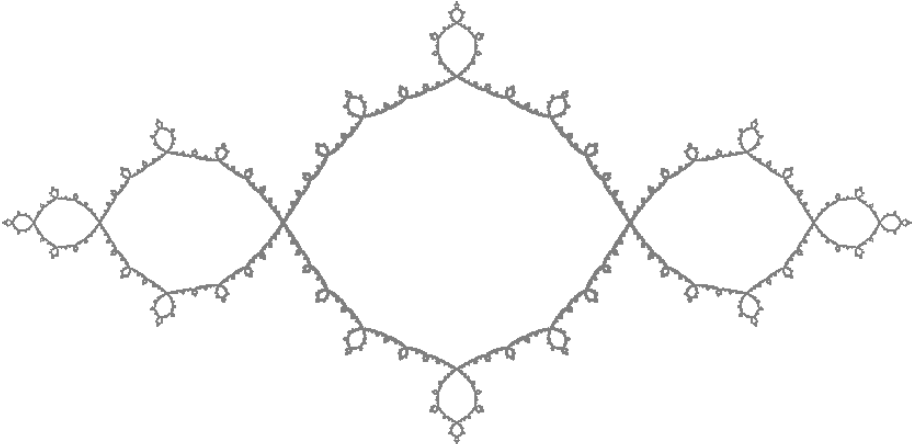
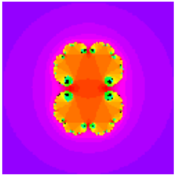
Julia sets

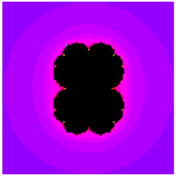


unbounded orbits	attraction basin of ∞	$A(\infty)$
bounded orbits	filled Julia set	K
common boundary	Julia set	J

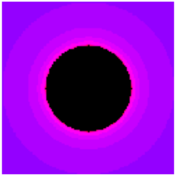
Julia set zoo



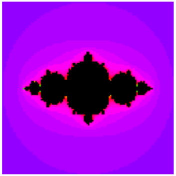
$c = 0.275$



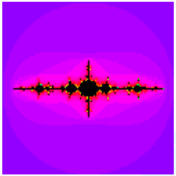
$c = \frac{1}{4}$



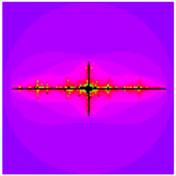
$c = 0$



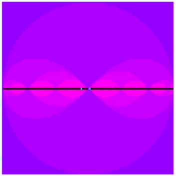
$c = -\frac{3}{4}$



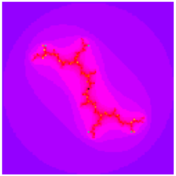
$c = -1.312$



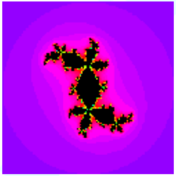
$c = -1.375$



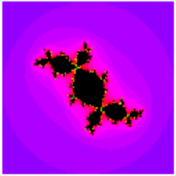
$c = -2$



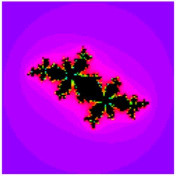
$c = i$



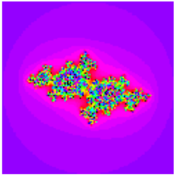
$c = (+0.285, +0.535)$



$c = (-0.125, +0.750)$



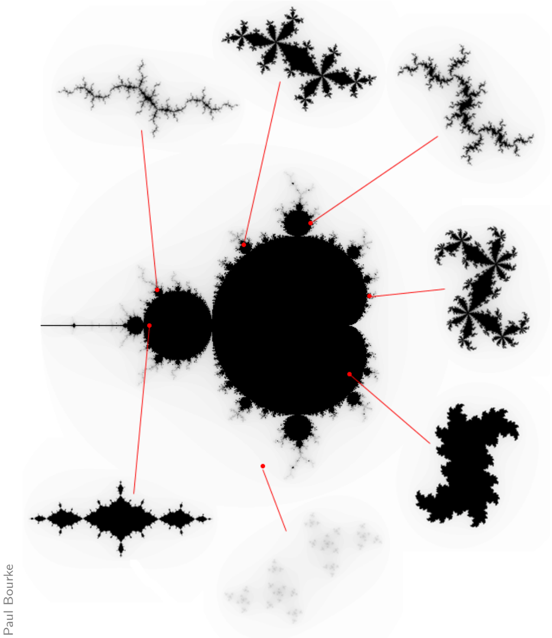
$c = (-0.500, +0.563)$



$c = (-0.687, +0.312)$

Glenn Elert

Julia set catalog: the Mandelbrot set



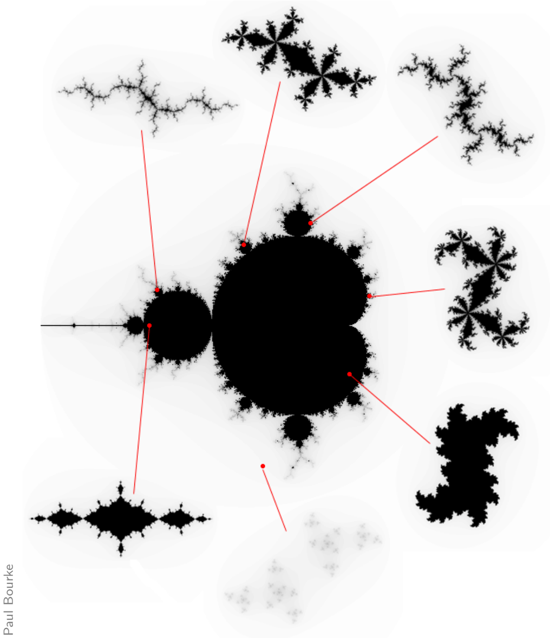
$$c \in \mathcal{M} \ := \ 0 \in K_c$$

Julia–Fatou dichotomy

$c \in \mathcal{M} \Rightarrow J_c$ is connected

$c \notin \mathcal{M} \Rightarrow J_c$ is a Cantor set

Julia set catalog: the Mandelbrot set demo. . .



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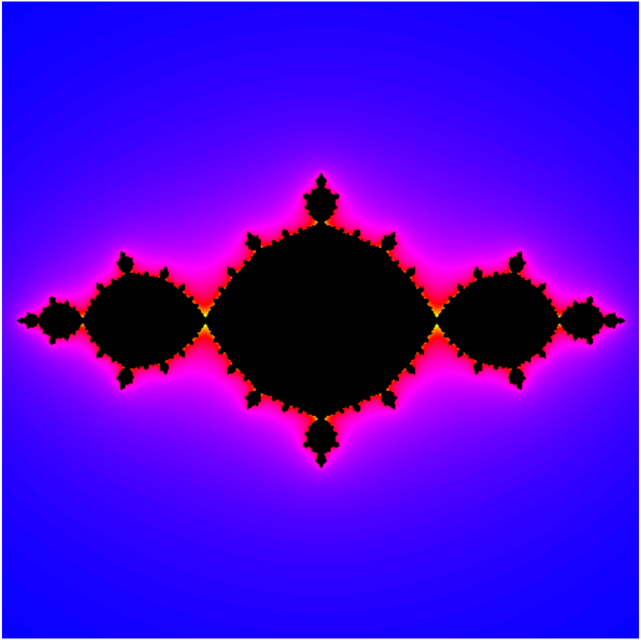

Curtis McMullen

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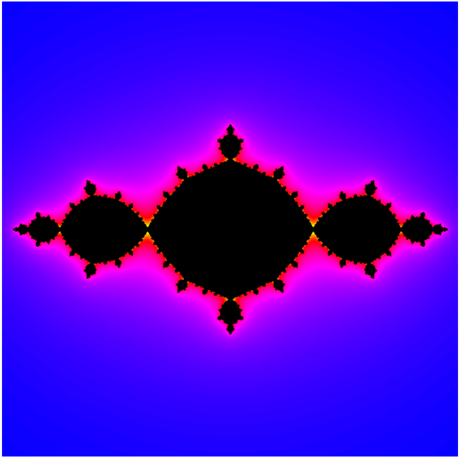

Curtis McMullen

Escape-time algorithm

for z_0 in a grid of points in Ω

$z \leftarrow z_0$

$n \leftarrow 0$

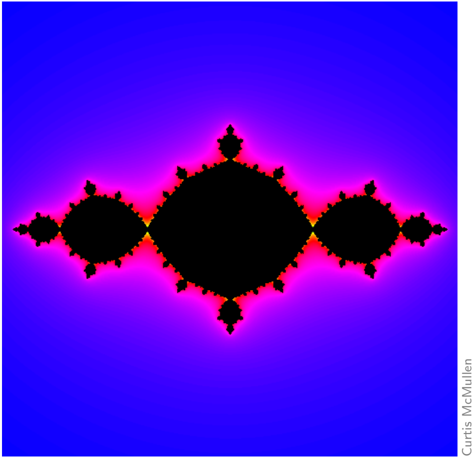
while $|z| \leq R$ and $n \leq N$ do

$z \leftarrow z^2 + c$

$n \leftarrow n + 1$

paint z_0 with color n

Why distrust this beautiful image?



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escape radius

$R = \max(|c|, 2)$ $J \subset B(0, R)$

Escape radius

Lemma. If $z \in \mathbb{C}$ and $|z| > R = \max(|c|, 2) \Rightarrow |f^n(z)| \to \infty$ as $n \to \infty$.

Proof. The triangle inequality gives

$$|z^2| = |z^2 + c - c| \leq |z^2 + c| + |c|$$

and so

$$|f(z)| = |z^2 + c| \geq |z^2| - |c| = |z|^2 - |c| > |z|^2 - |z| = |z|(|z| - 1) > |z| > R$$

Iterating, we get $|f^n(z)| > |z|(|z| - 1)^n \to \infty$ because $|z| - 1 > 1$. $\square$

Corollary. Every unbounded orbit escapes to ∞. $A(\infty)$

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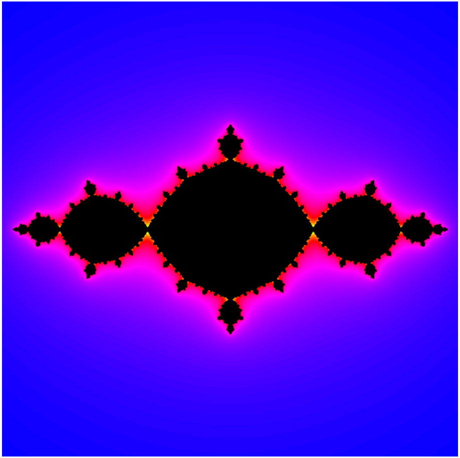

Curtis McMullen

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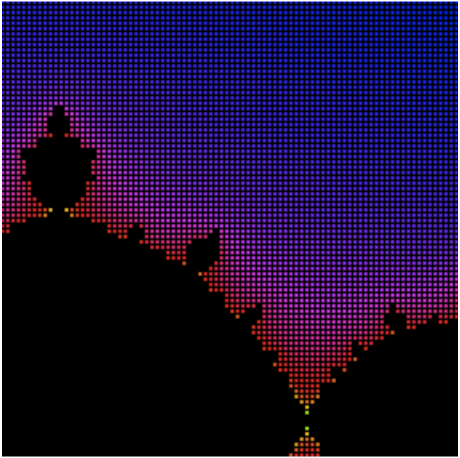

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Why distrust this beautiful image?

- Spatial sampling
  need fine grid
  what happens between samples?

Escape-time algorithm

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Why distrust this beautiful image?

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- Partial orbits
  program cannot run forever

Escape-time algorithm

for z_0 in a grid of points in Ω
    $z \leftarrow z_0$
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Why distrust this beautiful image?

- Spatial sampling
- Partial orbits
- Floating-point rounding errors
  squaring needs double digits

Escape-time algorithm

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Why distrust this beautiful image?

- Spatial sampling
  Compute color on grid points
  Cannot be sure grid is fine enough
  Cannot be sure behavior at sample points is typical
  Finer grid $\Rightarrow$ more detail
- Partial orbits
  Can only compute partial orbits
  Cannot be sure partial orbits are long enough
  Longer orbits $\Rightarrow$ more detail
- Floating-point errors
  z^2 needs twice the number of digits that z needs
  Do rounding errors during iteration influence classification of points?
  Multiple-precision $\Rightarrow$ more detail (deep zoom)

You can trust our method

- No spatial sampling
- No orbits
- No floating-point errors

You can trust our method

- No spatial sampling
  Classify entire rectangles in the complex plane
  Spatial resolution limited by available memory
  Deeper quadtree $\Rightarrow$ more detail

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You can trust our method

- No spatial sampling
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- No orbits
  Evaluate f once on each cell using interval arithmetic
  Perform no function iteration at all
  Use cell mapping and color propagation in graphs

- No floating-point errors

You can trust our method

- No spatial sampling
  Classify entire rectangles in the complex plane
  Spatial resolution limited by available memory
  Deeper quadtree $\Rightarrow$ more detail

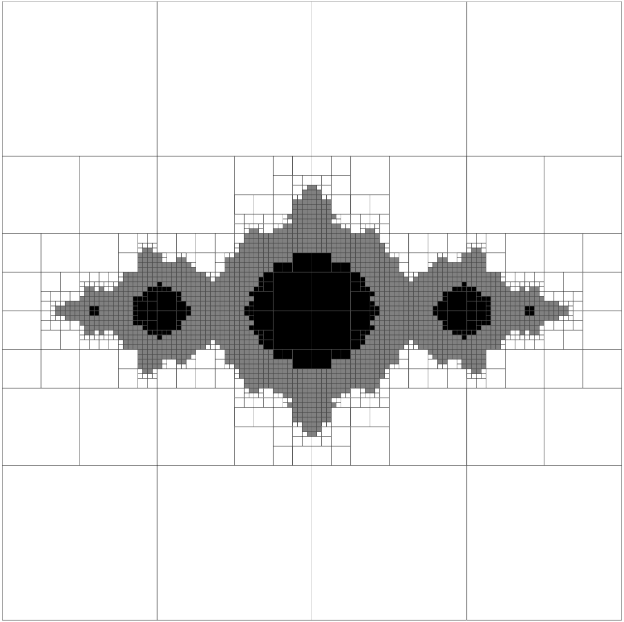
- No orbits
  Evaluate f once on each cell using interval arithmetic
  Perform no function iteration at all
  Use cell mapping and color propagation in graphs

- No floating-point errors
  All numbers are dyadic fractions with restricted range and precision
  Use error-free fixed-point arithmetic
  Precision depends only on spatial resolution
  Standard double-precision floating-point enough for huge images

Our algorithm

quadtree for
$\Omega = [-R, R] \times [-R, R]$

- white rectangles contained in $A(\infty)$
- black rectangles contained in K
- gray rectangles contain J



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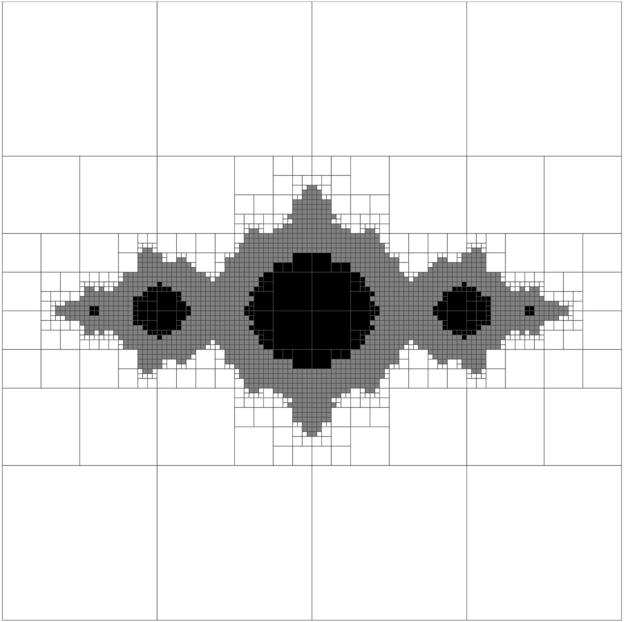
certified decomposition



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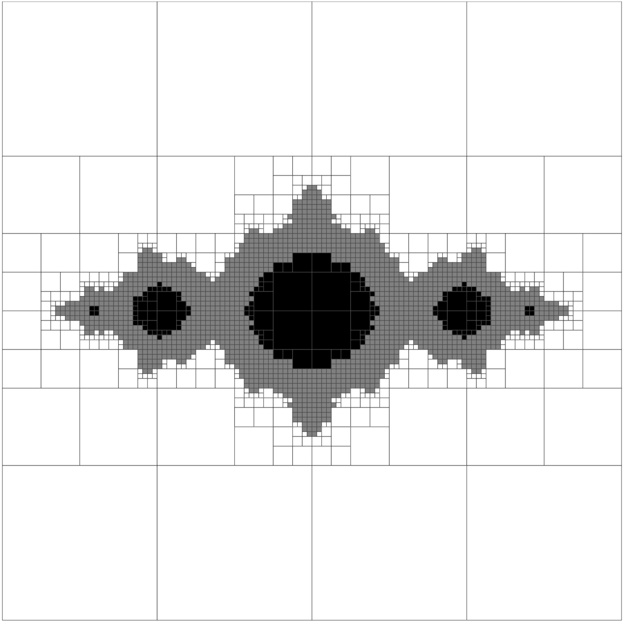
- refinement
- cell mapping
- color propagation



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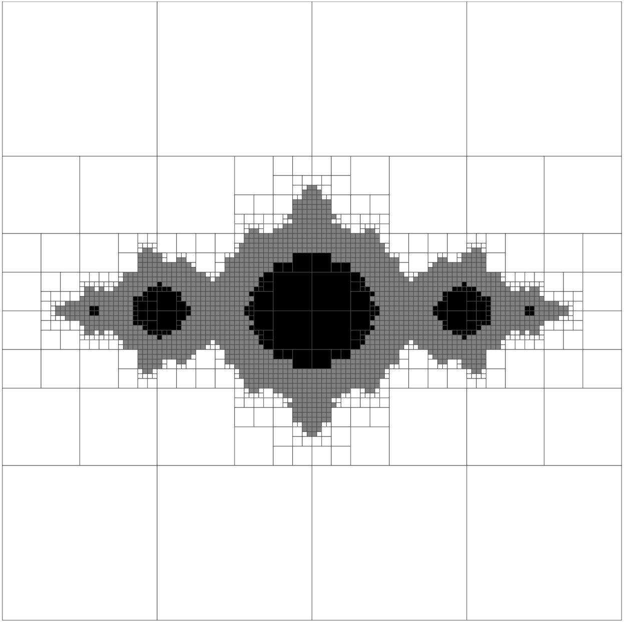
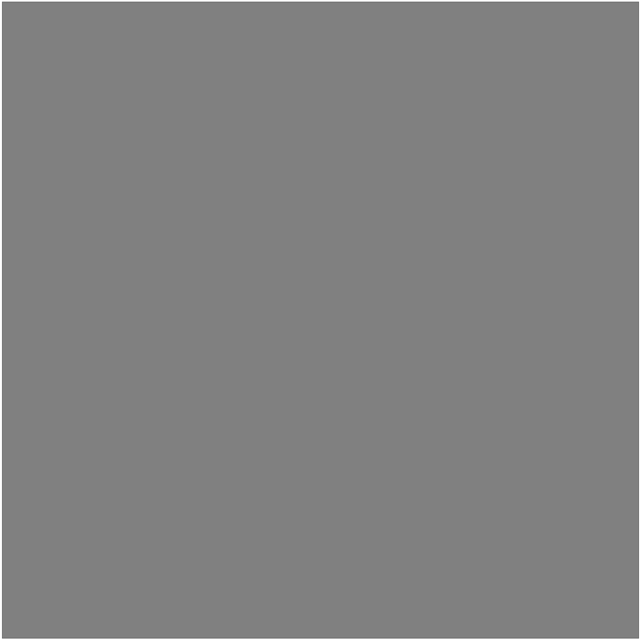
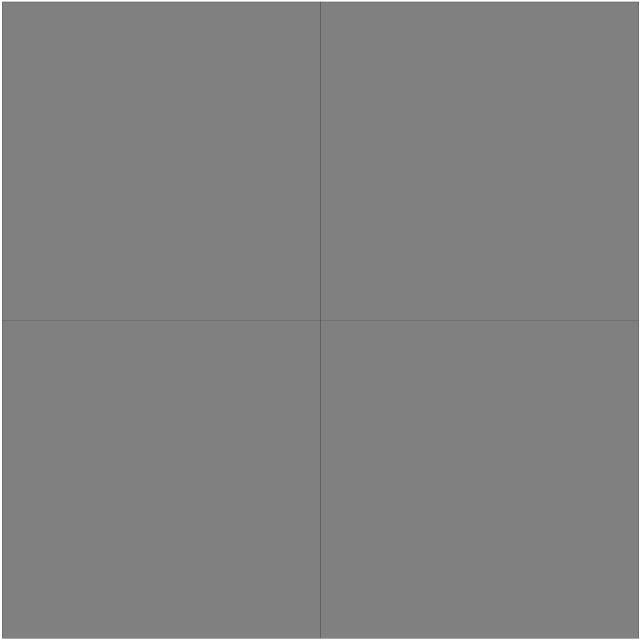


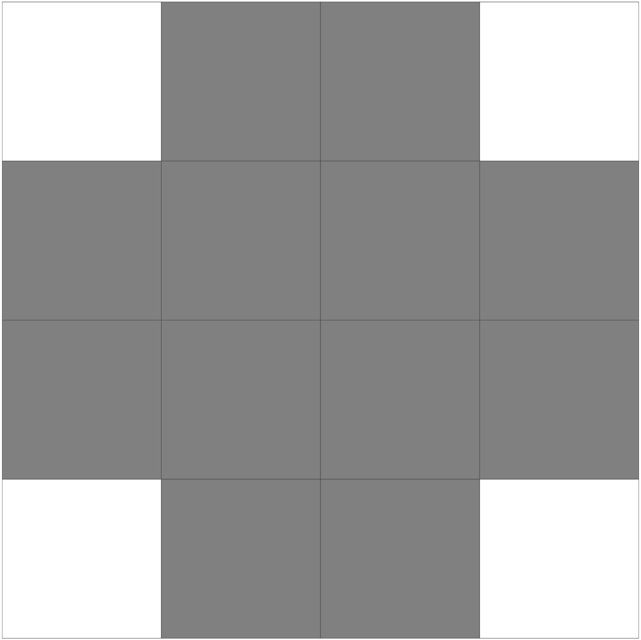
Quadtree $c = -1$ level 0



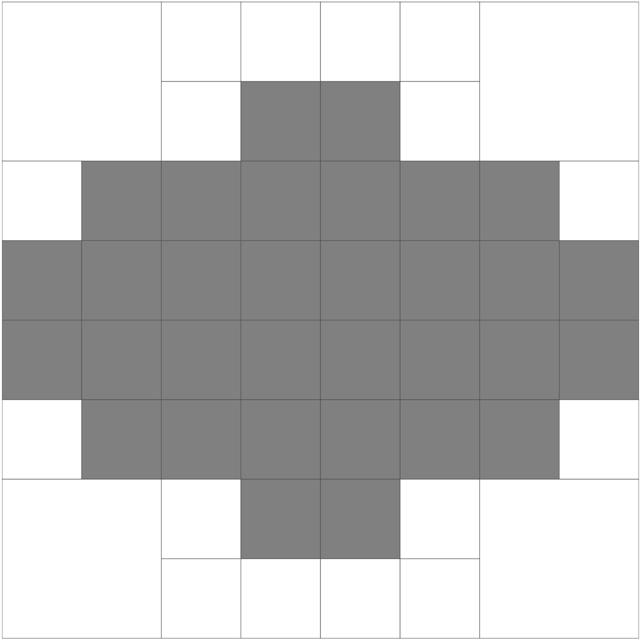
Quadtree $c = -1$ level 1



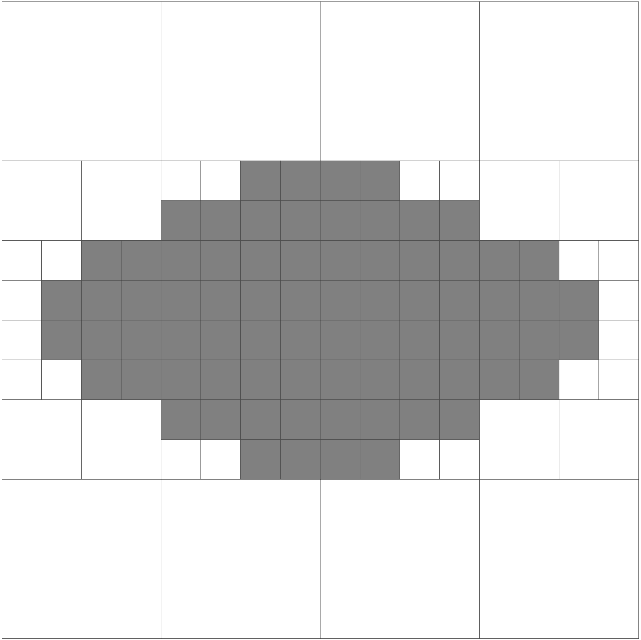
Quadtree $c = -1$ level 2



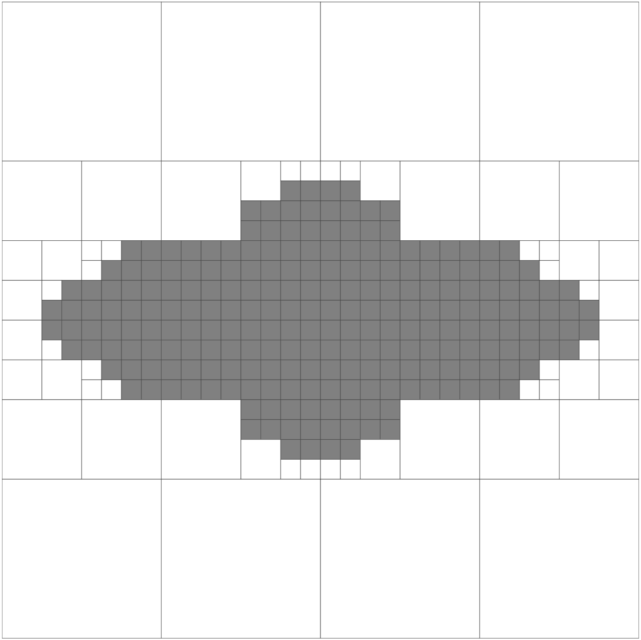
Quadtree $c = -1$ level 3



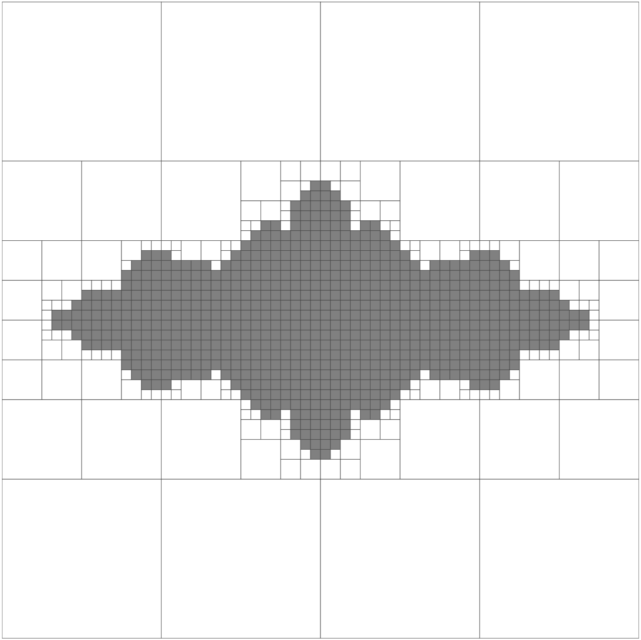
Quadtree $c = -1$ level 4



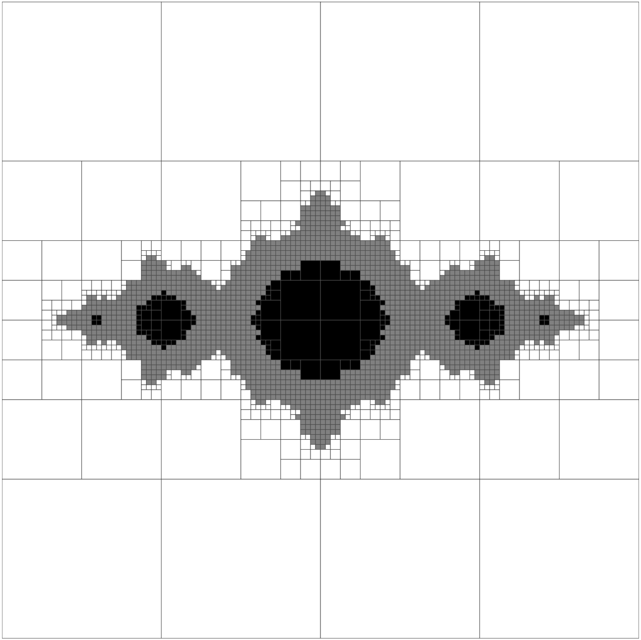
Quadtree $c = -1$ level 5



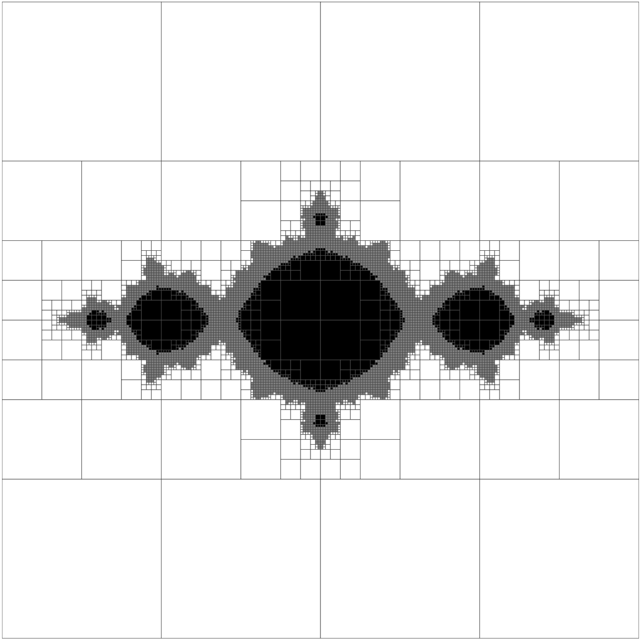
Quadtree $c = -1$ level 6



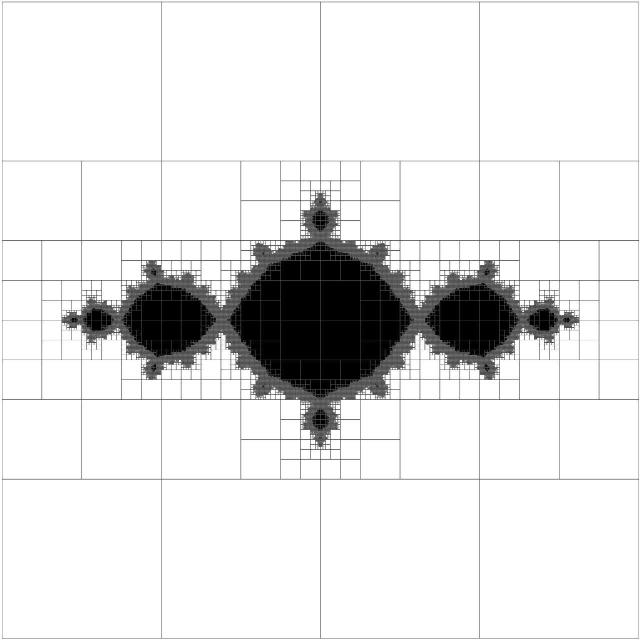
Quadtree $c = -1$ level 7



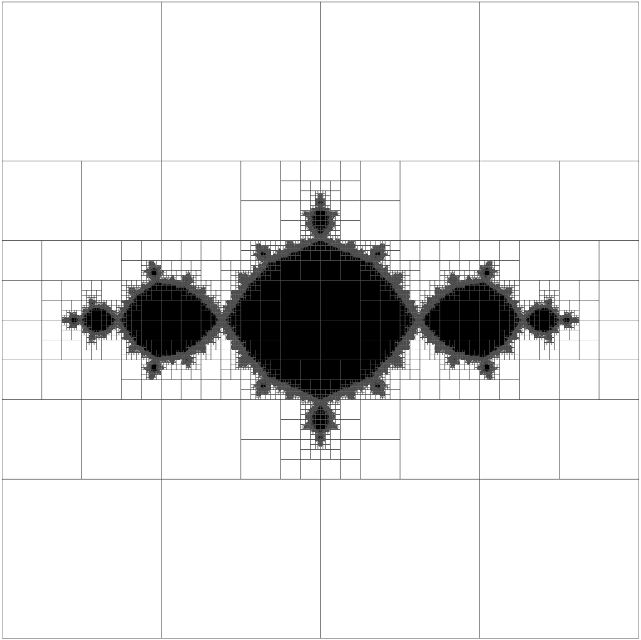
Quadtree $c = -1$ level 8



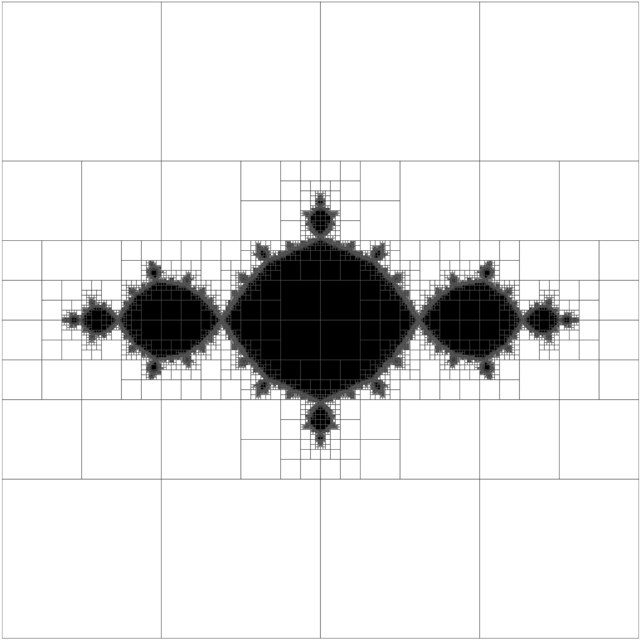
Quadtree $c = -1$ level 9



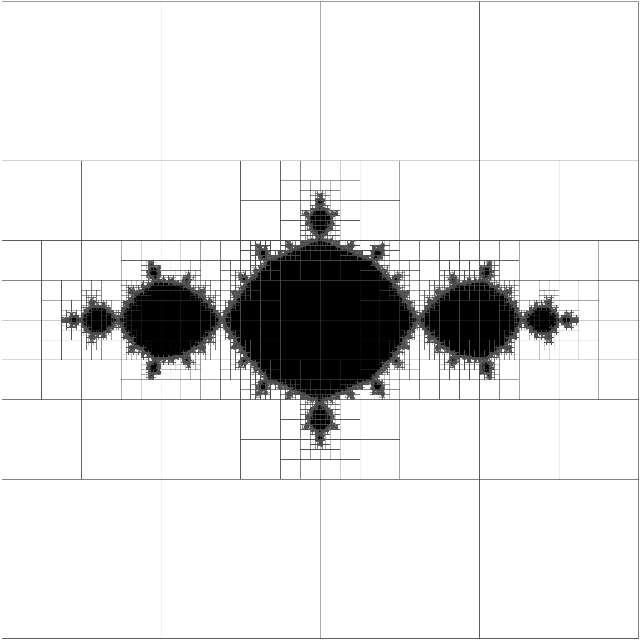
Quadtree $c = -1$ level 10



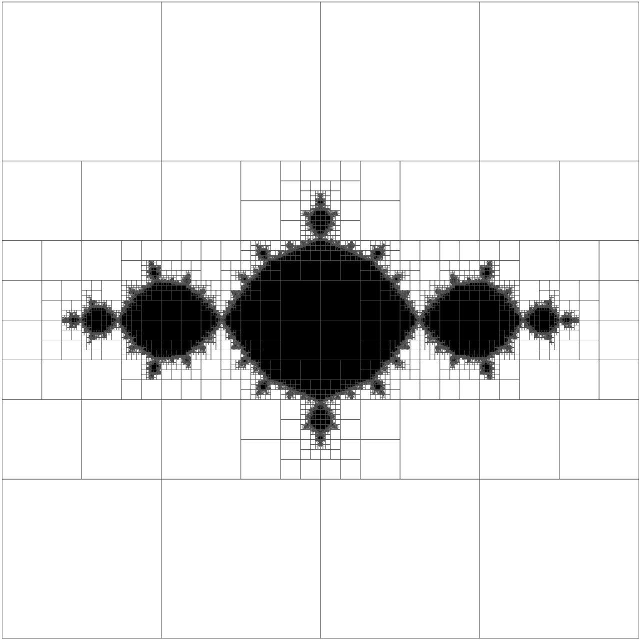
Quadtree $c = -1$ level 11



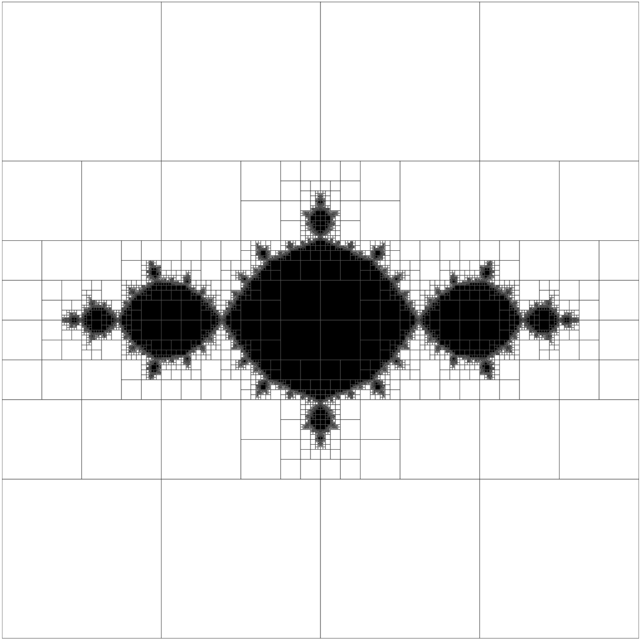
Quadtree $c = -1$ level 12



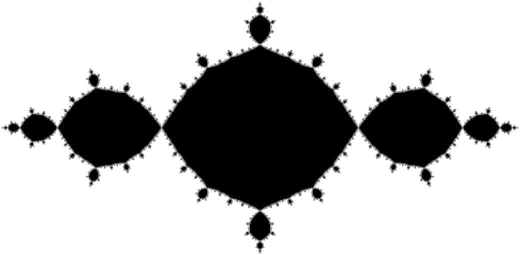
Quadtree $c = -1$ level 13



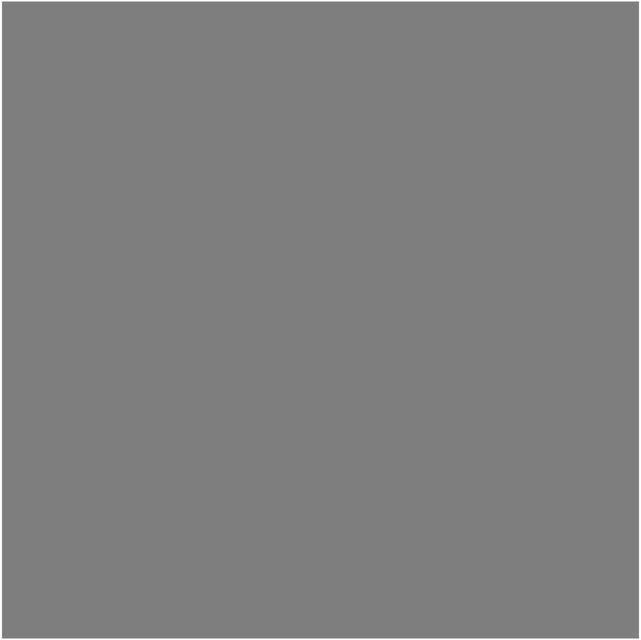
Quadtree $c = -1$ level 14



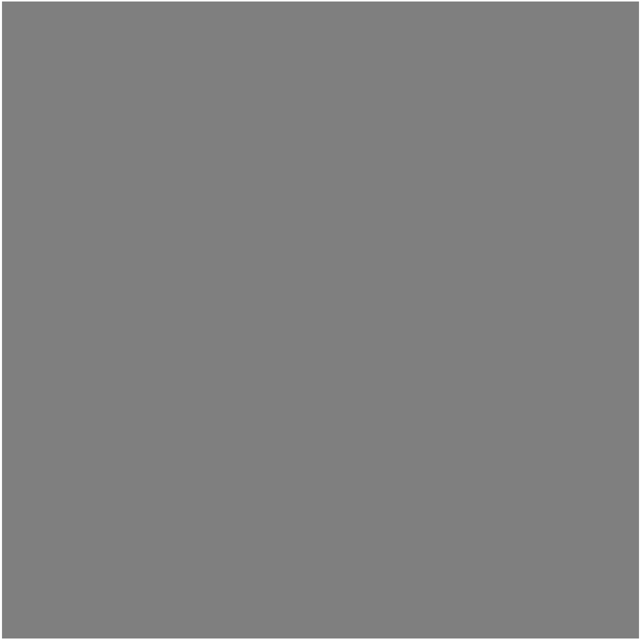
Adaptive approximation $c = -1$ level 14



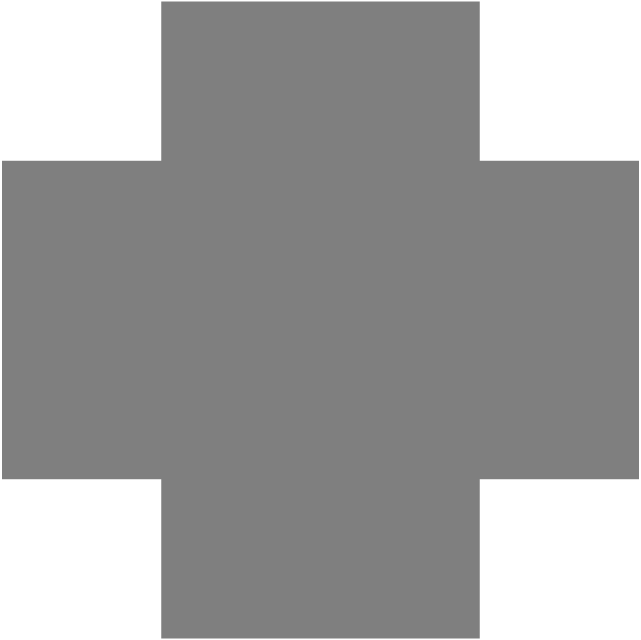
Adaptive approximation $c = -1$ level 0



Adaptive approximation $c = -1$ level 1



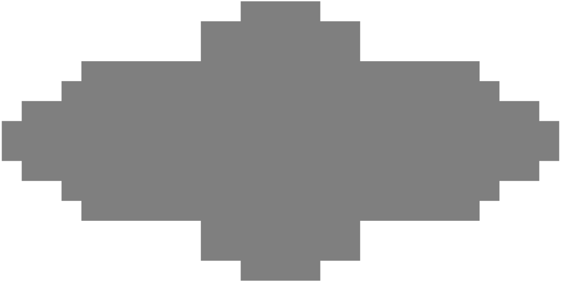
Adaptive approximation $c = -1$ level 2



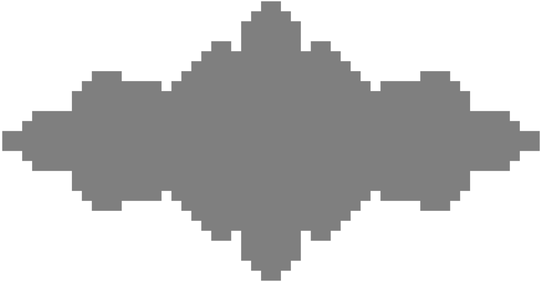
Adaptive approximation $c = -1$ level 3

Adaptive approximation $c = -1$ level 4

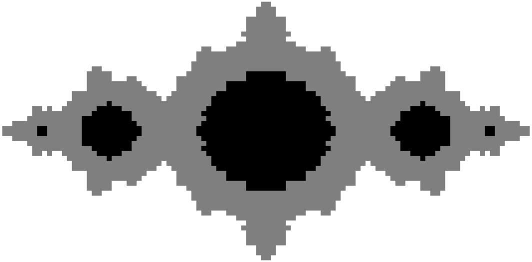
Adaptive approximation $c = -1$ level 5



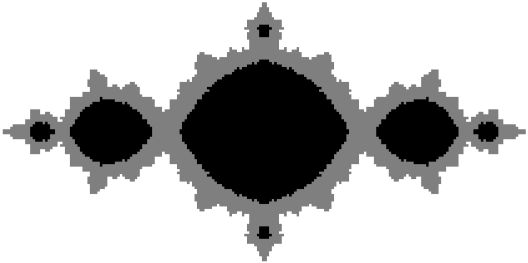
Adaptive approximation $c = -1$ level 6



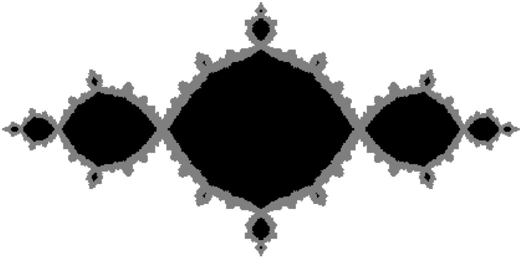
Adaptive approximation $c = -1$ level 7



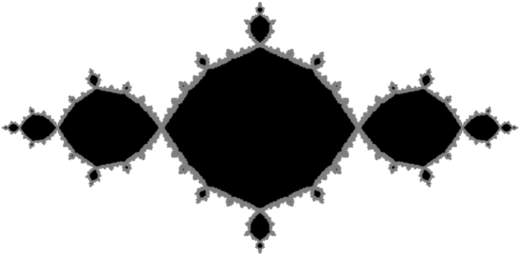
Adaptive approximation $c = -1$ level 8



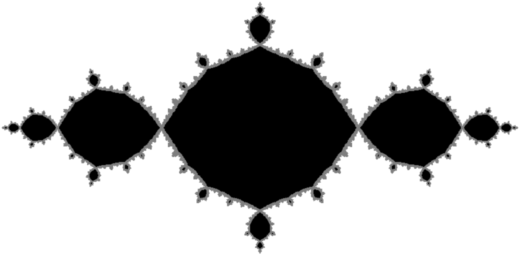
Adaptive approximation $c = -1$ level 9



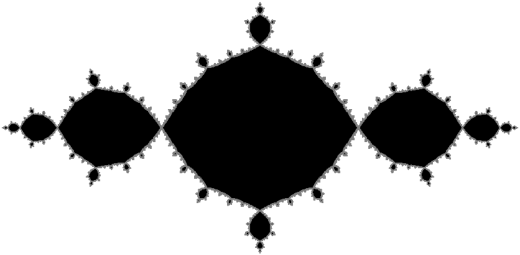
Adaptive approximation $c = -1$ level 10



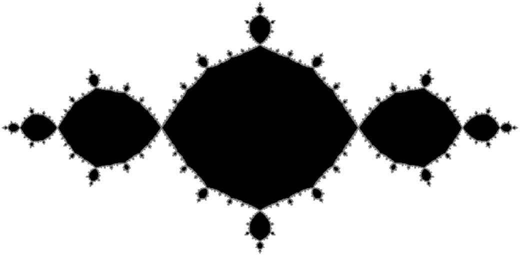
Adaptive approximation $c = -1$ level 11



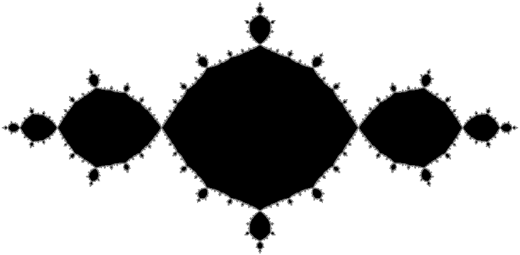
Adaptive approximation $c = -1$ level 12



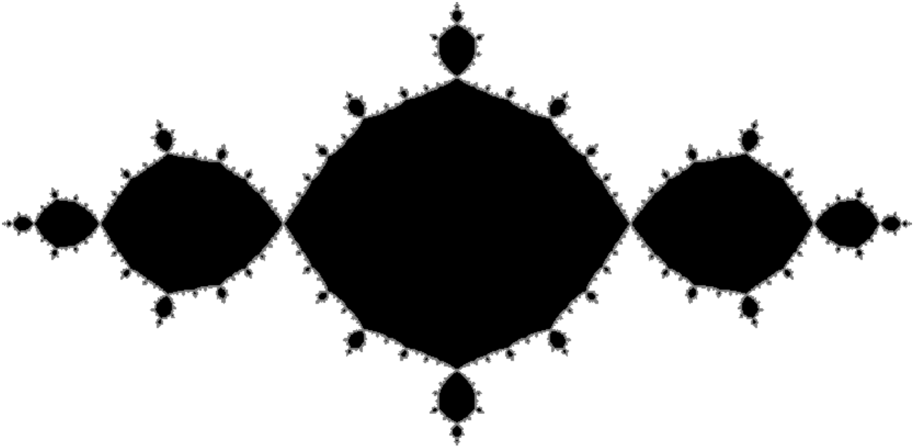
Adaptive approximation $c = -1$ level 13



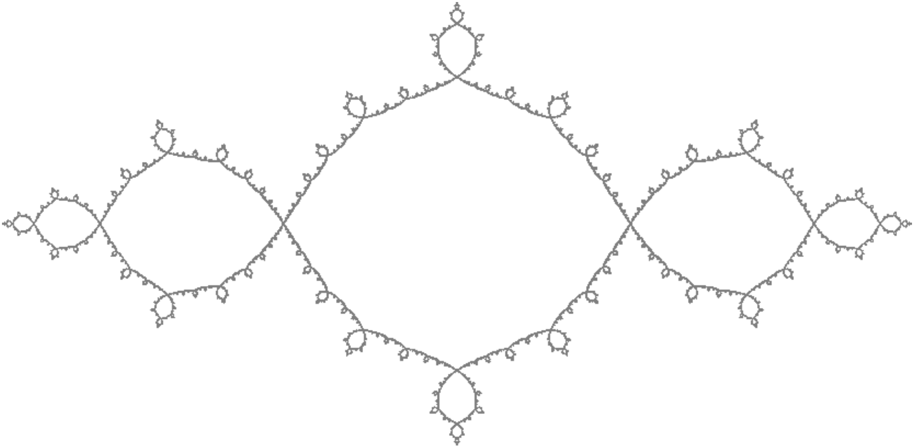
Adaptive approximation $c = -1$ level 14



Adaptive approximation $c = -1$



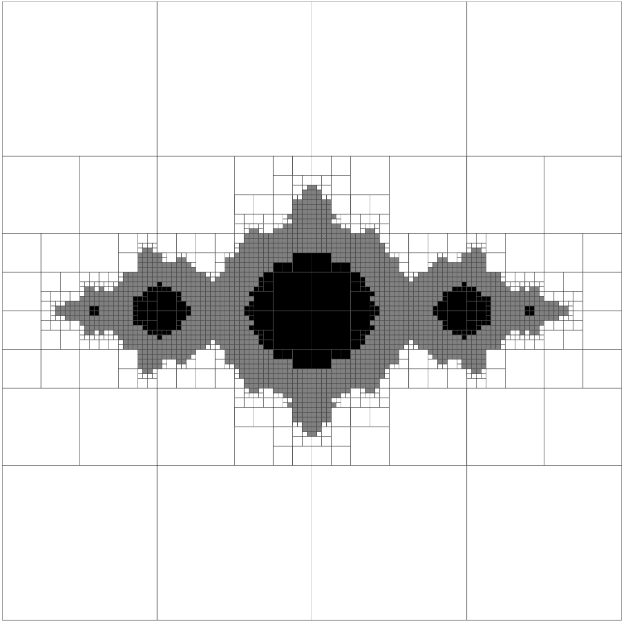
Adaptive approximation $c = -1$



Our algorithm

quadtree for
$\Omega = [-R, R] \times [-R, R]$

- refinement
- cell mapping
- color propagation



Cell mapping

Directed graph on the leaves of the quadtree and exterior

- edges emanate from each leaf gray cell A
- add edge $A \to B$ for each leaf cell B that intersects $f(A)$

$$f(A) \subseteq \bigcup_{A \to B} B$$

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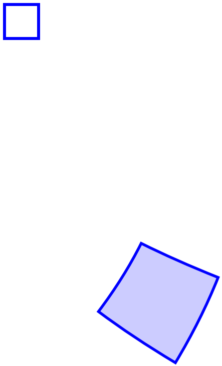
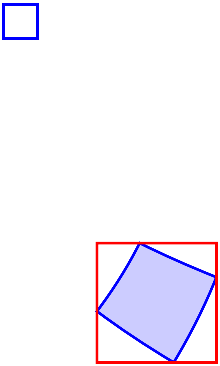
Conservative estimate of the dynamics

Avoid point sampling

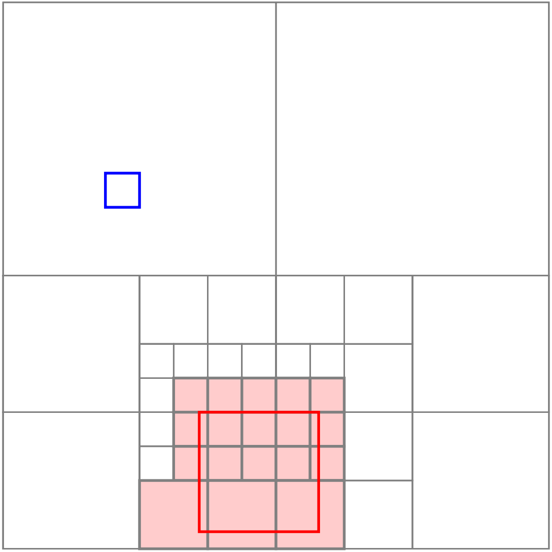
Cell mapping source cell leaf gray cell

Cell mapping exact image under f

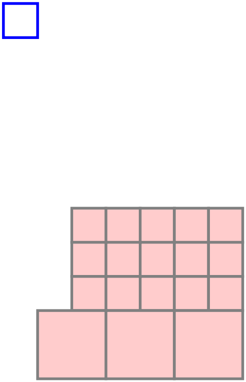




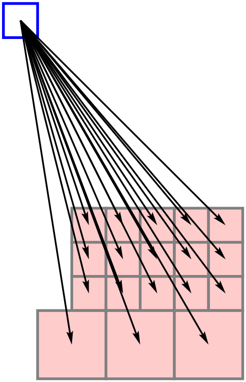
Cell mapping quadtree traversal

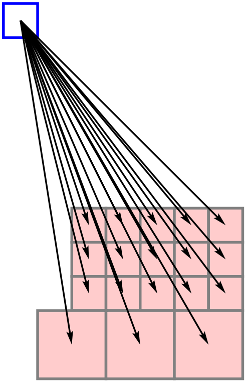


Cell mapping target cells contain exact image



Cell mapping edges





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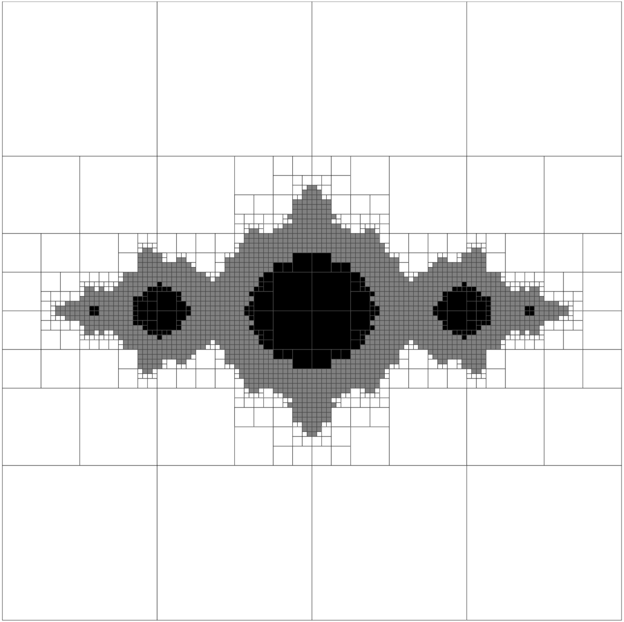

Color propagation

Propagate white and black to gray cells

- new white cells
  gray cells for which all paths end in white cells
- new black cells
  gray cells for which no path ends in a white cell

Color propagation

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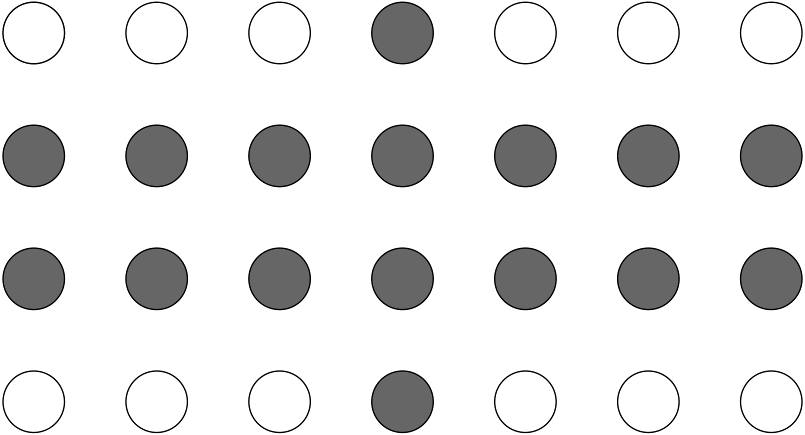
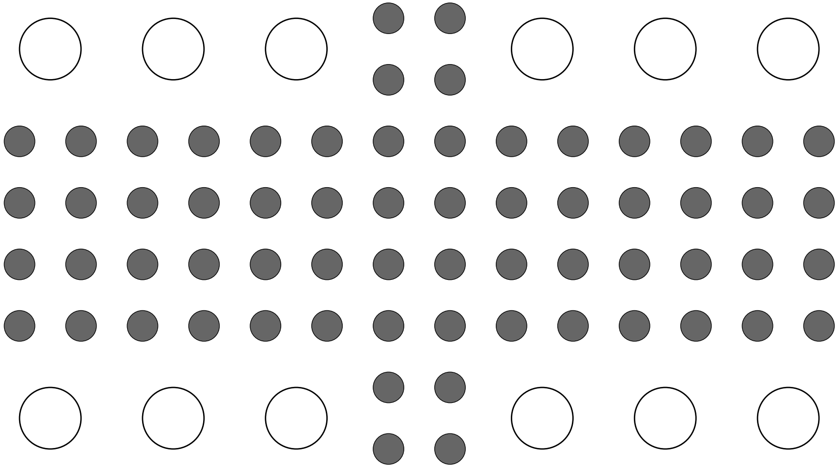
Graph traversals replace function iteration

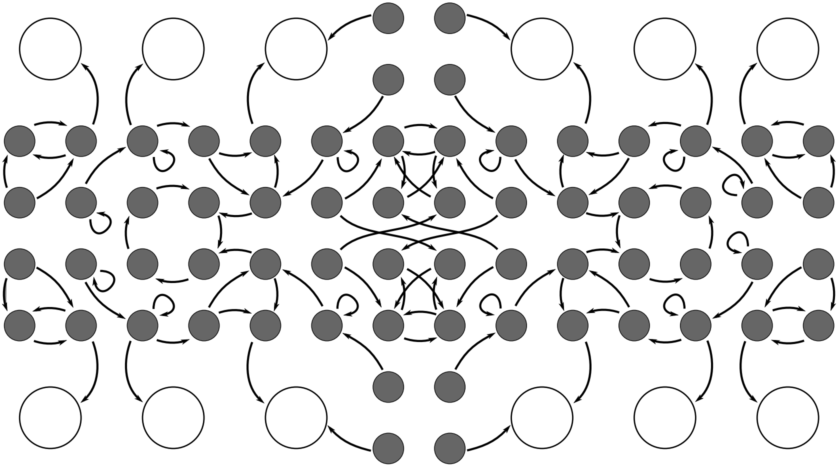
Avoid floating-point errors



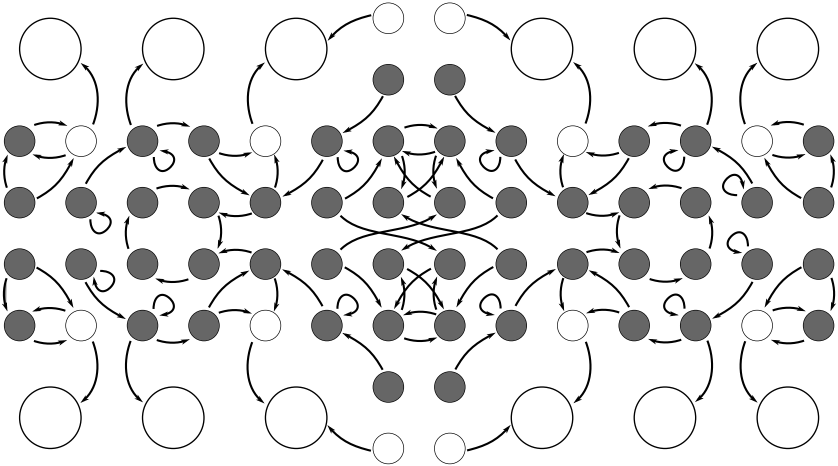
The algorithm refinement



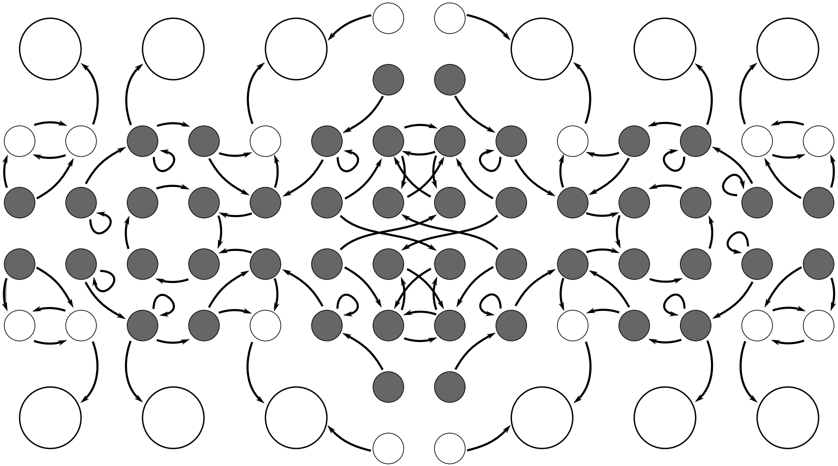
The algorithm   cell mapping

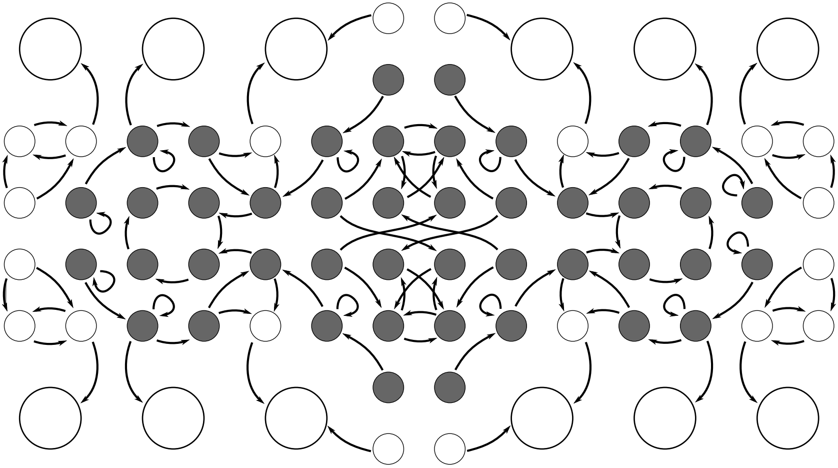


The algorithm new white cells. . .

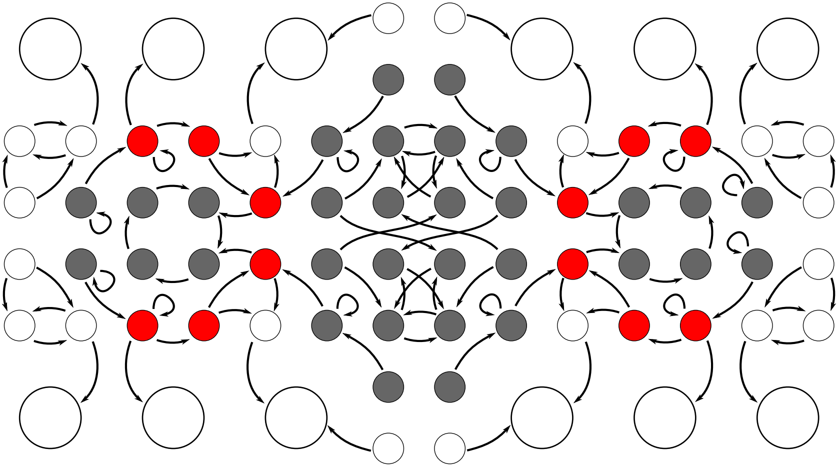


The algorithm new white cells. . .

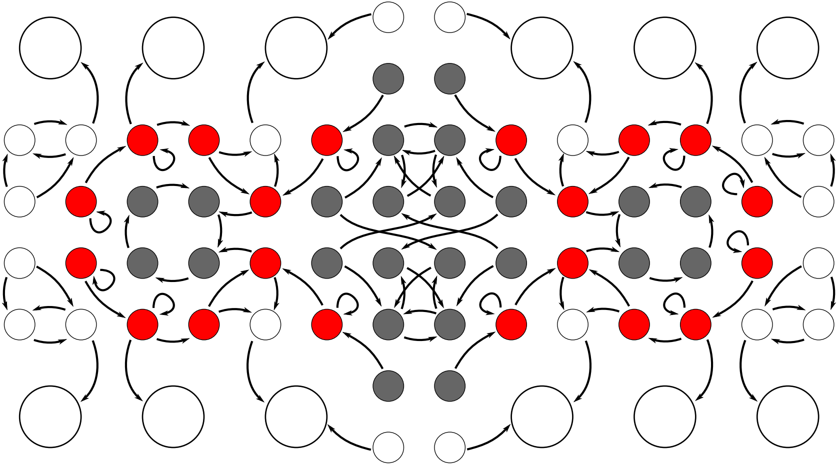


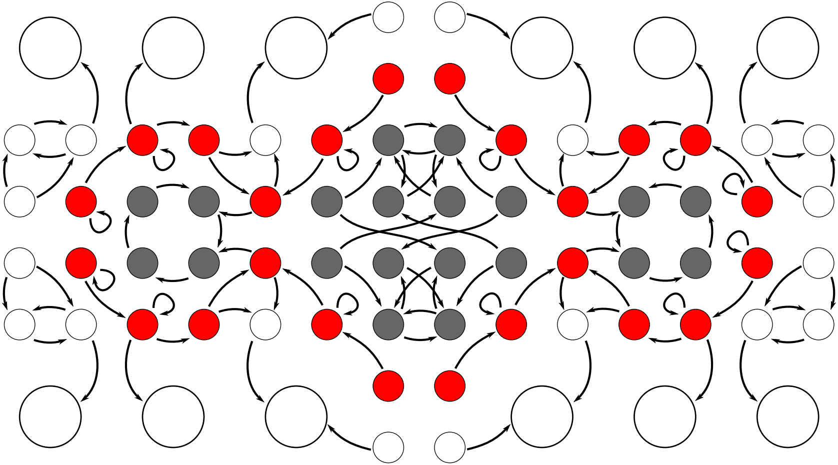


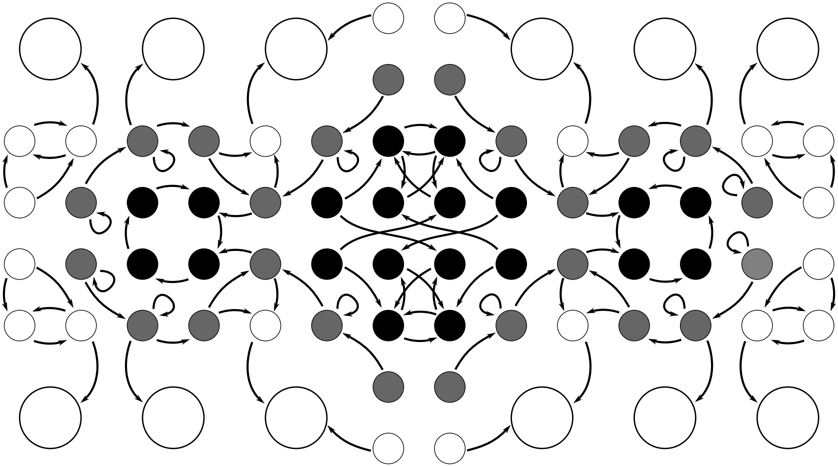
The algorithm gray cells that reach white. . .



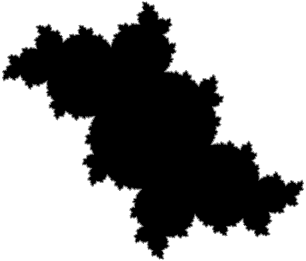
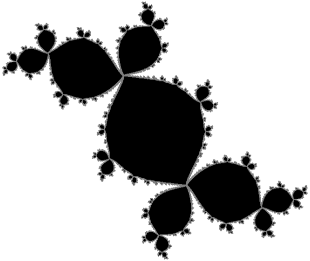
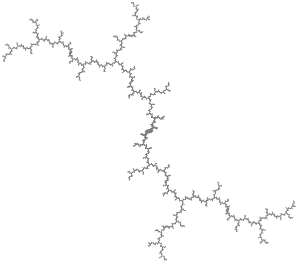
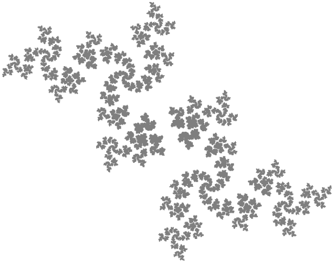
gray cells that reach white. . .



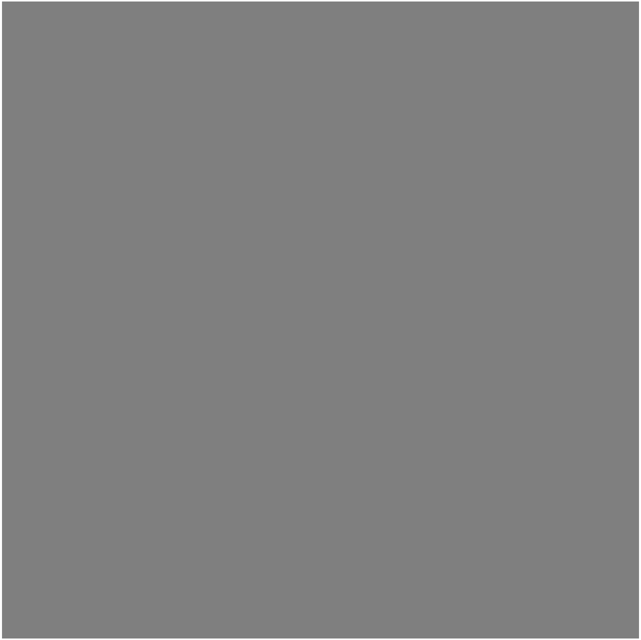




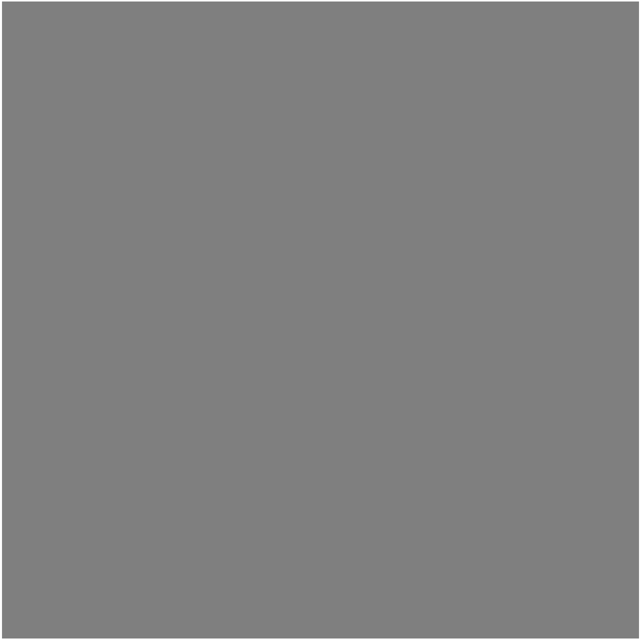
Adaptive approximation examples

Adaptive approximation $c = 0.12 + 0.30\,i$ level 0



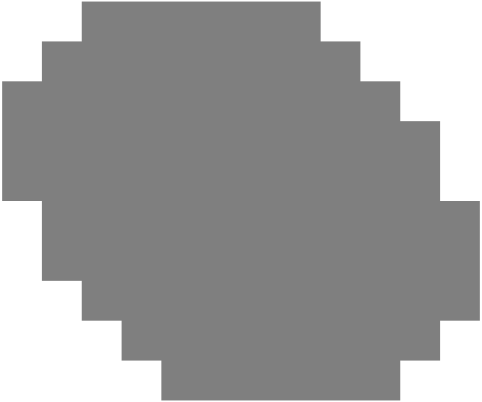
Adaptive approximation $c = 0.12 + 0.30\,i$ level 1



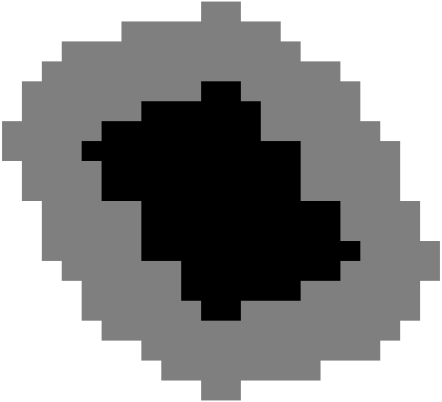
Adaptive approximation $c = 0.12 + 0.30\,i$ level 2

Adaptive approximation $c = 0.12 + 0.30\,i$ level 3

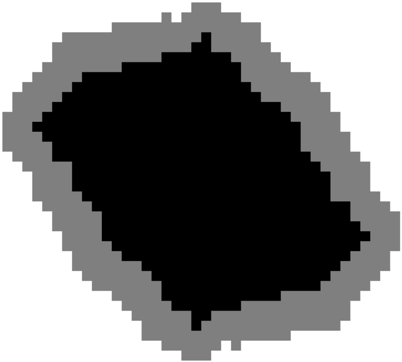
Adaptive approximation $c = 0.12 + 0.30\,i$ level 4



Adaptive approximation $\quad c = 0.12 + 0.30\,i \quad$ level 5

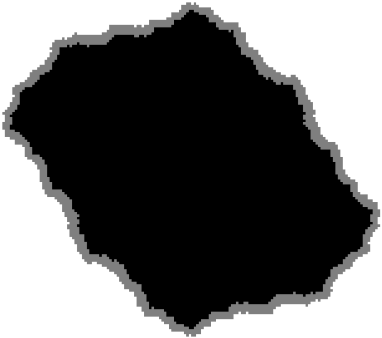


Adaptive approximation $c = 0.12 + 0.30\,i$ level 6

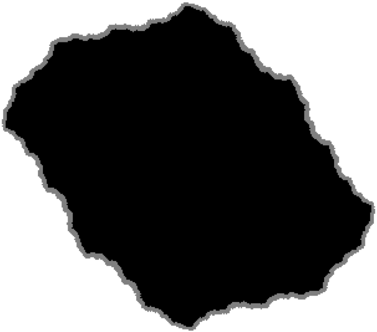


Adaptive approximation $c = 0.12 + 0.30\,i$ level 7

Adaptive approximation $c = 0.12 + 0.30\,i$ level 8



Adaptive approximation $c = 0.12 + 0.30\,i$ level 9



Adaptive approximation $c = 0.12 + 0.30\,i$ level 10

Adaptive approximation $c = 0.12 + 0.30\, i$ level 11

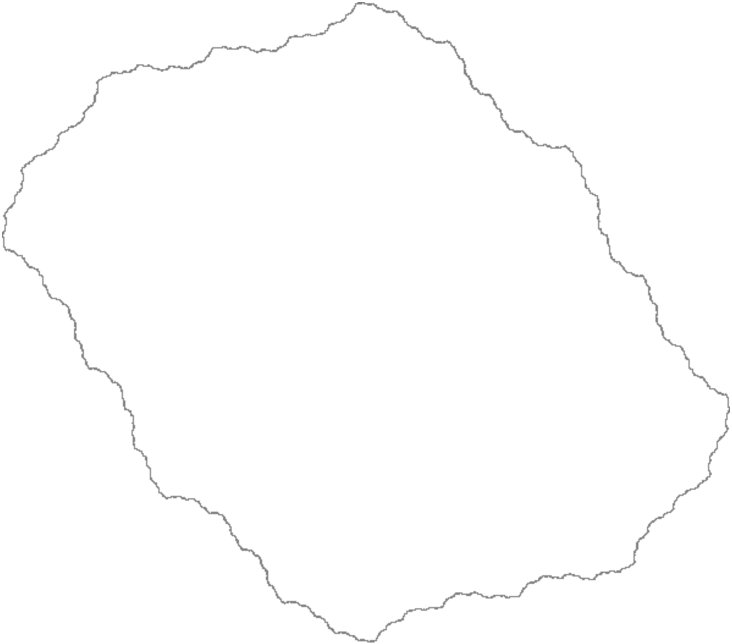
Adaptive approximation $c = 0.12 + 0.30\,i$ level 12

Adaptive approximation $c = 0.12 + 0.30\,i$ level 13

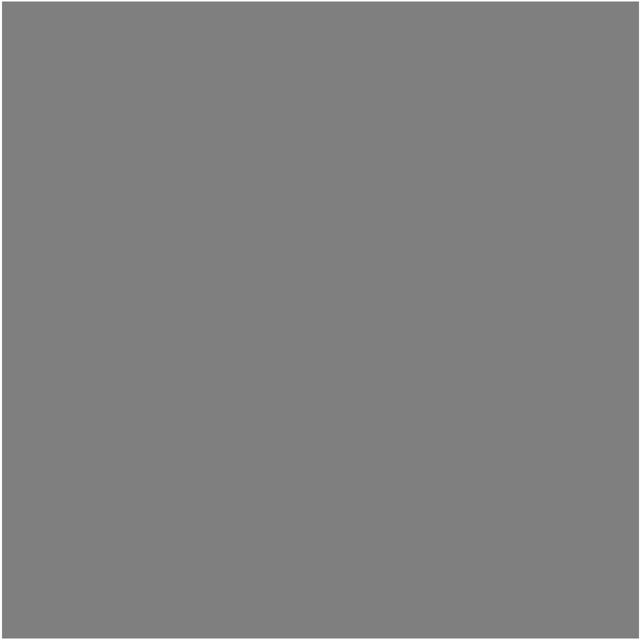
Adaptive approximation $\quad c = 0.12 + 0.30\,i \quad$ level 14

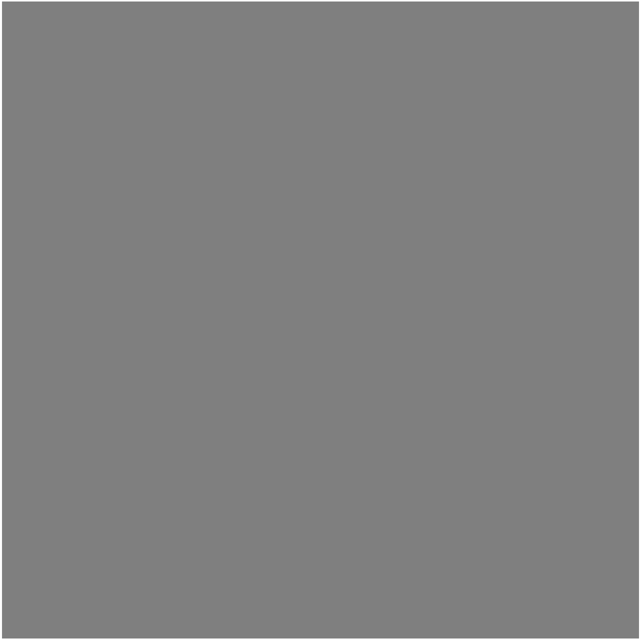
Adaptive approximation $c = 0.12 + 0.30\,i$

Adaptive approximation $c = 0.12 + 0.30\,i$

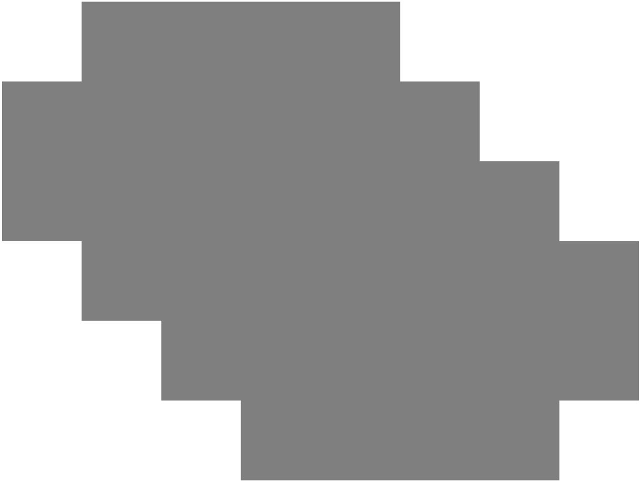


Adaptive approximation $c = -0.12 + 0.60\,i$ level 0

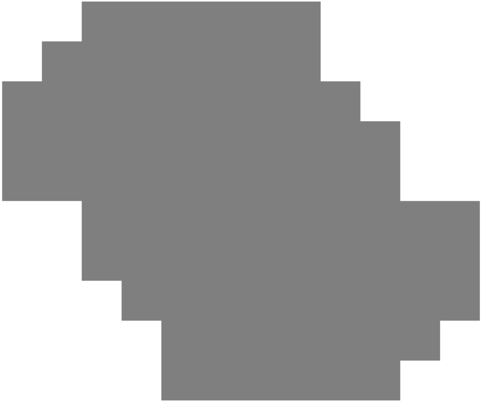




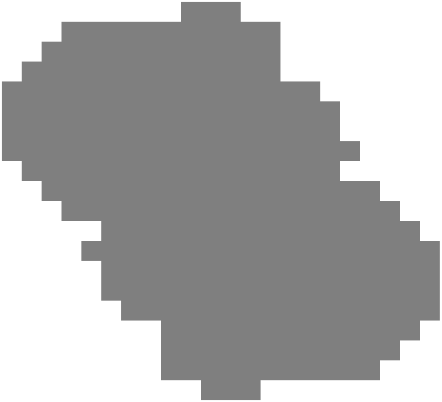
Adaptive approximation $c = -0.12 + 0.60\,i$ level 3



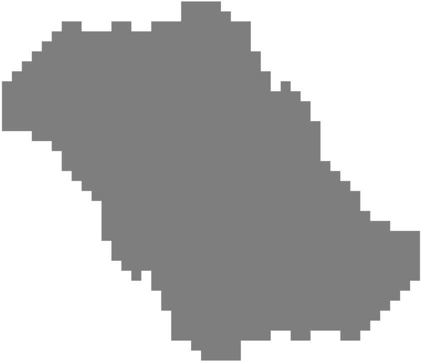
Adaptive approximation $c = -0.12 + 0.60\,i$ level 4



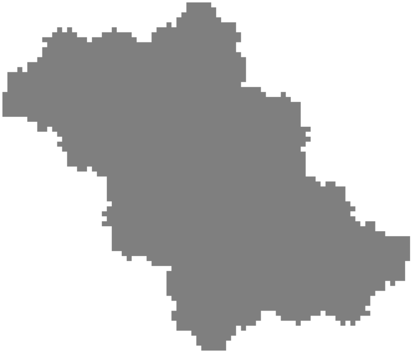
Adaptive approximation $c = -0.12 + 0.60\,i$ level 5



Adaptive approximation $c = -0.12 + 0.60\,i$ level 6

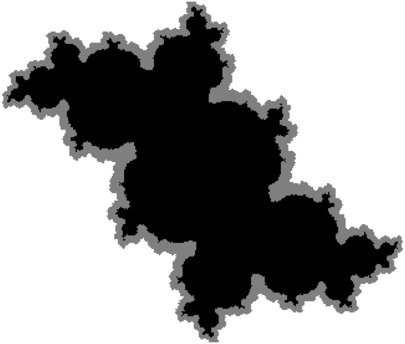


Adaptive approximation $c = -0.12 + 0.60\,i$ level 7

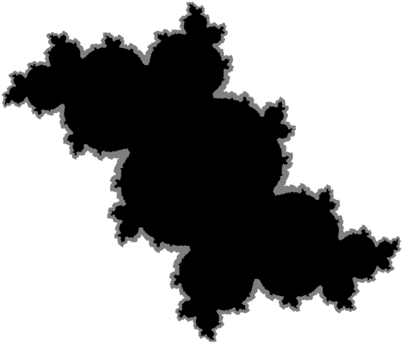


Adaptive approximation $c = -0.12 + 0.60\,i$ level 8

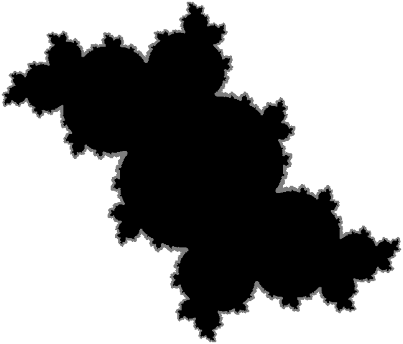
Adaptive approximation $c = -0.12 + 0.60\,i$ level 9



Adaptive approximation $c = -0.12 + 0.60\,i$ level 10



Adaptive approximation $c = -0.12 + 0.60\,i$ level 11



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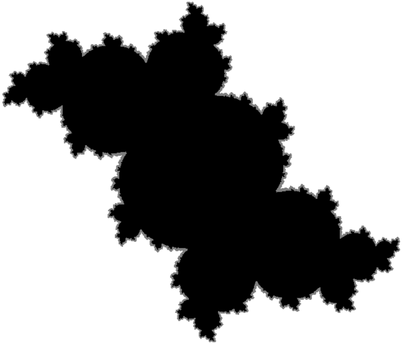
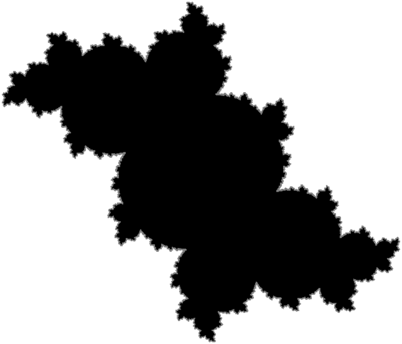
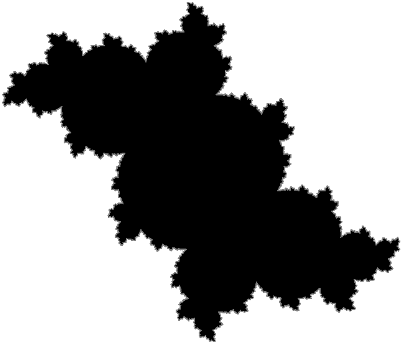


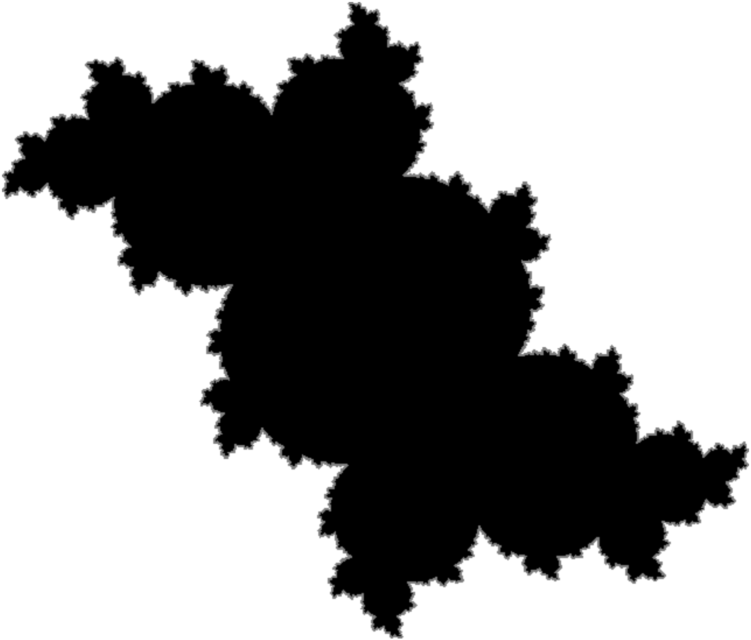
Adaptive approximation $c = -0.12 + 0.60\,i$ level 13



Adaptive approximation $c = -0.12 + 0.60\,i$ level 14



Adaptive approximation $\quad c = -0.12 + 0.60\,i$



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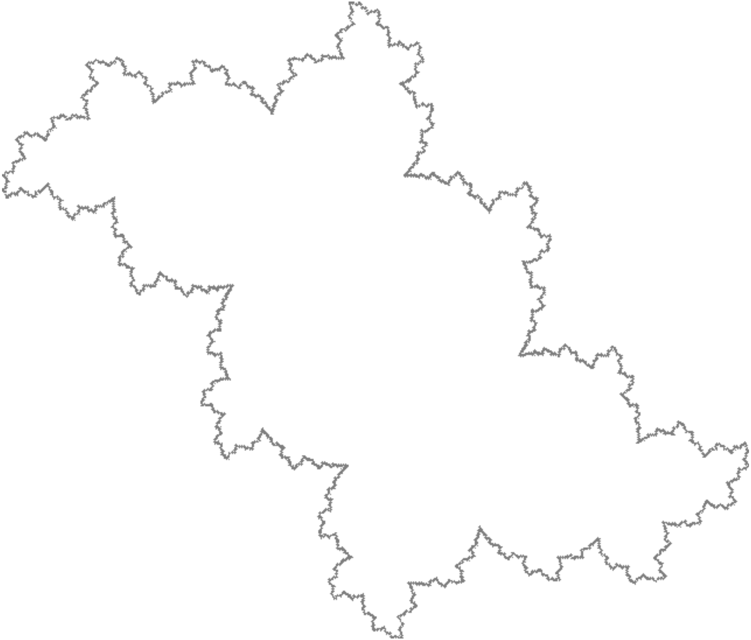
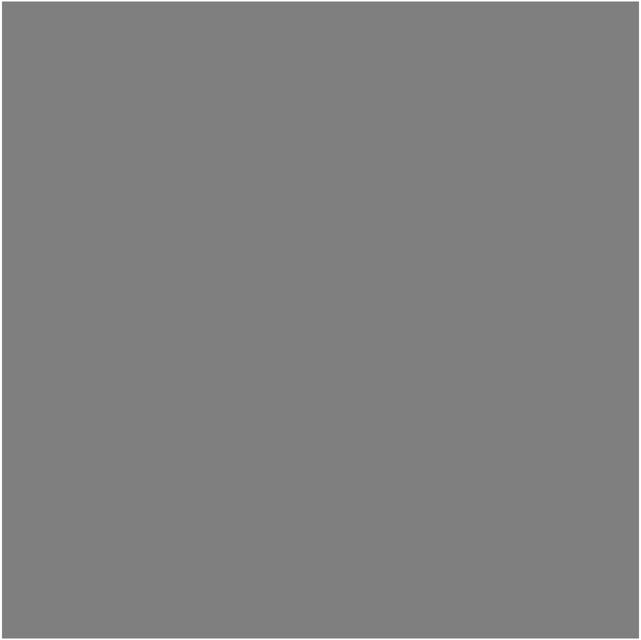
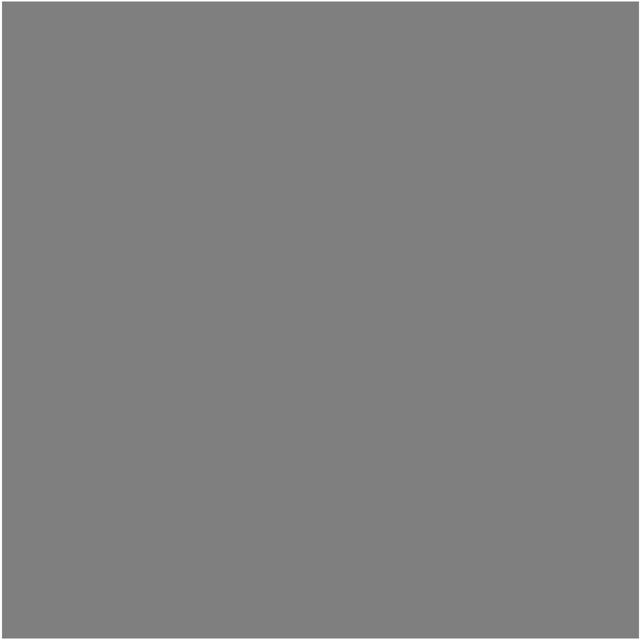


Adaptive approximation $\quad c = -0.12 + 0.74\,i \quad$ level 0

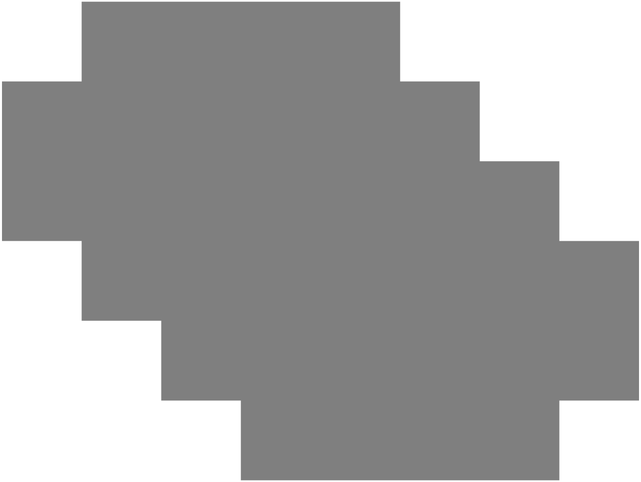


Adaptive approximation $\quad c = -0.12 + 0.74\, i \quad$ level 1

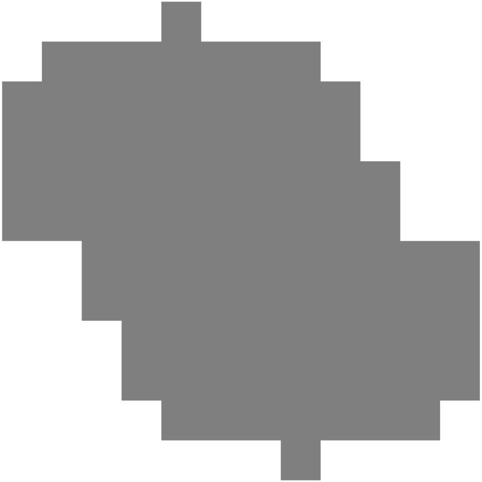


Adaptive approximation $c = -0.12 + 0.74\,i$ level 2

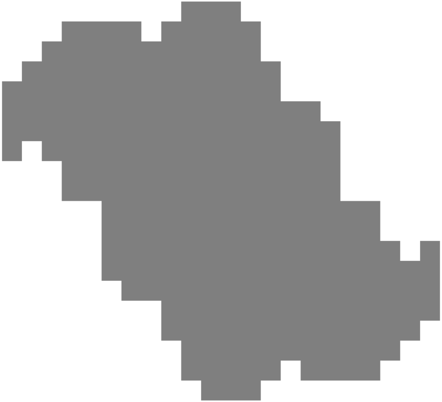
Adaptive approximation $c = -0.12 + 0.74\,i$ level 3



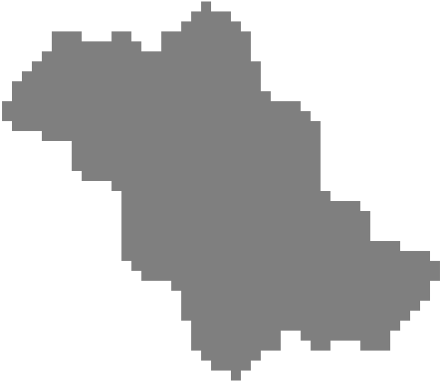
Adaptive approximation $c = -0.12 + 0.74\,i$ level 4



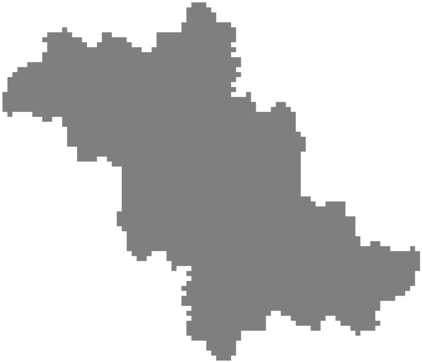
Adaptive approximation $c = -0.12 + 0.74\,i$ level 5



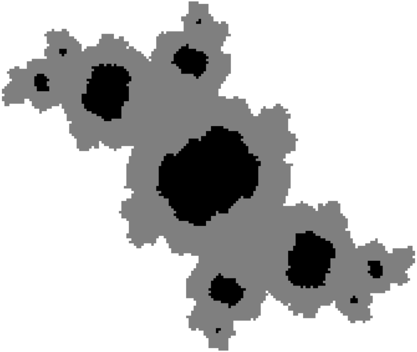
Adaptive approximation $c = -0.12 + 0.74\,i$ level 6



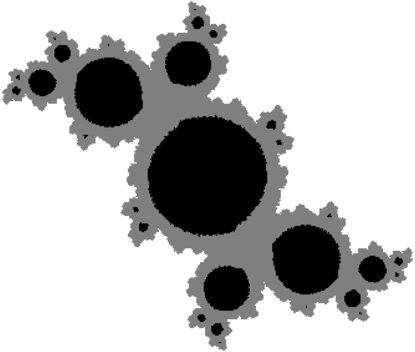
Adaptive approximation $\quad c = -0.12 + 0.74\,i \quad$ level 7



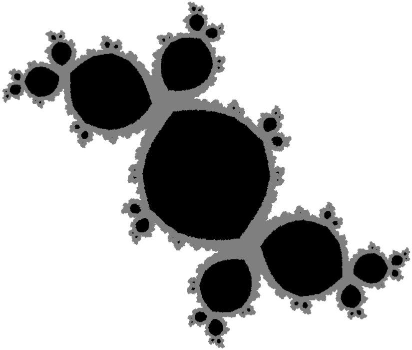
Adaptive approximation $c = -0.12 + 0.74\,i$ level 8



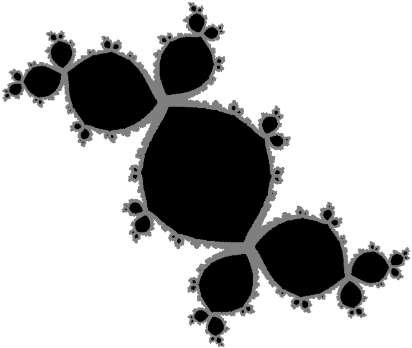
Adaptive approximation $\quad c = -0.12 + 0.74\, i \quad$ level 9



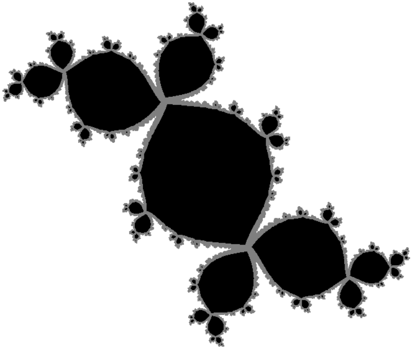
Adaptive approximation $c = -0.12 + 0.74\,i$ level 10



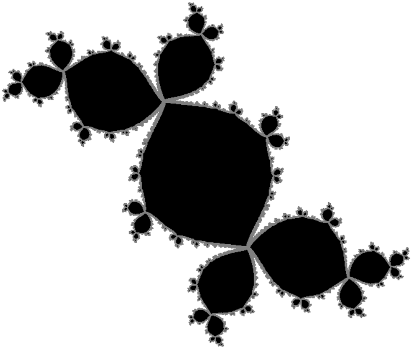
Adaptive approximation $c = -0.12 + 0.74\,i$ level 11



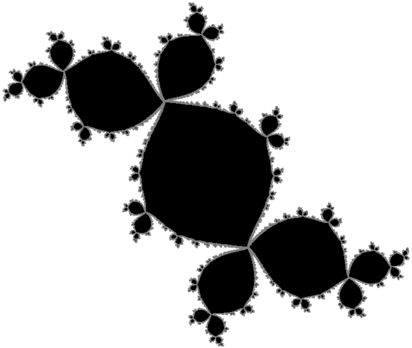
Adaptive approximation $\quad c = -0.12 + 0.74\,i \quad$ level 12



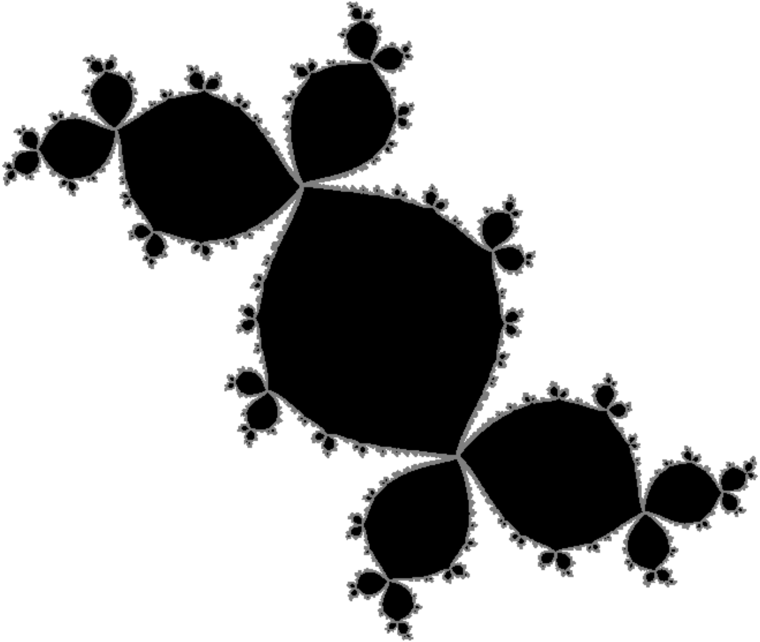
Adaptive approximation $c = -0.12 + 0.74\,i$ level 13



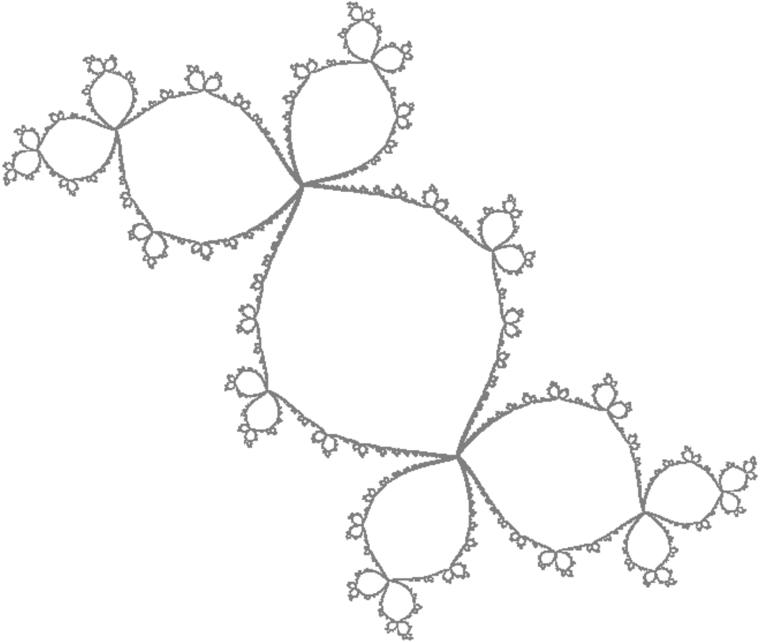
Adaptive approximation $c = -0.12 + 0.74\,i$ level 14



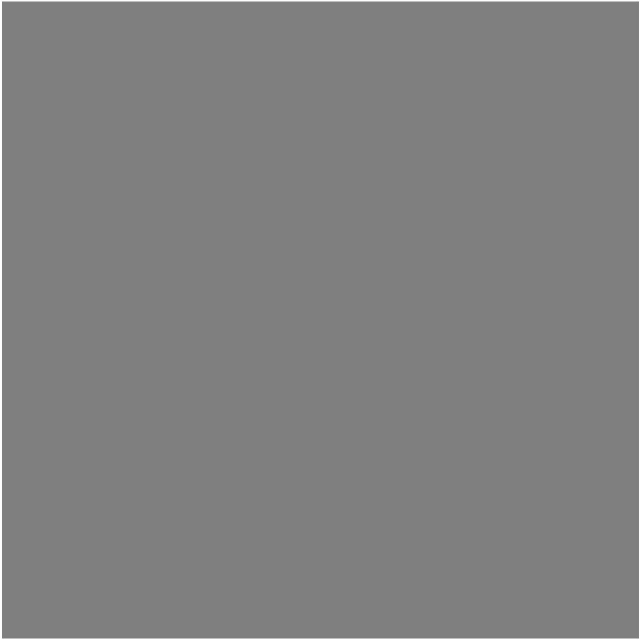
Adaptive approximation $c = -0.12 + 0.74\,i$



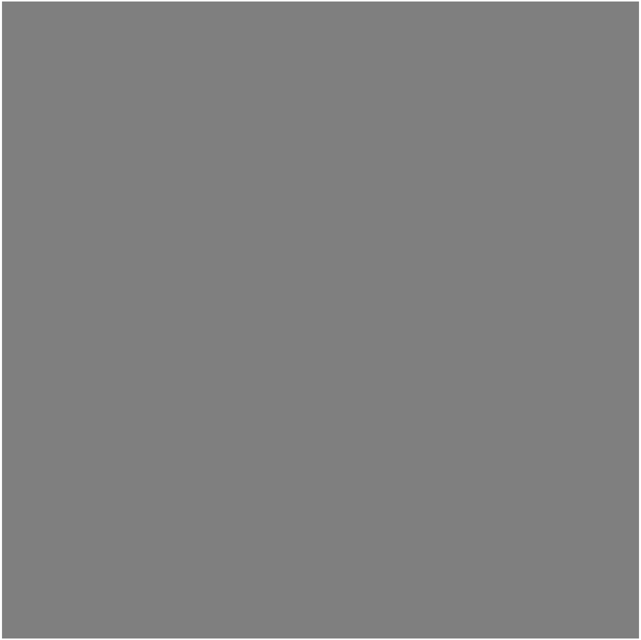
Adaptive approximation $c = -0.12 + 0.74\,i$



Adaptive approximation $c = i$ level 0

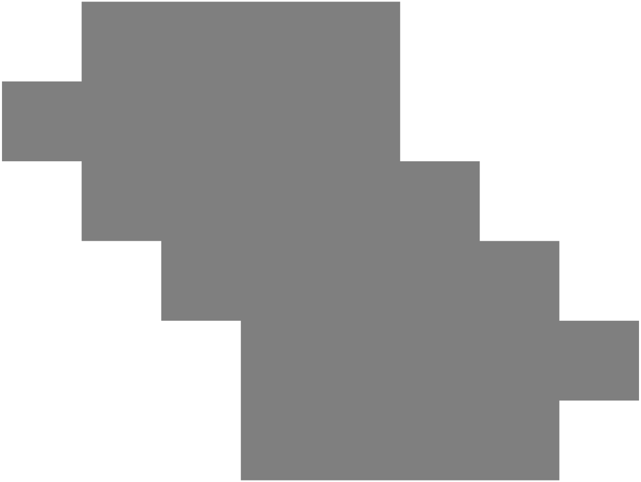


Adaptive approximation $c = i$ level 1

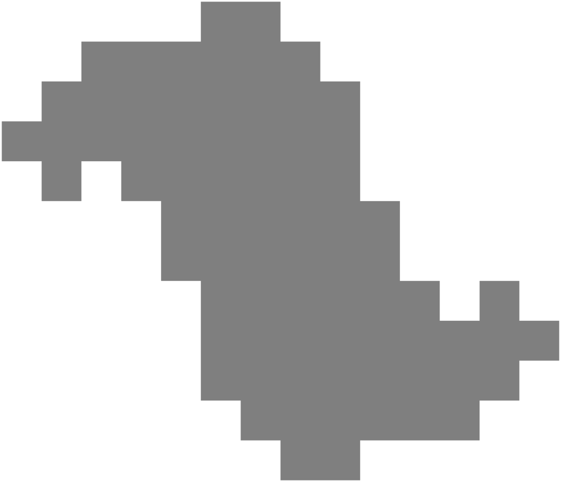


Adaptive approximation $c = i$ level 2

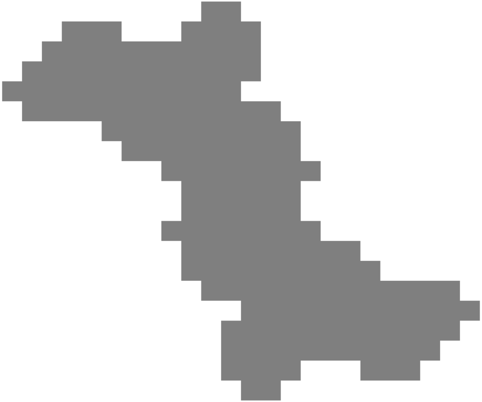
Adaptive approximation $c = i$ level 3



Adaptive approximation $c = i$ level 4

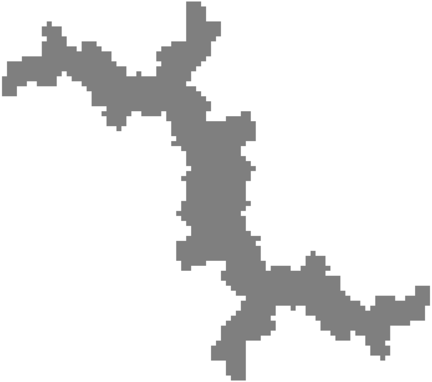


Adaptive approximation $c = i$ level 5

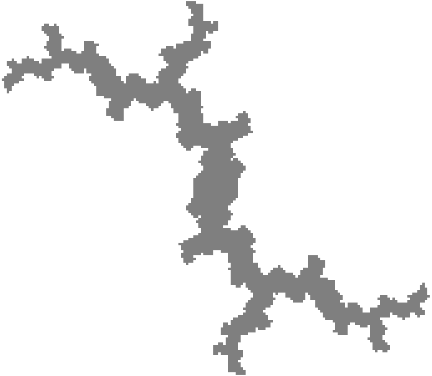


Adaptive approximation $c = i$ level 6

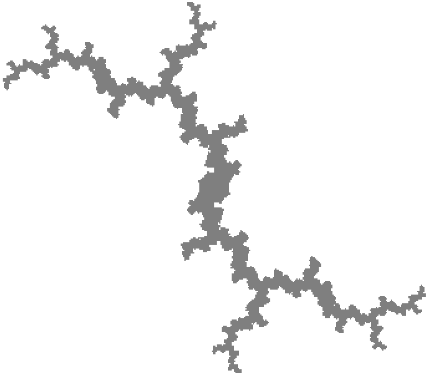
Adaptive approximation $c = i$ level 7



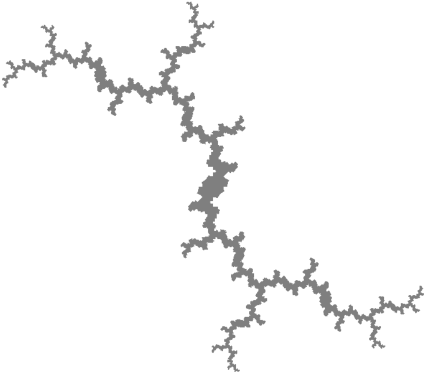
Adaptive approximation $c = i$ level 8



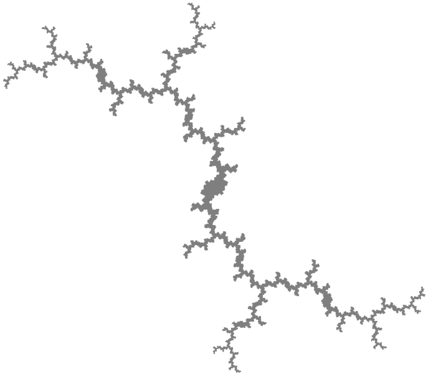
Adaptive approximation $\quad c = i \quad$ level 9



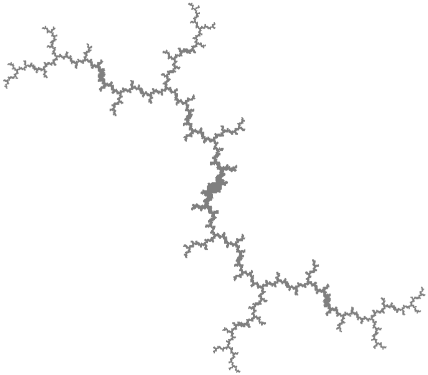
Adaptive approximation $c = i$ level 10



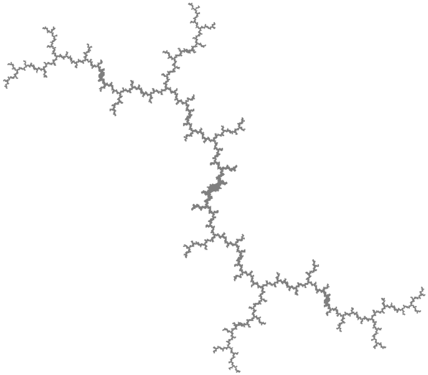
Adaptive approximation $c = i$ level 11



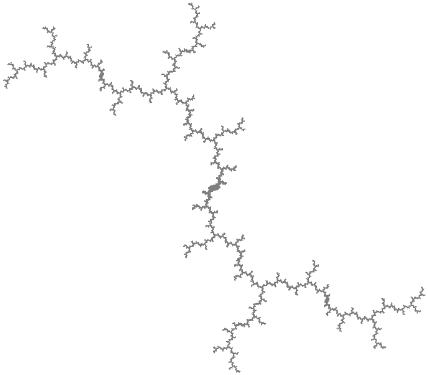
Adaptive approximation $c = i$ level 12



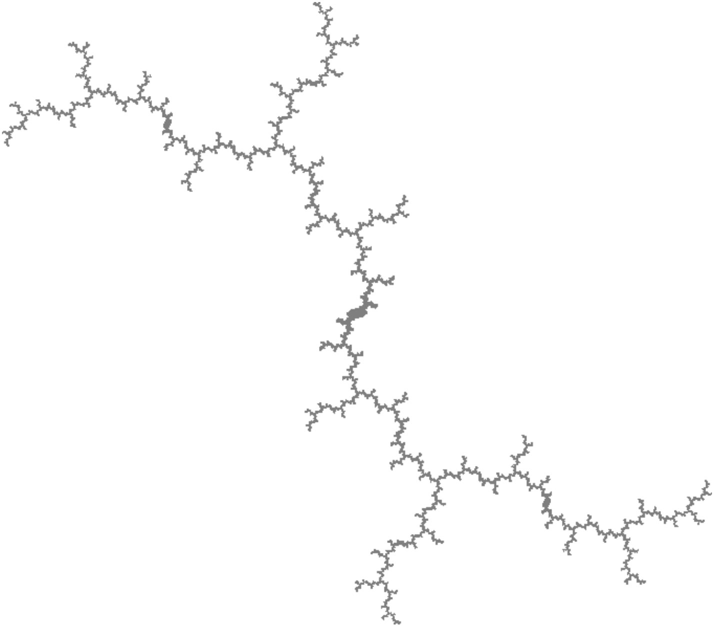
Adaptive approximation $\quad c = i \quad$ level 13



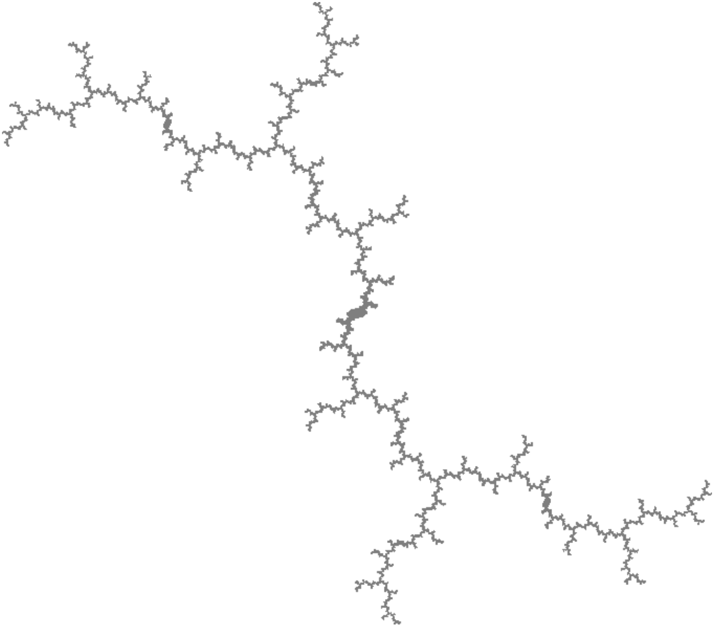
Adaptive approximation $c = i$ level 14



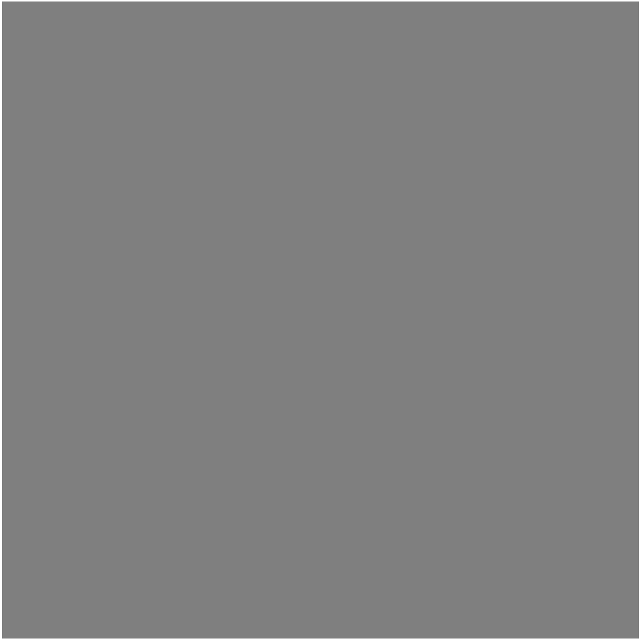
Adaptive approximation $c = i$



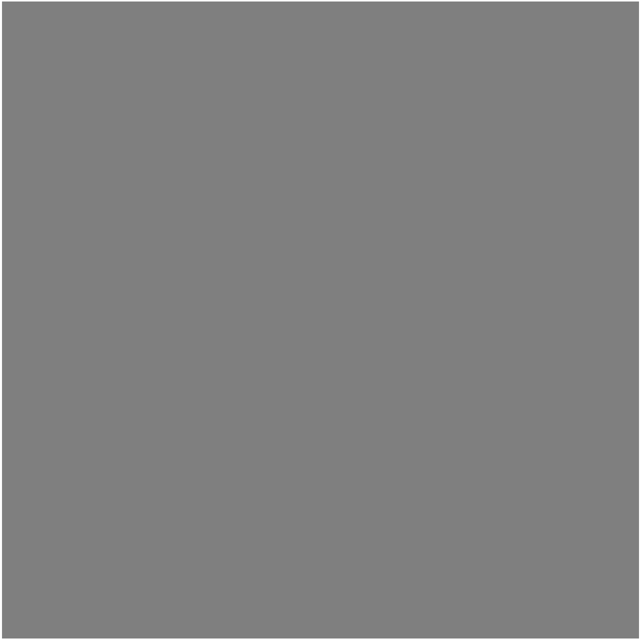
Adaptive approximation $c = i$



Adaptive approximation $c = -0.25 + 0.74\,i$ level 0

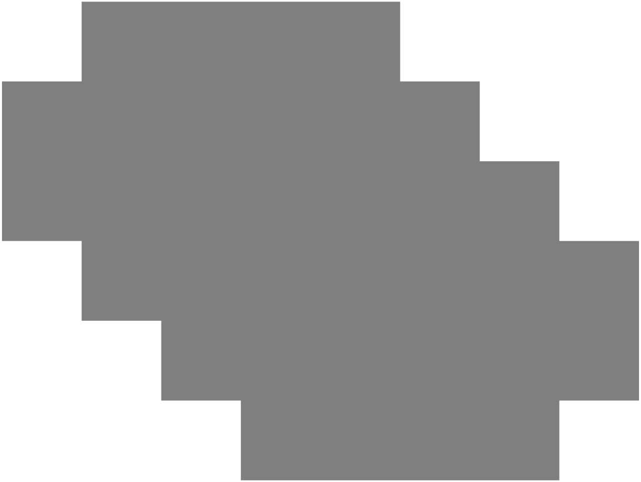


Adaptive approximation $c = -0.25 + 0.74\,i$ level 1

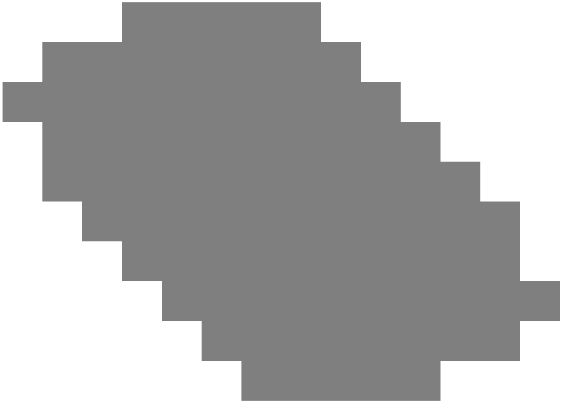


Adaptive approximation $c = -0.25 + 0.74\,i$ level 2

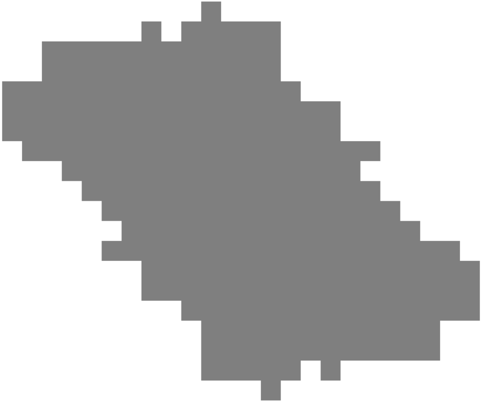
Adaptive approximation $c = -0.25 + 0.74\,i$ level 3



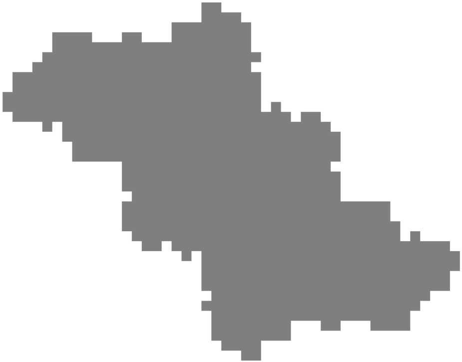
Adaptive approximation $c = -0.25 + 0.74\,i$ level 4



Adaptive approximation $c = -0.25 + 0.74\,i$ level 5

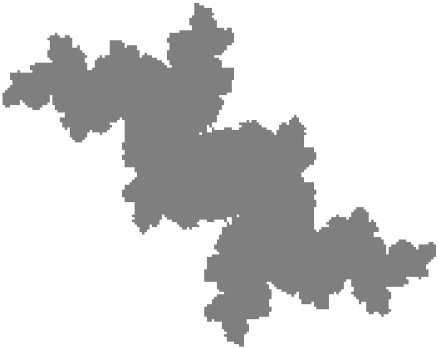


Adaptive approximation $c = -0.25 + 0.74\,i$ level 6

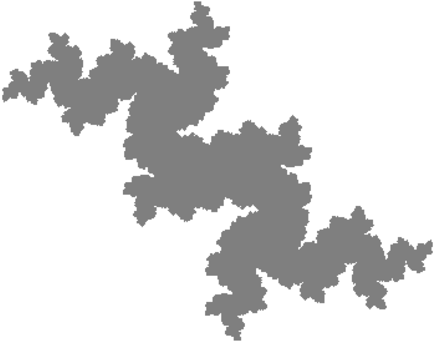


Adaptive approximation $c = -0.25 + 0.74\,i$ level 7

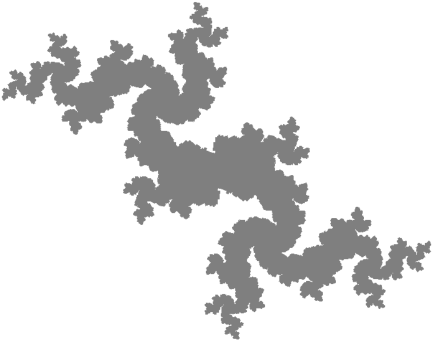
Adaptive approximation $c = -0.25 + 0.74\,i$ level 8



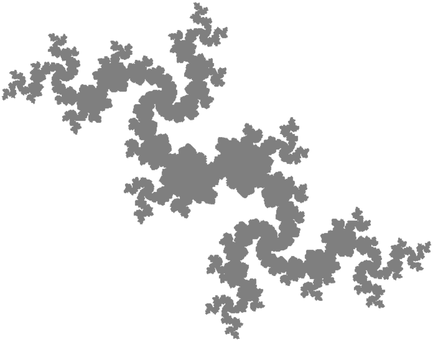
Adaptive approximation $c = -0.25 + 0.74\,i$ level 9



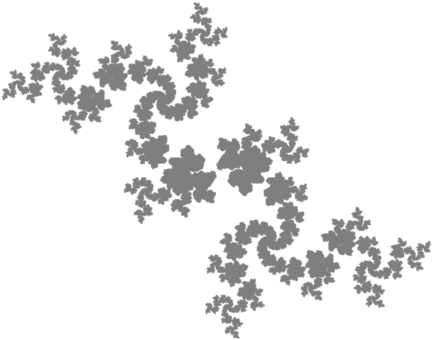
Adaptive approximation $c = -0.25 + 0.74\,i$ level 10



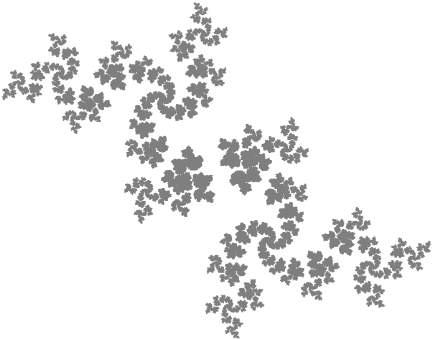
Adaptive approximation $c = -0.25 + 0.74\,i$ level 11



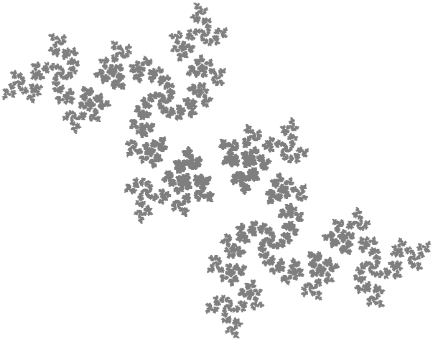
Adaptive approximation $c = -0.25 + 0.74\,i$ level 12



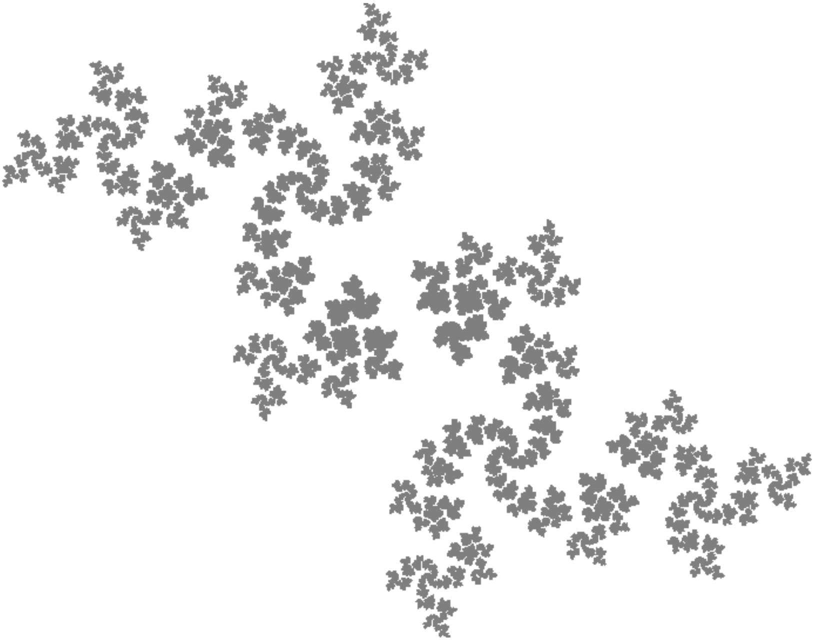
Adaptive approximation $c = -0.25 + 0.74\,i$ level 13



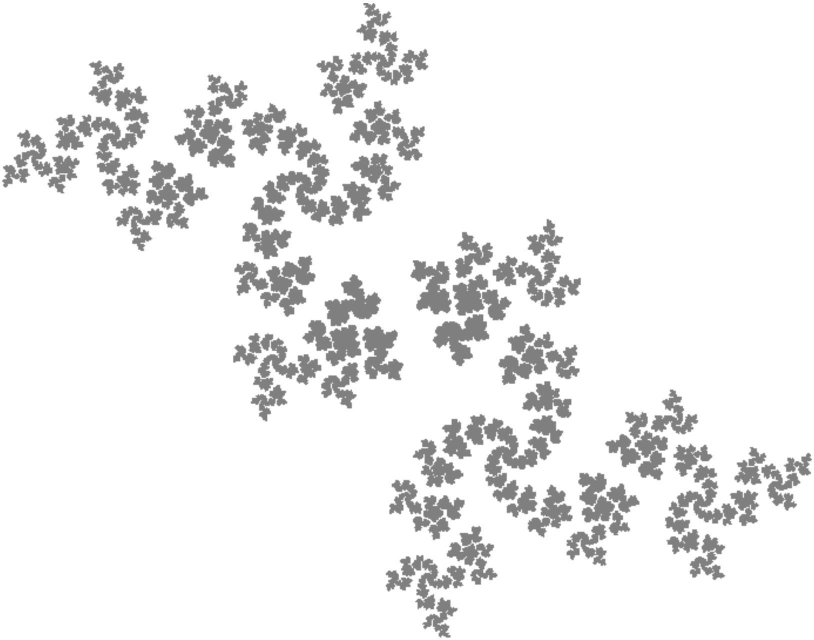
Adaptive approximation $c = -0.25 + 0.74\,i$ level 14



Adaptive approximation $c = -0.25 + 0.74\, i$



Adaptive approximation $c = -0.25 + 0.74\,i$



Applications

- Image generation
- Point and box classification
- Fractal dimension of Julia set
- Area of filled Julia set
- Diameter of Julia set

Applications certified numerical results

- Image generation
  large images
  smaller images with anti-aliasing

- Point and box classification
  quadtree traversal + one function evaluation if gray

- Fractal dimension of Julia set (Ruelle)
  upper bound $\dim_H = 1 + \frac{|c|^2}{4 \log 2} + \cdots$

- Area of filled Julia set (Milnor)
  lower and upper bounds $\pi(1 - |p_1(c)|^2 - 3|p_2(c)|^2 - 5|p_3(c)|^2 - \cdots)$

- Diameter of Julia set
  lower and upper bounds

Area of filled Julia sets after Milnor

Inverse Böttcher map $\psi\colon \mathbb{C} \setminus \mathbb{D} \to \mathbb{C} \setminus K$

$$\psi(w^2) = \psi(w)^2 + c$$

Laurent series near ∞

$$\psi(w) = w\left(1 + \frac{a_2}{w^2} + \frac{a_4}{w^4} + \frac{a_6}{w^6} + \cdots\right)$$

$$a_2 = -\frac{c}{2} \qquad a_{2n} = \frac{1}{2}(a_n - a_n^2) - \sum_{\substack{2 \le j < n \\ j \text{ even}}} a_j a_{2n-j} \qquad a_{2n+1} = 0$$

Gronwall's area theorem

$$\text{area}(K) = \pi(1 - |a_2|^2 - 3|a_4|^2 - 5|a_6|^2 - \cdots)$$

Truncating series gives upper bounds

series converges slowly

Quadtree gives both lower and upper bounds

Area of filled Julia sets after Milnor

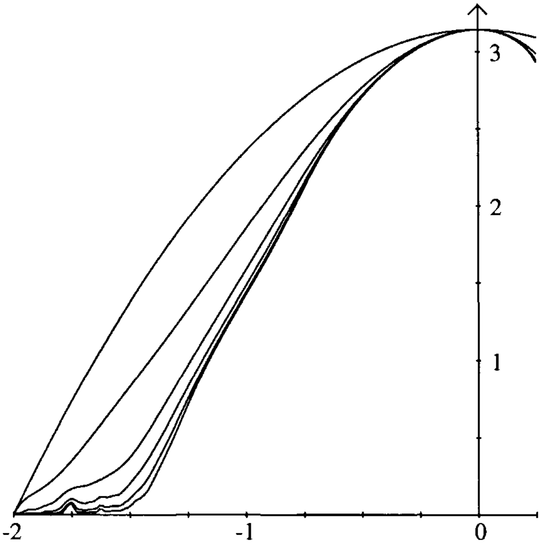
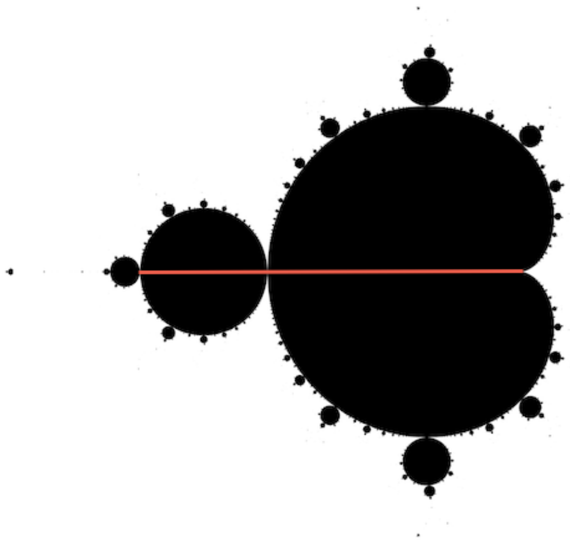
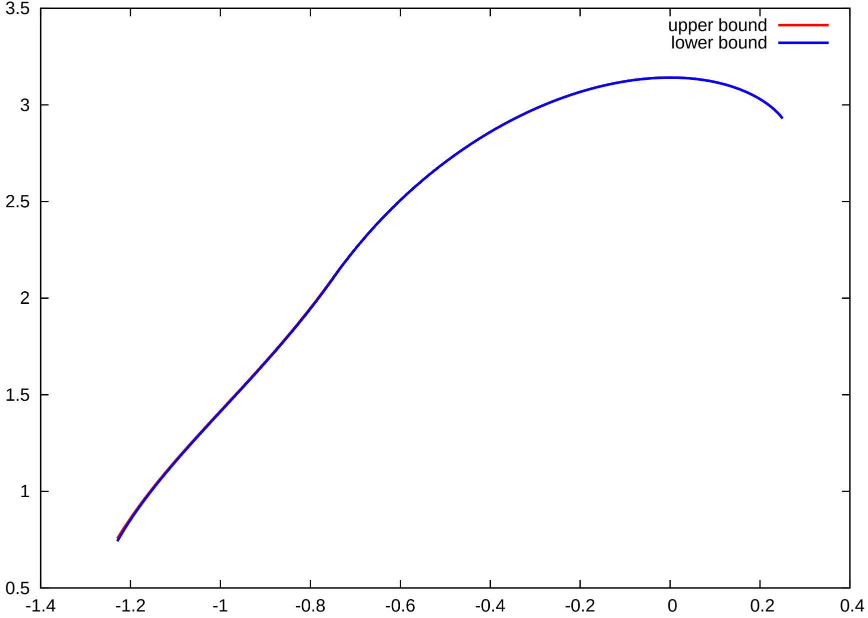


Figure 45. Upper bounds for the area of the filled Julia set for $f_c(z) = z^2 + c$ *in the range* $-2 \leq c \leq .25$.

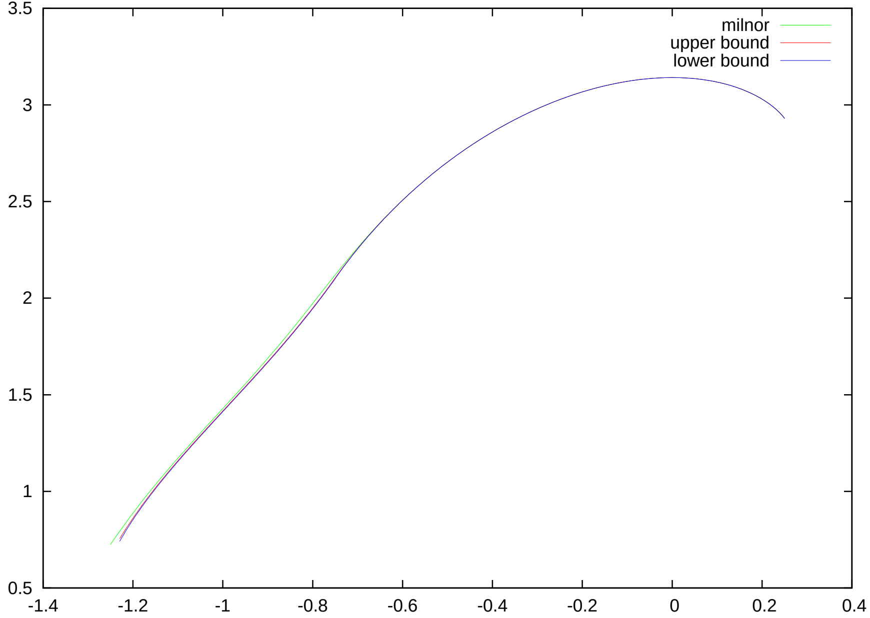
Area of filled Julia set $\quad -1.25 \le c \le 0.25$



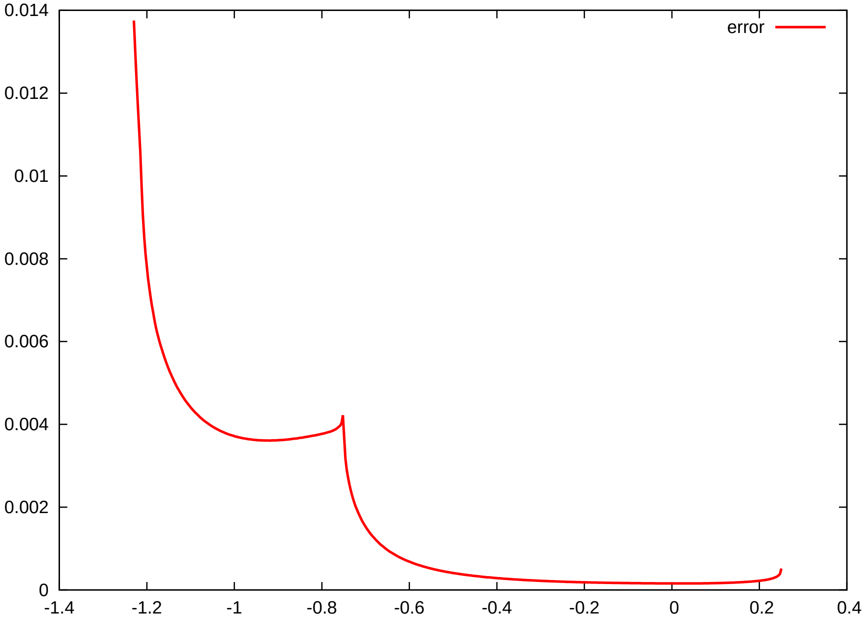
Area of filled Julia set $\quad -1.25 \le c \le 0.25 \quad$ level 19



Area of filled Julia set $-1.25 \leq c \leq 0.25$ level 19



Area of filled Julia set $-1.25 \leq c \leq 0.25$ level 19



Limitations

- Memory

- Need to explore $\Omega \supseteq [-R, R] \times [-R, R]$

Limitations

- Memory
  depth of quadtree and size of cell graph limited by available memory
  currently spatial resolution $\approx 4 \times 10^{-6}$
  cannot reach 20 levels

- Need to explore $\Omega \supseteq [-R, R] \times [-R, R]$
  even if region of interest is smaller
  limited amount of zoom
  limitation inherent to using cell mapping because f is transitive on J

Future work   higher-degree polynomials

- Escape radius
- Bounding box

Future work higher-degree polynomials

- Escape radius

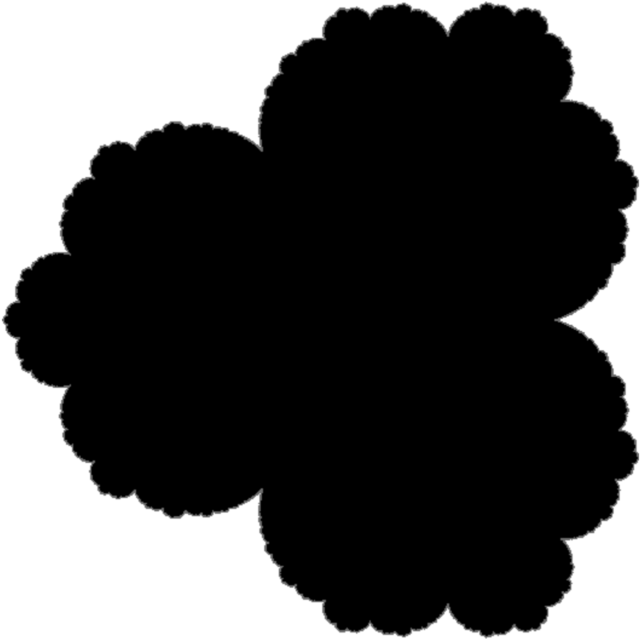
  $$R = \frac{1 + |a_d| + \cdots + |a_0|}{|a_d|}$$

  is an escape radius for $f(z) = a_d z^d + \cdots + a_0$ (Douady)

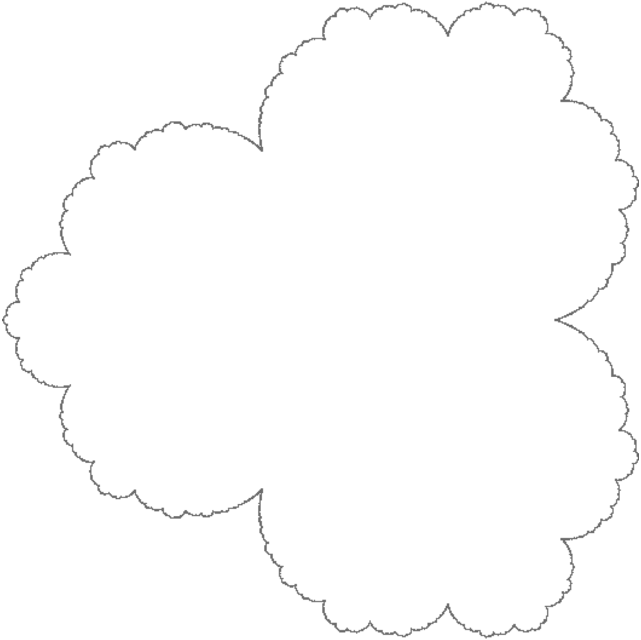
- Bounding box

  needs interval arithmetic with directed rounding

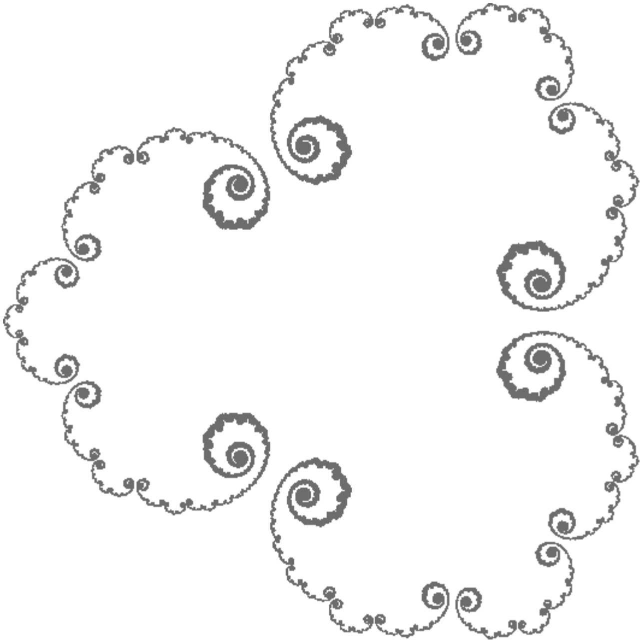
Cubic Julia set $z^3 + 0.38$



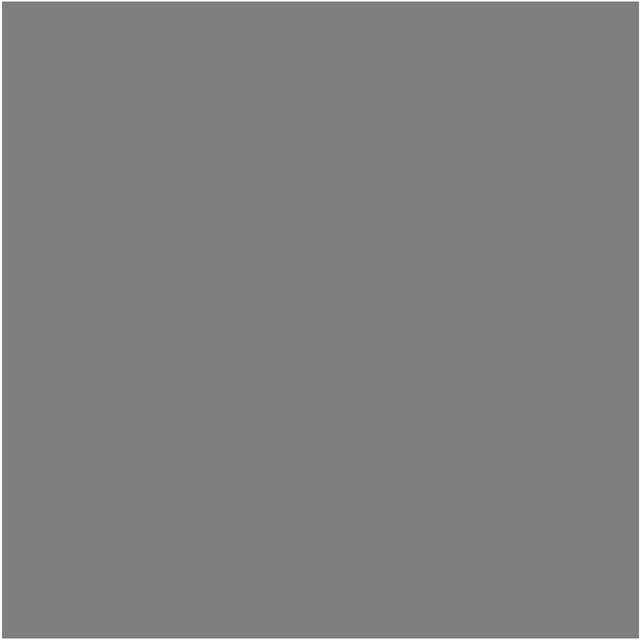
Cubic Julia set $z^3 + 0.38$

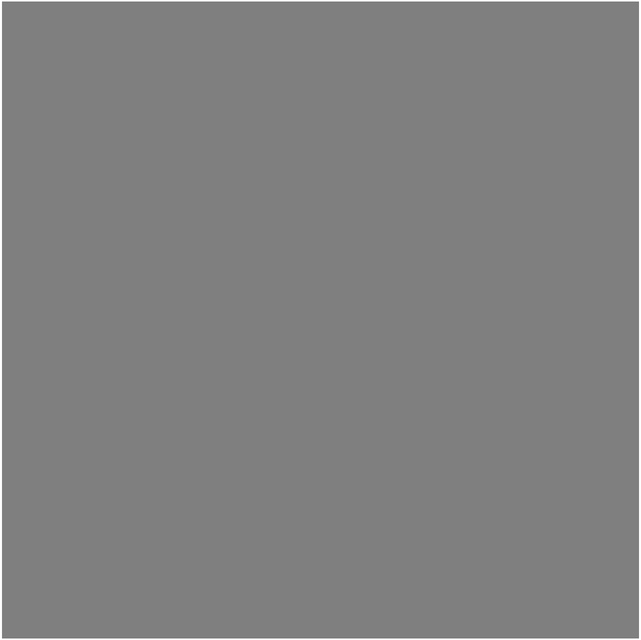


Cubic Julia set $z^3 + 0.41$

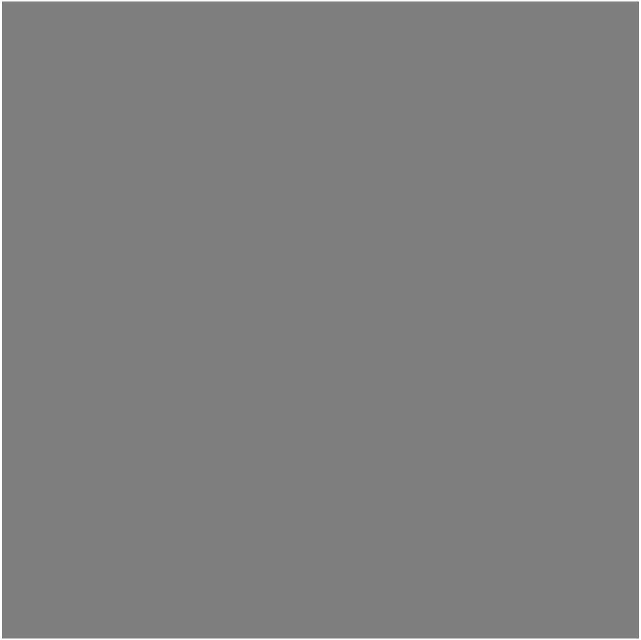


Cubic Julia set $z^3 - 3a^2 + b$ level 0



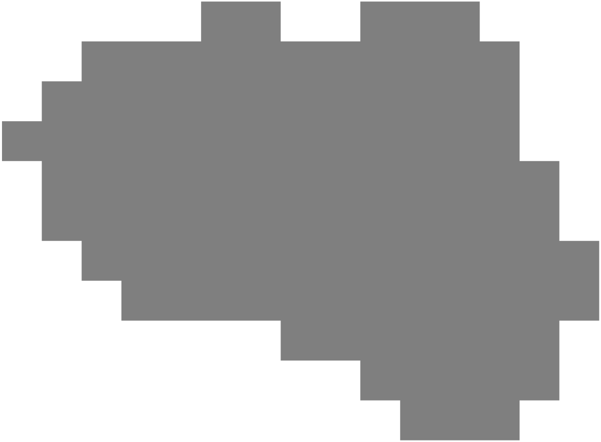


Cubic Julia set $z^3 - 3a^2 + b$ level 2



Cubic Julia set $z^3 - 3a^2 + b$ level 3

Cubic Julia set $z^3 - 3a^2 + b$ level 4

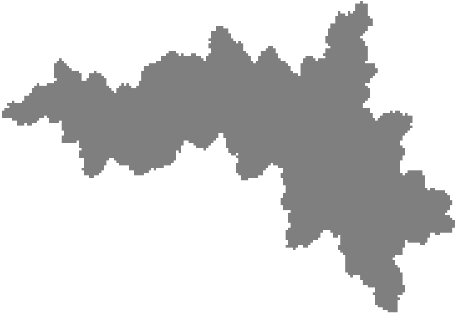


Cubic Julia set $z^3 - 3a^2 + b$ level 5

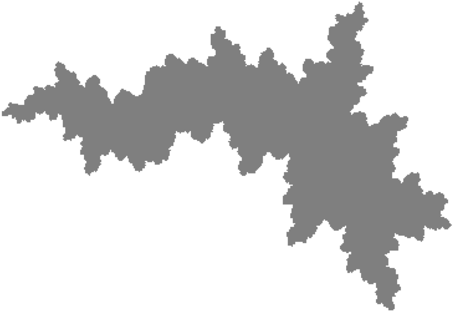
Cubic Julia set $z^3 - 3a^2 + b$ level 6

Cubic Julia set $z^3 - 3a^2 + b$ level 7

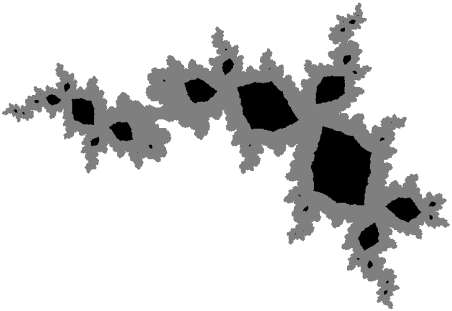
Cubic Julia set $z^3 - 3a^2 + b$ level 8



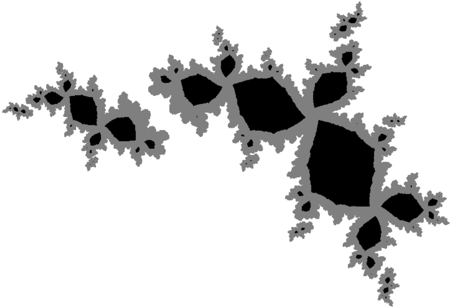
Cubic Julia set $z^3 - 3a^2 + b$ level 9



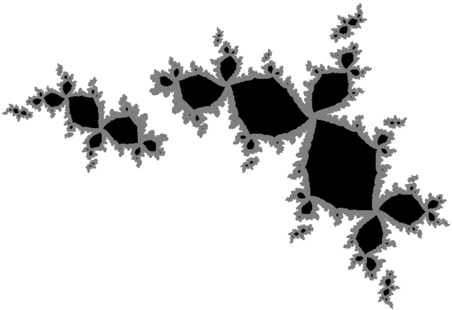
Cubic Julia set $z^3 - 3a^2 + b$ level 10



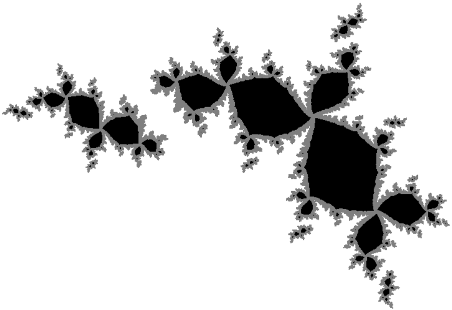
Cubic Julia set $z^3 - 3a^2 + b$ level 11



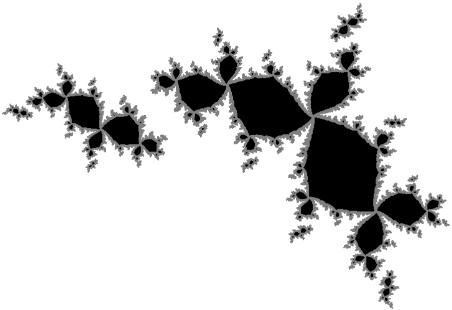
Cubic Julia set $z^3 - 3a^2 + b$ level 12



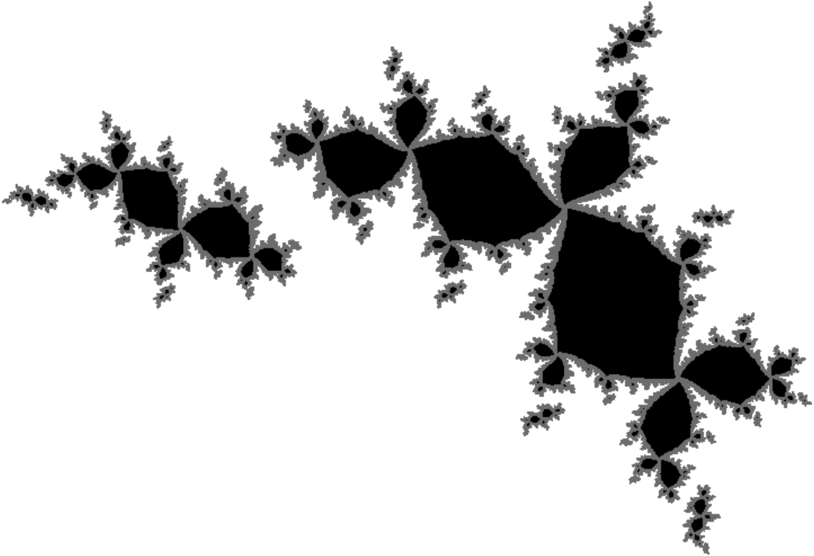
Cubic Julia set $z^3 - 3a^2 + b$ level 13



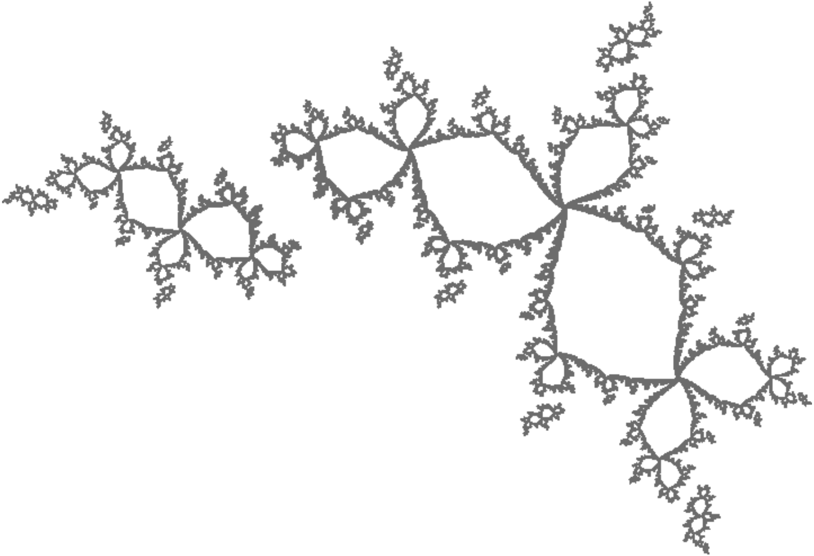
Cubic Julia set $z^3 - 3a^2 + b$ level 14



Cubic Julia set $z^3 - 3a^2 + b$



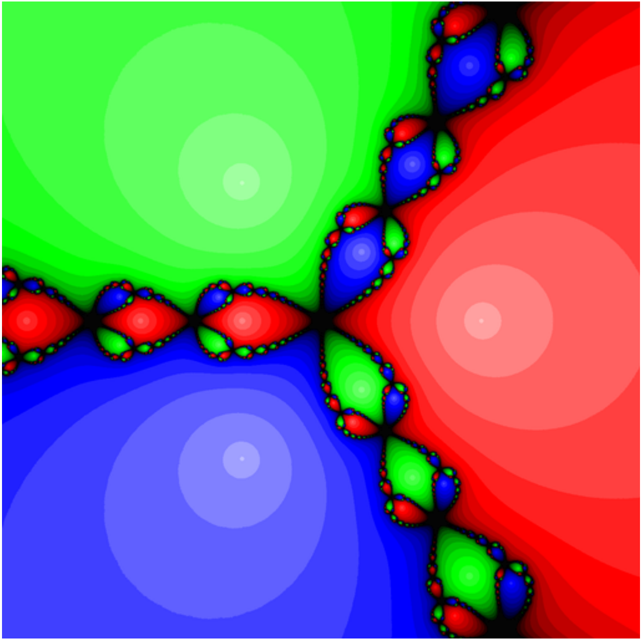
Cubic Julia set $z^3 - 3a^2 + b$



Newton's method

- Which points converge to which root? Cayley (1879)
- Points that do not converge form the Julia set
- No escape radius
- Need to find explicit attracting regions around roots?

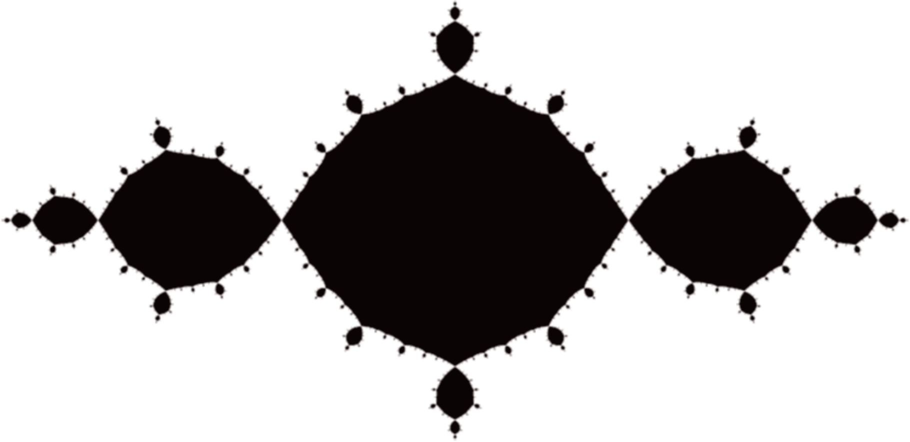
Newton's method $z^3 = 1$



Julia set panorama

`http://monge.visgraf.impa.br/panorama/viewer/index.html?img=../julia-256GP/julia.xml`

Images of Julia sets that you can trust

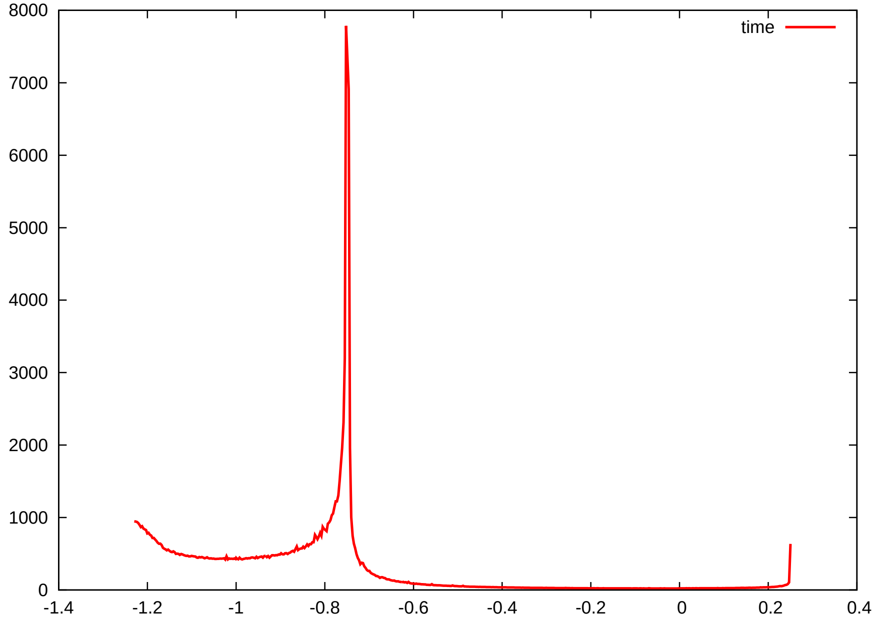


Thanks!

Related work

- M. Braverman and M. Yampolsky. *Computability of Julia sets*, volume 23 of *Algorithms and Computation in Mathematics*. Springer-Verlag, 2009.
- M. Dellnitz and A. Hohmann. A subdivision algorithm for the computation of unstable manifolds and global attractors. *Numerische Mathematik*, 75(3):293–317, 1997.
- C. S. Hsu. *Cell-to-cell mapping: A method of global analysis for nonlinear systems*. Springer-Verlag, 1987.
- J. Milnor. *Dynamics in one complex variable*, volume 160 of *Annals of Mathematics Studies*. Princeton University Press, third edition, 2006.
- R. E. Moore. *Interval Analysis*. Prentice-Hall, 1966.
- R. Rettinger and K. Weihrauch. The computational complexity of some Julia sets. In *Proceedings of the 35th Annual ACM Symposium on Theory of Computing*, pages 177–185. ACM, 2003.
- D. Saupe. Efficient computation of Julia sets and their fractal dimension. *Phys. D*, 28(3):358–370, 1987.

Area of filled Julia set $-1.25 \leq c \leq 0.25$ level 19



Interval arithmetic $f(z) = z^2 + c$

$$(x, y) \mapsto (x^2 - y^2 + a, 2xy + b)$$

```
function f(xmin,xmax,ymin,ymax)
        local x2min,x2max=isqr(xmin,xmax)
        local y2min,y2max=isqr(ymin,ymax)
        local xymin,xymax=imul(xmin,xmax,ymin,ymax)
        return x2min-y2max+a,x2max-y2min+a,2*xymin+b,2*xymax+b
end

function imul(xmin,xmax,ymin,ymax)
        local mm=xmin*ymin
        local mM=xmin*ymax
        local Mm=xmax*ymin
        local MM=xmax*ymax
        local m,M=mm,mm
        if m>mM then m=mM elseif M<mM then M=mM end
        if m>Mm then m=Mm elseif M<Mm then M=Mm end
        if m>MM then m=MM elseif M<MM then M=MM end
        return m,M
end
```

Interval arithmetic $f(z) = z^2 + c$

$$(x, y) \mapsto (x^2 - y^2 + a, 2xy + b)$$

```
function f(xmin,xmax,ymin,ymax)
        local x2min,x2max=isqr(xmin,xmax)
        local y2min,y2max=isqr(ymin,ymax)
        local xymin,xymax=imul(xmin,xmax,ymin,ymax)
        return x2min-y2max+a,x2max-y2min+a,2*xymin+b,2*xymax+b
end

function isqr(xmin,xmax)
        local u=xmin^2
        local v=xmax^2
        if xmin<=0 and 0<=xmax then
                if u<v then return 0,v else return 0,u end
        else
                if u<v then return u,v else return v,u end
        end
end
```

Interval arithmetic $f(z) = z^3 + c$

$$(x, y) \mapsto (x^3 - 3xy^2 + a, -y^3 + 3x^2y + b)$$

```
function f(xmin,xmax,ymin,ymax)
        local x2min,x2max=isqr(xmin,xmax)
        local y2min,y2max=isqr(ymin,ymax)
        local xy2min,xy2max=imul(xmin,xmax,y2min,y2max)
        local x2ymin,x2ymax=imul(x2min,x2max,ymin,ymax)
        local x3min,x3max=icub(xmin,xmax)
        local y3min,y3max=icub(ymin,ymax)
        return   x3min-3*xy2max+a, x3max-3*xy2min+a,
                -y3max+3*x2ymin+b,-y3min+3*x2ymax+b
end

function icub(xmin,xmax)
        return xmin^3,xmax^3
end
```