Instituto de Matemática
Pura e Aplicada

Interval methods for computer graphics and geometric modeling

Luiz Henrique de Figueiredo

Motivation

Basic problems in computer graphics and geometric modeling typically reduce to solving systems of nonlinear equations:

$$f_1(x_1, \ldots, x_n) = 0$$
$$\ldots$$
$$f_m(x_1, \ldots, x_n) = 0$$

Motivation – rendering an implicit surface with ray casting

Implicit surface

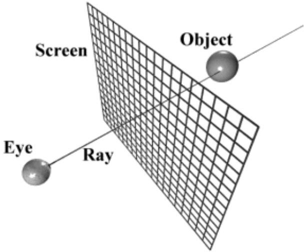
$$h(x, y, z) = 0, \qquad h\colon \mathbf{R}^3 \to \mathbf{R}$$

Ray

$$r(t) = e + t \cdot v = (x(t), y(t), z(t)), \quad t \in [0, \infty)$$

Ray intersects surface when

$$f(t) = h(r(t)) = 0$$



First intersection occurs at smallest zero of f in $[0, \infty)$

Need all zeros for CSG models

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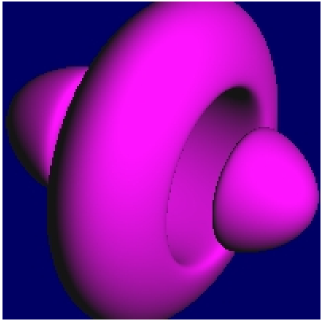

$$4(x^4 + (y^2 + z^2)^2) + 17x^2(y^2 + z^2) - 20(x^2 + y^2 + z^2) + 17 = 0$$

Motivation – plotting an implicit curve

Implicit curve

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$$0.004 + 0.110x - 0.177y - 0.174x^2 + 0.224xy - 0.303y^2$$
$$-0.168x^3 + 0.327x^2y - 0.087xy^2 - 0.013y^3 + 0.235x^4$$
$$-0.667x^3y + 0.745x^2y^2 - 0.029xy^3 + 0.072y^4 = 0$$

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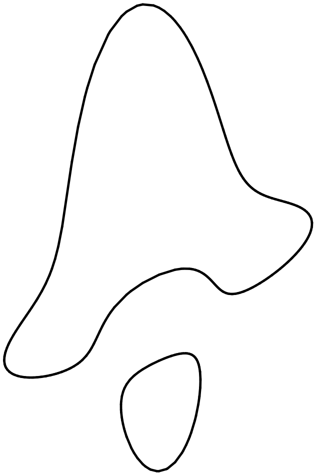

Motivation – intersecting two parametric surfaces

Parametric surfaces

$$g_1 \colon D_1 \subset \mathbf{R}^2 \to \mathbf{R}^3$$

$$g_2 \colon D_2 \subset \mathbf{R}^2 \to \mathbf{R}^3$$

Intersection

$$g_1(u_1, v_1) - g_2(u_2, v_2) = 0$$

$$x_1(u_1, v_1) - x_2(u_2, v_2) = 0$$

$$y_1(u_1, v_1) - y_2(u_2, v_2) = 0$$

$$z_1(u_1, v_1) - z_2(u_2, v_2) = 0$$

Motivation – intersecting two parametric surfaces

Parametric surfaces

$$g_1 : D_1 \subset \mathbf{R}^2 \to \mathbf{R}^3$$

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$$g_1(u_1, v_1) - g_2(u_2, v_2) = 0$$

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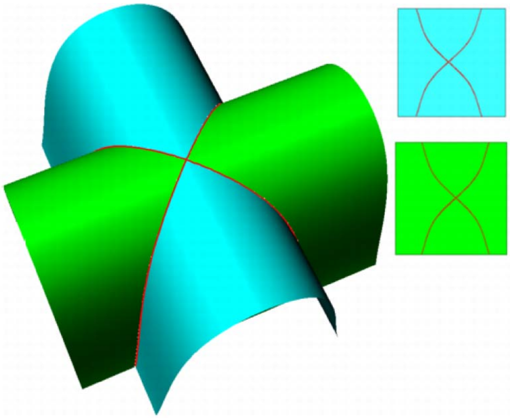

Hongwei Lin, Yang Qin, Hongwei Liao, Yunyang Xiong
"Affine Arithmetic-Based B-Spline Surface Intersection with GPU Acceleration"
IEEE TVCG, 2014

Motivation

Basic problems in computer graphics and geometric modeling typically reduce to solving systems of nonlinear equations:

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\begin{gathered}
f_1(x_1, \ldots, x_n) = 0 \\
\cdots \\
f_m(x_1, \ldots, x_n) = 0
\end{gathered}
$$

Low-dimensional solutions

Motivation

Basic problems in computer graphics and geometric modeling typically reduce to solving systems of nonlinear equations:

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Low-dimensional solutions $\implies$ sampling unreliable

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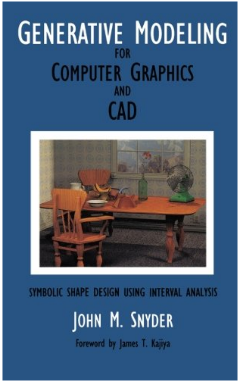

1992

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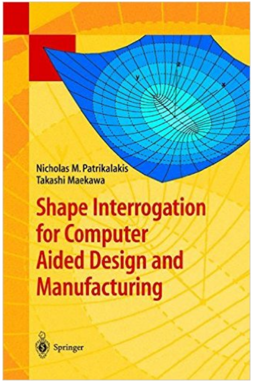

2002

interval arithmetic

Interval arithmetic

Moore (1960)

Introduced to improve reliability of numerical computations through automated a posteriori error analysis of both rounding errors in floating-point arithmetic and measurement errors in input data

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IA can probe the global behavior of mathematical functions

IA provides reliable bounds for the values of a function over whole regions of its domain

Avoid costly and unreliable point sampling

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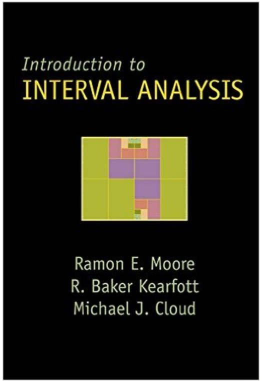

2009

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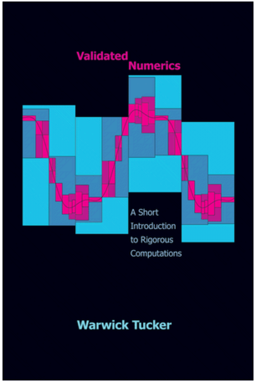

2011

Interval arithmetic

Represent quantities as intervals

$$x \sim [a, b] \implies x \in [a, b]$$

Operate with intervals generating other intervals

$$\begin{aligned}
[a, b] + [c, d] &= [a + c, b + d] \\
[a, b] \times [c, d] &= [\min\{ac, ad, bc, bd\}, \max\{ac, ad, bc, bd\}] \\
[a, b] \,/\, [c, d] &= [a, b] \times [1/d, 1/c] \\
[a, b]^2 &= [0, \max(a^2, b^2)] \quad \text{when} \quad 0 \in [a, b] \\
\exp\,[a, b] &= [\exp(a), \exp(b)]
\end{aligned}$$

Automatic extensions for complicated expressions with operator overloading

Interval arithmetic

Every expression f has an interval extension F :

$$x_i \in X_i \implies f(x_1, \ldots, x_n) \in F(X_1, \ldots, X_n)$$

Reliable range estimates without point sampling

$$F(X) \supseteq f(X) = \{f(x) : x \in X\}$$

In particular:

$$\begin{aligned} 0 \notin F(X) &\implies 0 \notin f(X) \\ &\implies f = 0 \text{ has no solution in } X \end{aligned}$$

This is a computational proof!

Interval arithmetic

Given a system of nonlinear equations

$$f_1(x_1, \ldots, x_n) = 0$$
$$\cdots$$
$$f_m(x_1, \ldots, x_n) = 0$$

and interval extensions

$$F_1, \ldots, F_m$$

there are no solutions in a box $X = X_1 \times \cdots \times X_n \subseteq \mathbf{R}^n$ if

$$0 \notin F_k(X) \quad \text{for some } k$$

There may be solutions in X if

$$0 \in F_k(X) \quad \text{for all } k$$

Interval probing of implicit curve

$$y^2 - x^3 + x = 0$$

$$\begin{aligned} X &= [-2, -1] \\ Y &= [1, 2] \end{aligned}$$

Interval probing of implicit curve

$$y^2 - x^3 + x = 0$$

$$\begin{aligned}
X &= [-2, -1] \\
Y &= [1, 2] \\
X^3 &= [-8, -1] \\
-X^3 &= [1, 8] \\
-X^3 + X &= [-1, 7]
\end{aligned}$$

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Y^2 &= [1, 4] \\
Y^2 - X^3 + X &= [0, 11]
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Y^2 &= [1, 4] \\
Y^2 - X^3 + X &= [0, 11] \qquad \text{exact} = [1, 10]
\end{aligned}$$

Interval estimates not tight, but improve as intervals shrink

Interval probing of implicit curve

$$y^2 - x^3 + x = 0$$

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Y^2 - X^3 + X &= [0, 11] \qquad \text{exact} = [1, 10]
\end{aligned}$$

Interval estimates not tight, but improve as intervals shrink $\implies$ divide-and-conquer

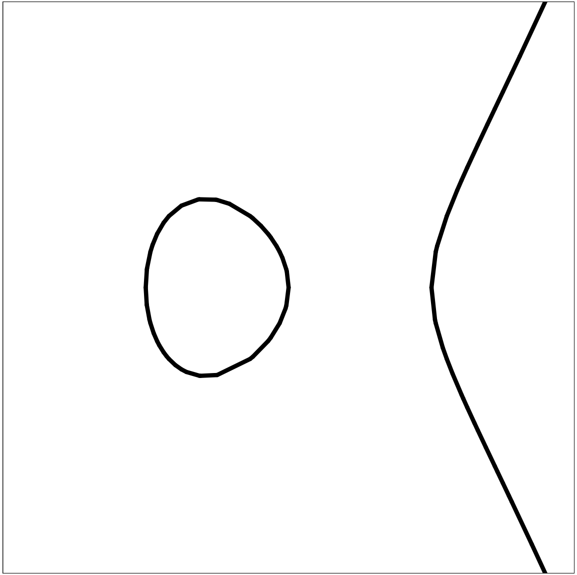
Interval probing of implicit curve

$y^2 - x^3 + x = 0$

$X \times Y = [-2, -1] \times [1, 2]$

$F(X, Y) = [0, 11]$ maybe

$f(X, Y) = [1, 10]$ no



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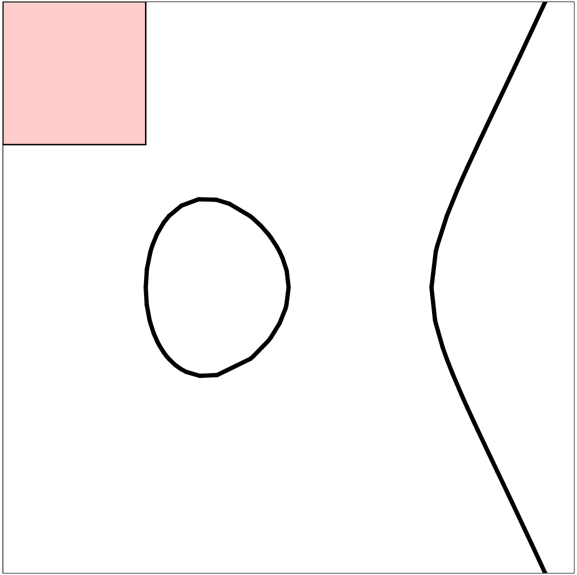

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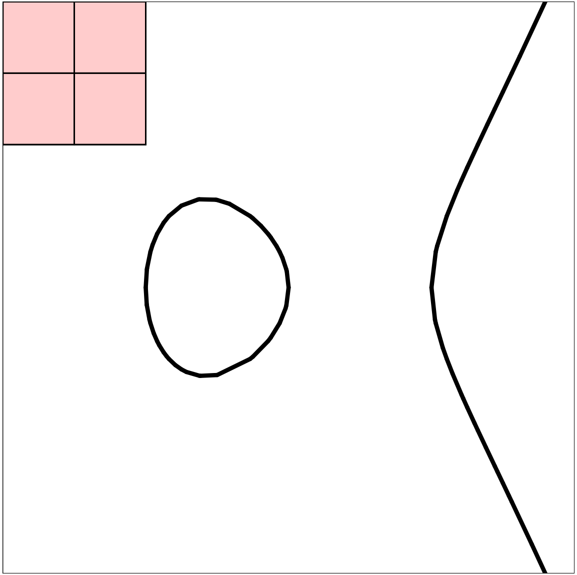
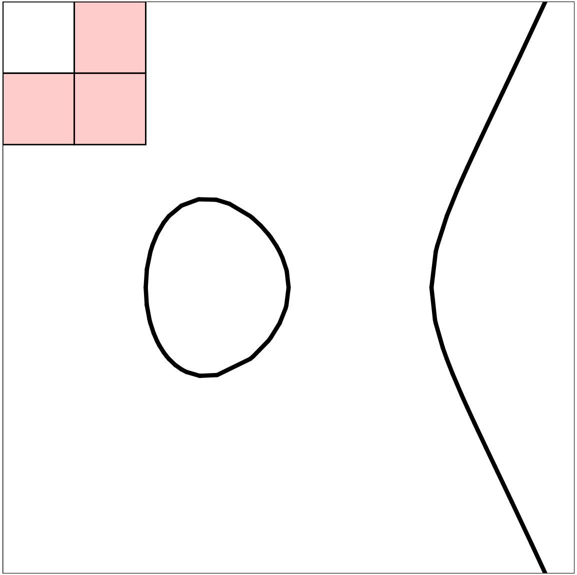


Interval probing of implicit curve

$y^2 - x^3 + x = 0$

$X \times Y = [-2, -1.5] \times [1.5, 2]$

$F(X, Y) = [3.625, 10.5]$ no

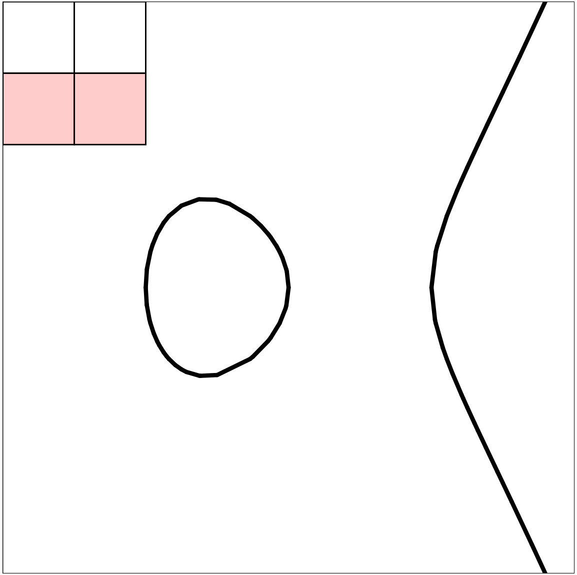


Interval probing of implicit curve

$$y^2 - x^3 + x = 0$$

$$X \times Y = [-1.5, -1] \times [1.5, 2]$$

$$F(X, Y) = [1.75, 6.375]$$ no

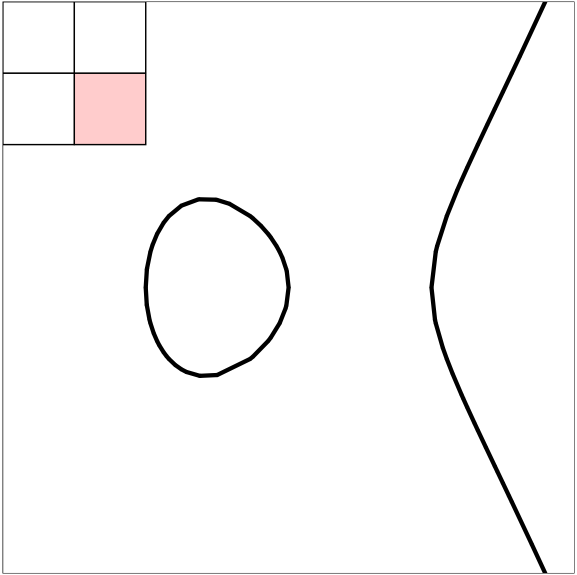


Interval probing of implicit curve

$$y^2 - x^3 + x = 0$$

$$X \times Y = [-2, -1.5] \times [1, 1.5]$$

$$F(X, Y) = [2.375, 8.75]$$ no



Interval probing of implicit curve

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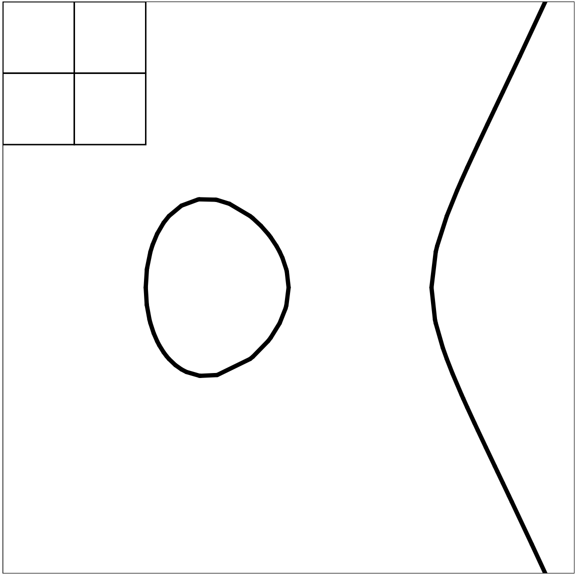
$F(X, Y) = [0.5, 4.625]$ no



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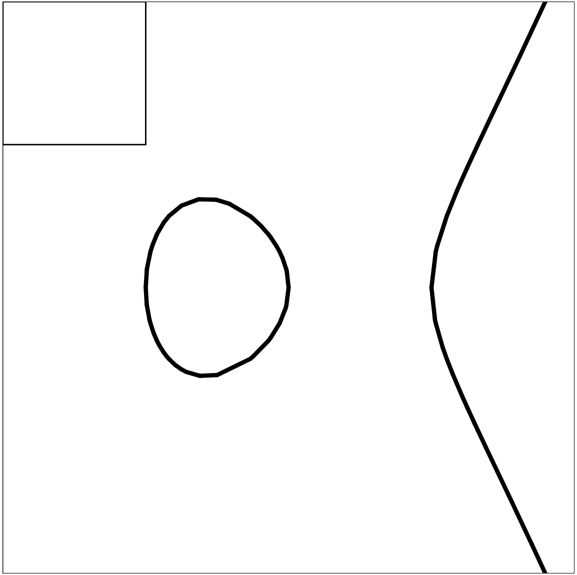
$f(X, Y) = [1, 10]$ no



Adaptive domain subdivision

To solve $f(x) = 0$ in $\Omega \subseteq \mathbf{R}^n$

call *explore*(Ω)

```
procedure explore(X)
   if 0 ∉ F(X) then
         discard X
   elseif small(X) then
         output X
   else
         X_1, ..., X_k ← subdivide(X)
         for each i do explore(X_i)
   end
end
```

Adaptive domain subdivision

To solve $f(x) = 0$ in $\Omega \subseteq \mathbf{R}^n$

call *explore*(Ω)

procedure *explore*(X)
  if $0 \notin F(X)$ **then**
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  end
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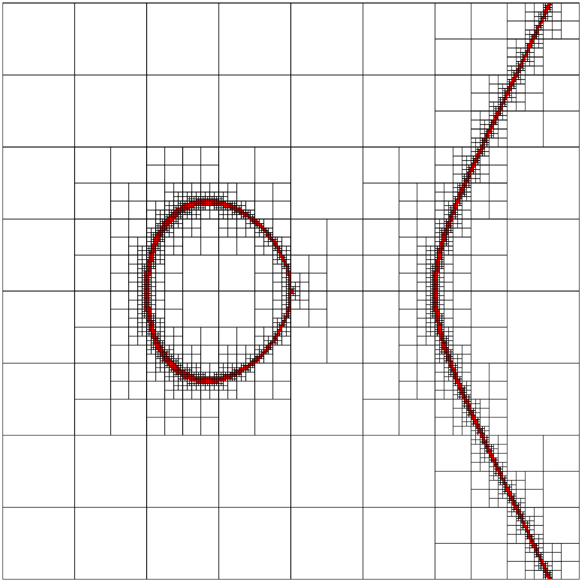
Suffern–Fackerell (1991), Snyder (1992)



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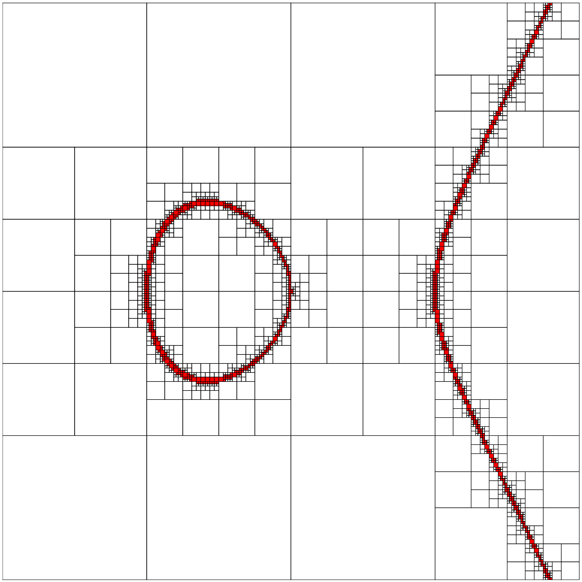

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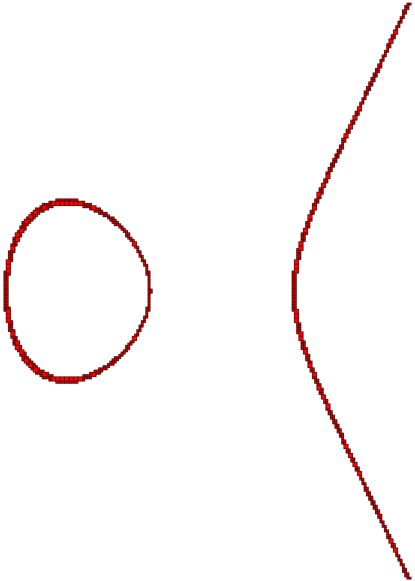
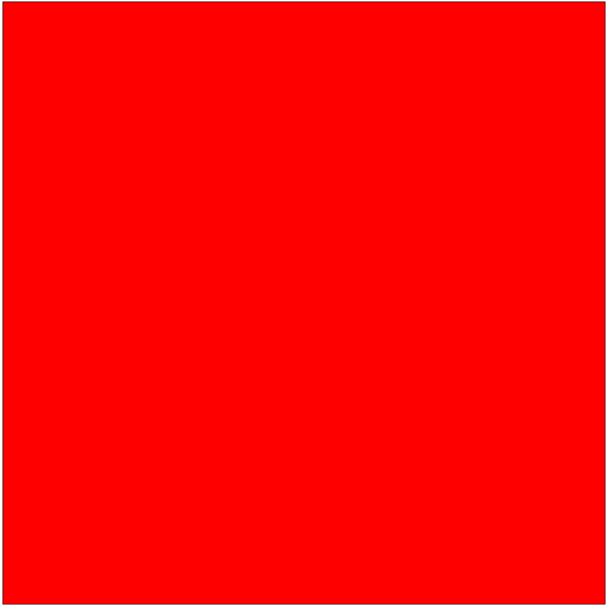


Implicit curves



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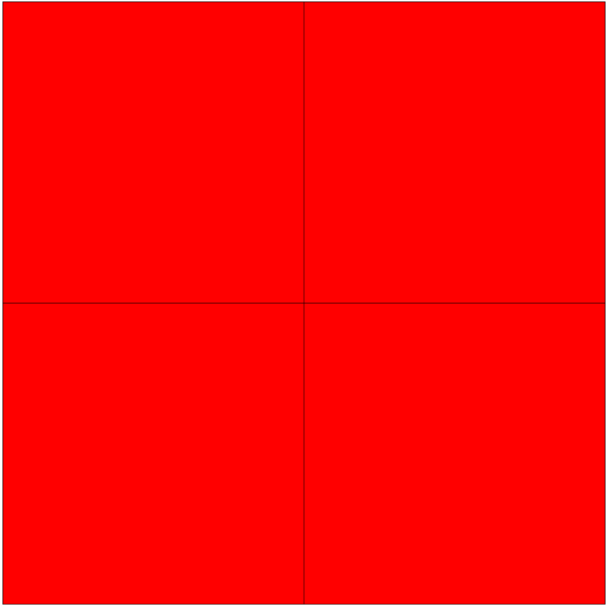

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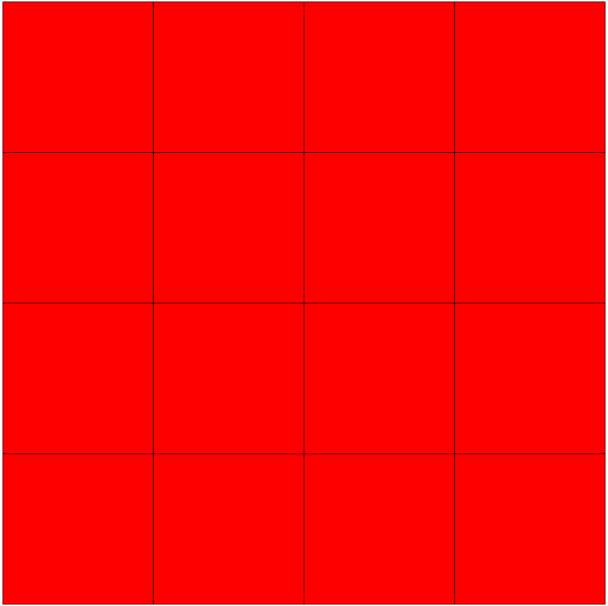

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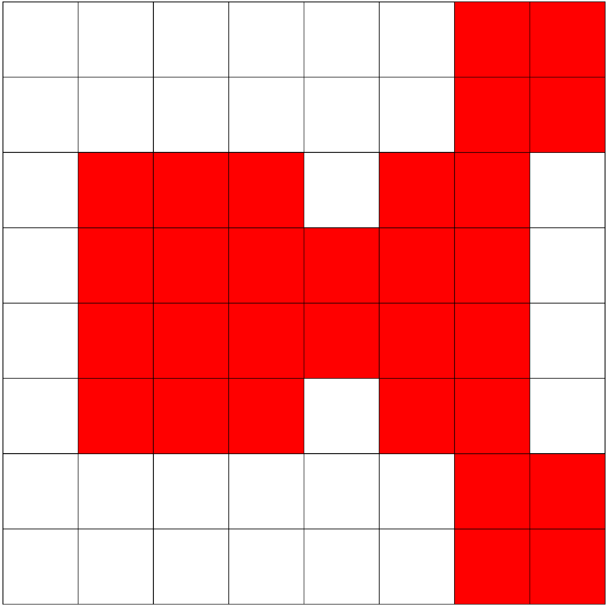

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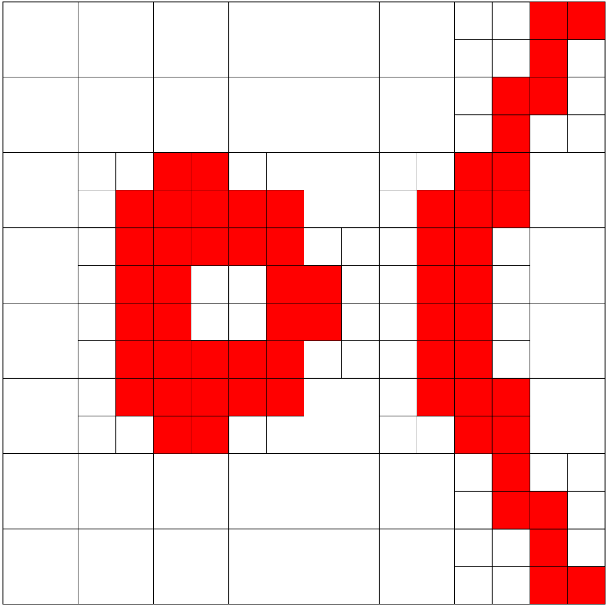

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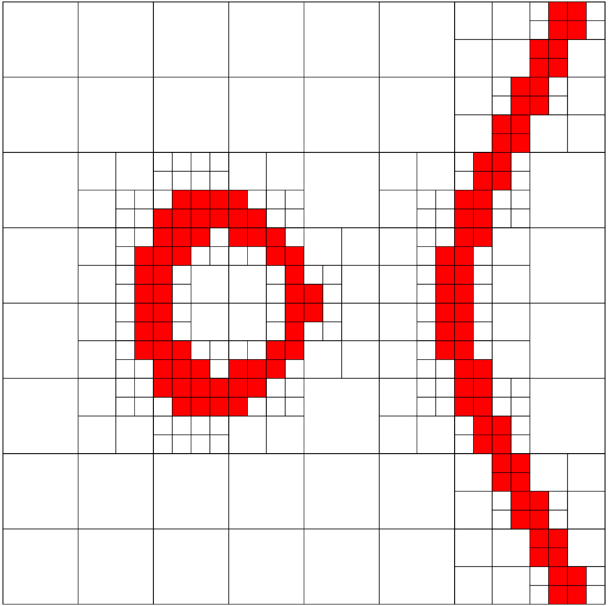

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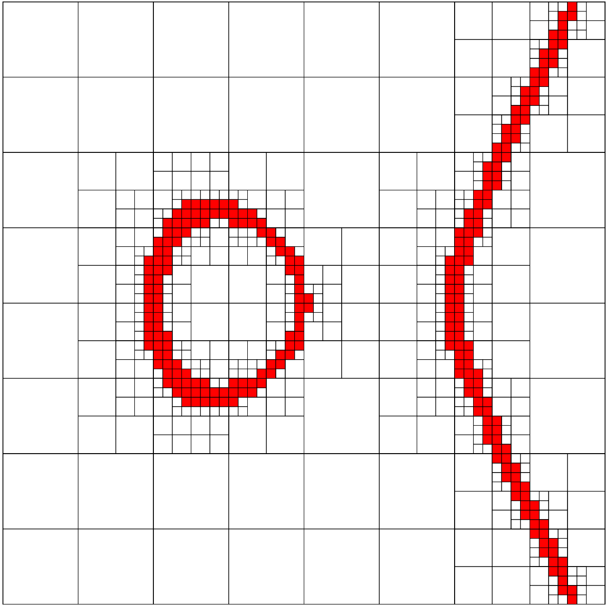

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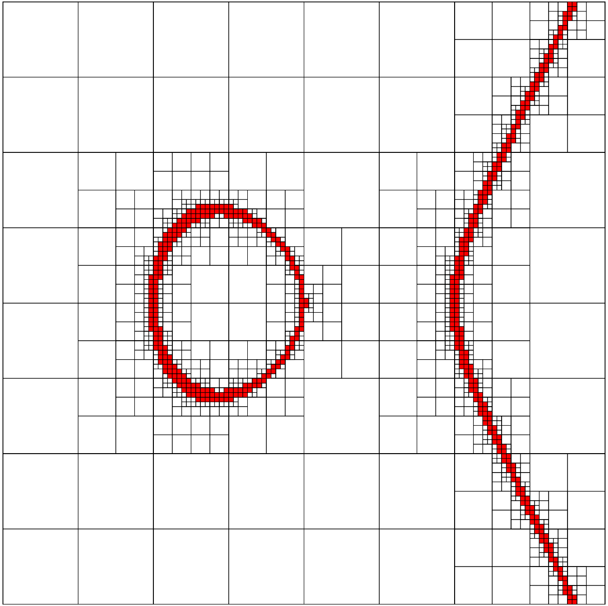

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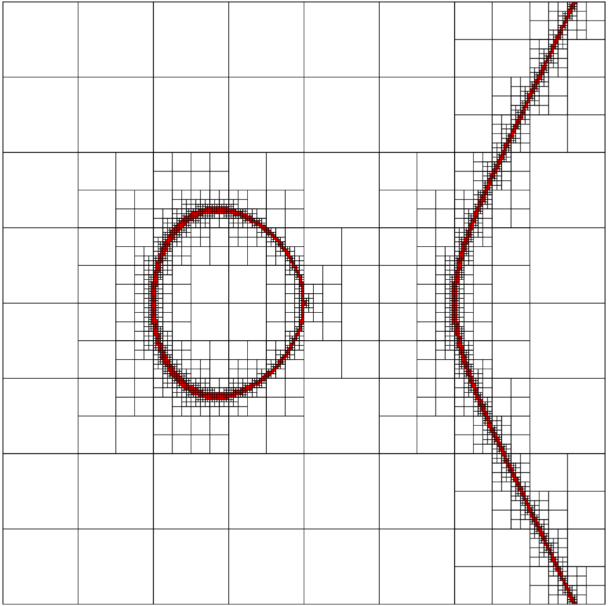

Implicit curves

F inclusion function for f

procedure *explore*(X)
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    output X
  else
    $X_1, \ldots, X_k \leftarrow$ *subdivide*(X)
    for each i **do** explore(X_i)
  end
end

spatial adaption

Suffern–Fackerell (1991), Snyder (1992)

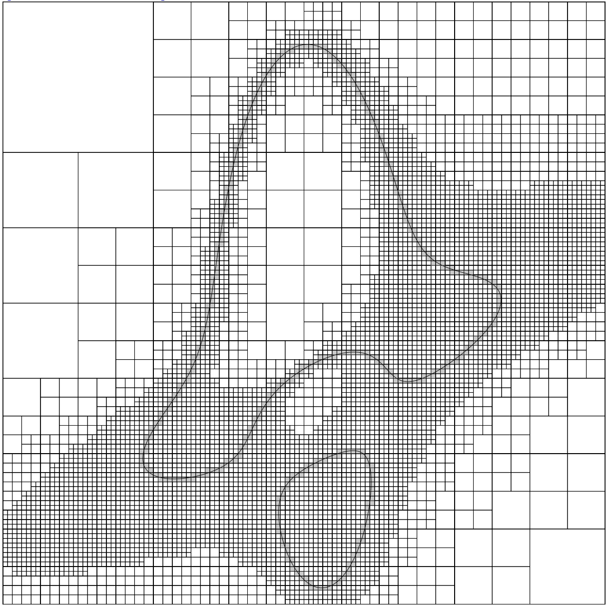
G inclusion function for $\operatorname{grad} f$

procedure *explore*(X)
  if $0 \notin F(X)$ **then**
    discard X
  elseif *small*(X) **or** *small*$(G(X))$ **then**
    approx(X)
  else
    $X_1, \ldots, X_k \leftarrow$ *subdivide*(X)
    for each i **do** explore(X_i)
  end
end

geometric adaption

Lopes–Oliveira–Figueiredo (2002)

Implicit curves – spatial adaption



Implicit curves – geometric adaption

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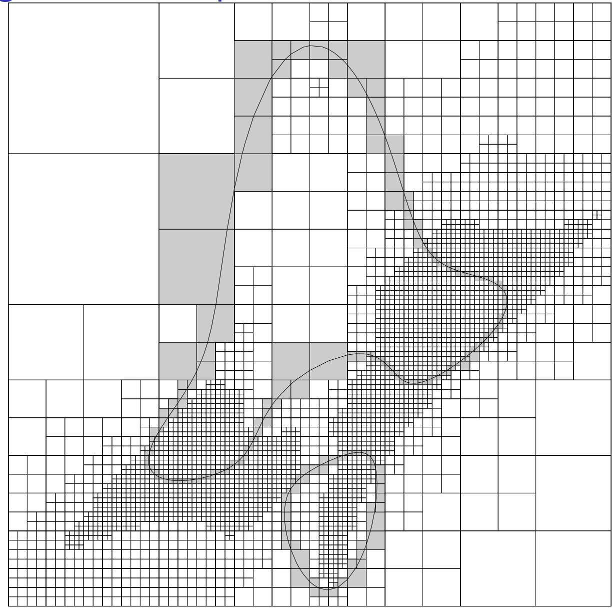

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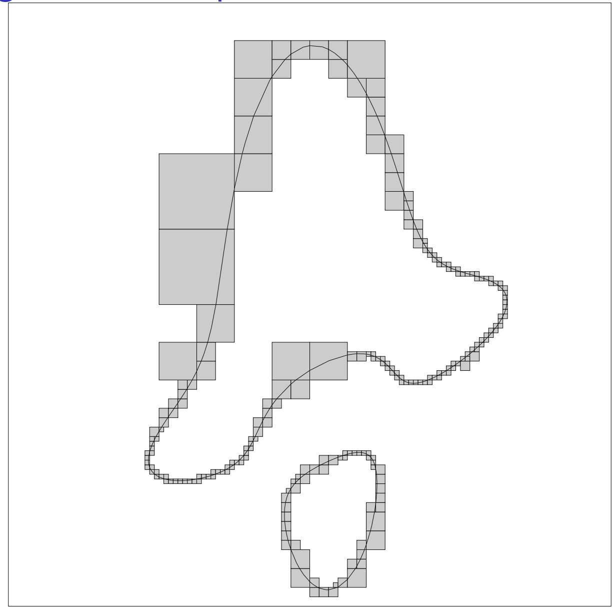

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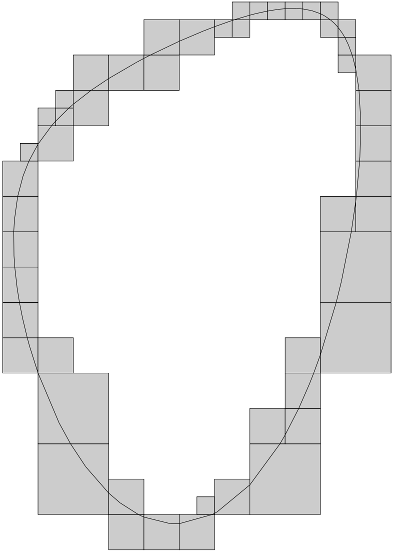
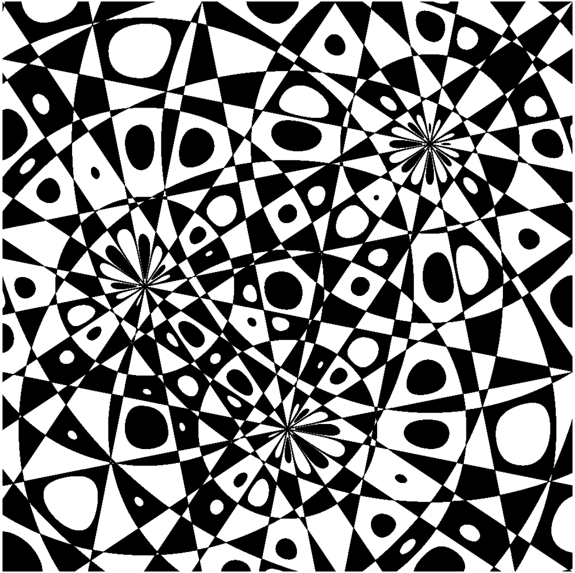


more applications

Implicit regions

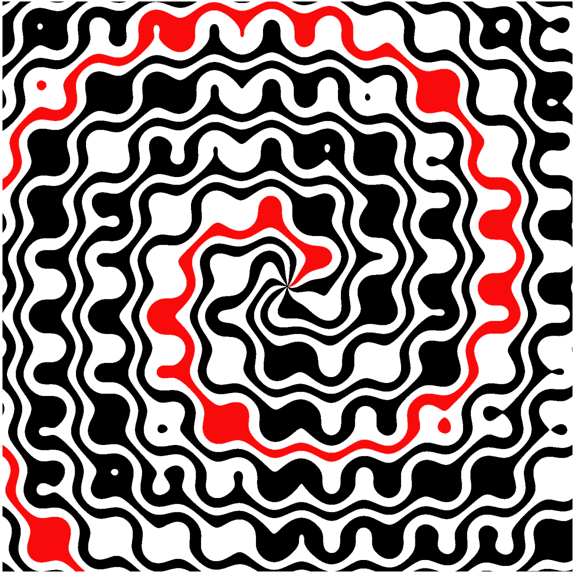
Tupper (2001)



GrafEq

Implicit regions

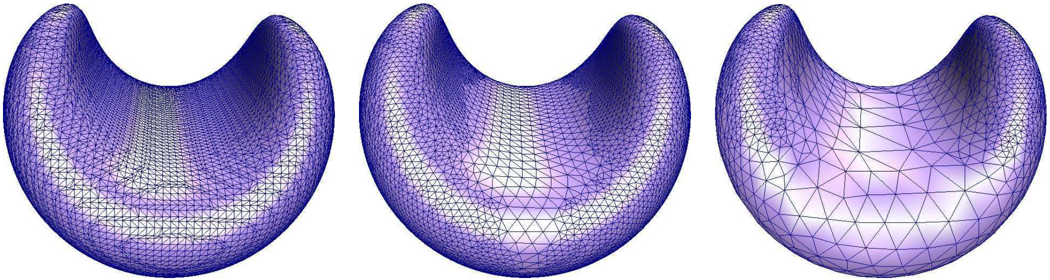
Tupper (2001)



GrafEq

Implicit surfaces

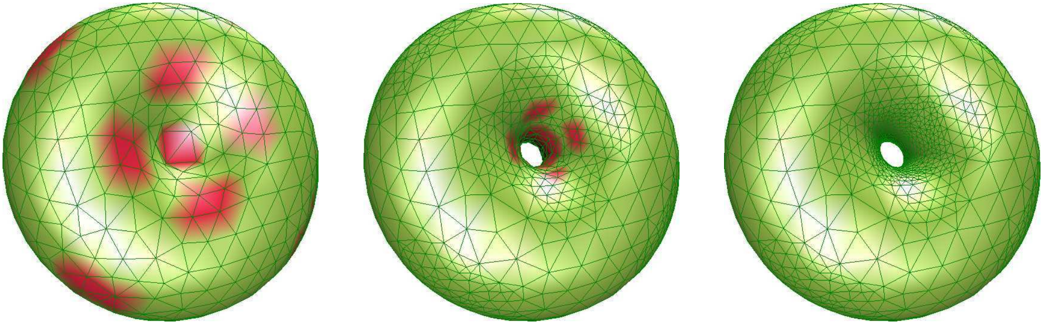
Paiva–Lopes–Lewiner–Figueiredo (2006)



track regions of high curvature

Implicit surfaces

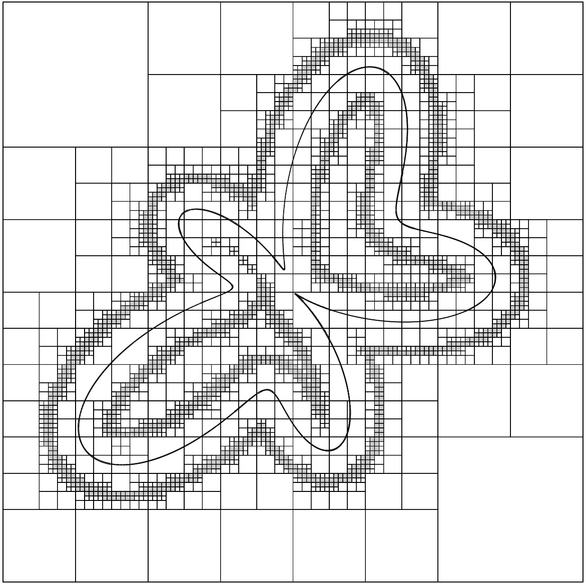
Paiva–Lopes–Lewiner–Figueiredo (2006)



flag regions of possible topological ambiguity

Offsets of parametric curves

Oliveira–Figueiredo (2003)



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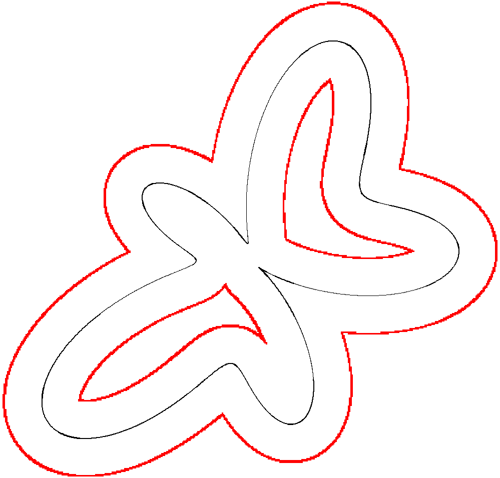
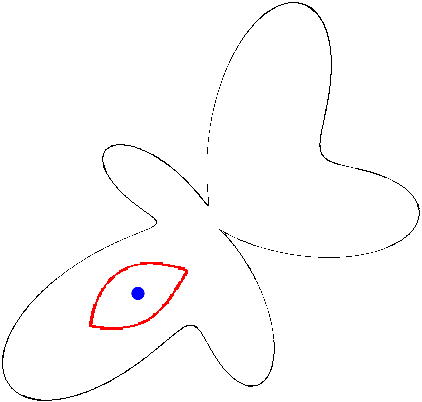


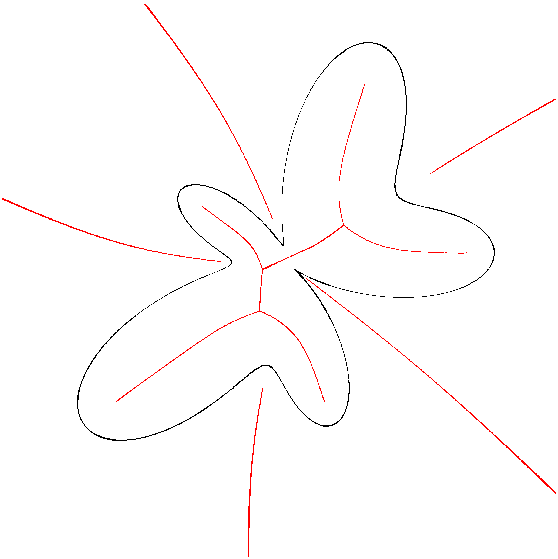
Bisectors of parametric curves

Oliveira–Figueiredo (2003)



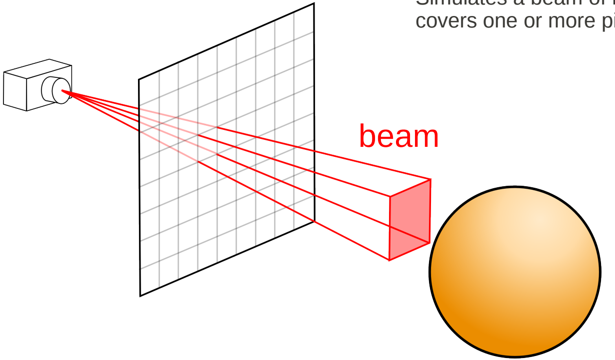
Medial axis of parametric curves

Oliveira–Figueiredo (2003)



Beam casting implicit surfaces

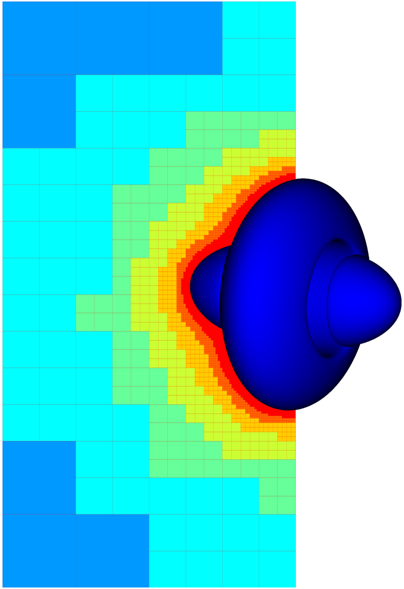
Ganacim–Figueiredo–Nehab (2011)



- Simulates a beam of rays that covers one or more pixels

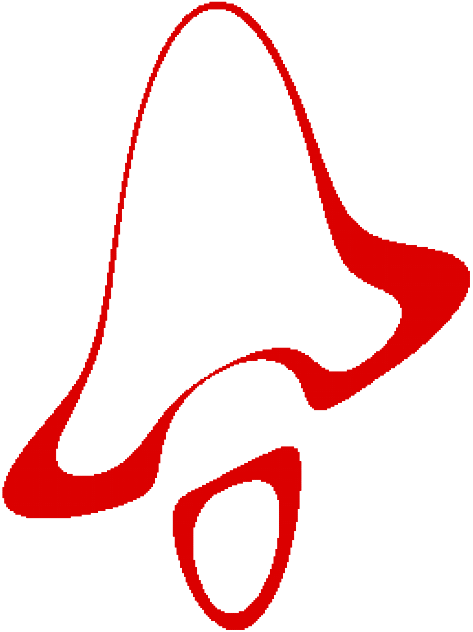
avoids sampling errors

also Flórez et al (2008)



Overestimation

$$0.004+0.110x-0.177y-0.174x^2+0.224xy-0.303y^2$$
$$-0.168x^3+0.327x^2y-0.087xy^2-0.013y^3+0.235x^4$$
$$-0.667x^3y+0.745x^2y^2-0.029xy^3+0.072y^4=0$$



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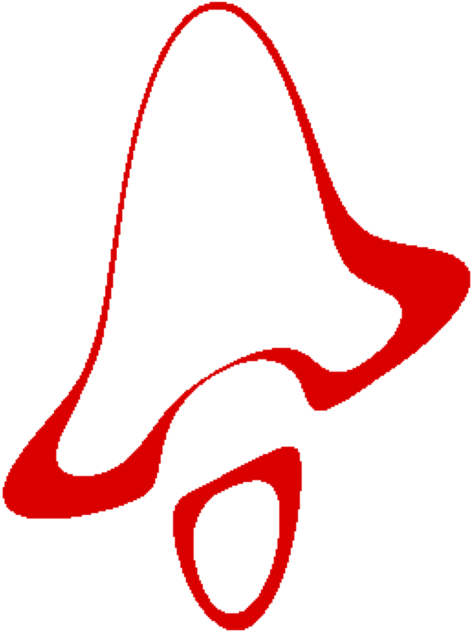
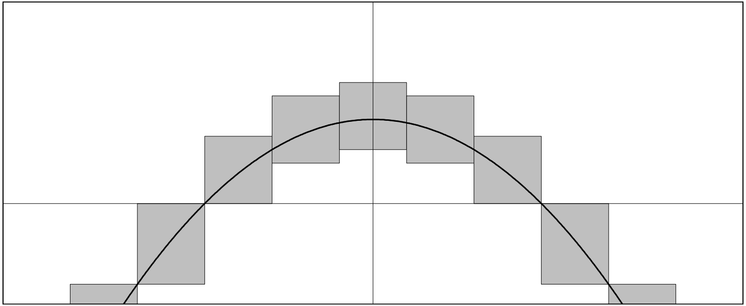
IA can't see correlations between operands



The dependency problem in interval arithmetic

$$f(x) = (10+x)(10-x) \text{ for } x \in [-2,2]$$

$$\begin{aligned} 10+x &= [8,12] \\ 10-x &= [8,12] \\ (10+x)(10-x) &= [64,144] \qquad \text{diam} = 80 \qquad \text{relative accuracy} = 20 \\ \text{exact range} &= [96,100] \qquad \text{diam} = 4 \end{aligned}$$



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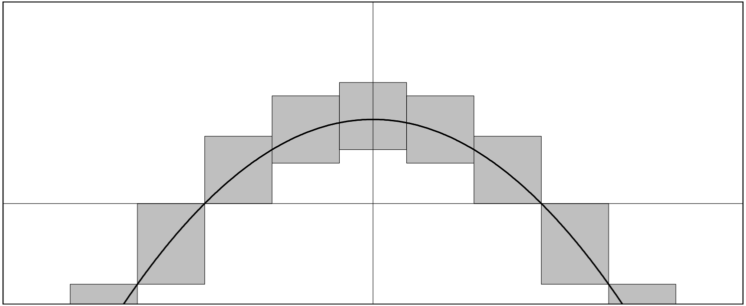
$$\begin{aligned}
10 + x &= [10-u, 10+u] \\
10 - x &= [10-u, 10+u] \\
(10+x)(10-x) &= [(10-u)^2, (10+u)^2] \qquad \text{diam} = 40u \qquad \text{relative accuracy} = 40/u \\
\text{exact range} &= [100-u^2, 100] \qquad \text{diam} = u^2
\end{aligned}$$



affine arithmetic

Affine arithmetic

Comba–Stolfi (1993)

AA represents a quantity x with an affine form

$$\hat{x} = x_0 + x_1\varepsilon_1 + \cdots + x_n\varepsilon_n$$

noise symbols ε_i : independent, vary in $[-1, +1]$ but are otherwise unknown

Can compute arbitrary formulas on affine forms
Use affine approximations for non-affine operations
New noise symbols created during computation

AA generalizes IA:

$$x \sim \hat{x} \quad \Longrightarrow \quad x \in [x_0 - \delta,\, x_0 + \delta] \quad \text{for} \quad \delta = |x_1| + \cdots + |x_n|$$

$$x \in [a, b] \quad \Longrightarrow \quad x \sim \hat{x} = x_0 + x_1\varepsilon_1 \quad \text{for} \quad x_0 = (b+a)/2,\; x_1 = (b-a)/2$$

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$$x \in [a, b] \quad \Longrightarrow \quad x \sim \hat{x} = x_0 + x_1\varepsilon_1 \quad \text{for} \quad x_0 = (b + a)/2,\ x_1 = (b - a)/2$$

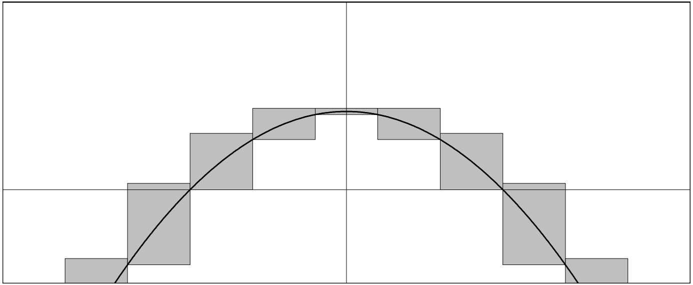
AA automatically exploits first-order correlations in complex expressions

$\Longrightarrow$ better interval estimates!

The dependency problem in interval arithmetic – with AA

$$f(x) = (10+x)(10-x) \text{ for } x \in [-u, u], \qquad x = 0 + u\,\varepsilon_1$$

$$\begin{aligned}
10 + x &= 10 - u\,\varepsilon_1 \\
10 - x &= 10 + u\,\varepsilon_1 \\
(10+x)(10-x) &= 100 - u^2\,\varepsilon_2 \\
\text{range} &= [100 - u^2, 100 + u^2] \qquad \text{diam} = 2u^2 \\
\text{exact range} &= [100 - u^2, 100] \qquad \text{diam} = u^2
\end{aligned}$$



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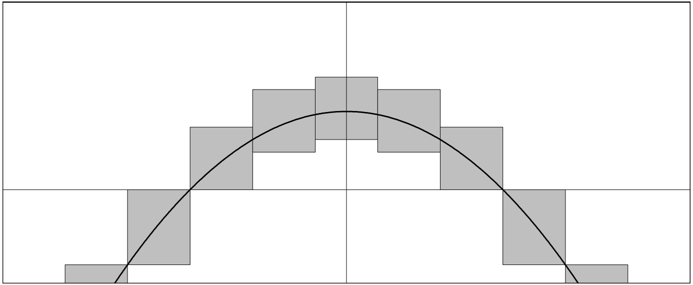


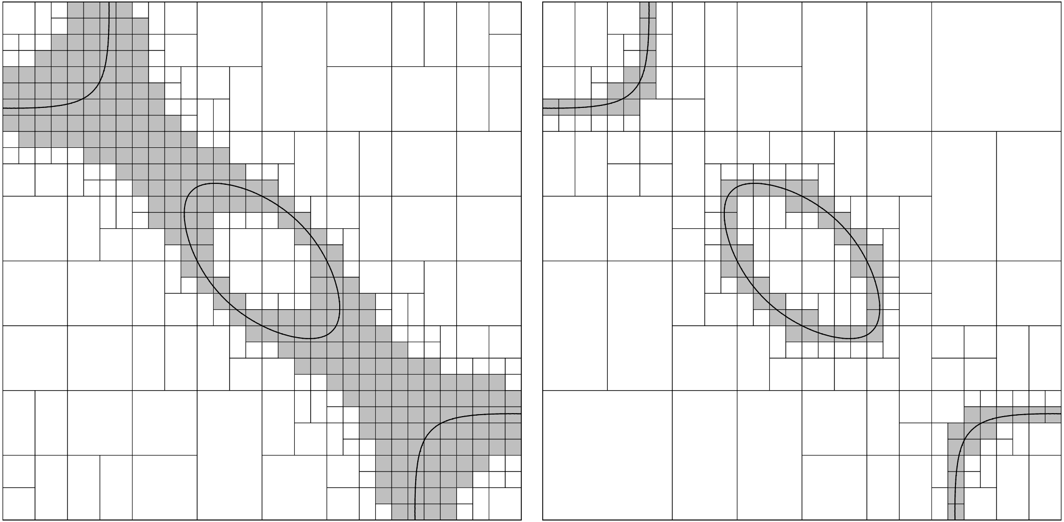
IA

replacing IA with AA

IA versus AA for plotting implicit curves

Comba–Stolfi (1993)

$$x^2 + y^2 + xy - (xy)^2/2 - 1/4 = 0$$



IA: 246 exact: 66 AA: 70

Interval method for intersecting two parametric surfaces

Parametric surfaces

$$g_1\colon D_1 \subset \mathbf{R}^2 \to \mathbf{R}^3$$

$$g_2\colon D_2 \subset \mathbf{R}^2 \to \mathbf{R}^3$$

Intersection

$$g_1(u_1, v_1) - g_2(u_2, v_2) = 0$$

Interval test

$$G_1(U_1, V_1) \cap G_2(U_2, V_2) \neq \varnothing$$

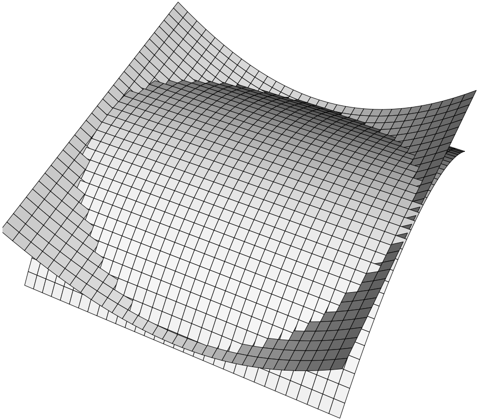
Gleicher–Kass (1992):

- intersect bounding boxes in space
- discard if no intersection
- subdivide until tolerance
- string boxes into curves

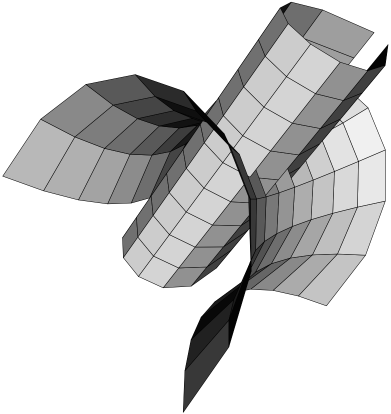
Replacing IA with AA for surface intersection

Tensor product Bézier surfaces of degree (p, q):

$$s(u,v) = \sum_{i=0}^{p}\sum_{j=0}^{q} a_{ij} B_i^p(u) B_j^q(v), \quad B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}, \quad u, v \in [0,1]$$



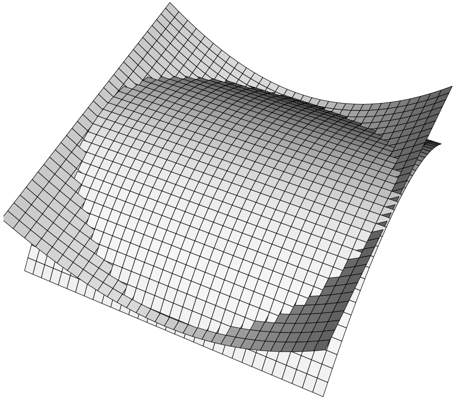
$(2, 1)$

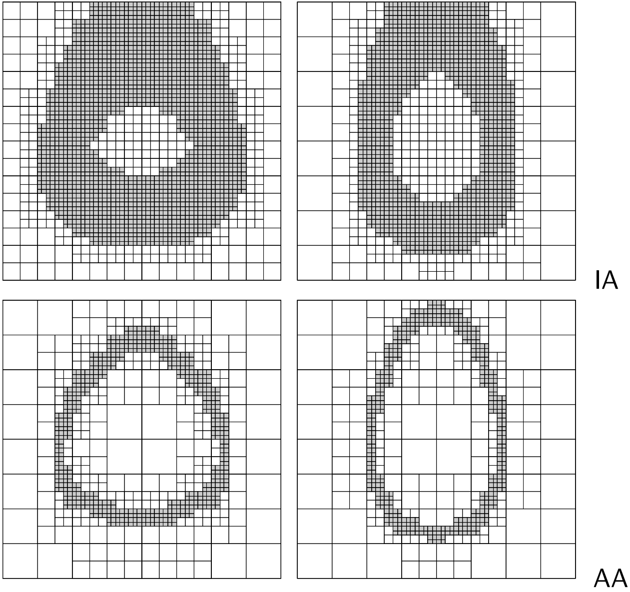


$(3, 3)$

Replacing IA with AA for surface intersection

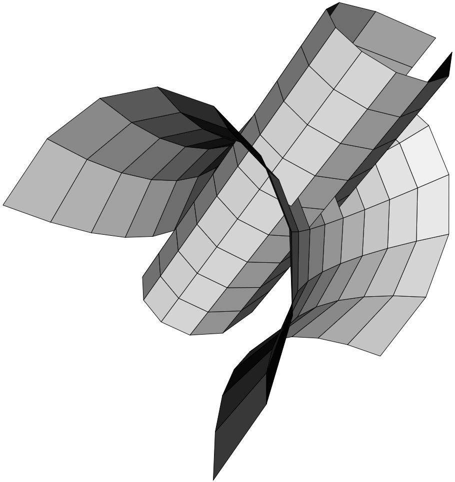
Figueiredo (1996)

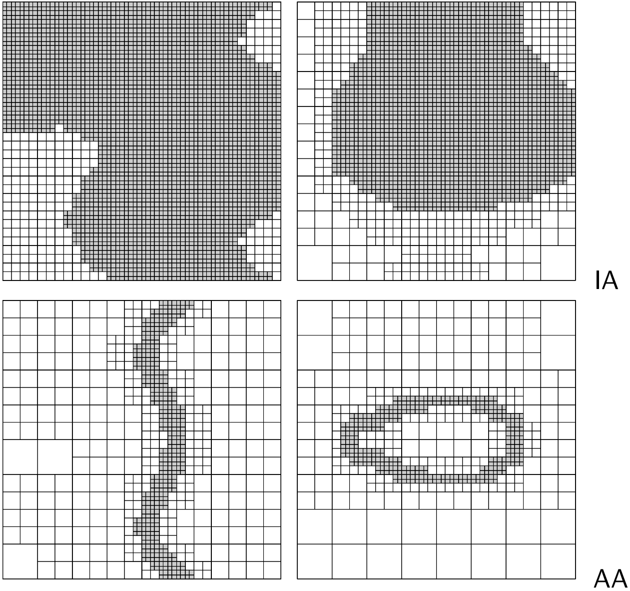




Replacing IA with AA for surface intersection

Figueiredo (1996)





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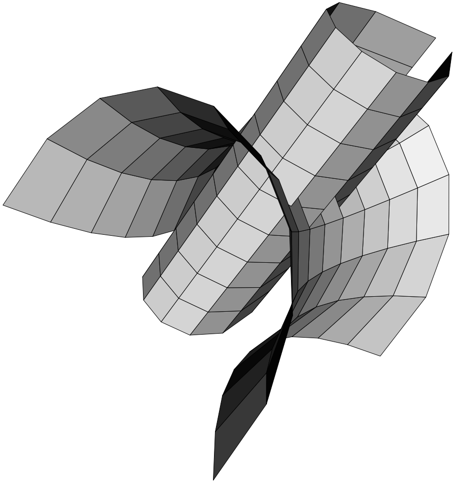
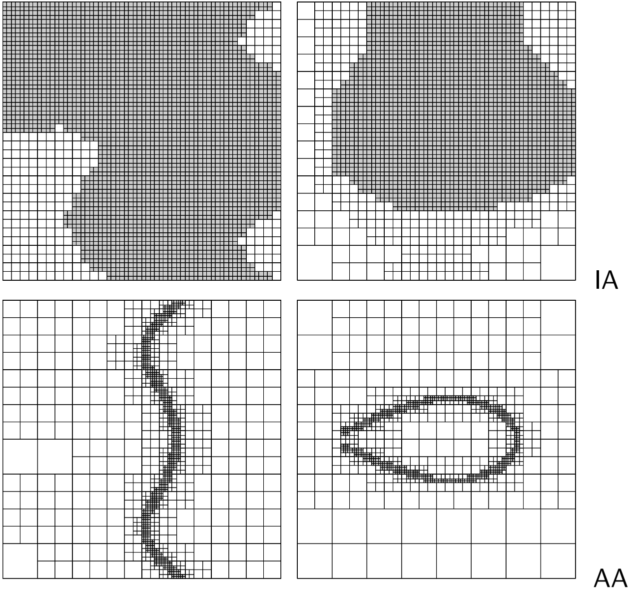
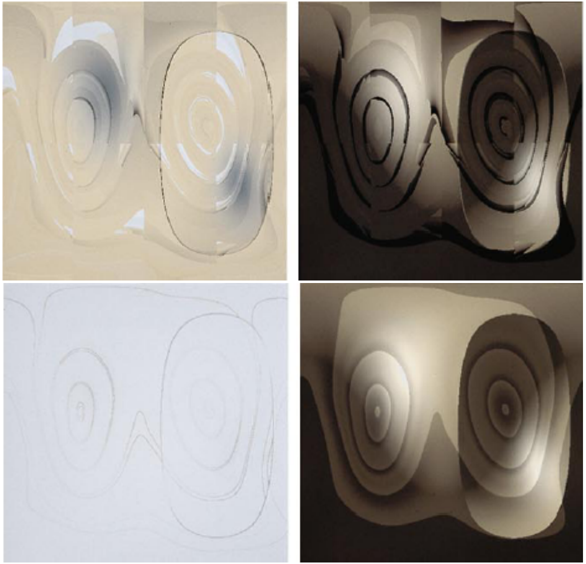




Sampling procedural shaders



IA

AA

exploiting geometry in AA

Geometry of affine forms

Affine forms that share noise symbols are not independent:

$$\begin{aligned}\hat{x} &= x_0 + x_1\varepsilon_1 + \cdots + x_n\varepsilon_n \\ \hat{y} &= y_0 + y_1\varepsilon_1 + \cdots + y_n\varepsilon_n\end{aligned}$$

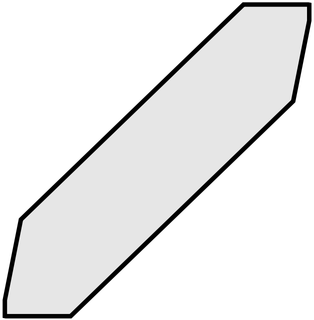
Joint range is a zonotope: centrally symmetric convex polygon

Image of hypercube $[-1,1]^n$ under affine transformation

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \begin{bmatrix} x_1 & \cdots & x_n \\ y_1 & \cdots & y_n \end{bmatrix} \cdot \begin{bmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_n \end{bmatrix}$$

Minkowski sum of point and line segments

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \varepsilon_1 + \cdots \begin{bmatrix} x_n \\ y_n \end{bmatrix} \varepsilon_n$$



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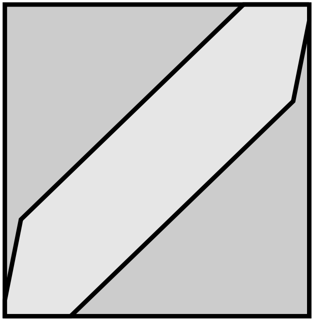

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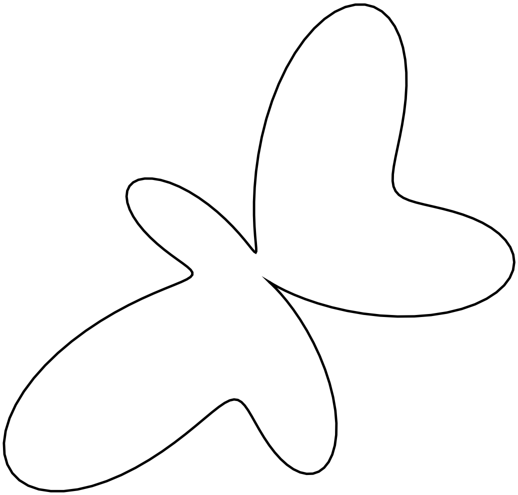
Approximating parametric curves

Parametric curve

$$\mathcal{C} = \gamma(I), \qquad \gamma\colon I \to \mathbf{R}^2$$

Compute good bounding rectangle for

$$\mathcal{P} = \gamma(T), \qquad T \subseteq I$$



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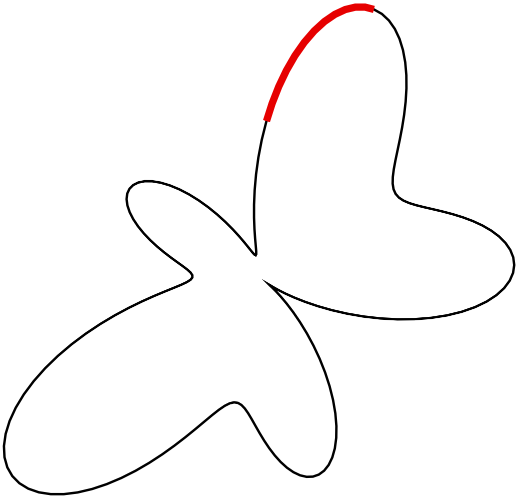

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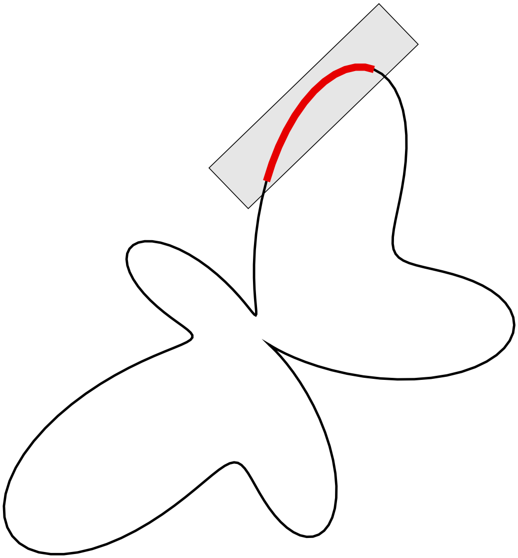

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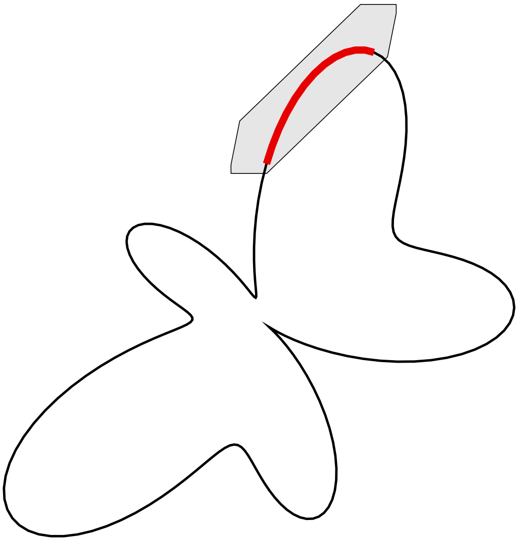
Write

$$\gamma(t) = (x(t), y(t))$$

Find joint range of $\hat{x}(\hat{t})$ and $\hat{y}(\hat{t})$ with AA



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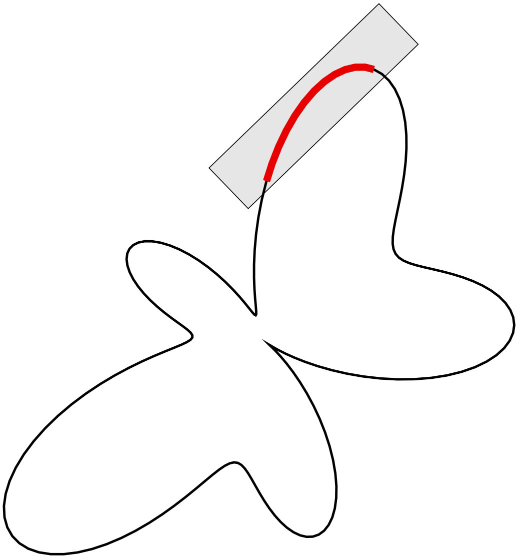
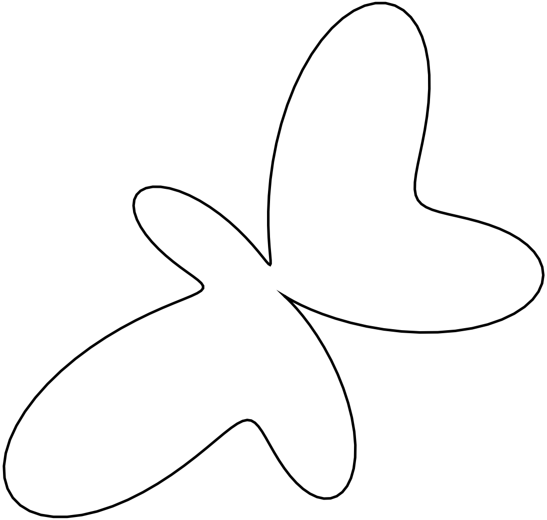
Use bounding rectangle of zonotope



Approximating parametric curves

Figueiredo–Stolfi–Velho (2003)



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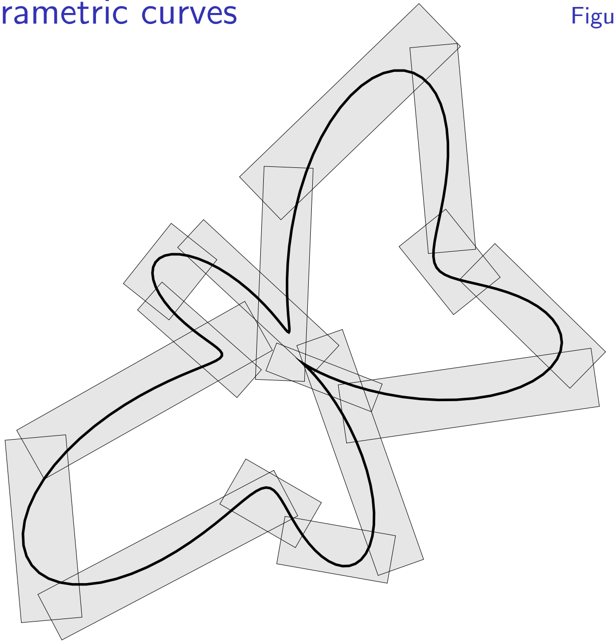

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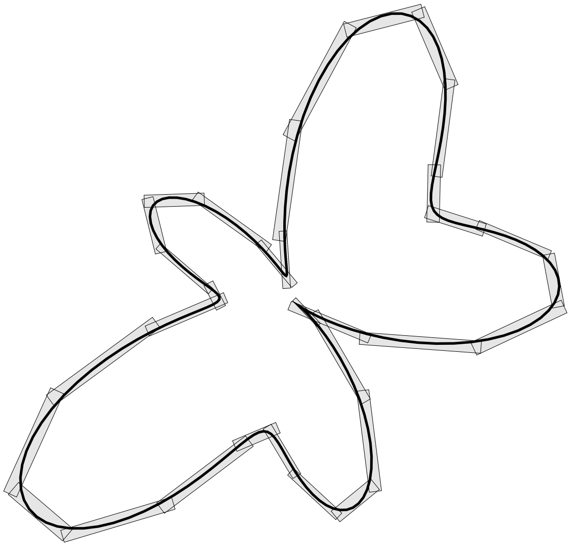

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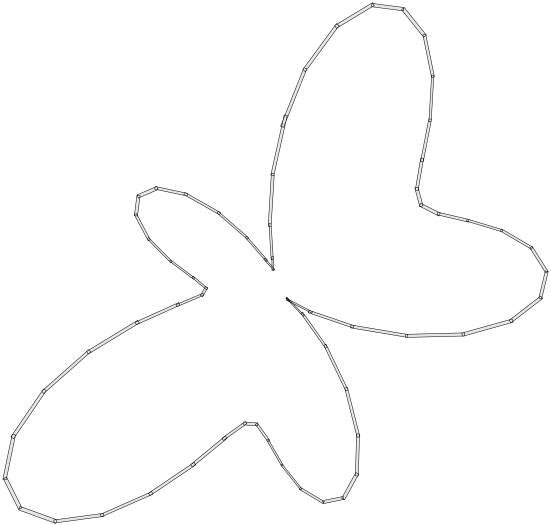
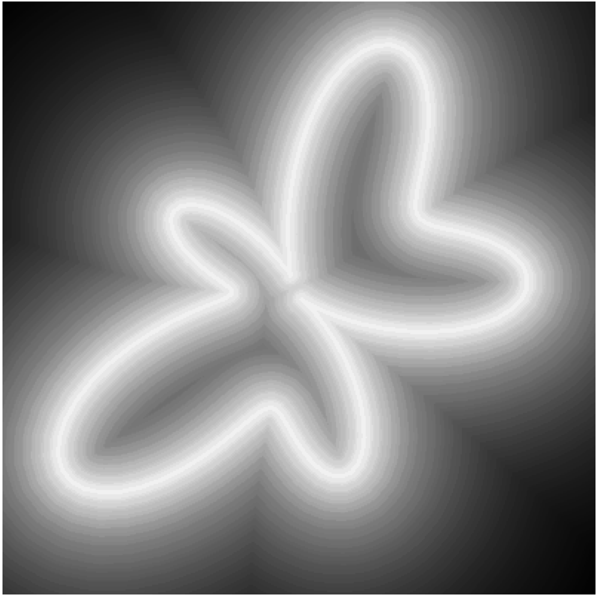


Distance fields for parametric curves

Figueiredo–Stolfi–Velho (2003)



Ray casting implicit surfaces

Implicit surface

$$h(x, y, z) = 0, \qquad h\colon \mathbf{R}^3 \to \mathbf{R}$$

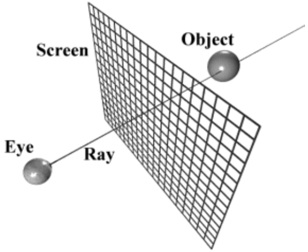
Ray

$$r(t) = e + t \cdot v = (x(t), y(t), z(t)), \quad t \in [0, \infty)$$

Ray intersects surface when

$$f(t) = h(r(t)) = 0$$

First intersection occurs at smallest zero of f in $[0, \infty)$



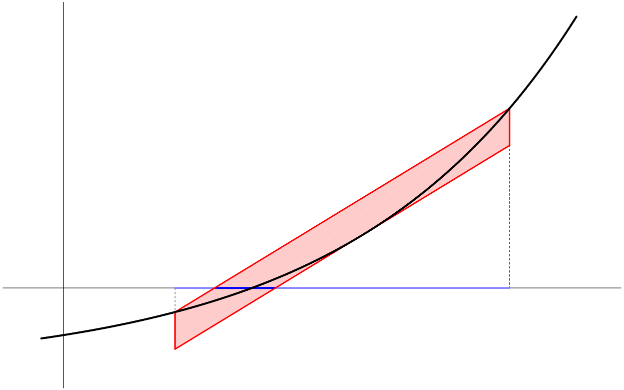
Ray casting implicit surfaces

```
procedure interval-bisection([a, b])
   if 0 ∈ F([a, b]) then
         c ← (a + b)/2
         if (b − a) < ε then
               return c
         else
               interval-bisection([a, c])        ← try left half first!
               interval-bisection([c, b])
         end
   end
end
```

Call interval-bisection($[0, t_\infty]$) to find the first zero.

Ray casting implicit surfaces

Custatis–Figueiredo–Gattass (1999)



AA exploits linear correlations in

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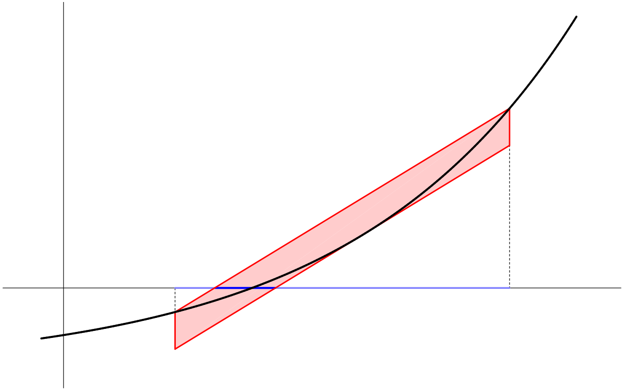
$$r(t) = (x(t), y(t), z(t))$$

Ray casting implicit surfaces



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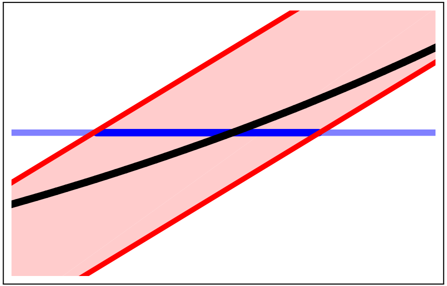
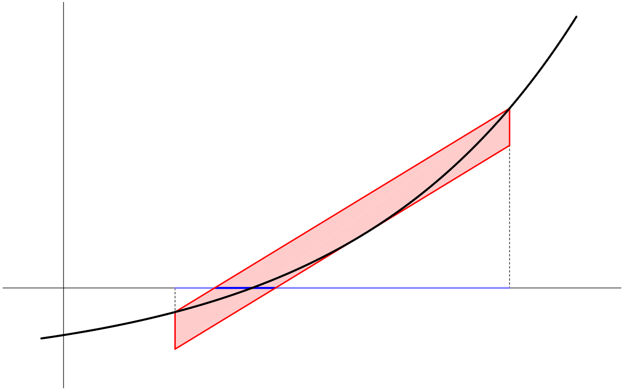
$$r(t) = (x(t), y(t), z(t))$$

root must lie in smaller interval



Ray casting implicit surfaces

Custatis–Figueiredo–Gattass (1999)

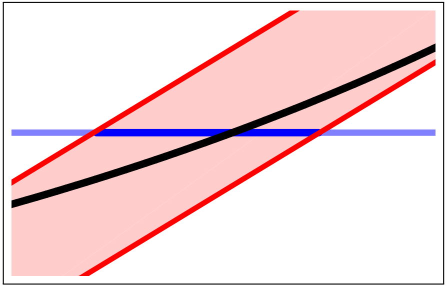


quadratic convergence

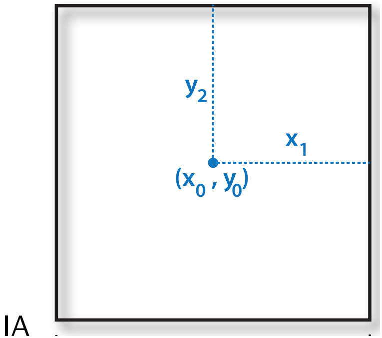
AA exploits linear correlations in

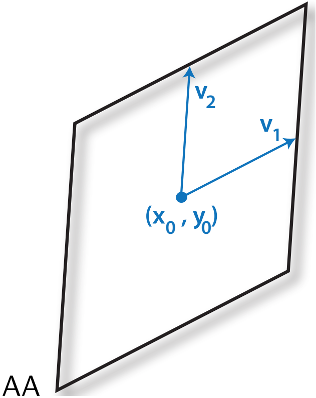
$$f(t) = h(r(t))$$
$$r(t) = (x(t), y(t), z(t))$$

root must lie in smaller interval



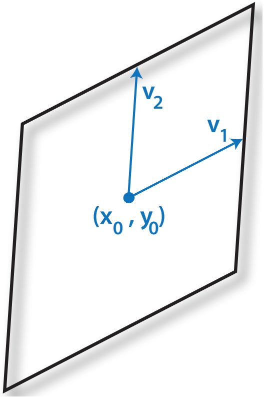
Natural domains





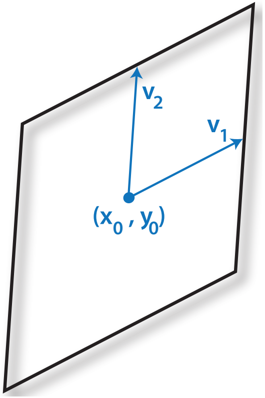
$$(\hat{x}, \hat{y}) = (x_0, y_0) + v_1\varepsilon_1 + v_2\varepsilon_2$$

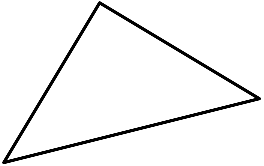
AA on triangles



$$(\hat{x}, \hat{y}) = (x_0, y_0) + v_1\varepsilon_1 + v_2\varepsilon_2$$

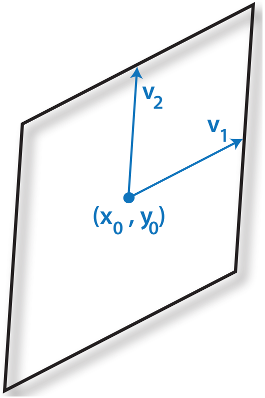
AA on triangles

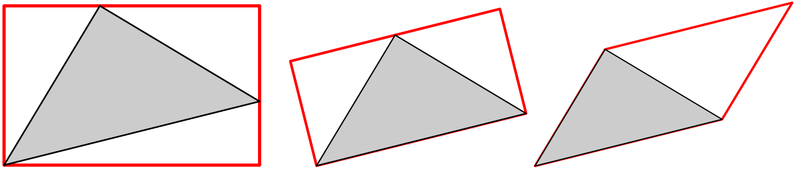




$$(\hat{x}, \hat{y}) = (x_0, y_0) + v_1\varepsilon_1 + v_2\varepsilon_2$$

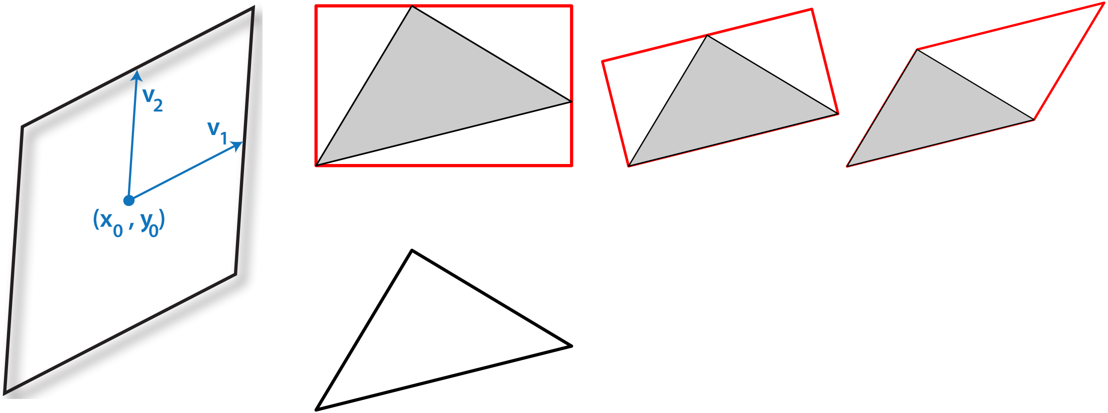
AA on triangles





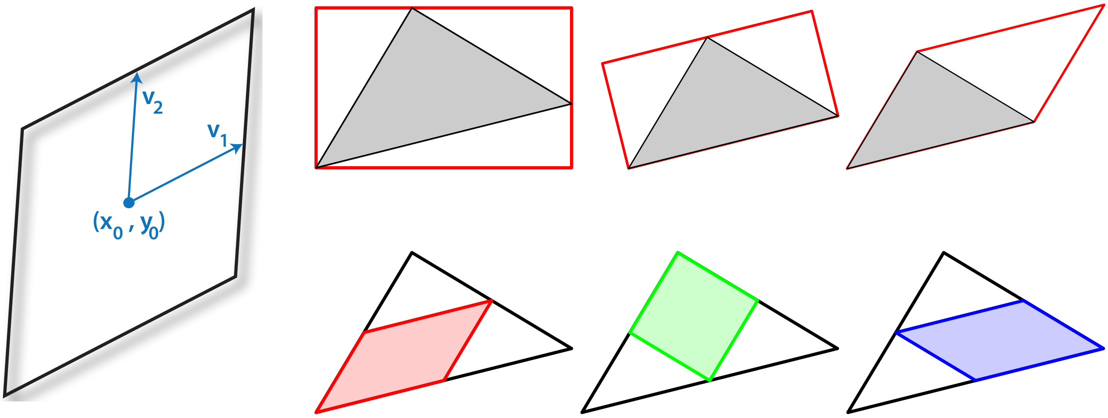
$$(\hat{x}, \hat{y}) = (x_0, y_0) + v_1\varepsilon_1 + v_2\varepsilon_2$$

AA on triangles



$$(\hat{x}, \hat{y}) = (x_0, y_0) + v_1\varepsilon_1 + v_2\varepsilon_2$$

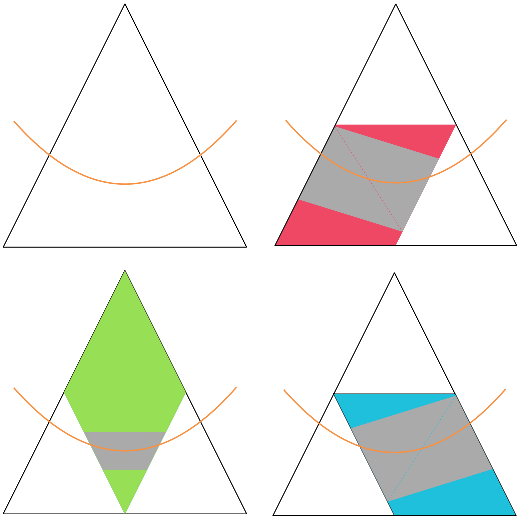
AA on triangles



$$(\hat{x}, \hat{y}) = (x_0, y_0) + v_1\varepsilon_1 + v_2\varepsilon_2$$

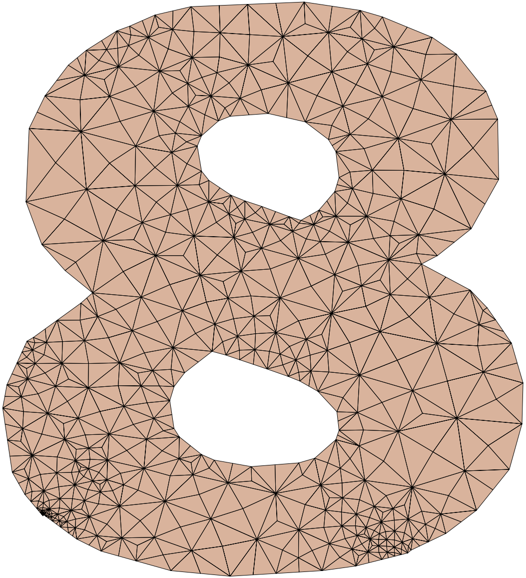
Implicit curves on triangles

Nascimento–Paiva–Figueiredo–Stolfi (2014)



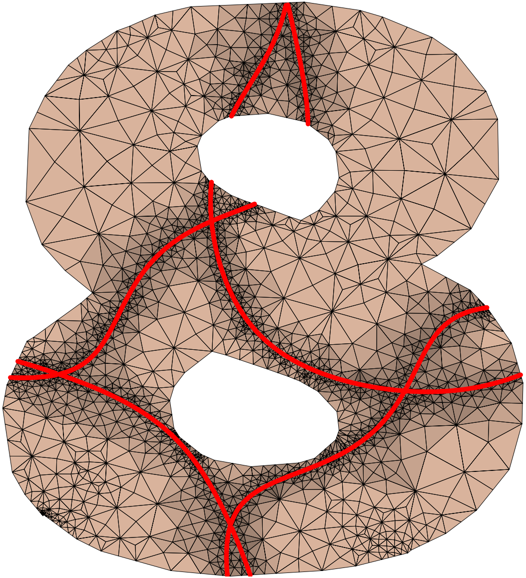
Implicit curves on triangulations

Nascimento–Paiva–Figueiredo–Stolfi (2014)



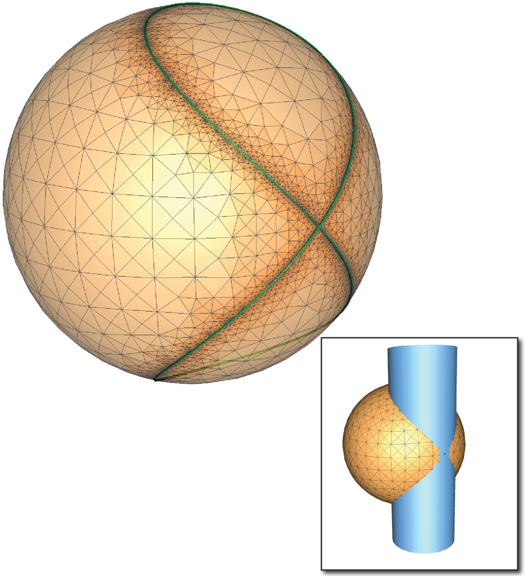
Implicit curves on triangulations

Nascimento–Paiva–Figueiredo–Stolfi (2014)



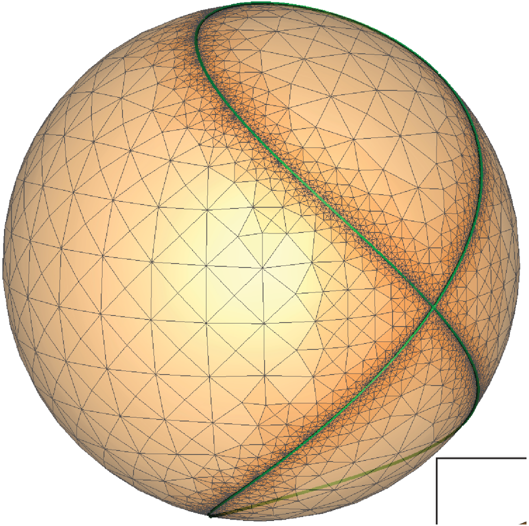
Implicit curves on triangulations

Nascimento–Paiva–Figueiredo–Stolfi (2014)



Implicit curves on triangulations

Nascimento–Paiva–Figueiredo–Stolfi (2014)



Conclusion

Interval methods

- can reliably probe the global behavior of functions without sampling
- lead naturally to robust, adaptive algorithms
- useful in many domains

Affine arithmetic is a useful tool for interval methods

- AA can replace IA transparently
- AA more accurate than IA
- AA locally more expensive than IA but globally more efficient
- AA provides geometric information that can be exploited

Lots more to be done!

Interval methods for computer graphics and geometric modeling

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