Workshop of Geometry Processing and Applications

Interval Methods in Computer Graphics

Luiz Henrique de Figueiredo (IMPA)

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Motivation

- How do I plot an implicit curve?
  - ◇ Must solve $f(x, y) = 0$
  - ◇ Solution is a curve, but where is it?

- How do I render an implicit surface?
  - ◇ Must solve $f(x, y, z) = 0$ for (x, y, z) on a ray
  - ◇ Solution is one or more points, but need point closest to eye!

- How do I intersect two parametric surfaces?
  - ◇ Must solve $f(u, v) = g(s, t)$
  - ◇ Solution is a set of curves in space and a set of curves in each parametric plane. Where are they? How do they match?

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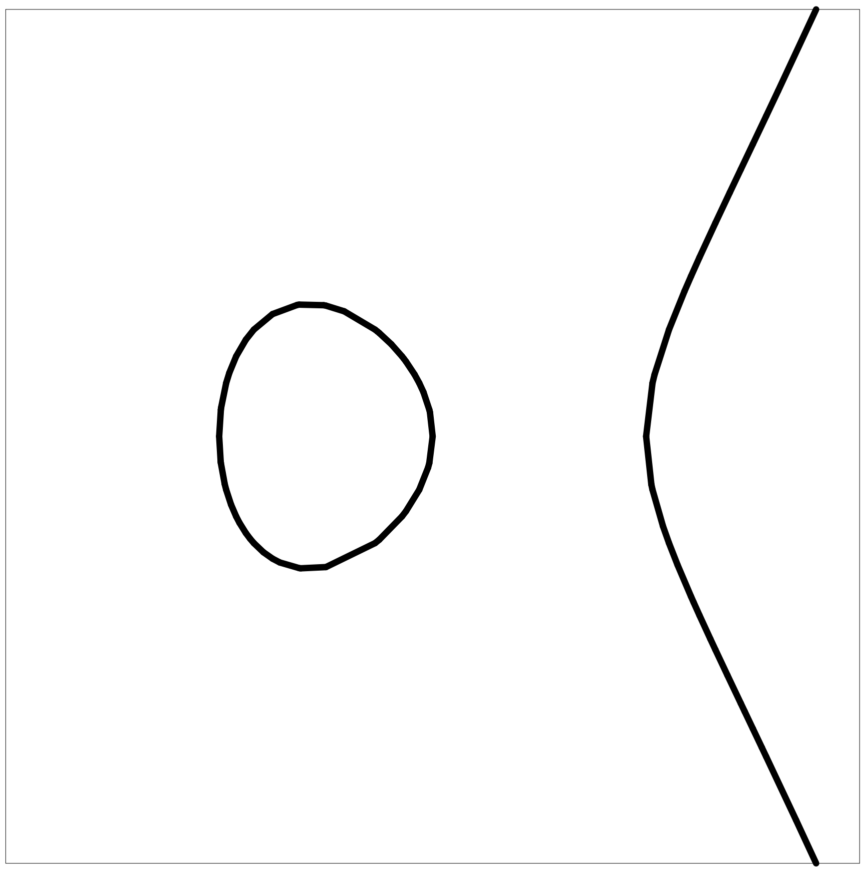
Plotting an implicit curve

$y^2 - x^3 + x = 0$ $\Omega = [-2, 2] \times [-2, 2]$



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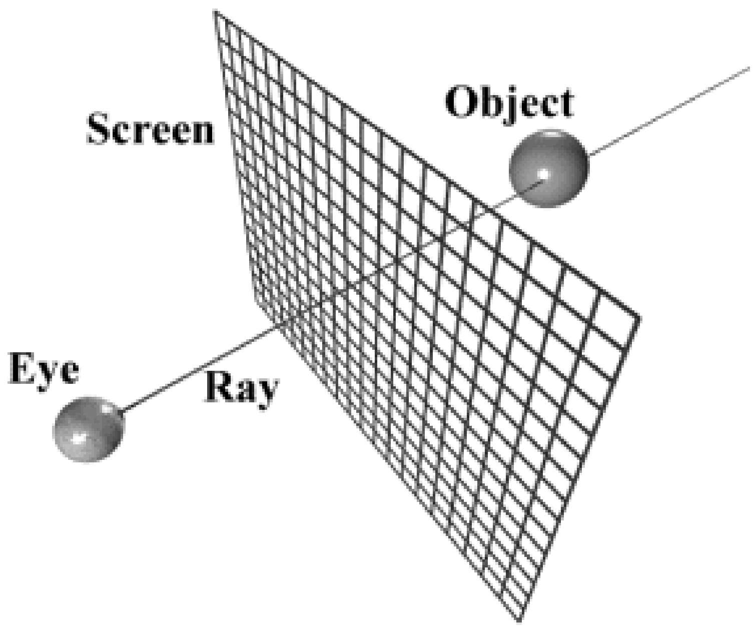
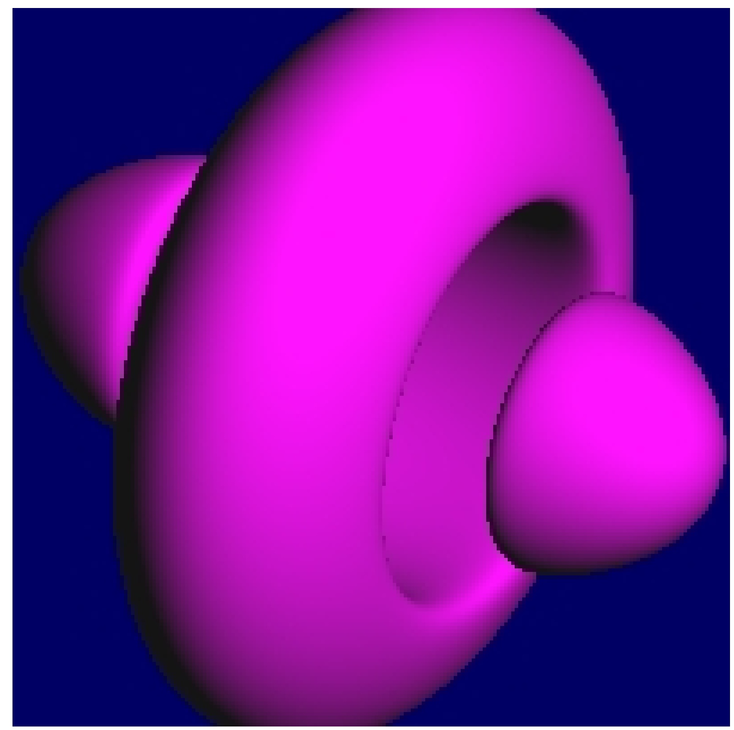
Rendering implicit surfaces

$$4(x^4 + (y^2 + z^2)^2) + 17x^2(y^2 + z^2) - 20(x^2 + y^2 + z^2) + 17 = 0$$





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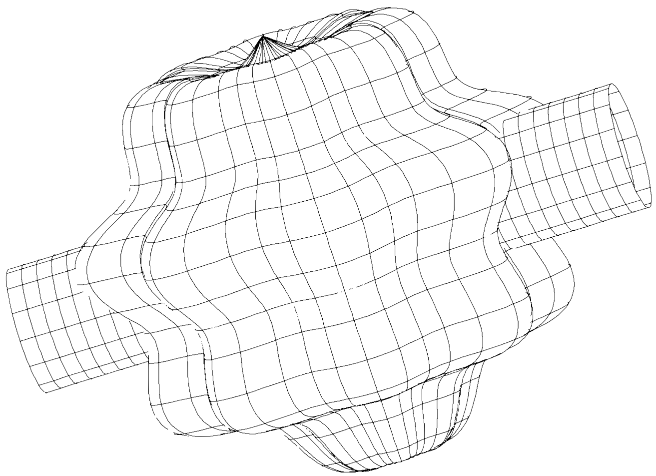
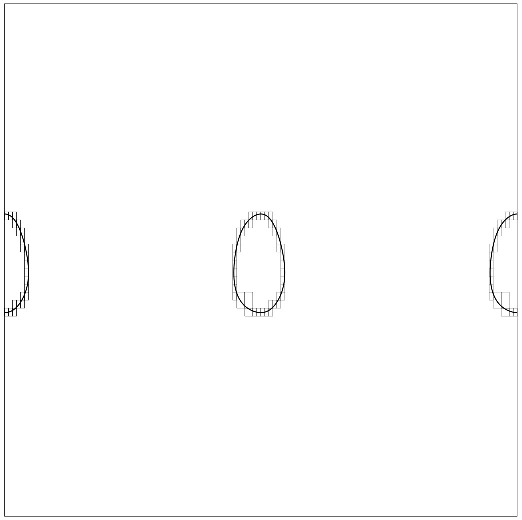
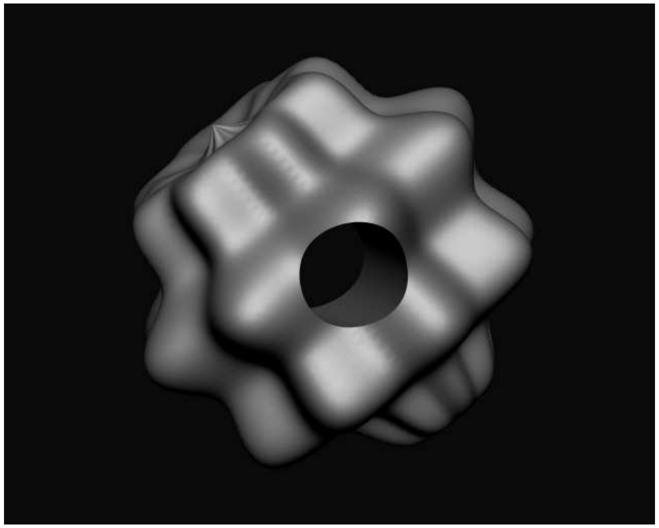
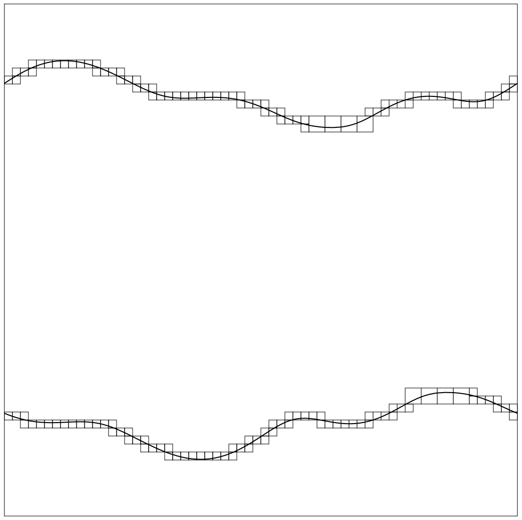
Intersecting two parametric surfaces

(Snyder, 1992)

Interval arithmetic

Can we trust floating-point arithmetic?

Rump's example – Evaluate this innocent-looking polynomial expression:

$$f = 333.75y^6 + x^2(11x^2y^2 - y^6 - 121y^4 - 2) + 5.5y^8 + x/(2y),$$

for $x = 77617$ and $y = 33096$.

```
f:=333.75*y^6+x^2*(11*x^2*y^2-y^6-121*y^4-2)+5.5*y^8+x/(2*y);
                                f := 1.172603940

f:=33375/100*y^6+x^2*(11*x^2*y^2-y^6-121*y^4-2)+55/10*y^8+x/(2*y);
                                       54767
                                f := - -----
                                       66192
evalf(f,10);
                                -0.8273960599
```

Not Maple's fault! Running gcc under Linux gives 5.76461×10^{17}.

Culprit is catastrophic cancellation of floating-point arithmetic!

Interval arithmetic

- To improve reliability of floating-point computations (Moore, 1960)
- Represent quantities as intervals:

$$x \sim [a, b] \Rightarrow x \in [a, b]$$

- Operate with intervals generating other intervals:
  - ◇ Simple formulas for elementary operations and functions:

$$\begin{aligned}
[a,b] + [c,d] &= [a+c, b+d] \\
[a,b] \times [c,d] &= [\mathrm{min}\{ac, ad, bc, bd\}, \mathrm{max}\{ac, ad, bc, bd\}] \\
[a,b] \,/\, [c,d] &= [a,b] \times [1/d, 1/c] \\
[a,b]^2 &= [0, \mathrm{max}(a^2, b^2)] \ \text{ when } \ 0 \in [a,b] \\
\exp\,[a,b] &= [\exp(a), \exp(b)] \\
&\cdots
\end{aligned}$$

  - ◇ Automatic extensions for complicated expressions
  - ◇ Rounding control available in modern floating-point units (IEEE 754)

Interval arithmetic

- Every expression f has an interval extension F :

$$x_i \in X_i \Rightarrow f(x_1, \ldots, x_n) \in F(X_1, \ldots, X_n)$$

- Interval computations not immune to roundoff errors
  Wide results alert user of catastrophic cancellation
- Roundoff errors are not our main motivation!
- Interval computations allow range estimates and avoid point sampling

$$F(X) \supseteq f(X) = \{f(x) : x \in X\}$$

For instance

$$\begin{aligned} 0 \notin F(X) &\Rightarrow 0 \notin f(X) \\ &\Rightarrow f = 0 \text{ has no solution in } X \end{aligned}$$

This is a computational proof!

Interval probing of implicit curve

$$y^2 - x^3 + x = 0$$

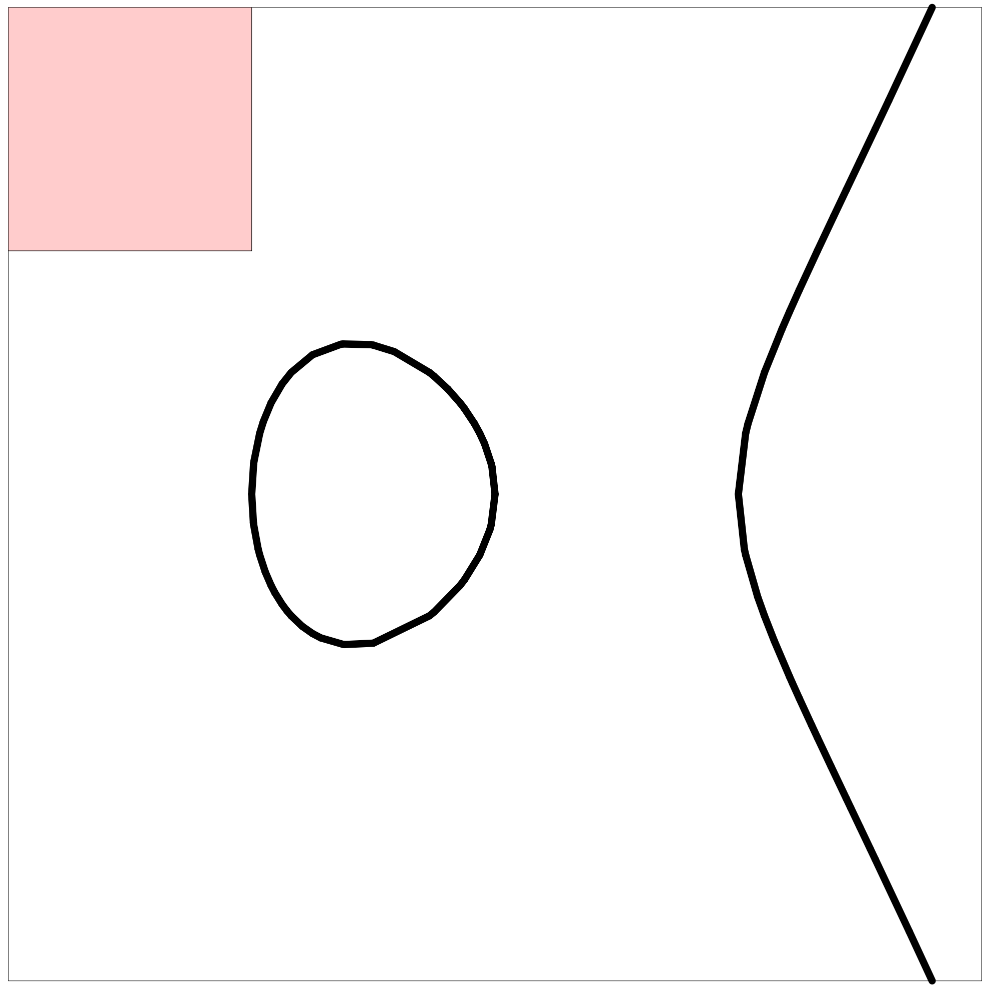
$$\begin{aligned} X &= [-2, -1] \\ Y &= [1, 2] \\ X^3 &= [-8, -1] \\ -X^3 &= [1, 8] \\ -X^3 + X &= [-1, 7] \\ Y^2 &= [1, 4] \\ Y^2 - X^3 + X &= [0, 11] \end{aligned}$$

- Interval estimates not tight

$$f(X, Y) = [1, 10] \subset [0, 11]$$

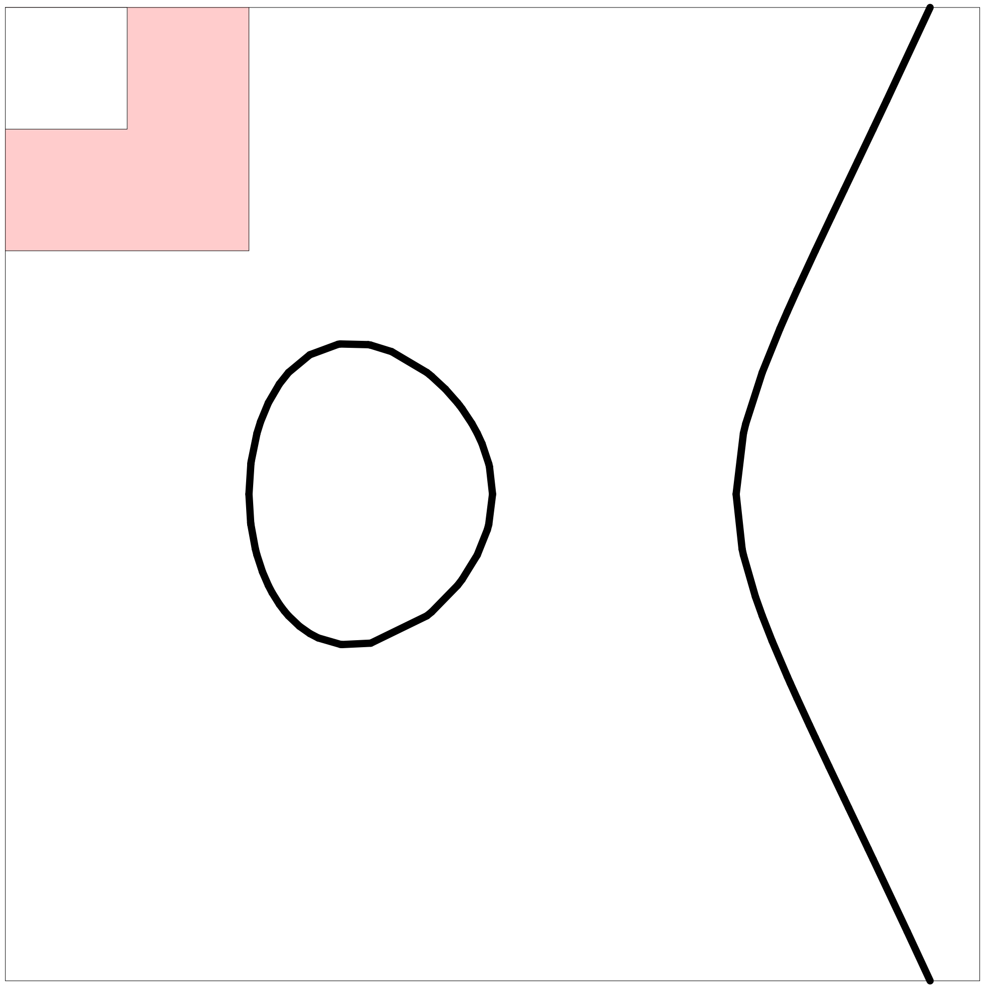
- Interval estimates improve as intervals shrink

Interval probing of implicit curve



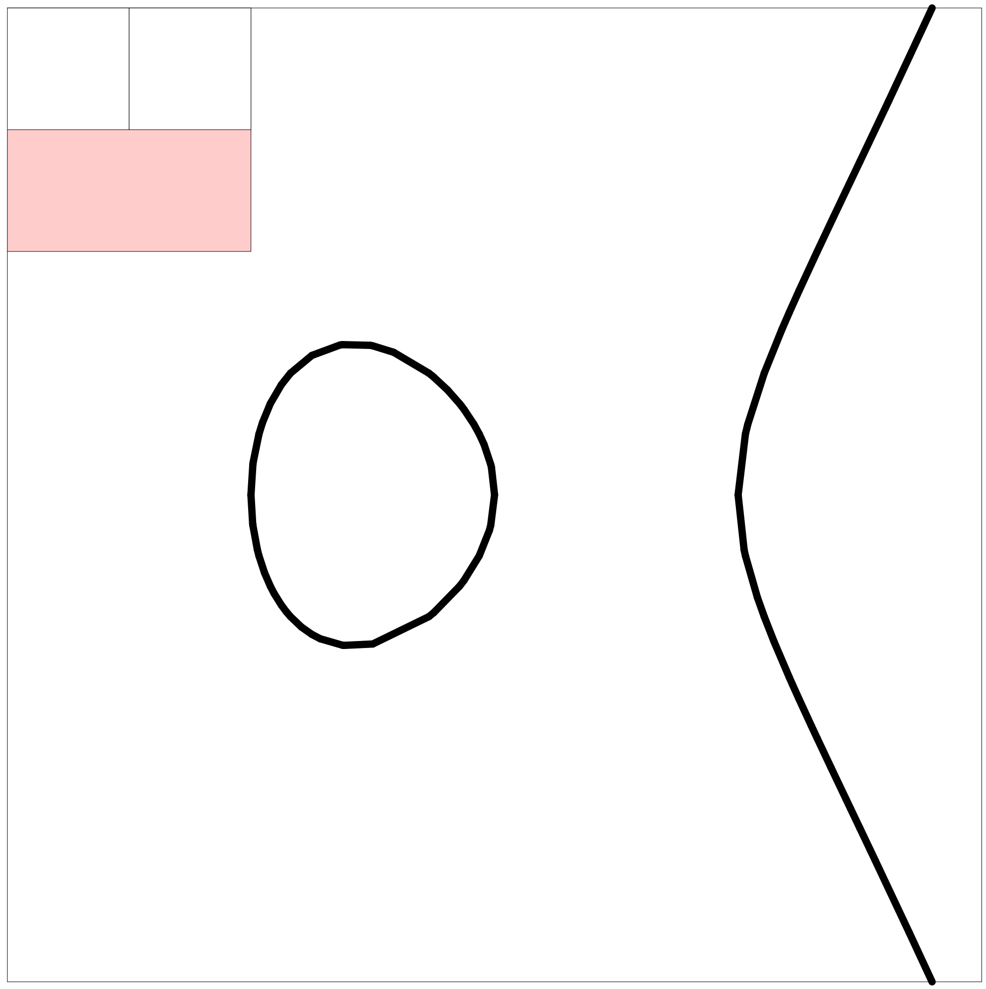
$[-2, -1] \times [1, 2]$ $[0, 11]$ yes?

Interval probing of implicit curve



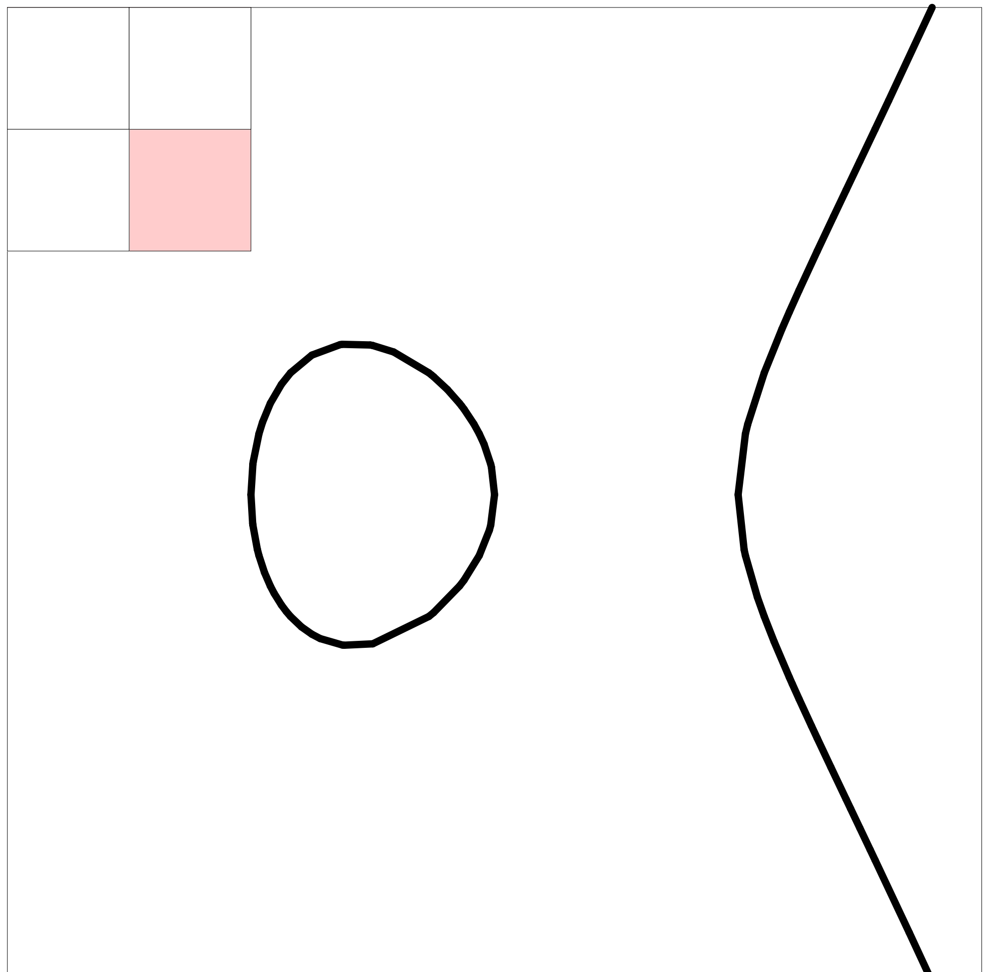
$[-2, -1.5] \times [1.5, 2]$ $[3.625, 10.5]$ no

Interval probing of implicit curve



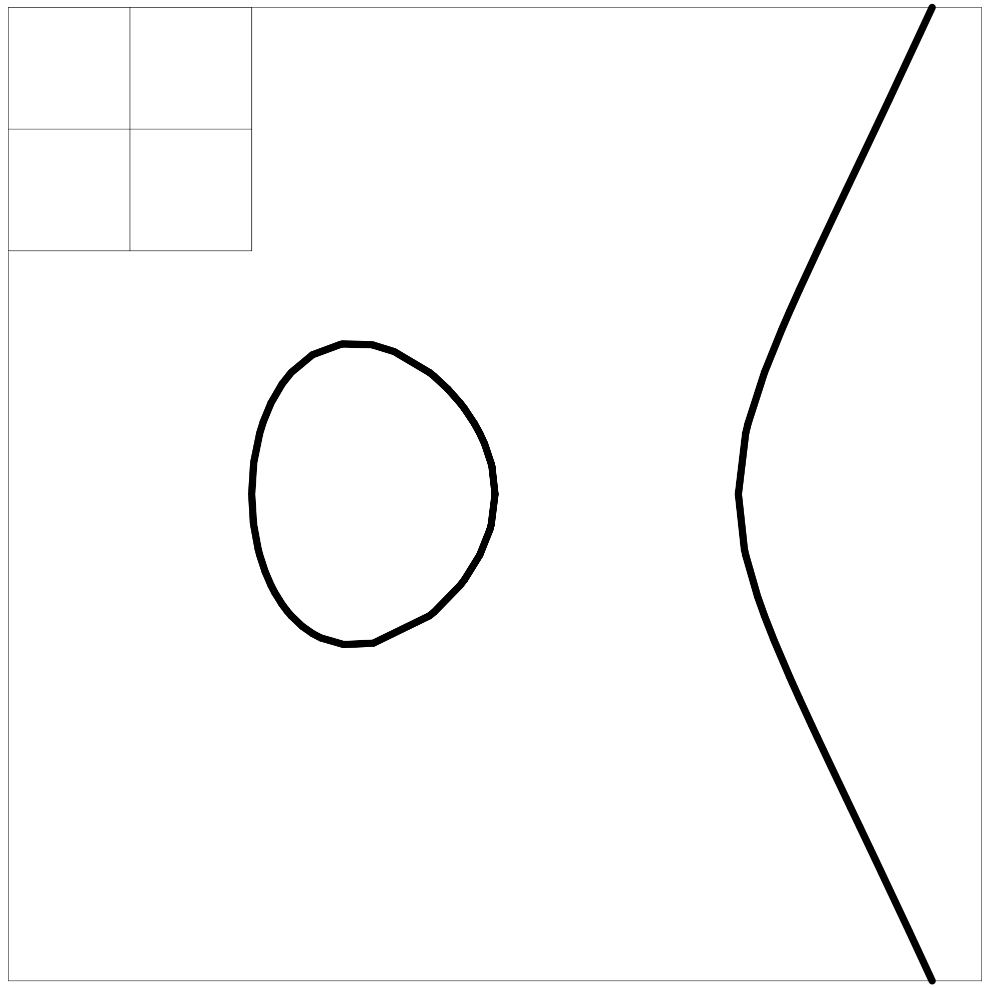
$[-1.5, -1] \times [1.5, 2]$ $[1.75, 6.375]$ no

Interval probing of implicit curve



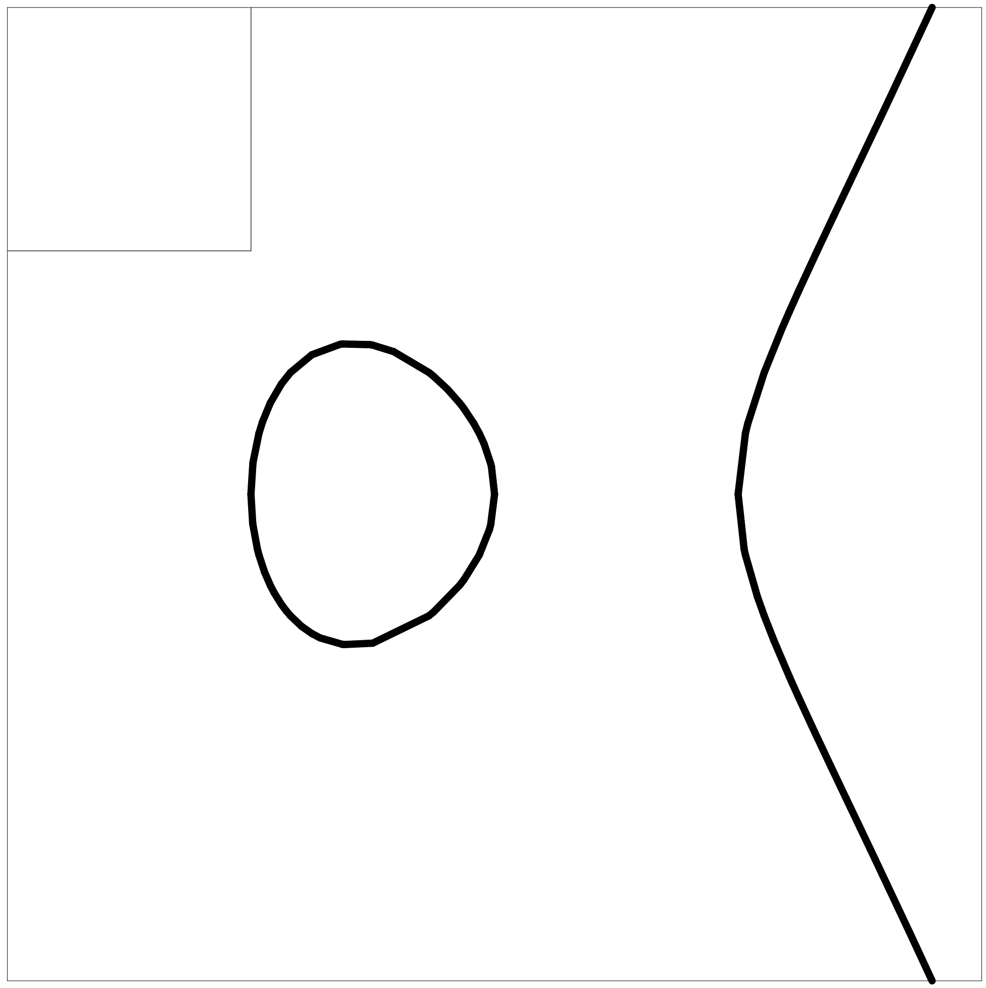
$[-2, -1.5] \times [1, 1.5]$ $[2.375, 8.75]$ no

Interval probing of implicit curve



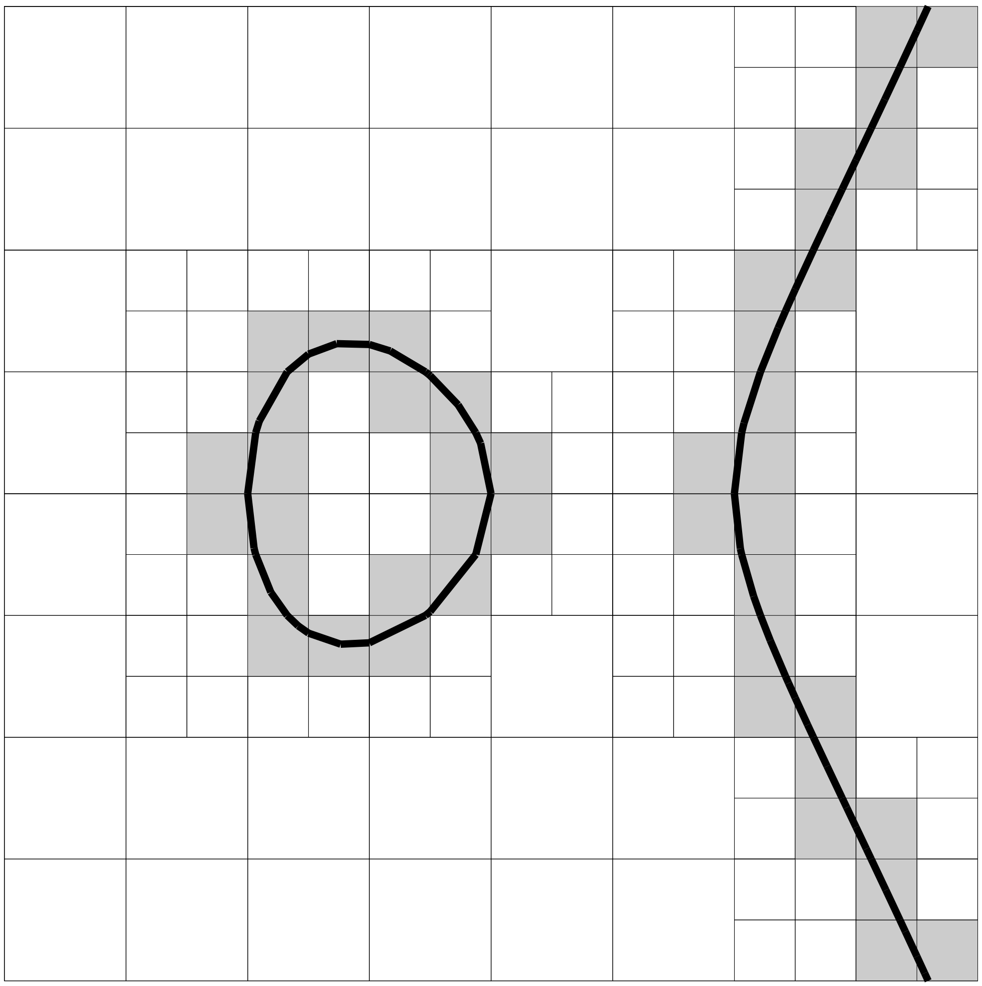
$[-1.5, -1] \times [1, 1.5]$ $[0.5, 4.625]$ no

Interval probing of implicit curve



$[-2, -1] \times [1, 2]$ $[0.5, 10.5]$ no!

Approximation of implicit curve



Robust adaptive enumeration

- Recursive exploration of domain Ω starts with explore(Ω)
- Discard subregions X of Ω when $0 \notin F(X)$
  = proof that X does not contain any part of the curve!

```
explore(X):
    if 0 ∉ F(X) then
        discard X
    elseif diam(X) < ε then
        output X
    else
        divide X into smaller pieces X_i
        for each i, explore(X_i)
```

- Output cells have the same size: only spatial adaption

Suffern–Fackerell (1991), Snyder (1992)

Robust adaptive approximation

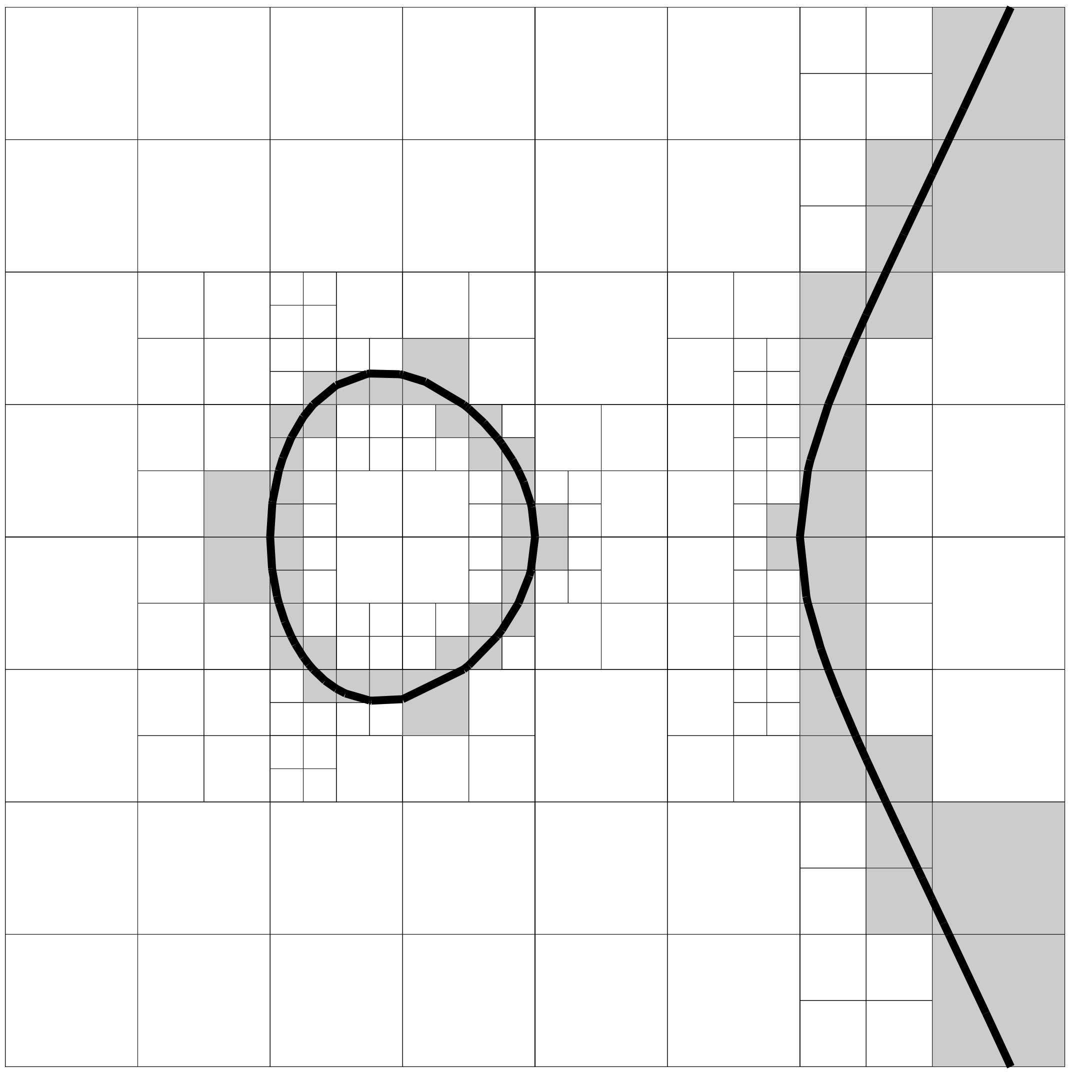
- Estimate curvature by gradient variation
- $G =$ inclusion function for the normalized gradient of f
- $G(X)$ small $\Rightarrow$ curve approximately flat inside X

```
explore(X):
    if 0 ∉ F(X) then
        discard X
    elseif diam(X) < ε or diam(G(X)) < δ then
        approx(X)
    else
        divide X into smaller pieces X_i
        for each i, explore(X_i)
```

- Output cells vary in size: spatial and geometrical adaption

Lopes–Oliveira–Figueiredo (2002)

Robust adaptive approximation



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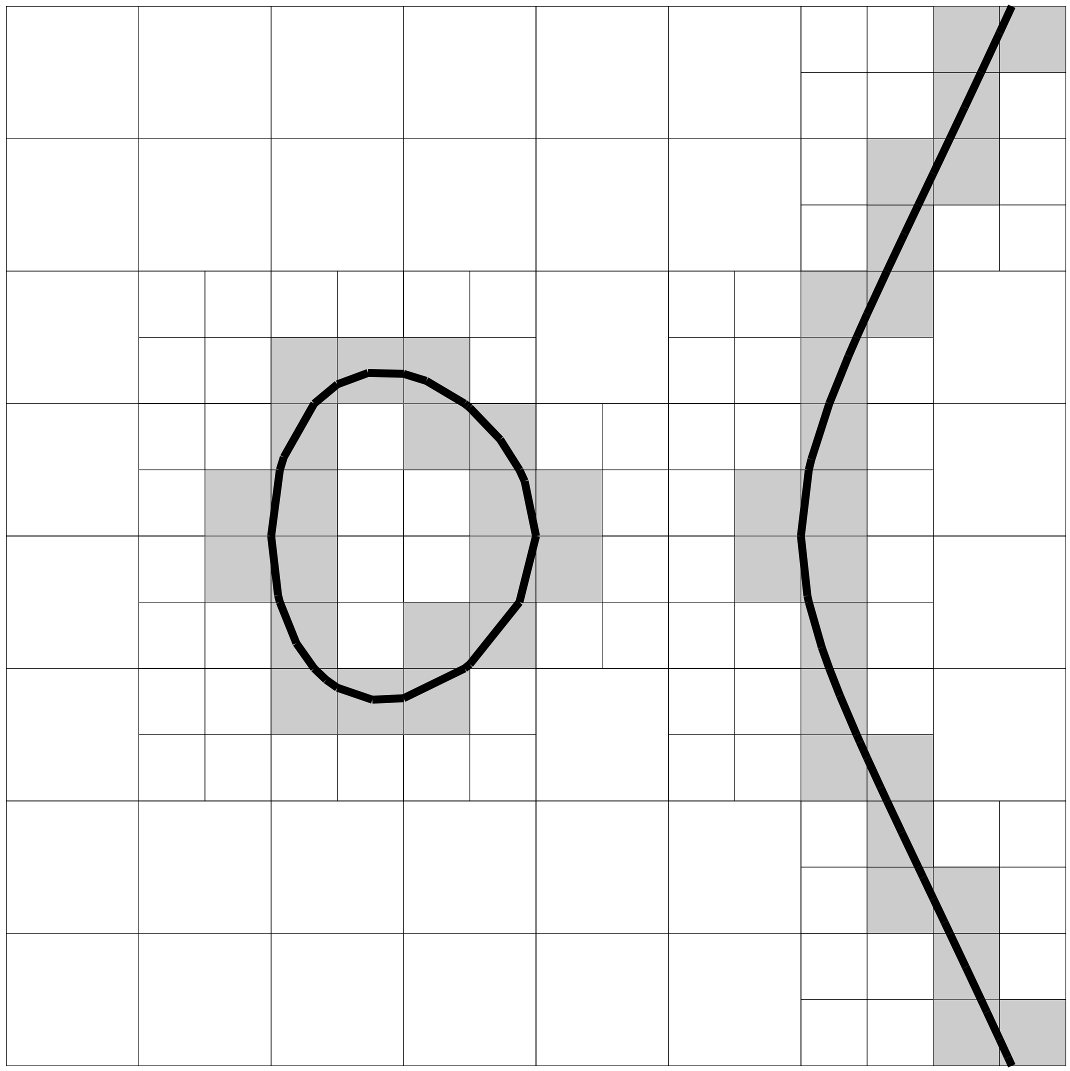
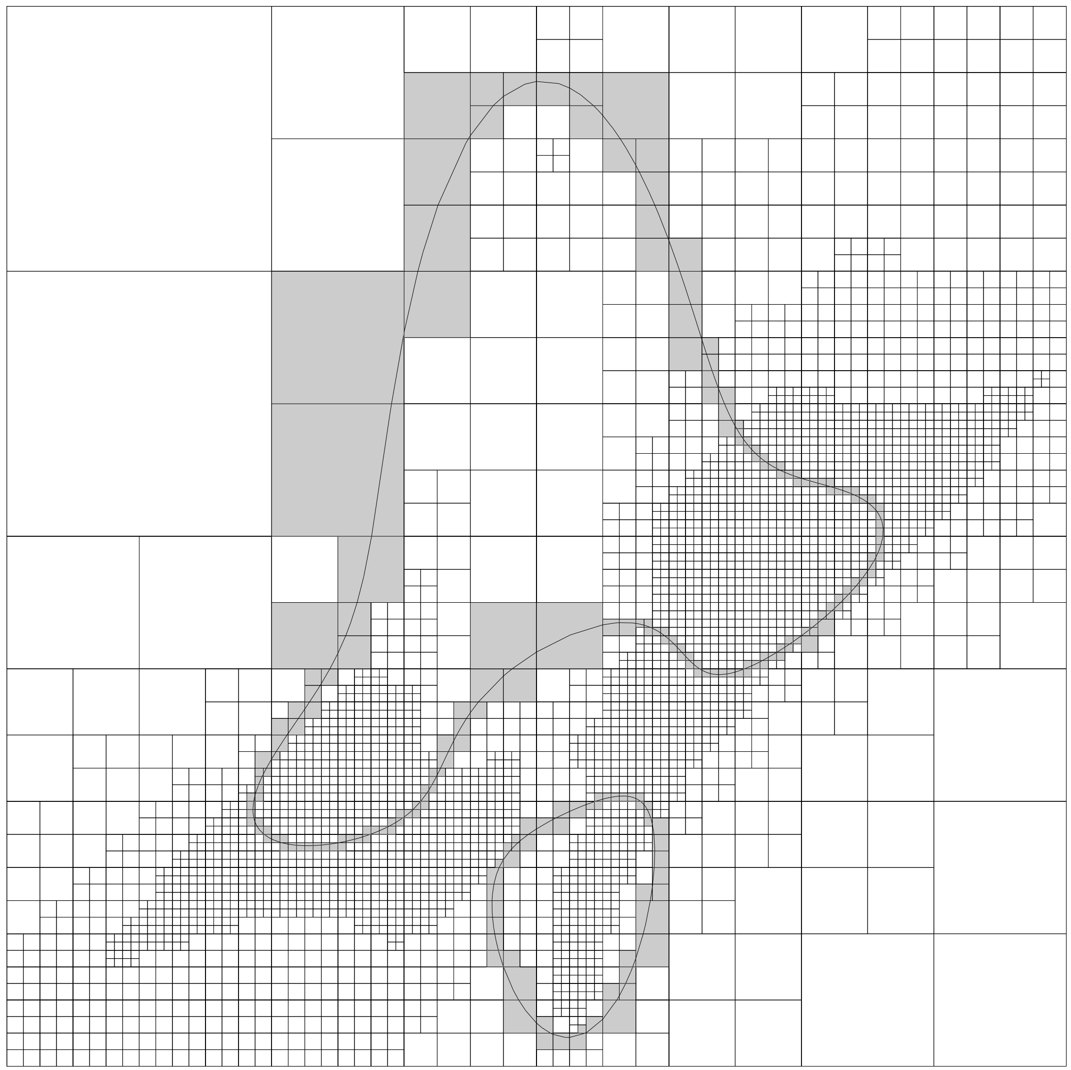
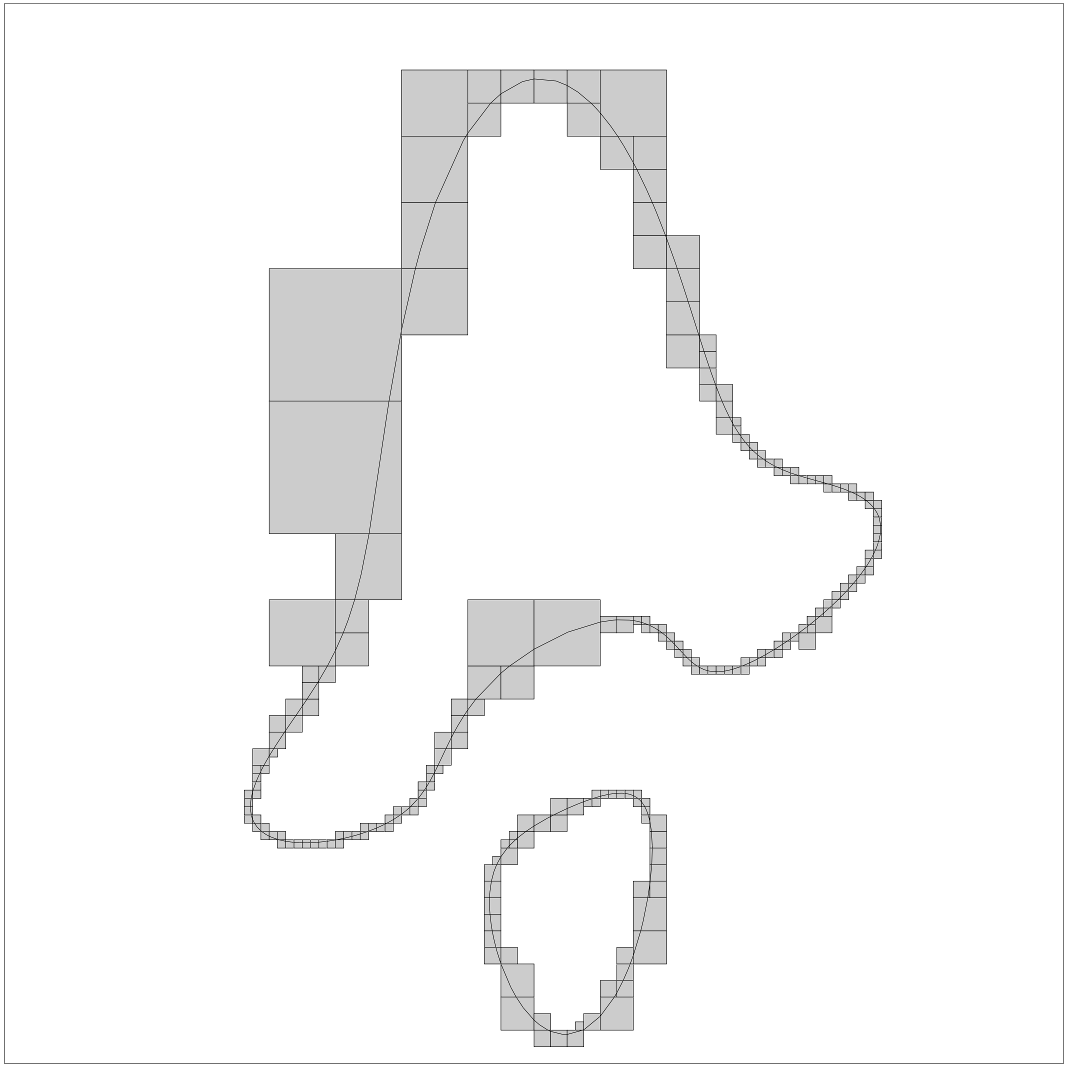


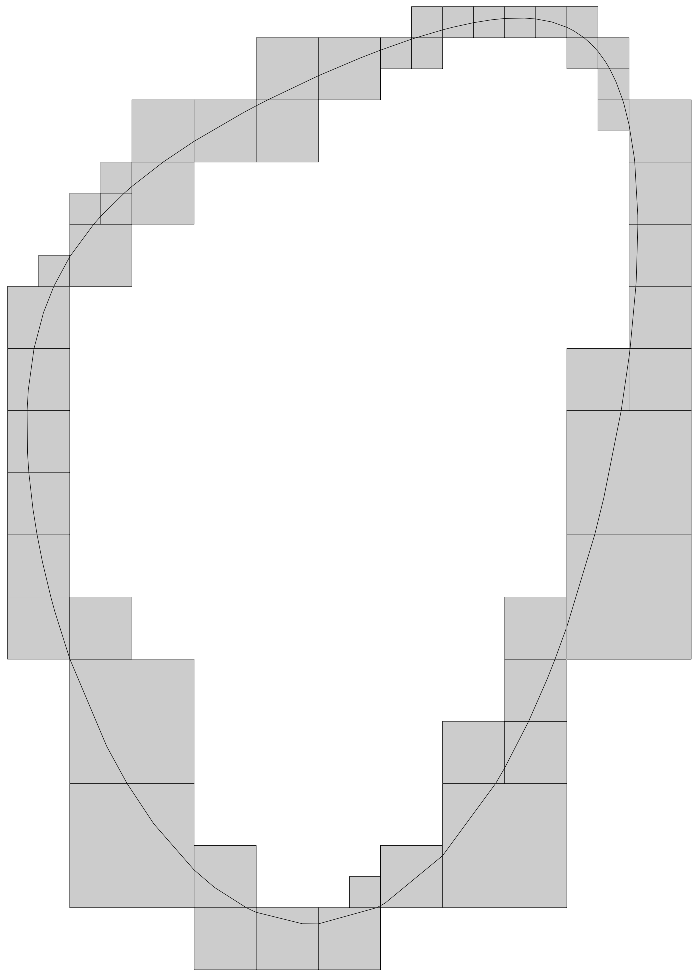
Robust adaptive approximation



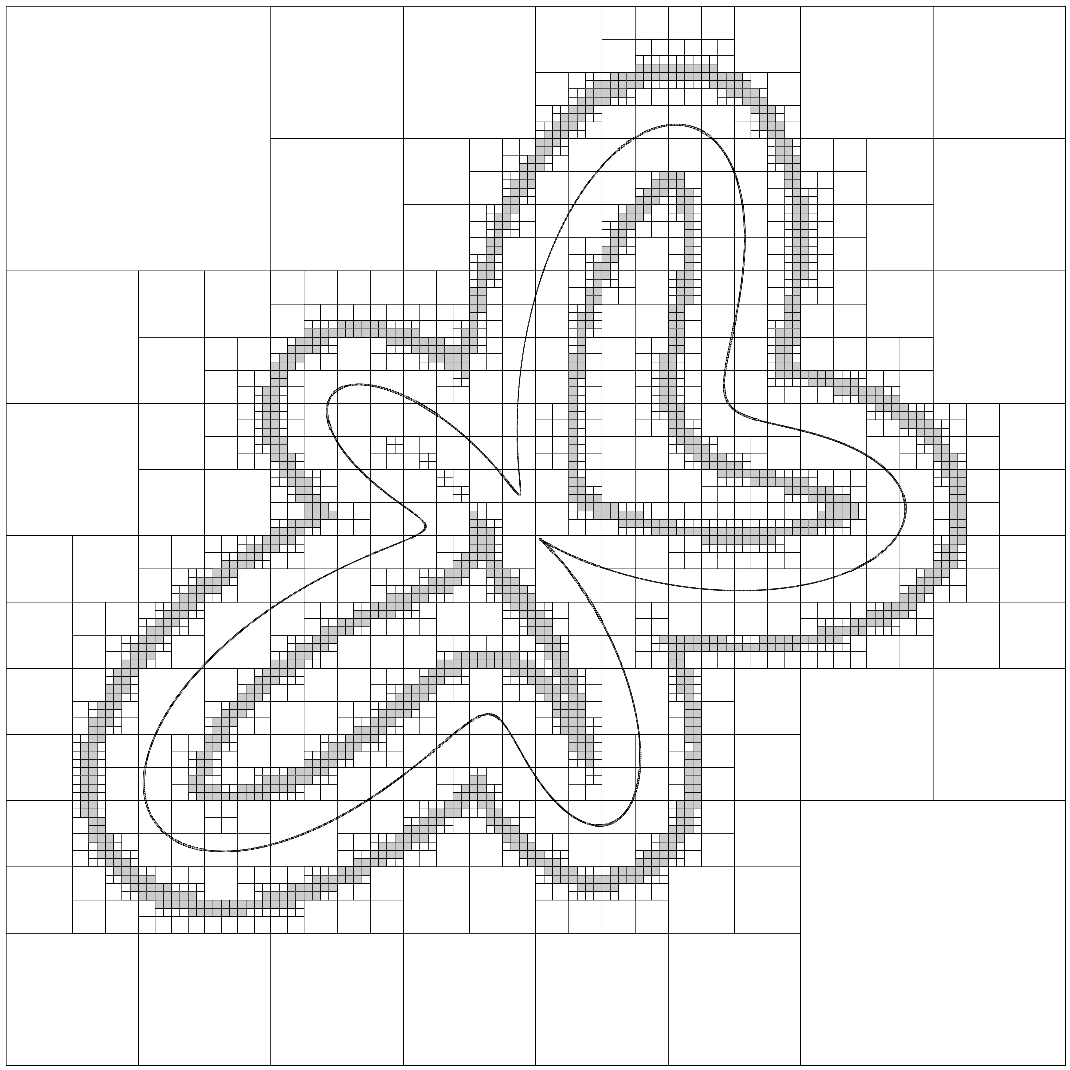
Robust adaptive approximation



Robust adaptive approximation

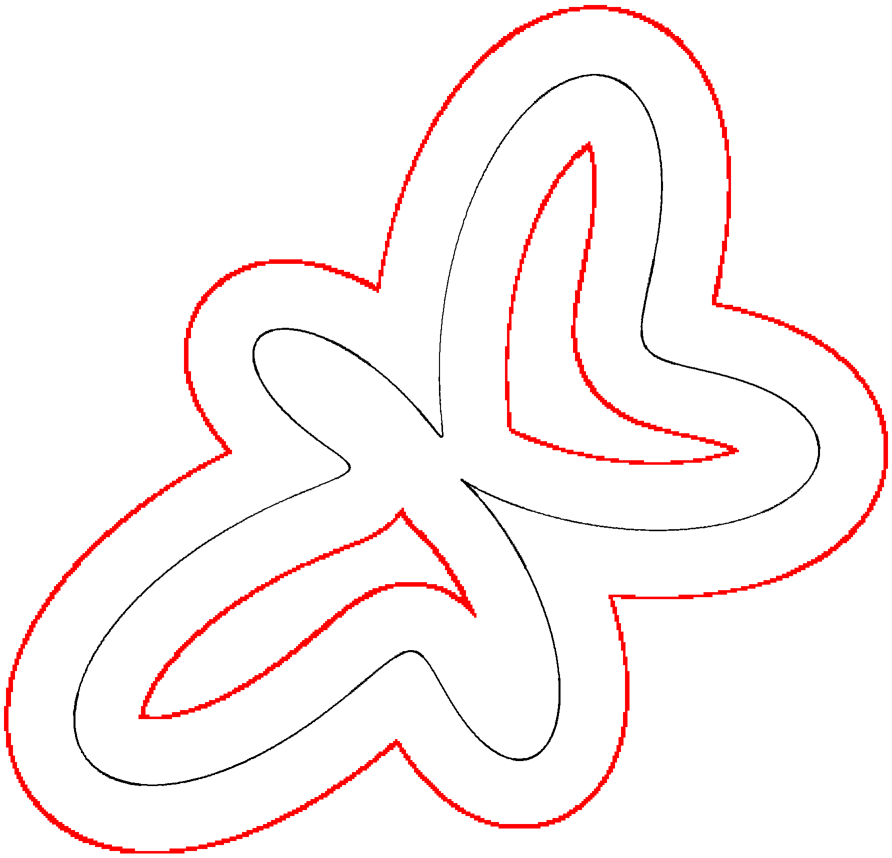


Offsets of parametric curves



Oliveira–Figueiredo (2003)

Offsets of parametric curves



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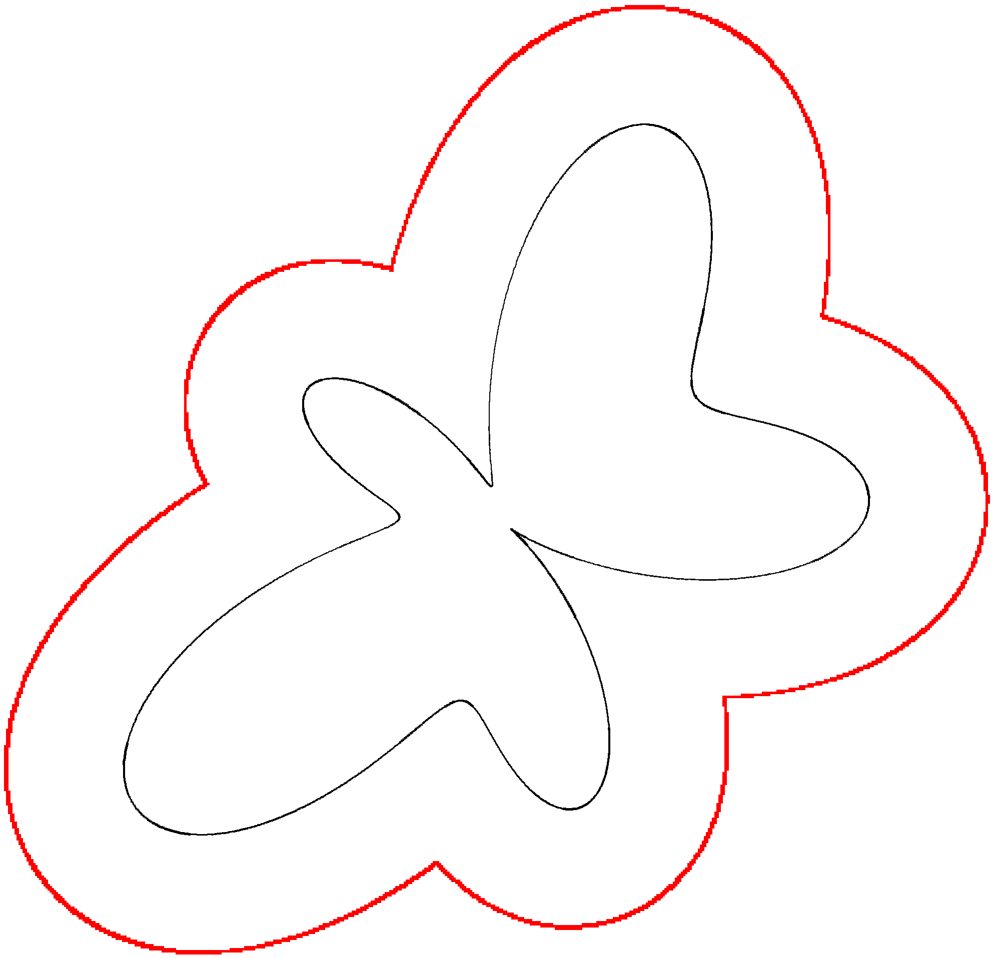

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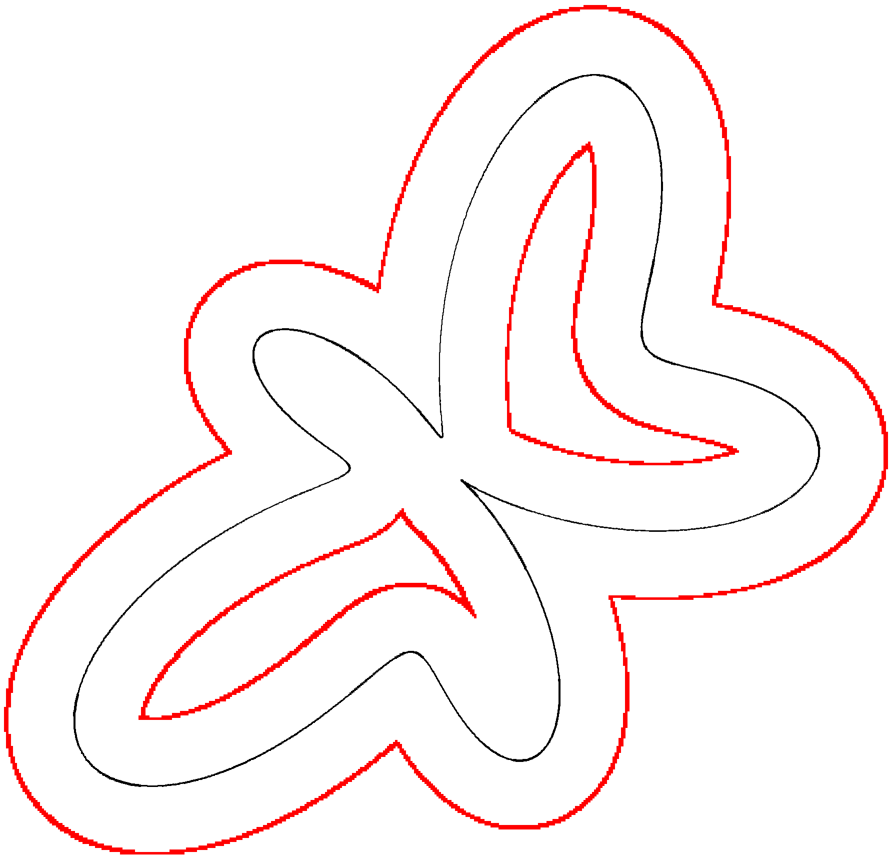

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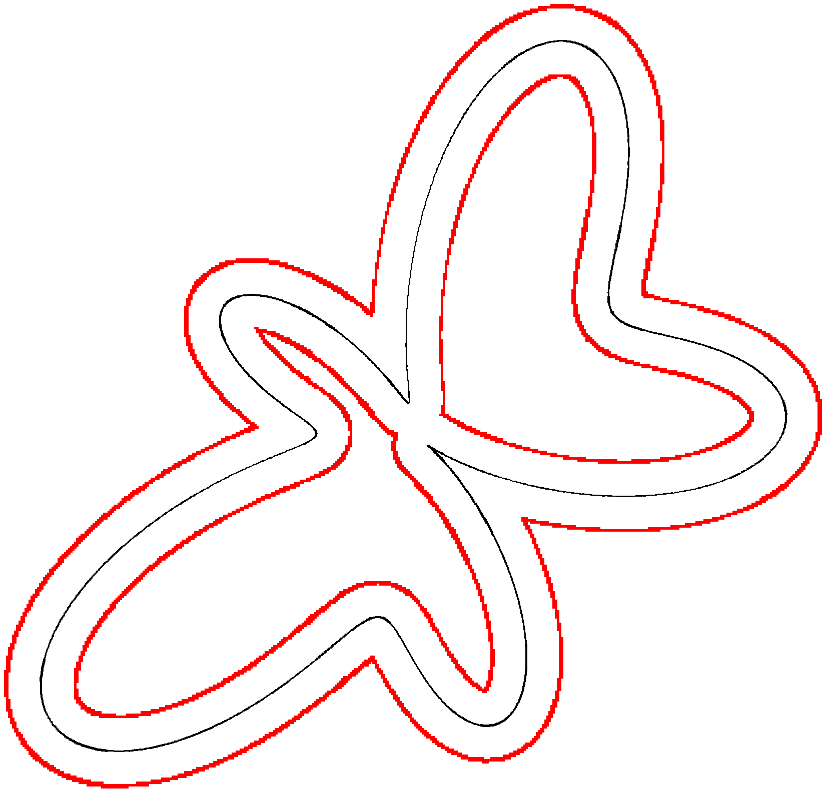

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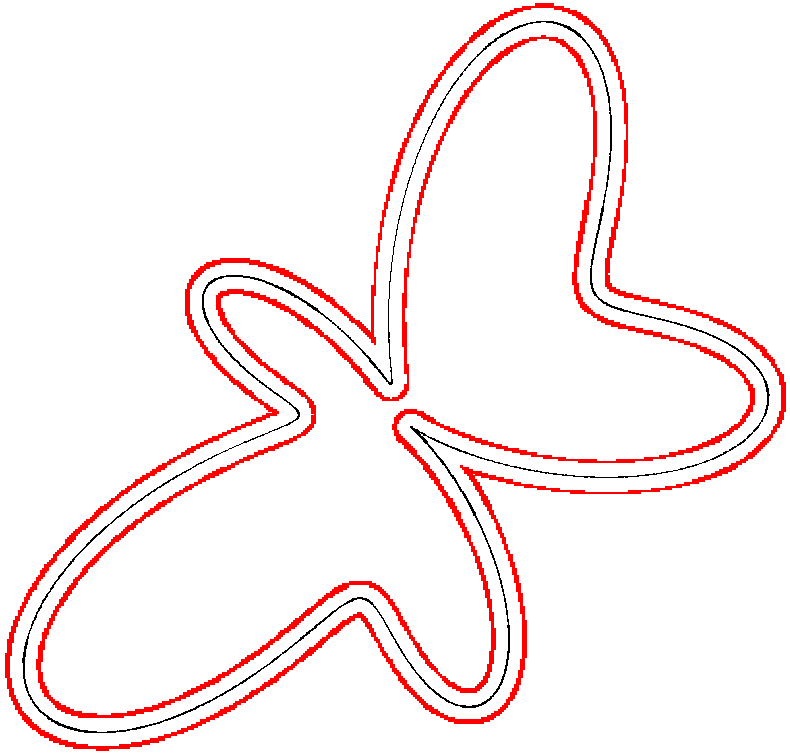
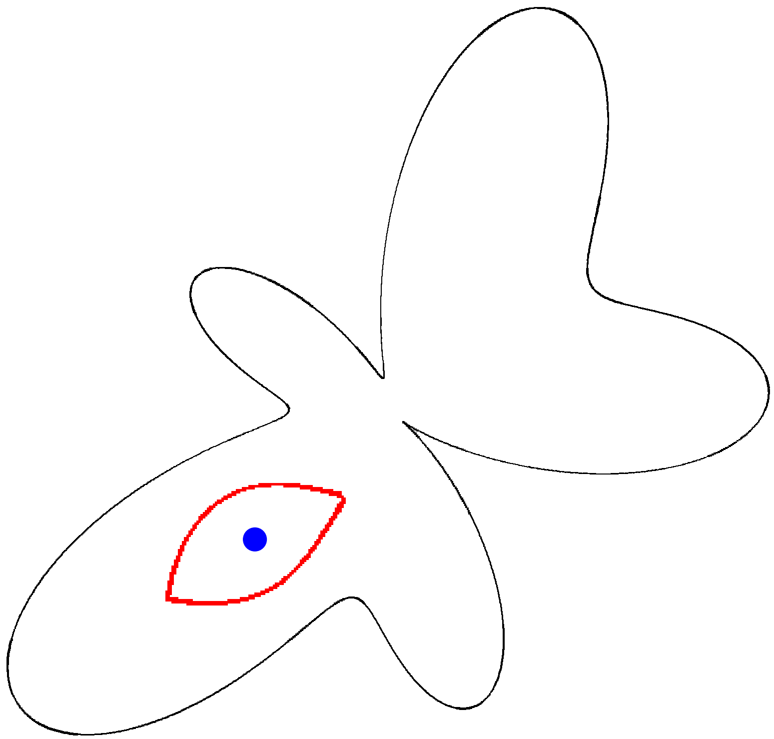


Bisectors of parametric curves



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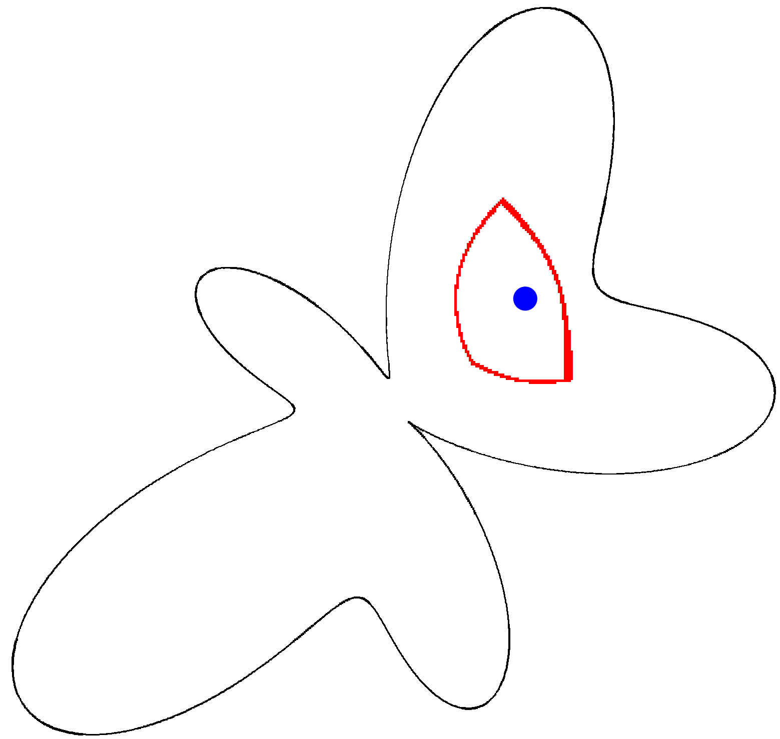

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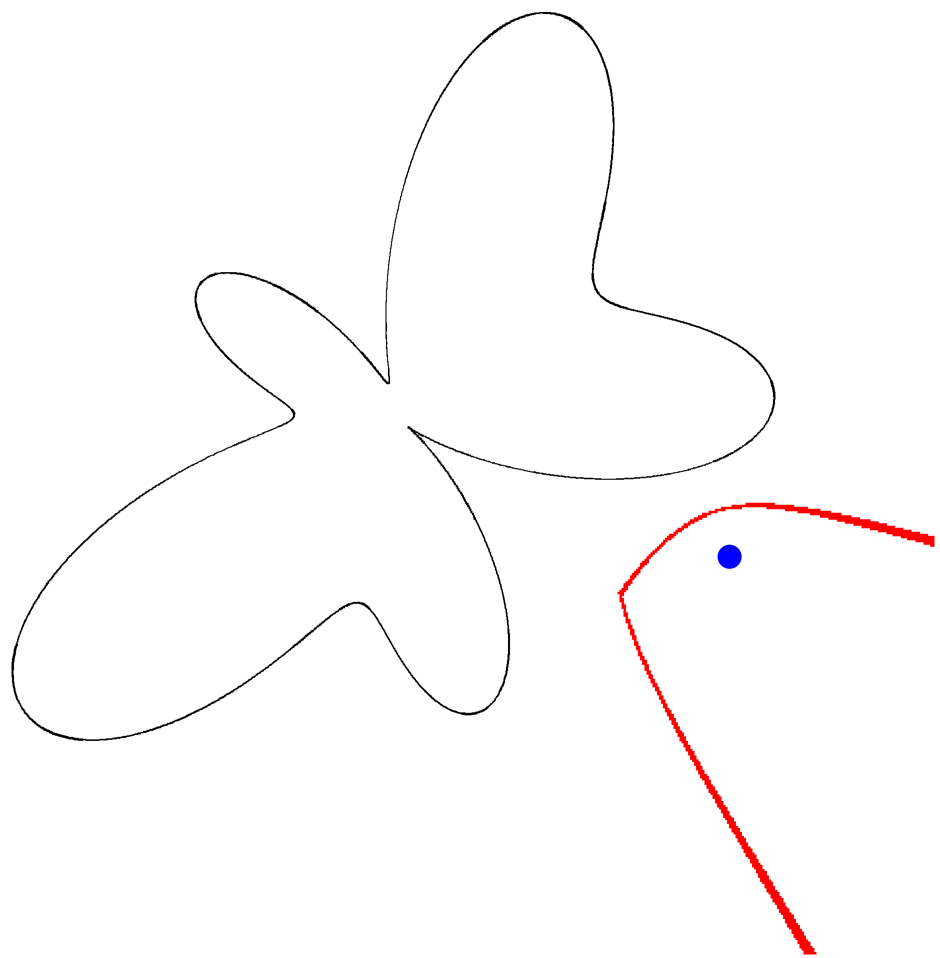

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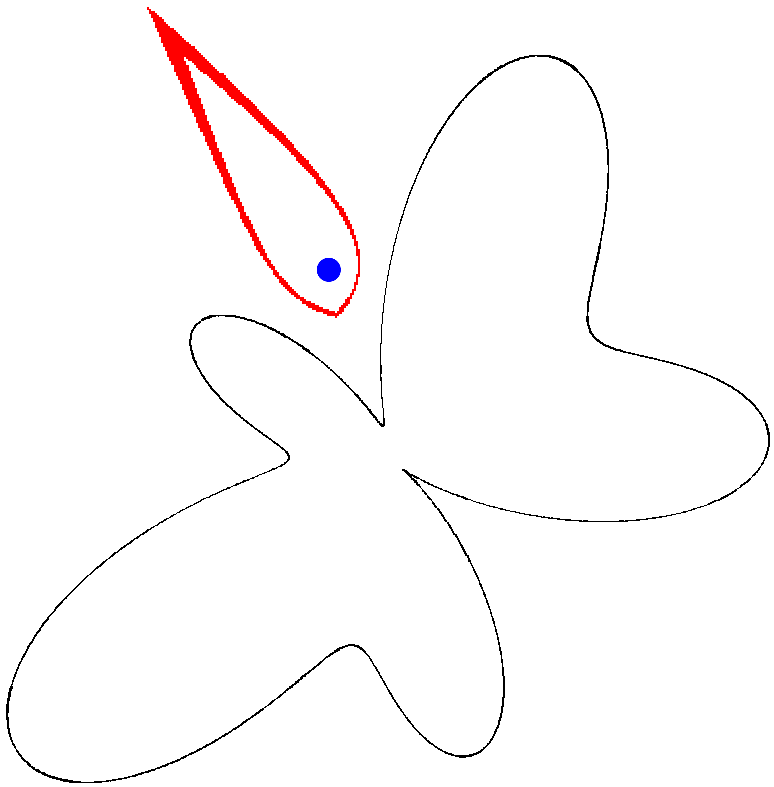
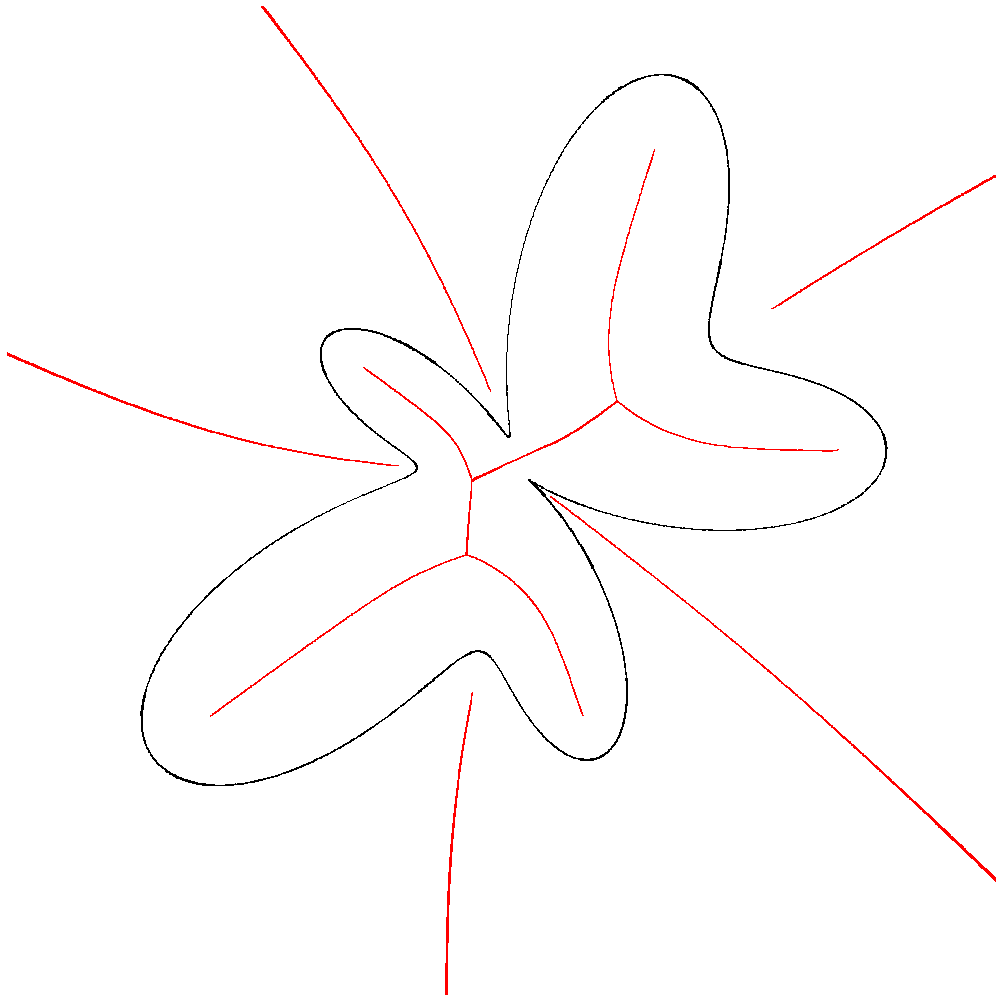


Medial axis of parametric curves



Interval methods

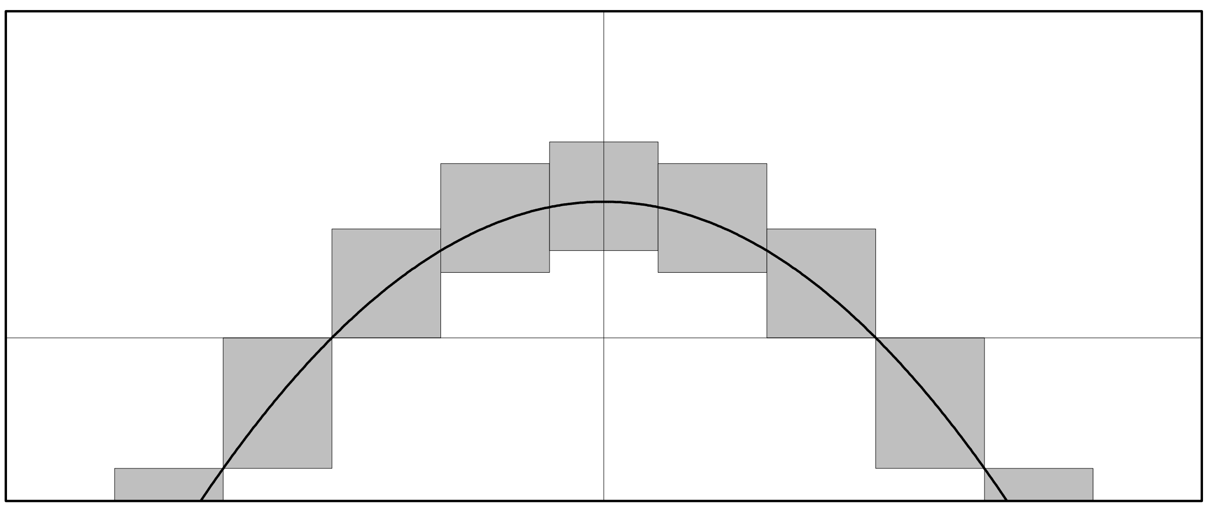
- Robust: they don't lie
  - ◇ correctness depends on $F(X) \supseteq f(X)$
  - ◇ can prove $0 \notin f(X)$, not that $0 \in f(X)$
- Converge: solutions get better
  - ◇ $F(X) \to \{f(x)\}$ as $X \to \{x\}$
- Conservative: they tend to exagerate
  - ◇ $f(x,y) = y^2 - x^3 + x \qquad X = [-2,-1] \times [1,2]$
    $F(X) = [0,11] \qquad f(X) = [1,10]$
  - ◇ gets worse in complicated expressions and iterative methods
- Efficient?
  - ◇ how much larger is $F(X)$?
  - ◇ better estimates imply faster methods

The dependency problem in interval arithmetic

IA can't see correlations between operands

$$g(x) = (10 + x)(10 - x) \text{ for } x \in [-2, 2]$$

$$\begin{aligned}
10 + x &= [8, 12] \\
10 - x &= [8, 12] \\
(10 + x)(10 - x) &= [64, 144] && \text{diam} = 80 \\
\text{Exact range} &= [96, 100] && \text{diam} = 4
\end{aligned}$$

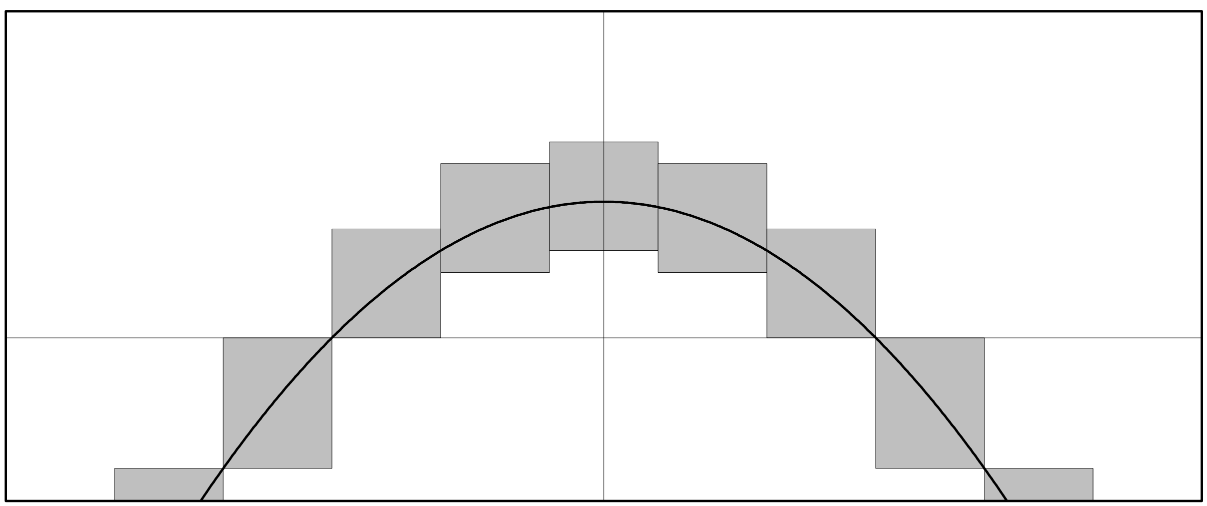


The dependency problem in interval arithmetic

IA can't see correlations between operands

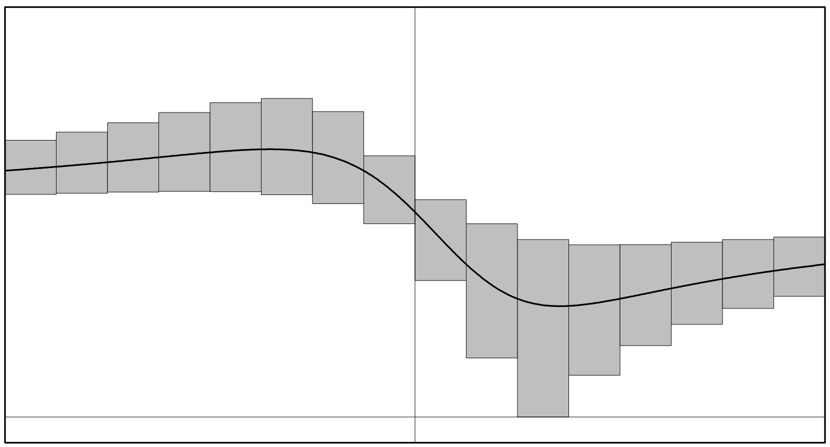
$$g(x) = (10 + x)(10 - x) \text{ for } x \in [-u, u]$$

$$\begin{aligned} 10 + x &= [10 - u, 10 + u] \\ 10 - x &= [10 - u, 10 + u] \\ (10 + x)(10 - x) &= [(10 - u)^2, (10 + u)^2] && \text{diam} = 40u \\ \text{Exact range} &= [100 - u^2, 100] && \text{diam} = u^2 \end{aligned}$$

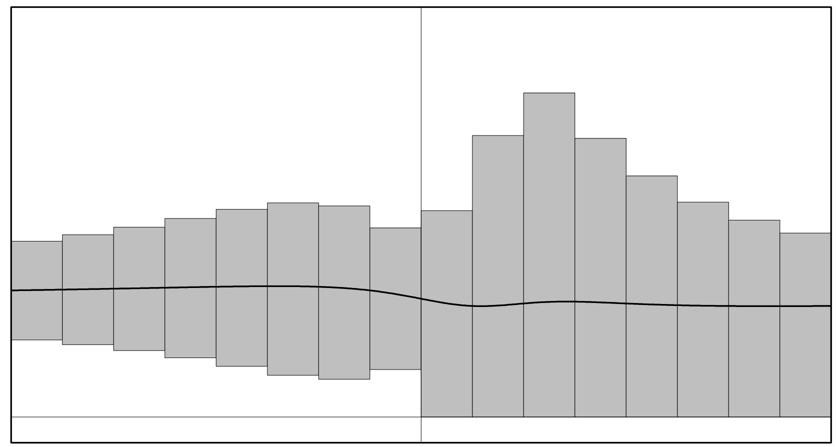


The dependency problem in interval arithmetic

$$g(x) = \sqrt{x^2 - x + 1/2}/\sqrt{x^2 + 1/2}$$



g



$g \circ g$

$g^n \to c =$ fixed point of $g \approx 0.5586$, but intervals diverge

Interval estimates may get too large in long computations

Affine arithmetic

Affine arithmetic

AA represents a quantity x with an *affine form*

$$\hat{x} = x_0 + x_1\varepsilon_1 + \cdots + x_n\varepsilon_n$$

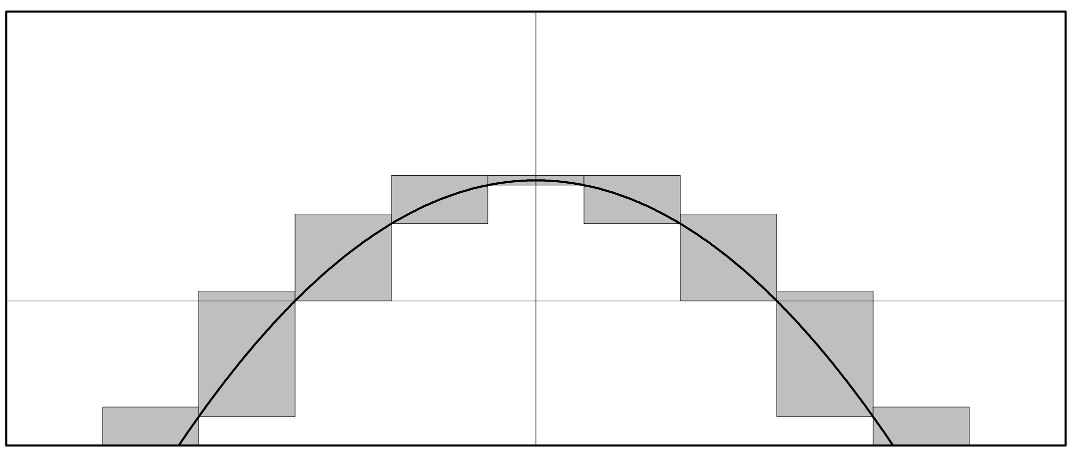
- *Noise symbols* $\varepsilon_i \in [-1, +1]$: independent, but otherwise unknown
- Can compute arbitrary formulas on affine forms
  - ◇ Need affine approximations for non-affine operations
  - ◇ New noise symbols created during computation due to approximation and rounding
- Can replace IA
  - ◇ $x \sim \hat{x} \;\Rightarrow\; x \in [x_0 - r,\, x_0 + r]$ for $r = |x_1| + \cdots + |x_n|$
  - ◇ $x \in [a, b] \;\Rightarrow\; x \sim \hat{x} = x_0 + x_1\varepsilon_1$
    $x_0 = (b + a)/2 \quad x_1 = (b - a)/2$

The dependency problem in interval arithmetic – AA version

AA can see correlations between operands

$$g(x) = (10 + x)(10 - x) \text{ for } x \in [-u, u], \qquad x = 0 + u\,\varepsilon$$

$$\begin{aligned}
10 + x &= 10 - u\,\varepsilon \\
10 - x &= 10 + u\,\varepsilon \\
(10 + x)(10 - x) &= 100 - u^2\,\varepsilon \\
\text{range} &= [100 - u^2, 100 + u^2] \qquad \text{diam} = 2u^2 \\
\text{Exact range} &= [100 - u^2, 100] \qquad \text{diam} = u^2
\end{aligned}$$



AA

The dependency problem in interval arithmetic – AA version

AA can see correlations between operands

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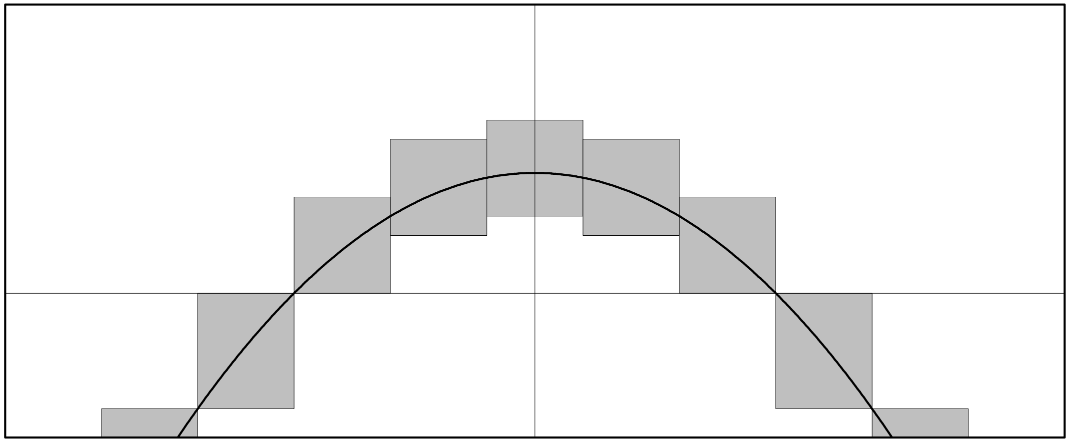
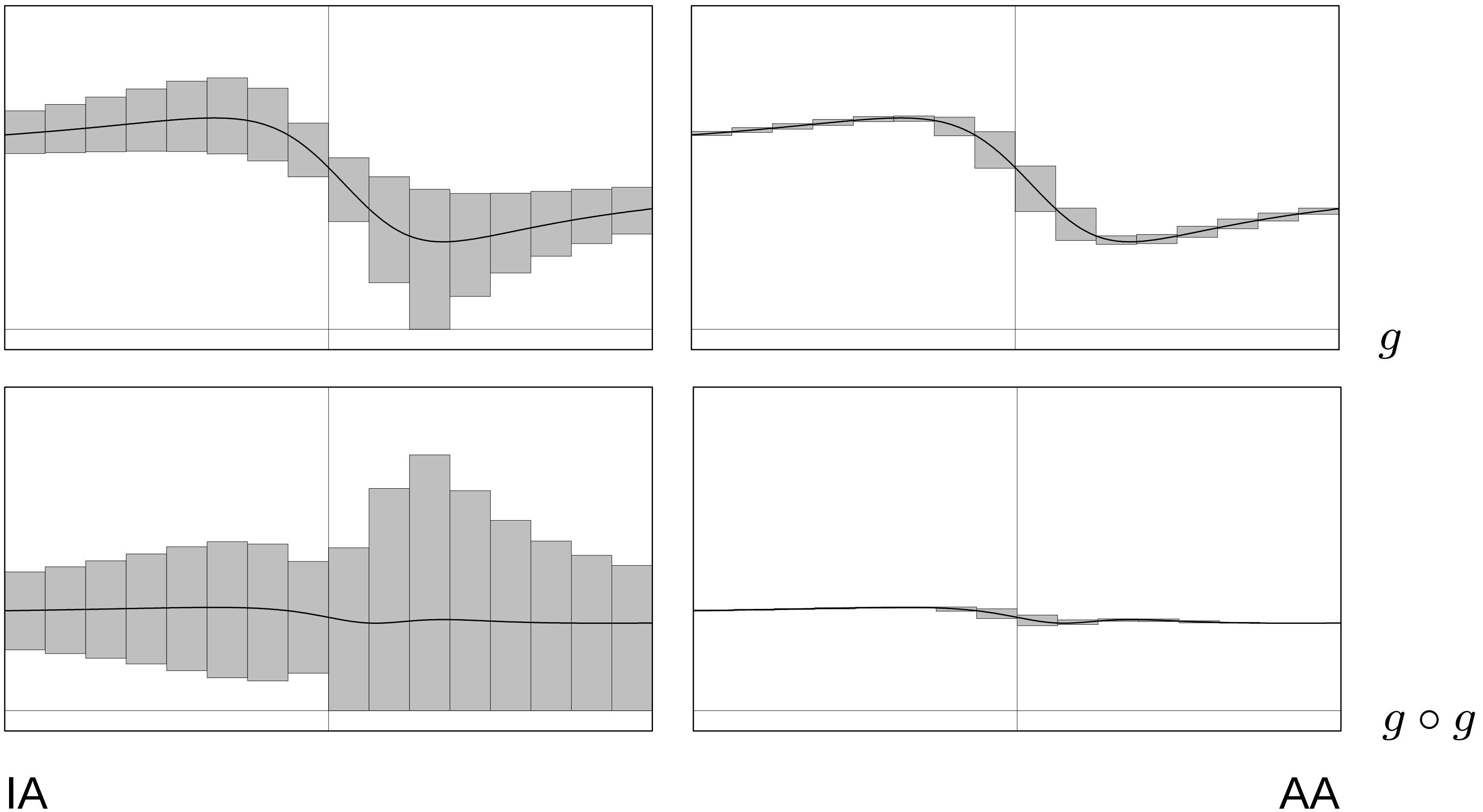


IA

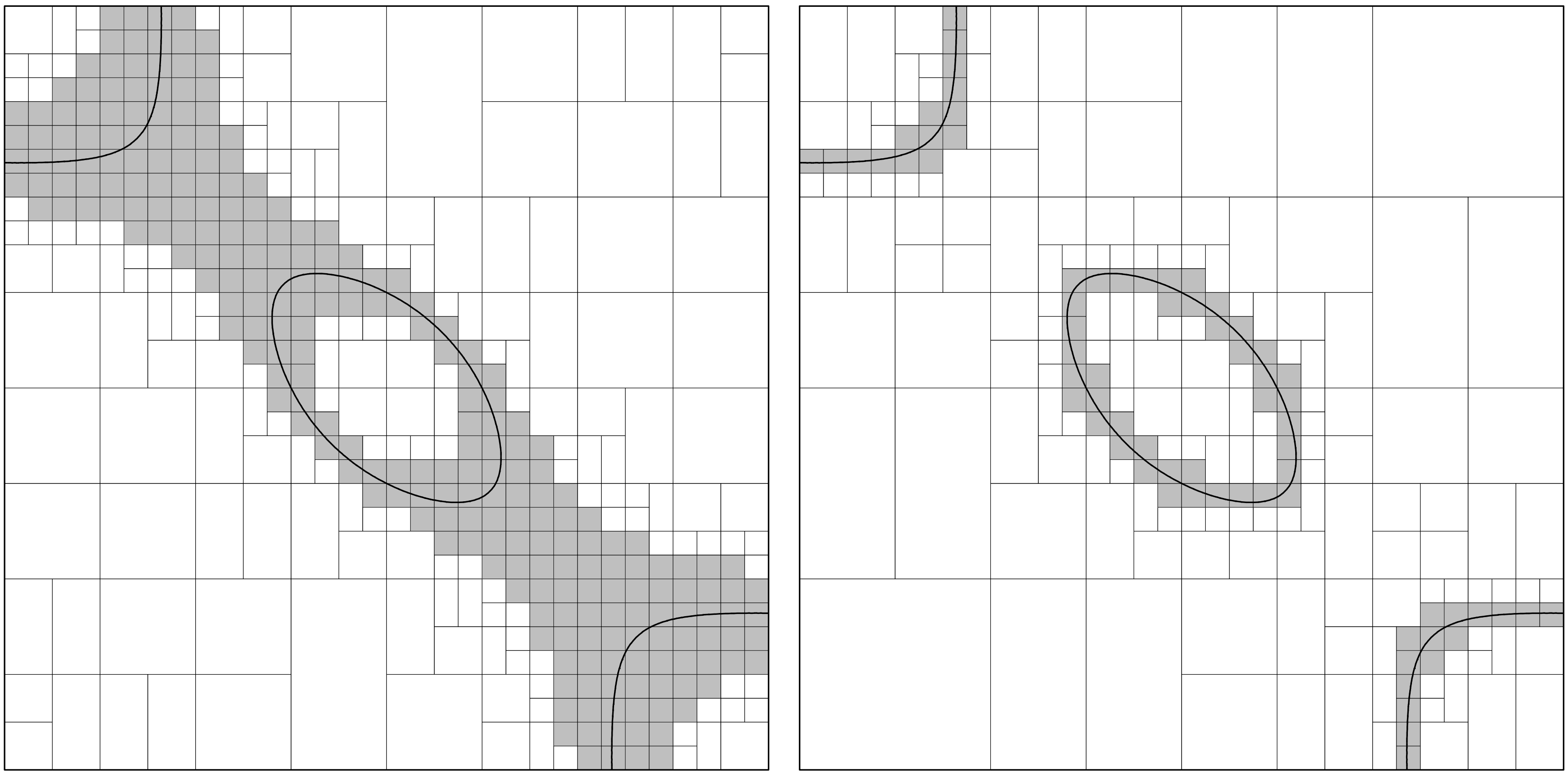
The dependency problem in interval arithmetic – AA version

$$g(x) = \sqrt{x^2 - x + 1/2}/\sqrt{x^2 + 1/2}$$



Replacing IA with AA for plotting implicit curves

$$x^2 + y^2 + xy - (xy)^2/2 - 1/4 = 0$$

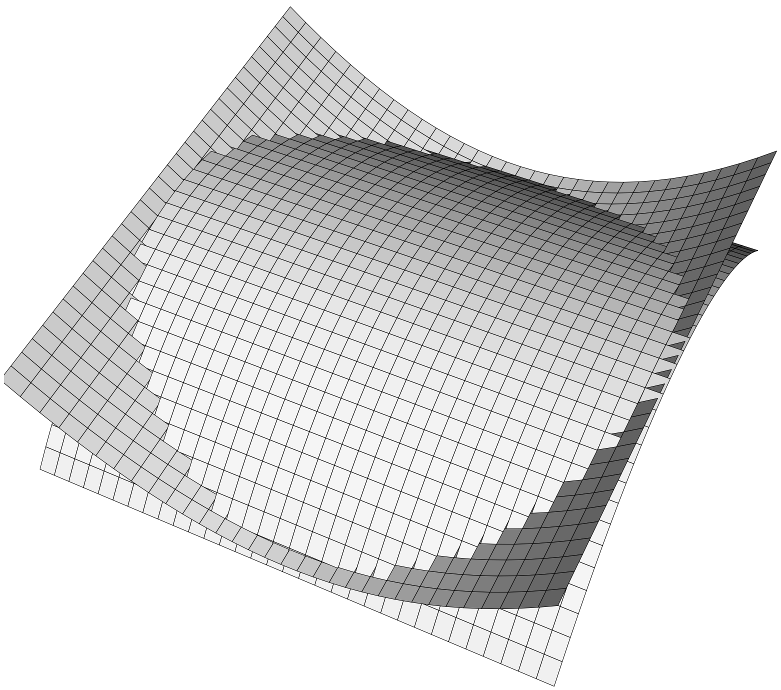


IA (246 cells, 66 exact) (Comba–Stolfi , 1993) (70 cells) AA

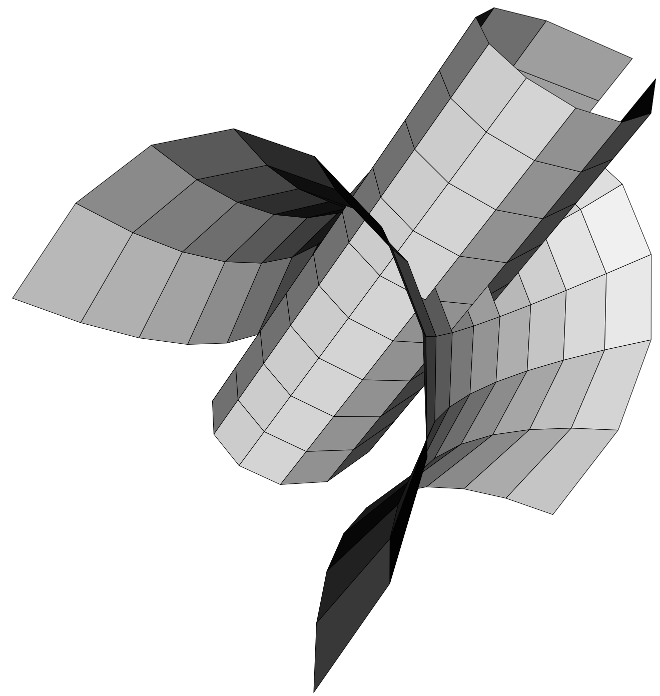
Replacing IA with AA for surface intersection

Tensor product Bézier surfaces of degree (p, q):

$$f(u,v) = \sum_{i=0}^{p} \sum_{j=0}^{q} a_{ij} B_i^p(u) B_j^q(v), \quad B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$$



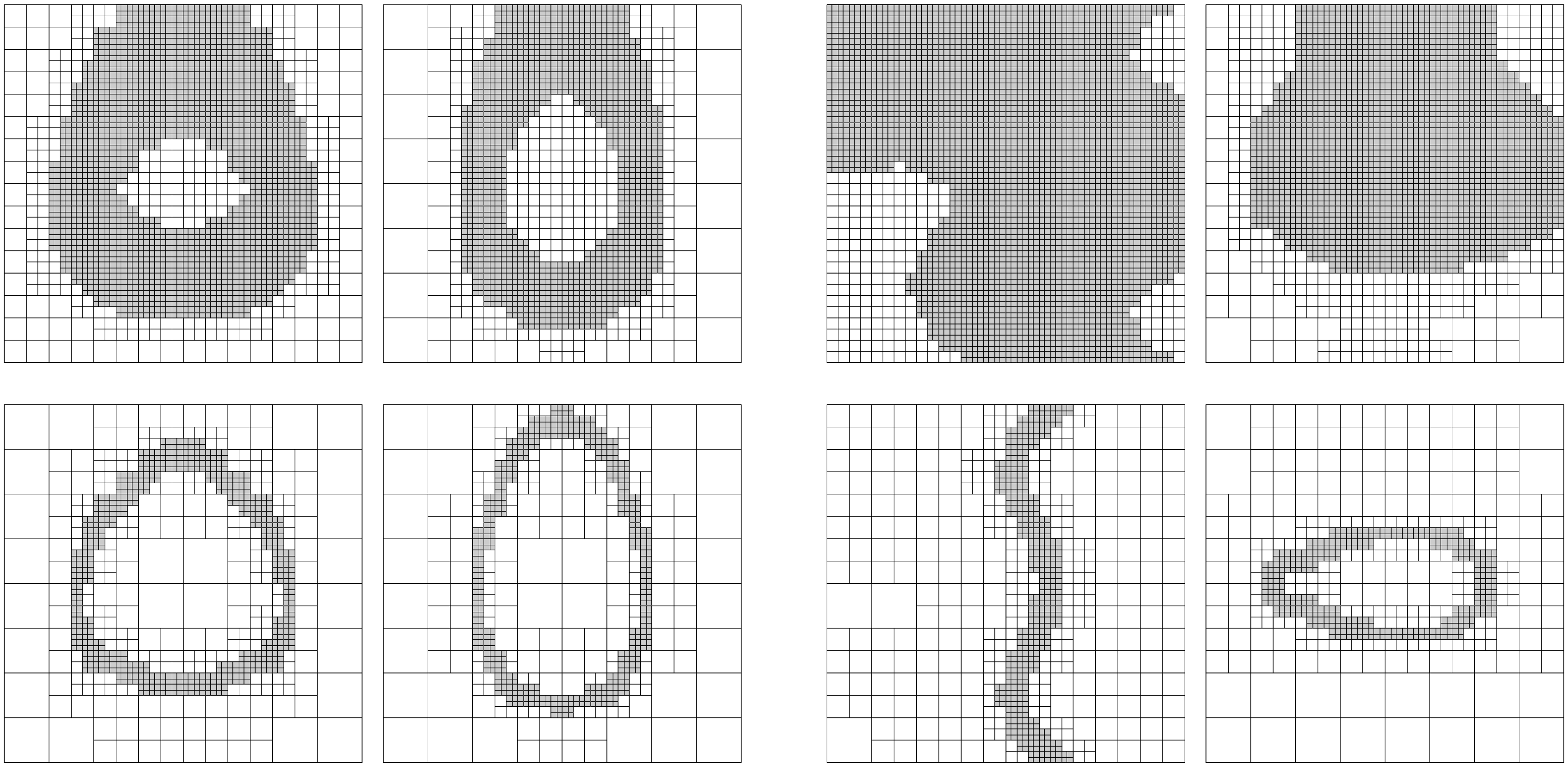
$(2, 1)$



$(3, 3)$

Replacing IA with AA for surface intersection

$(2,1)$ IA $(3,3)$



AA

(Figueiredo, 1996)

Exploiting the correlations given by AA

Geometry of affine arithmetic

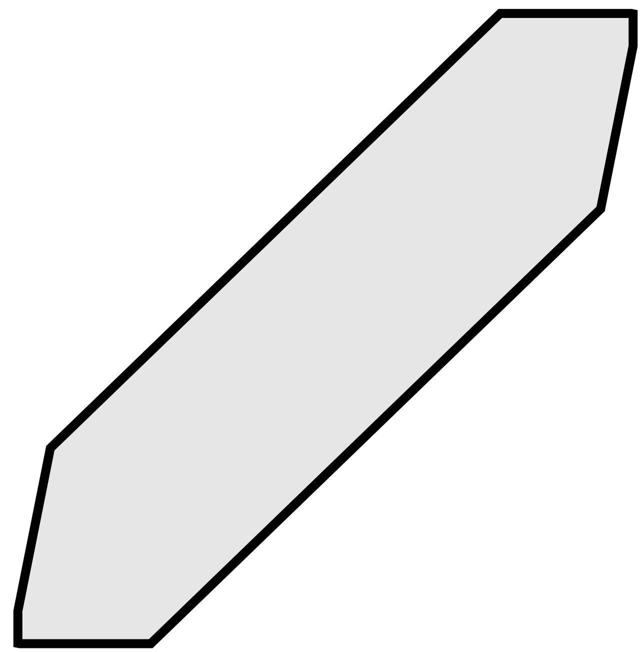
Affine forms that share noise symbols are not independent:

$$\hat{x} = x_0 + x_1\varepsilon_1 + \cdots + x_n\varepsilon_n$$
$$\hat{y} = y_0 + y_1\varepsilon_1 + \cdots + y_n\varepsilon_n$$

The region containing (x, y) is

$$Z = \{(x, y) : \varepsilon_i \in \mathbf{U}\}$$

This region is the image of $\mathbf{U}^n$ under an affine map $\mathbf{R}^n \to \mathbf{R}^2$. It's a centrally symmetric convex polygon, a *zonotope*.



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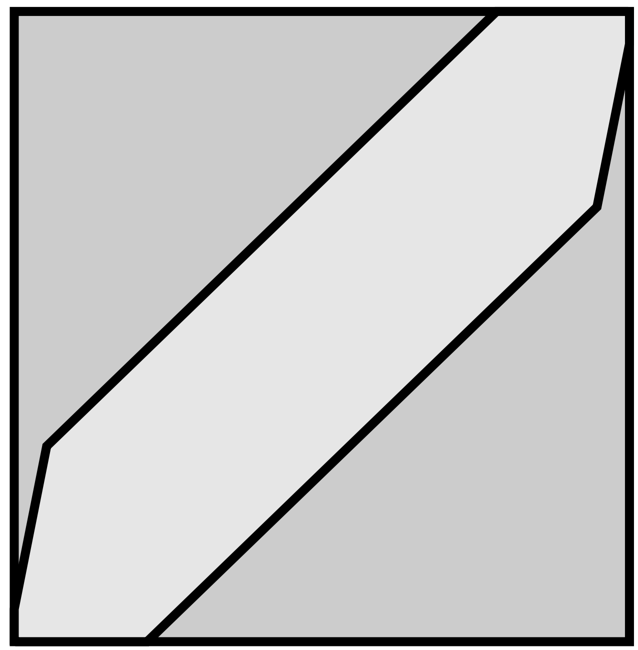
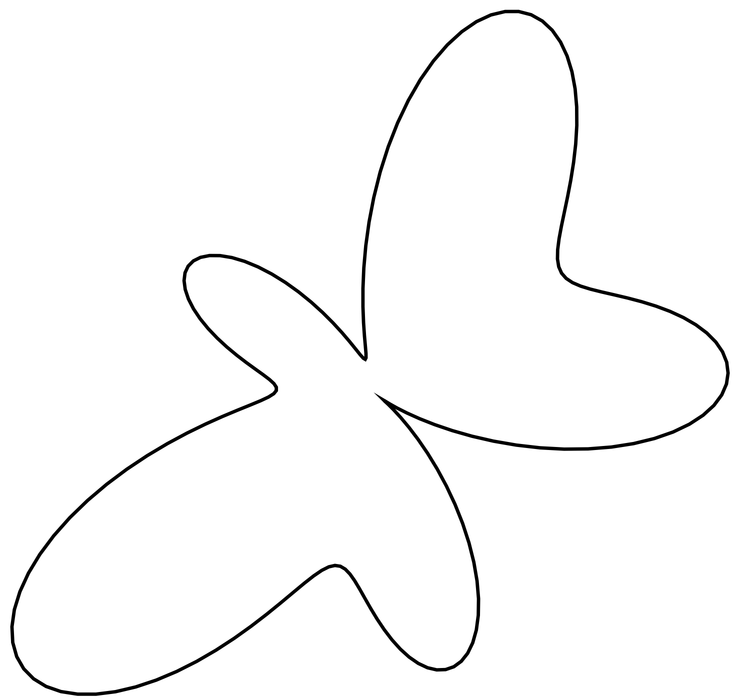
The region would be a rectangle if x and y were independent.



Approximating parametric curves

Given a parametric curve $\mathcal{C} = \gamma(I)$, where $\gamma\colon I \to \mathbf{R}^2$ and $T \subseteq I$, compute a bounding rectangle for $\mathcal{P} = \gamma(T)$.



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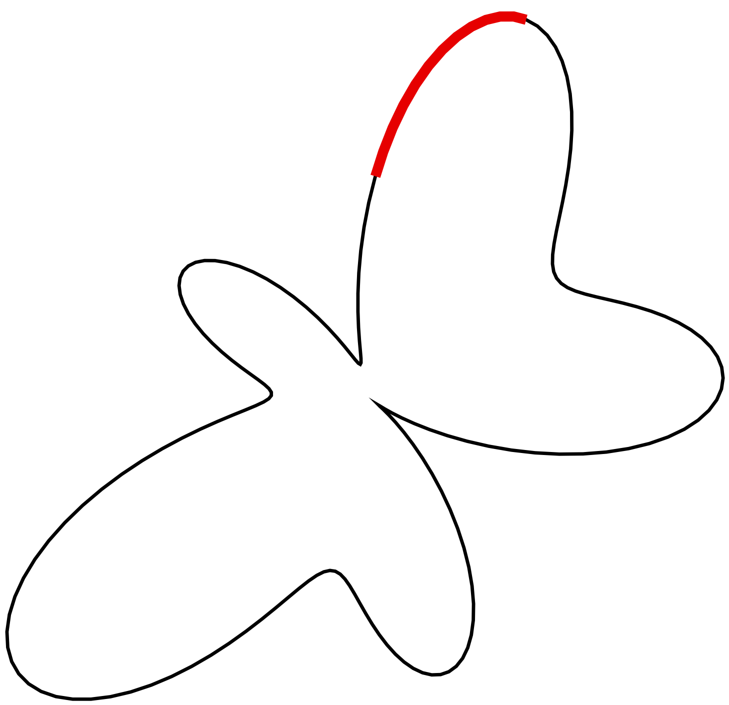

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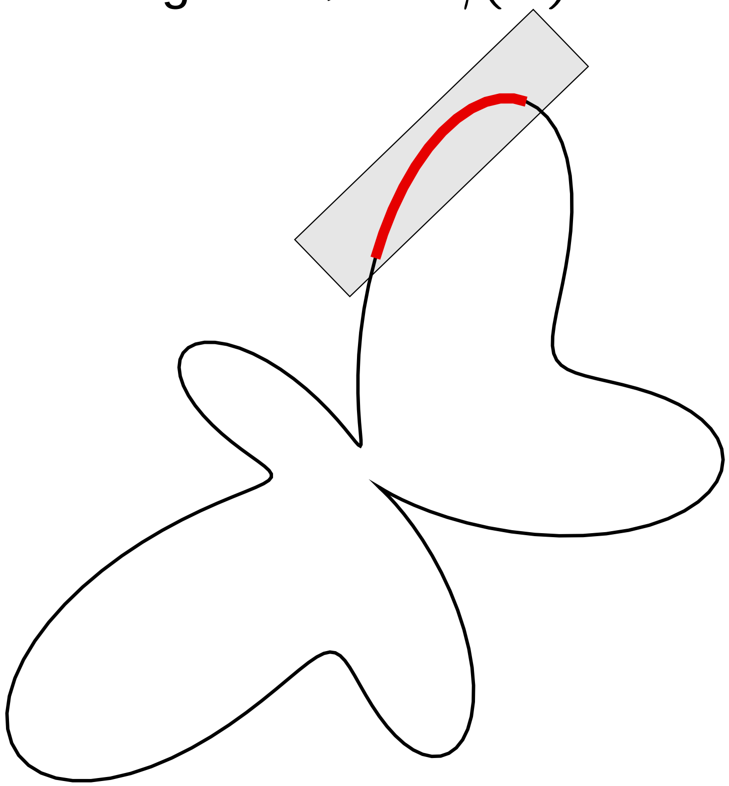

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Solution:

- Write $\gamma(t) = (x(t), y(t))$.
- Represent $t \in T$ with an affine form:

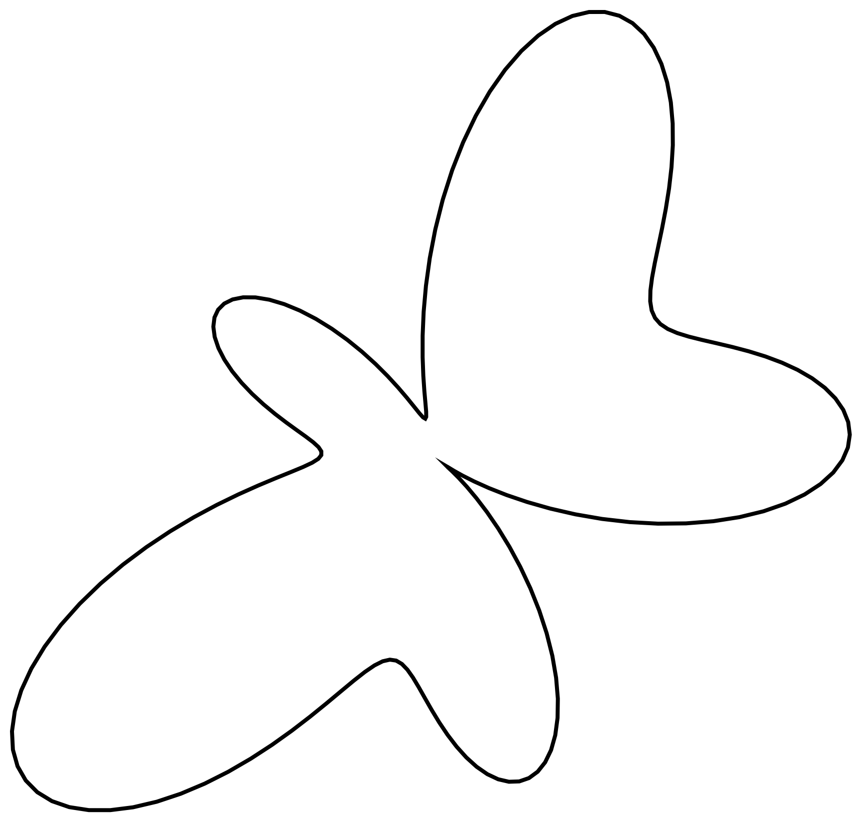
$$\hat{t} = t_0 + t_1\,\varepsilon_1, \qquad t_0 = (b+a)/2, \quad t_1 = (b-a)/2$$

- Compute coordinate functions x and y at $\hat{t}$ using AA:

$$\begin{aligned} \hat{x} &= x_0 + x_1\varepsilon_1 + \cdots + x_n\varepsilon_n \\ \hat{y} &= y_0 + y_1\varepsilon_1 + \cdots + y_n\varepsilon_n \end{aligned}$$

- Use bounding rectangle of the xy zonotope.

Approximating parametric curves



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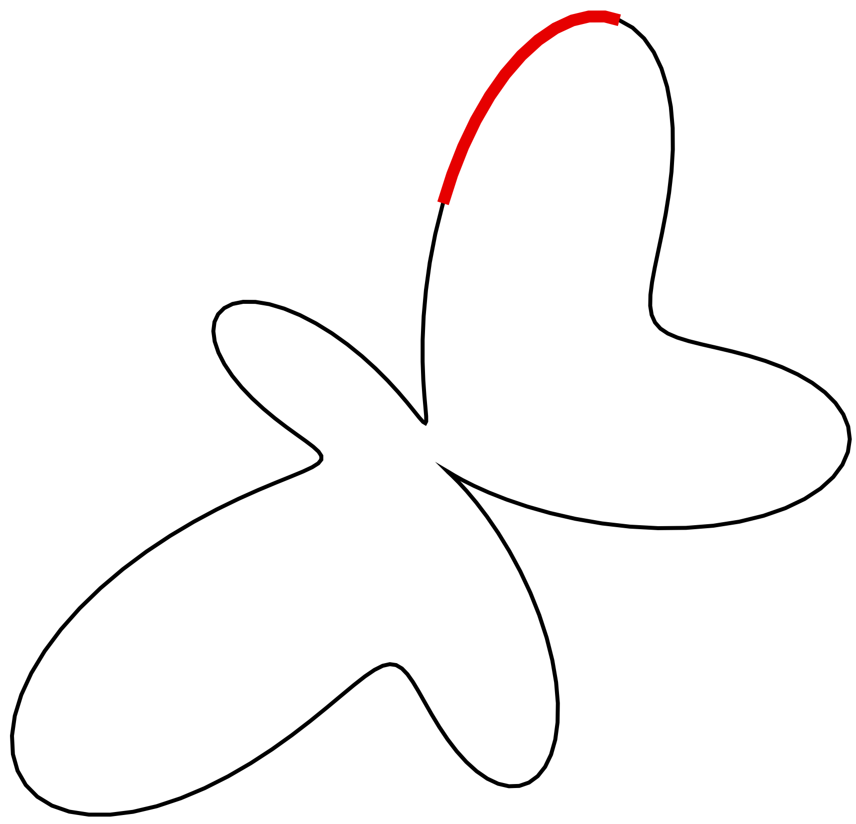
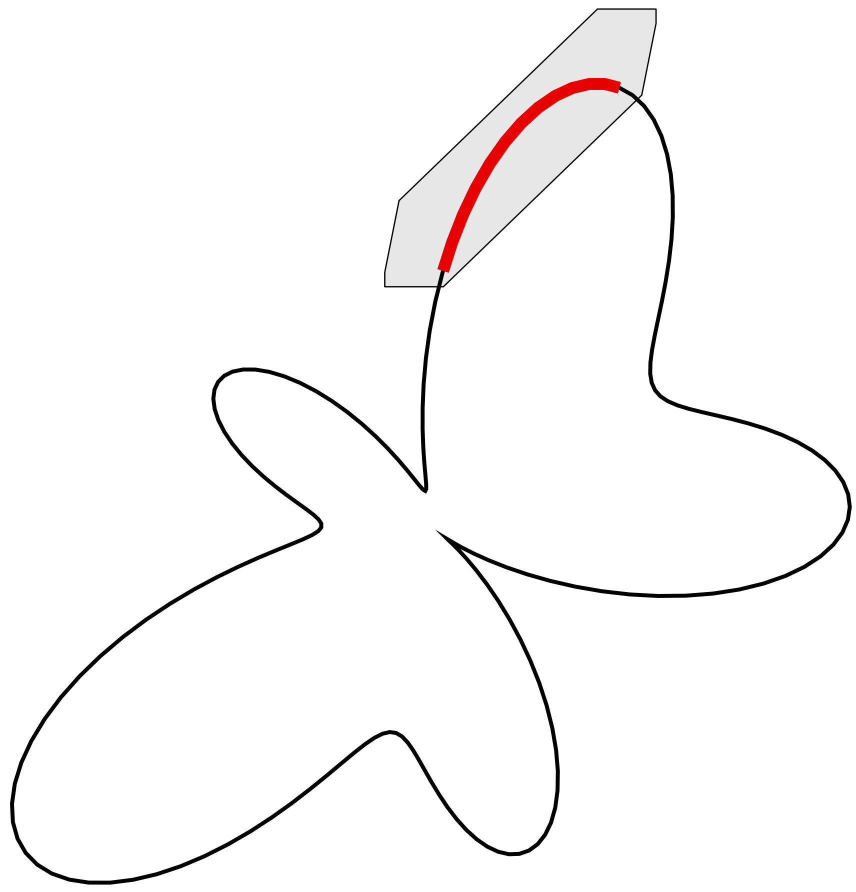


Approximating parametric curves



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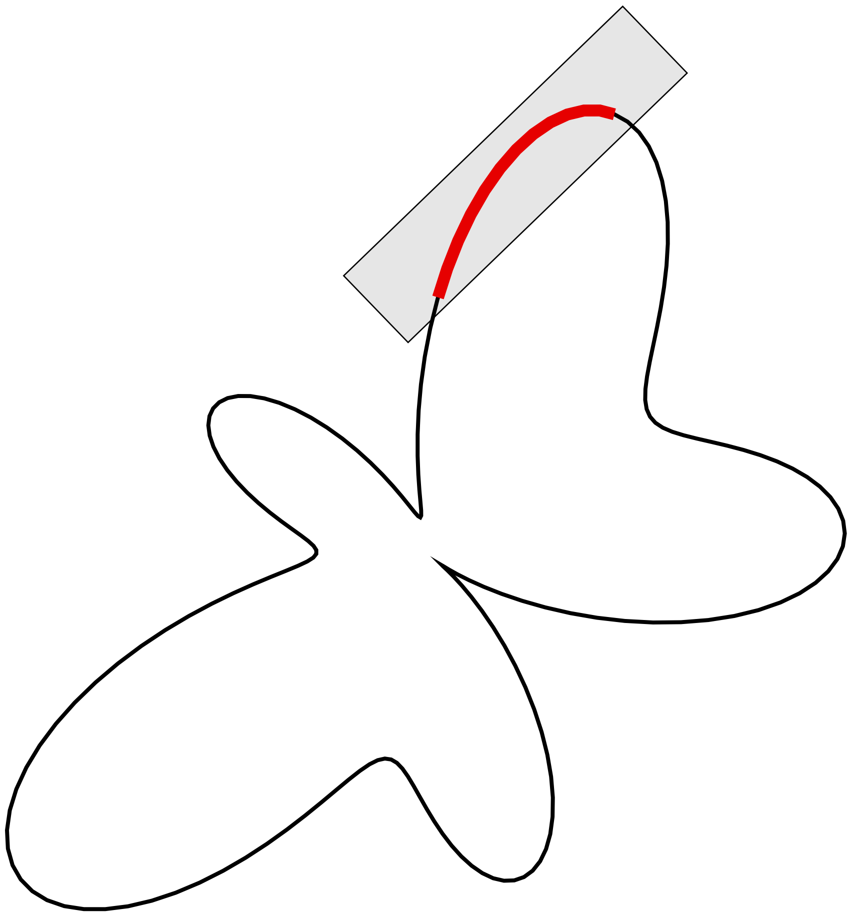

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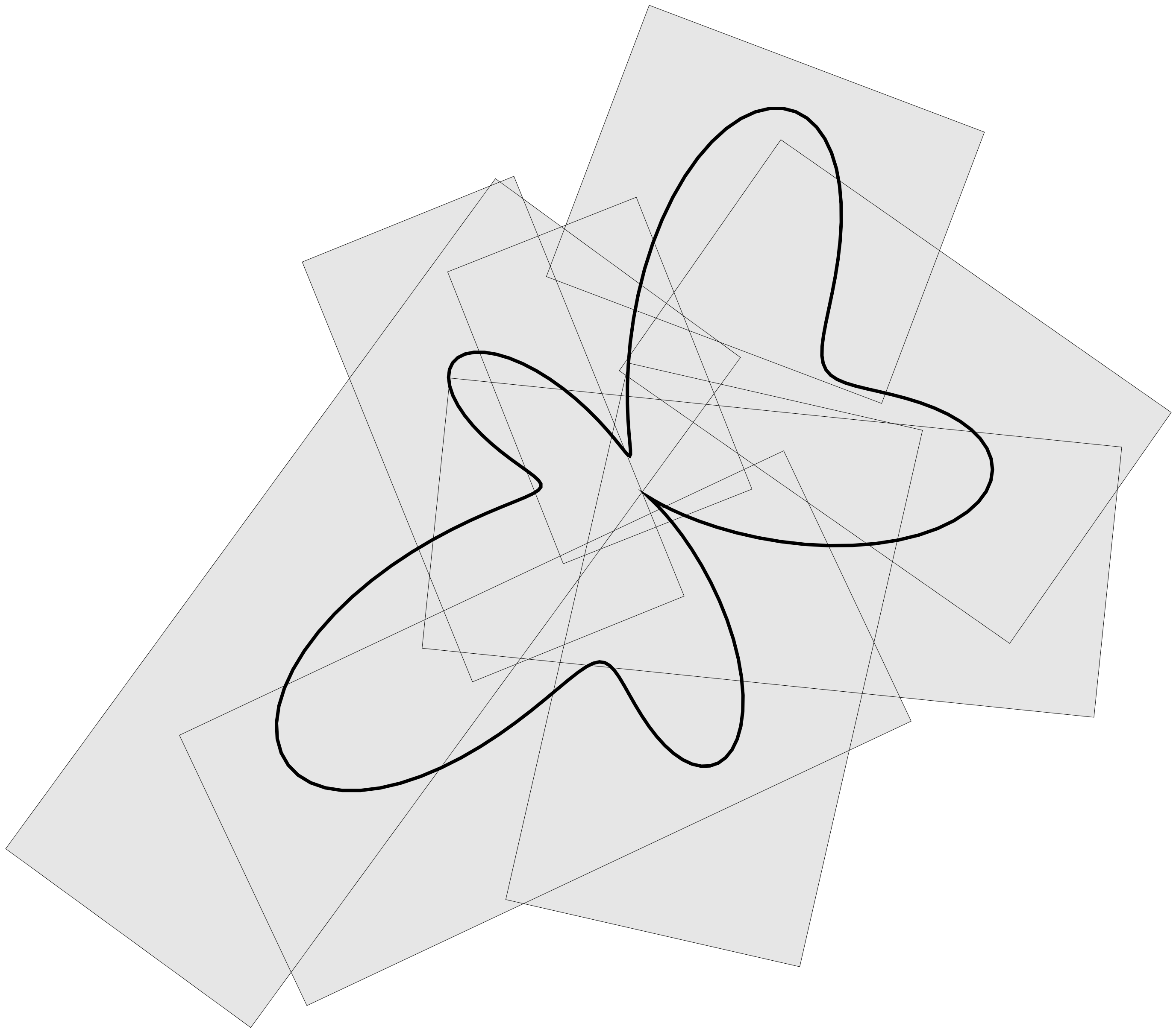

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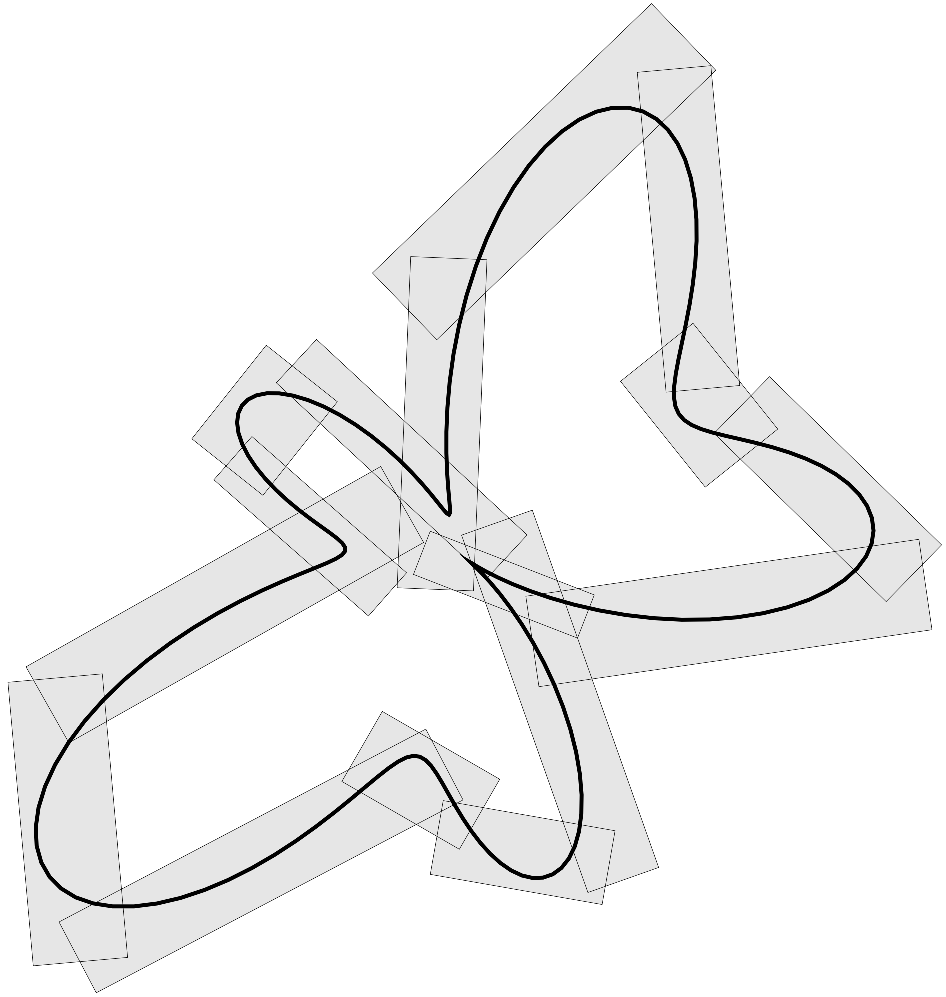

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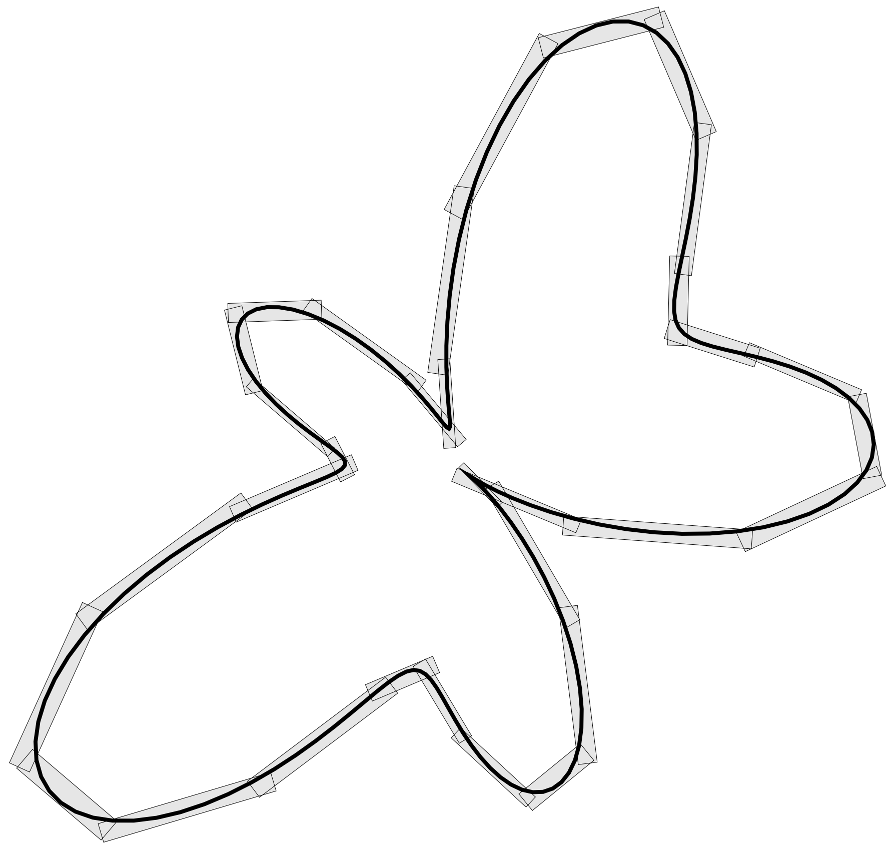
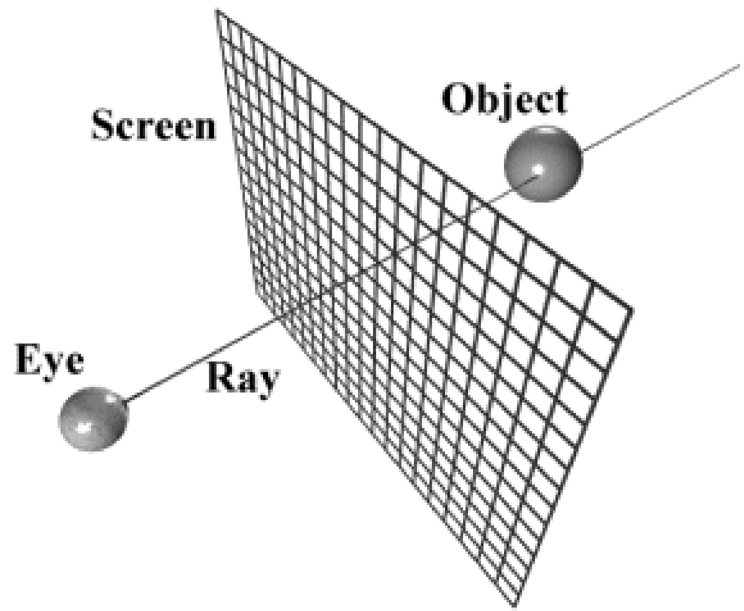


Approximating parametric curves

(Figueiredo–Stolfi –Velho, 2003)

Ray casting implicit surfaces

- Implicit surface
  $h: \mathbf{R}^3 \to \mathbf{R}$
  $S = \{p \in \mathbf{R}^3 : h(p) = 0\}$
- Ray
  $r(t) = E + t \cdot v, \quad t \in [0, \infty)$
- Ray intersects S when
  $f(t) = h(r(t)) = 0$
- First intersection occurs at *smallest* zero of f in $[0, \infty)$.
- Paint pixel with color based on normal at first intersection point



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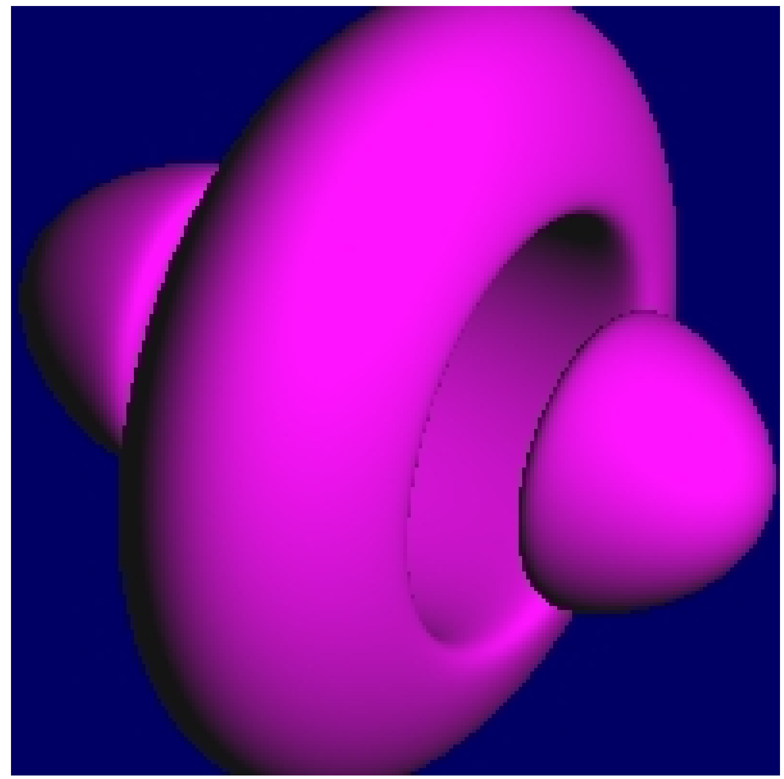


$4(x^4 + (y^2 + z^2)^2) + 17x^2(y^2 + z^2) - 20(x^2 + y^2 + z^2) + 17 = 0$

(Custatis–Figueiredo–Gattass, 1999)

Interval bisection

- Solve $f(t) = 0$ using inclusion function F for f:

$$F(T) \supseteq f(T) = \{f(t) : t \in T\}, \quad T \subseteq I$$

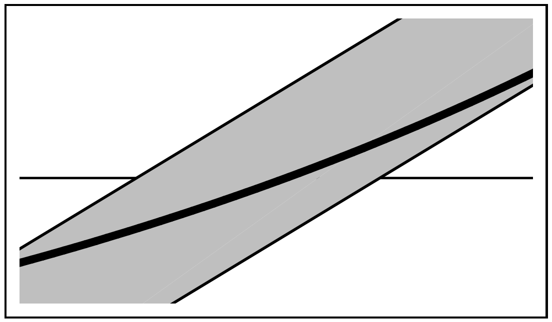
- $0 \notin F(T) \Rightarrow$ *no* solutions of $f(t) = 0$ in T
- $0 \in F(T) \Rightarrow$ there *may* be solutions in T

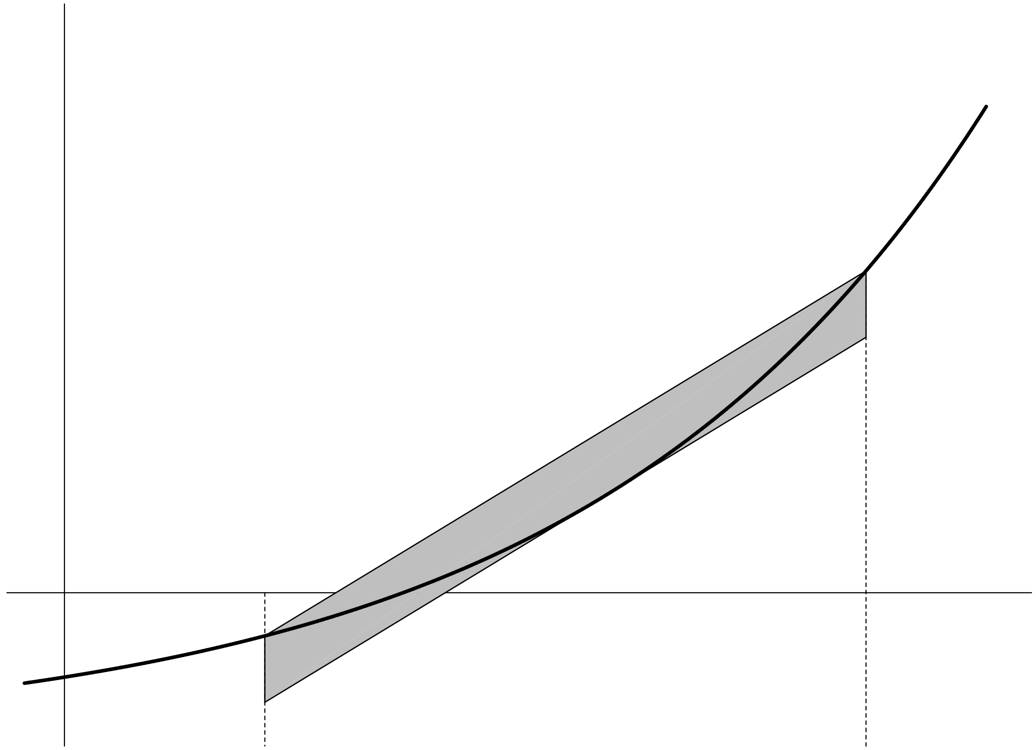
```
interval-bisection([a, b]):
    if 0 ∈ F([a, b]) then
        c ← (a + b)/2
        if (b − a) < ε then
            return c
        else
            interval-bisection([a, c])        ← try left half first!
            interval-bisection([c, b])
```

Start with interval-bisection($[0, t_\infty]$) to find the *first* zero.

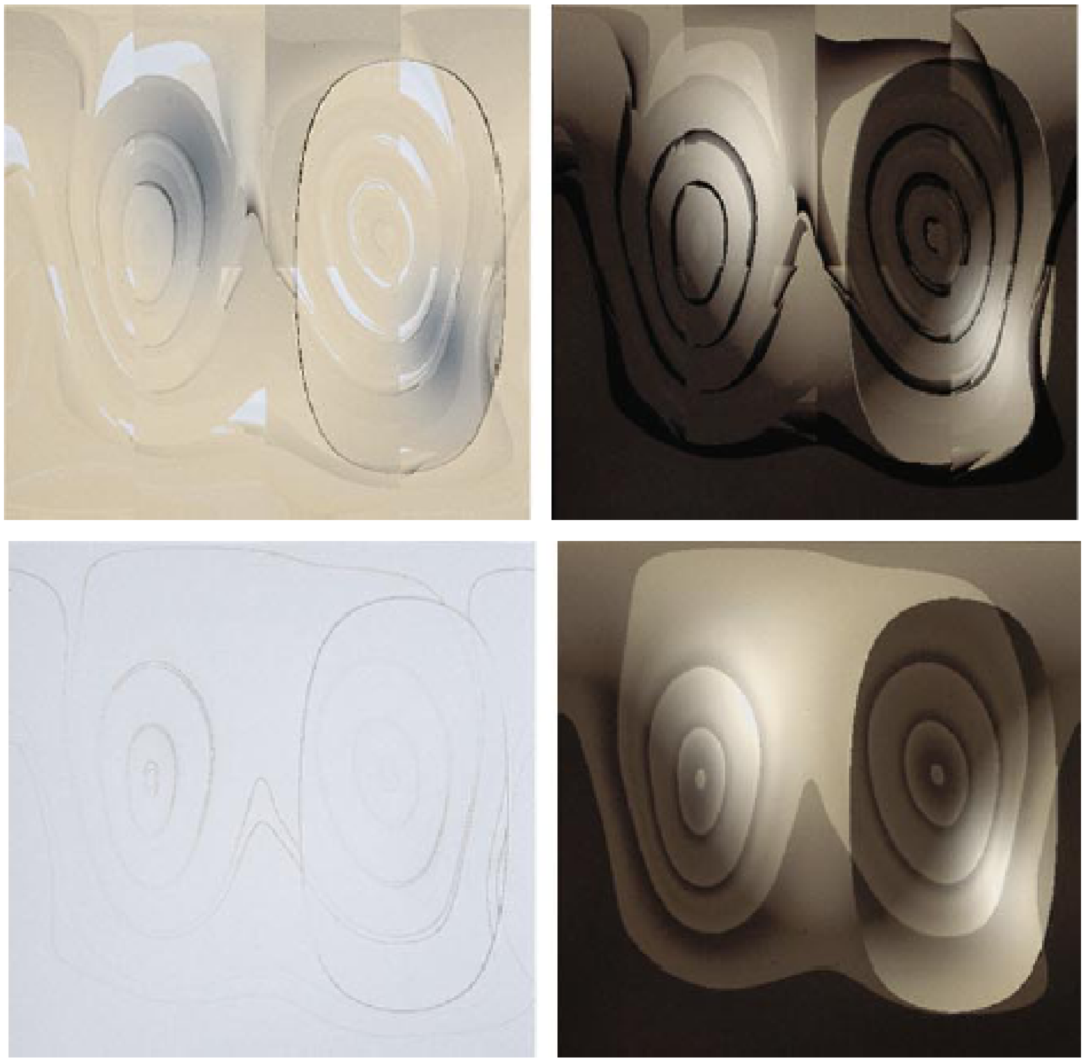
Ray casting implicit surfaces with affine arithmetic

- AA exploits linear correlations of x, y, z in $f(t) = h(r(t))$
- AA provides additional information
  - ◇ root must lie in smaller interval
  - ◇ quadratic convergence near simple zeros





Sampling procedural shaders



IA

AA (Heidrich–Slusallek–Seidel)

Conclusion

Interval methods have a place for solving computer graphics problems:

- Give reliable way to probe the global behavior of functions
- Lead naturally to robust, adaptive algorithms
- Several good libraries available on the internet

Affine arithmetic is a useful tool for interval methods

- AA more accurate than IA
- AA provides additional information that can be exploited
- AA locally more expensive than IA but globally more eficient
- AA has geometric flavor

Lots more to be done!

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