Acquiring Periodic Tilings of Regular Polygons from Images

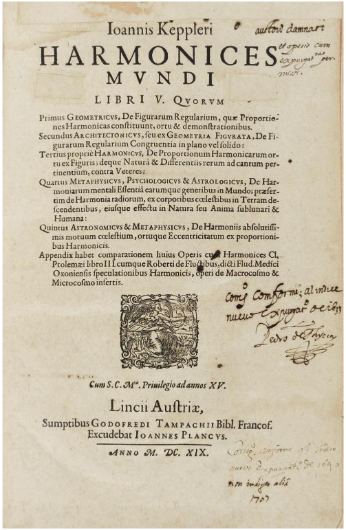
José Ezequiel Soto Sánchez · IMPA
Asla Medeiros e Sá · FGV
Luiz Henrique de Figueiredo · IMPA

Visgraf Vision and Graphics Laboratory

impa Instituto de Matemática Pura e Aplicada

Motivation: tile the plane with regular polygons

Kepler (1619)



Ioannis Keppleri

HARMONICES MVNDI

LIBRI V. QVORVM

Primus GEOMETRICVS, De Figurarum Regularium, quæ Proportiones Harmonicas constituunt, ortu & demonstrationibus.

Secundus ARCHITECTONICVS, seu ex GEOMETRIA FIGVRATA, De Figurarum Regularium Congruentia in plano vel solido:

Tertius propriè HARMONICVS, De Proportionum Harmonicarum ortu ex Figuris; deque Naturâ & Differentiis rerum ad cantum pertinentium, contra Veteres:

Quartus METAPHYSICVS, PSYCHOLOGICVS & ASTROLOGICVS, De Harmoniarum mentali Essentiâ earumque generibus in Mundo; præsertim de Harmonia radiorum, ex corporibus cœlestibus in Terram descendentibus, eiusque effectu in Natura seu Anima sublunari & Humana:

Quintus ASTRONOMICVS & METAPHYSICVS, De Harmoniis absolutissimis motuum cœlestium, ortuque Eccentricitatum ex proportionibus Harmonicis.

Appendix habet comparationem huius Operis cum Harmonices Cl. Ptolemæi libro III. cumque Roberti de Fluctibus, dicti Flud. Medici Oxoniensis speculationibus Harmonicis, operi de Macrocosmo & Microcosmo insertis.

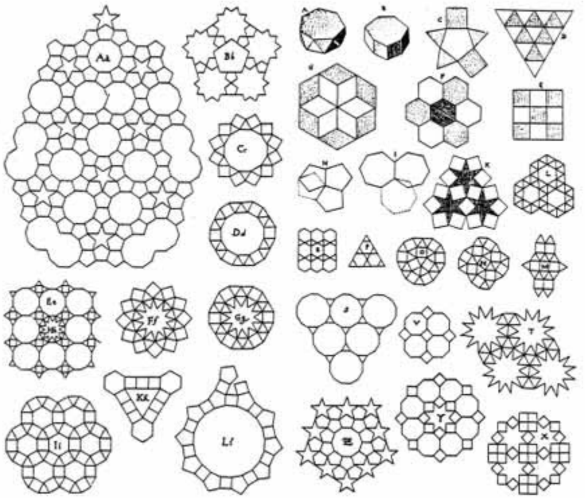
Cum S. C. Mtis. Priuilegio ad annos XV.

Lincii Austriæ,

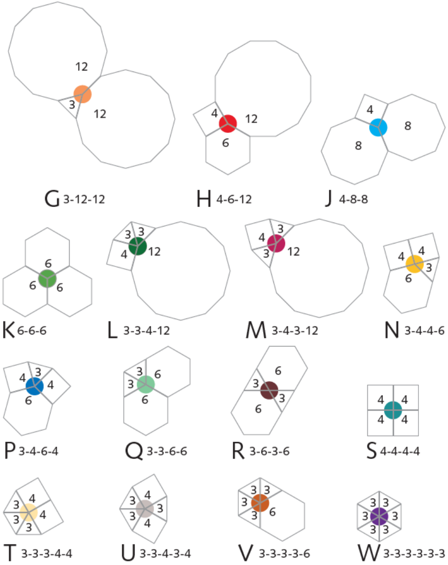
Sumptibus GODOFREDI TAMPACHII Bibl. Francof.

Excudebat IOANNES PLANCVS.

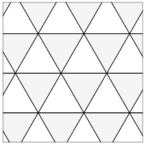
ANNO M. DC. XIX.

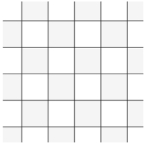


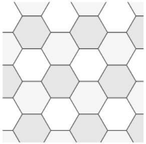
Rigidity: only 15 vertex neighborhoods

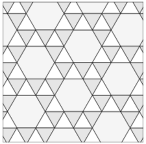


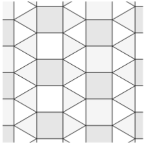
Rigidity: only 11 tilings are 1-uniform

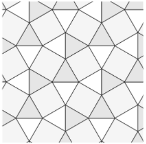

(3^6)

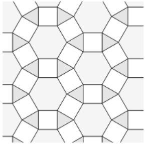

(4^4)

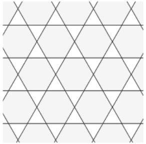

(6^3)

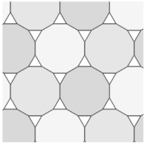

$(3^4.6)$

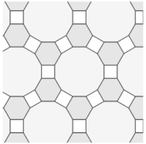

$(3^3.4^2)$


$(3^2.4.3.4)$


(3.4.6.4)


(3.6.3.6)


(3.12^2)


(4.6.12)

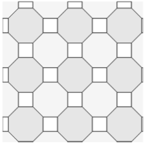
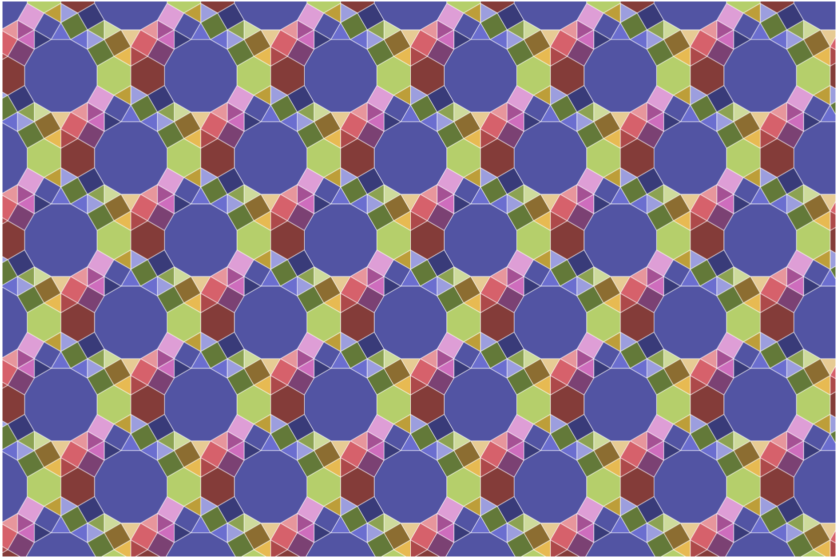

(4.8.8)

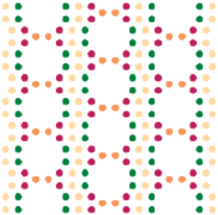
image by C. Kaplan

Goal: represent, synthesize, and analyze complex k-uniform tilings



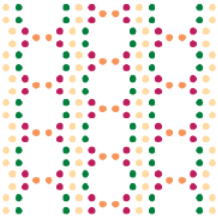
Outline

image



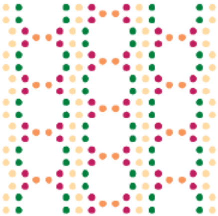
Outline

image



Outline

image



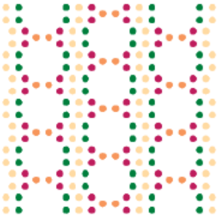
$\longrightarrow$

symbol

$$\left(\begin{array}{cccc} 0 & 2 & 3 & 1 \\ 2 & 6 & 0 & -3 \\ \hdashline 0 & 2 & 1 & 0 \\ 0 & 3 & 1 & 0 \\ & & \vdots & \\ 2 & 1 & 1 & 3 \end{array}\right)$$

Outline

image



$\longrightarrow$

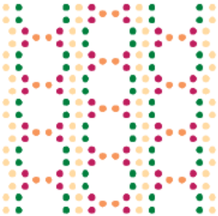
symbol

$$\begin{pmatrix} 0 & 2 & 3 & 1 \\ 2 & 6 & 0 & -3 \\ \hdashline 0 & 2 & 1 & 0 \\ 0 & 3 & 1 & 0 \\ & & \vdots & \\ 2 & 1 & 1 & 3 \end{pmatrix}$$

- Tile arbitrarily large areas

Outline

image



$\longrightarrow$

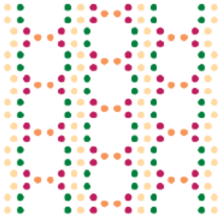
symbol

$$\begin{pmatrix} 0 & 2 & 3 & 1 \\ 2 & 6 & 0 & -3 \\ \hline 0 & 2 & 1 & 0 \\ 0 & 3 & 1 & 0 \\ & & \vdots & \\ 2 & 1 & 1 & 3 \end{pmatrix}$$

- Tile arbitrarily large areas
- Establish properties of the symbol

Outline

image



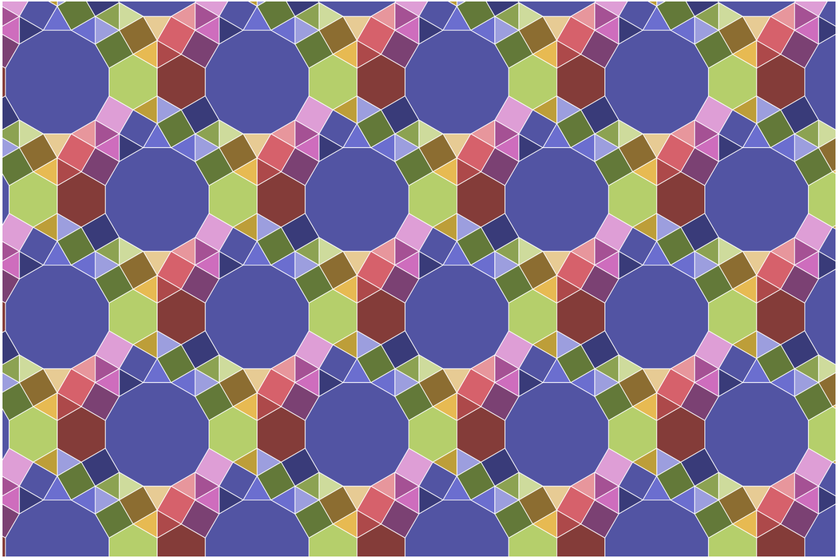
$\longrightarrow$

symbol

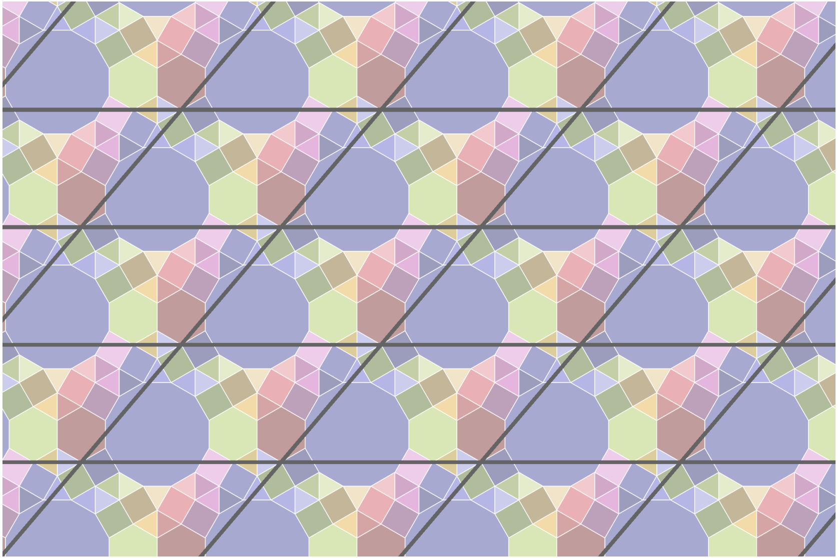
$$\begin{pmatrix} 0 & 2 & 3 & 1 \\ 2 & 6 & 0 & -3 \\ \hdashline 0 & 2 & 1 & 0 \\ 0 & 3 & 1 & 0 \\ & & \vdots & \\ 2 & 1 & 1 & 3 \end{pmatrix}$$

- Tile arbitrarily large areas
- Establish properties of the symbol
- Allow further analysis of the tilings

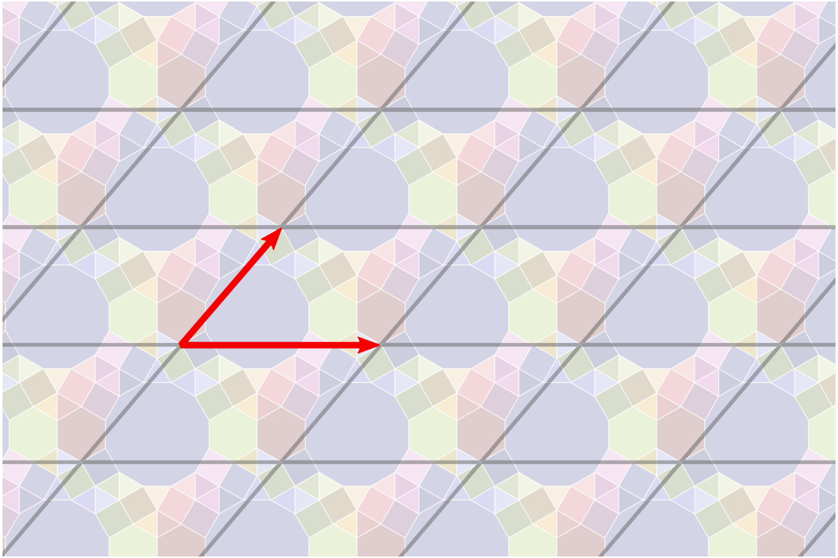
Understanding tilings: many symmetries



Understanding tilings: translation symmetries (group p1)

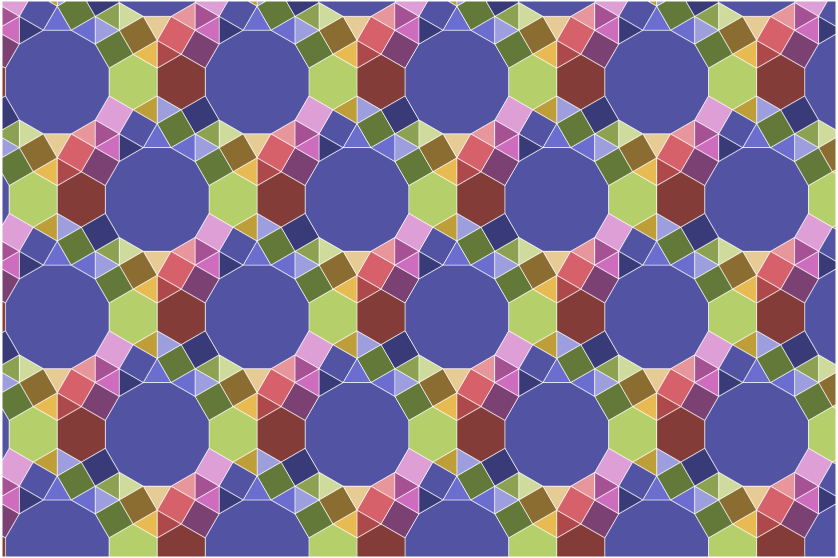


Understanding tilings: fundamental domain



Regular systems of points

Hilbert & Cohn-Vossen (1952)



Regular systems of points

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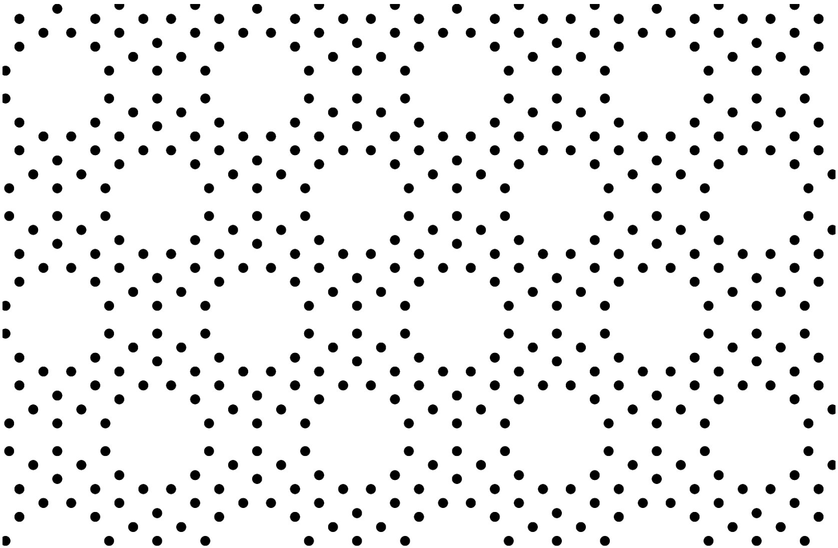
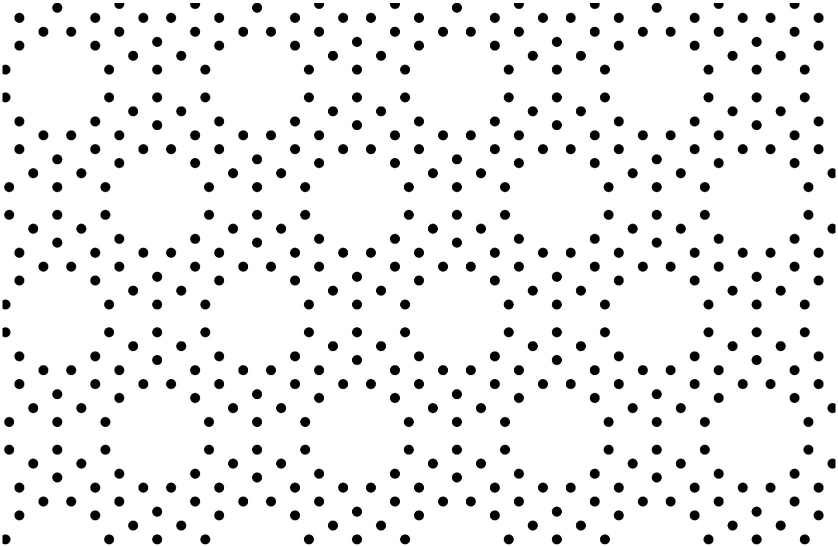
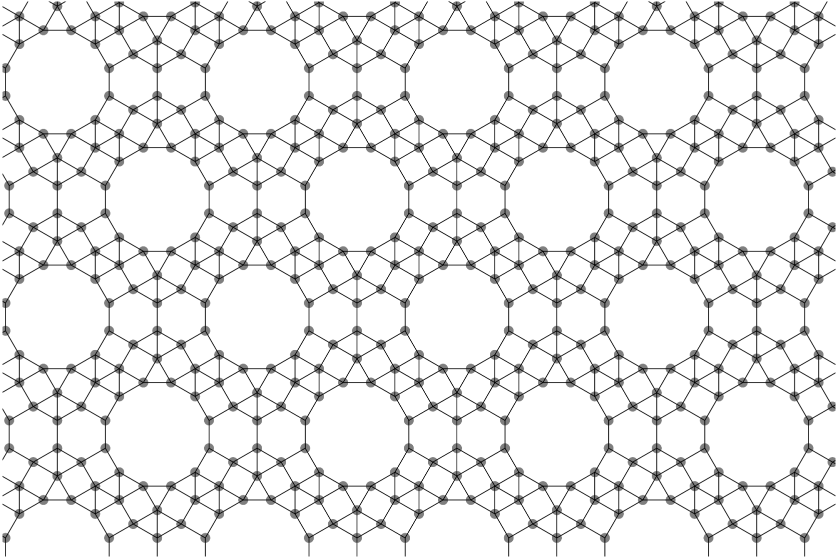


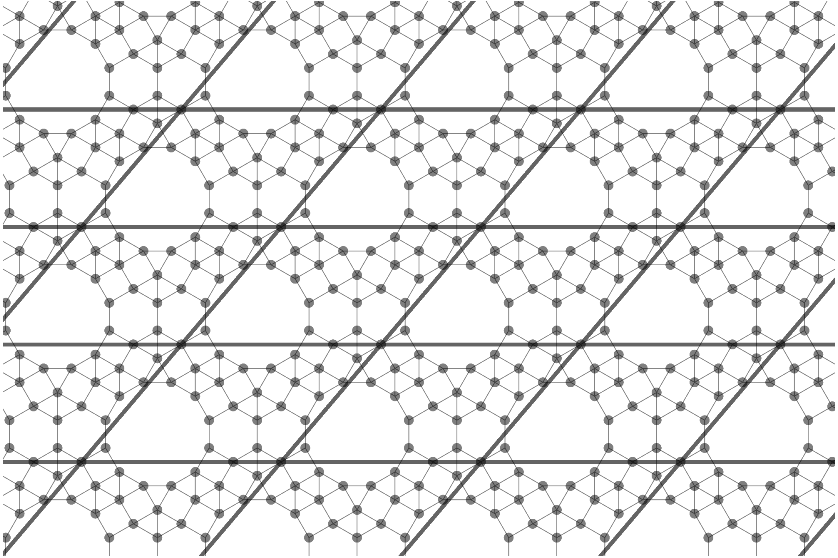
Reconstruct tiling from vertices



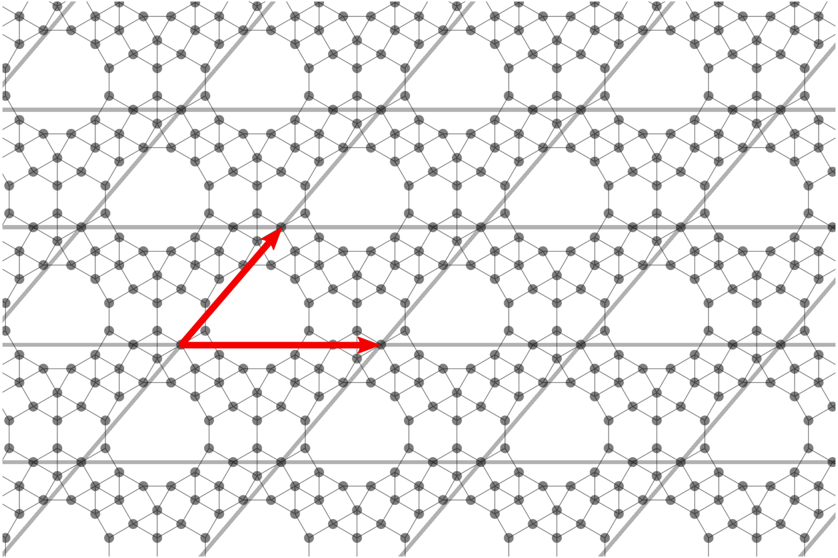
Reconstruct tiling from vertices: edges



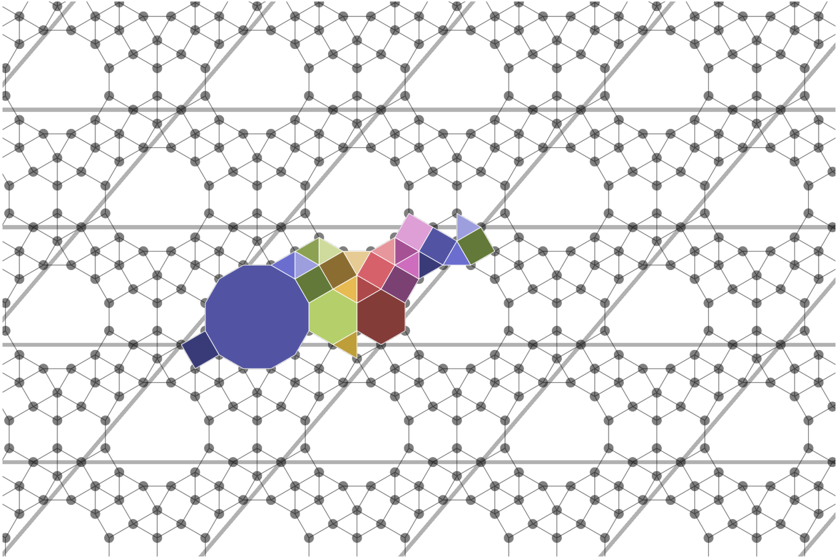
Reconstruct tiling from vertices: translation grid



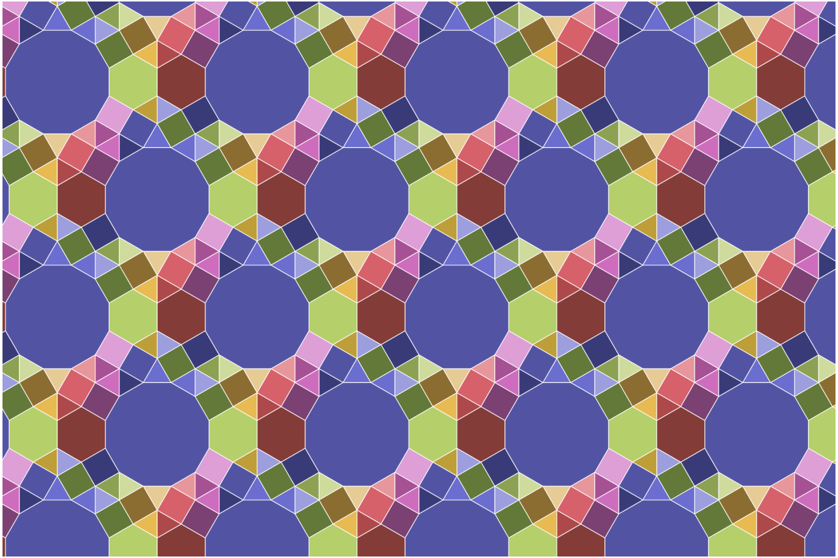
Reconstruct tiling from vertices: fundamental domain



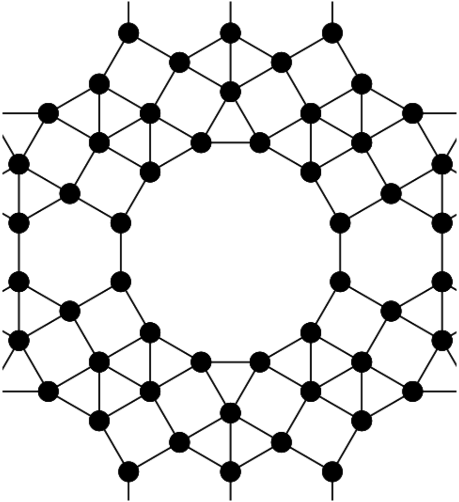
Reconstruct tiling from vertices: patch



Reconstruct tiling from vertices: full tiling



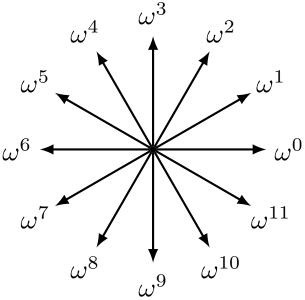
Edges aligned to a few basic directions



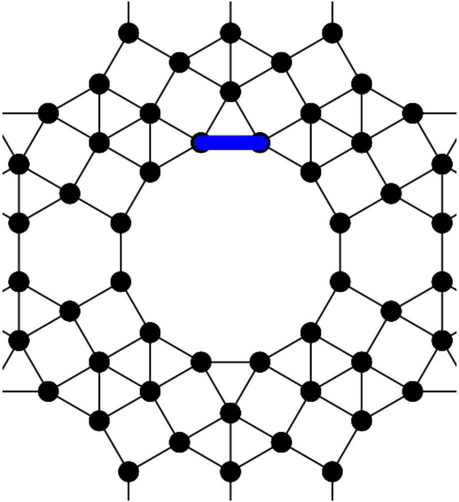
roots of unity

$$\omega^{12} = 1, \quad \omega = e^{\frac{2\pi i}{12}}$$

$$\omega^{n} = e^{\frac{2\pi i}{12} n}, \quad n \in \{0, 1, \ldots, 11\}$$



Edges aligned to a few basic directions



roots of unity

$$\omega^{12} = 1, \quad \omega = e^{\frac{2\pi i}{12}}$$

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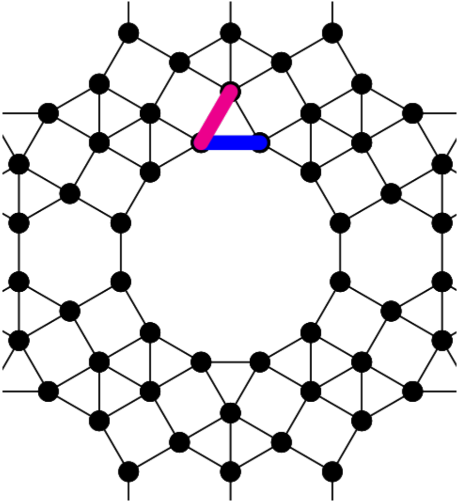
- ω^0

Edges aligned to a few basic directions



roots of unity

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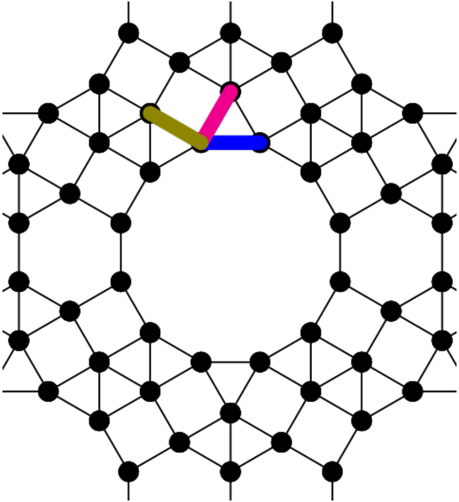
- ω^0
- ω^2

Edges aligned to a few basic directions



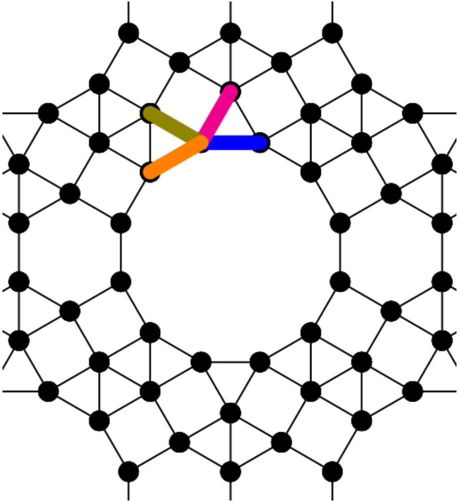
roots of unity

$$\omega^{12} = 1, \quad \omega = e^{\frac{2\pi i}{12}}$$

$$\omega^n = e^{\frac{2\pi i}{12} n}, \quad n \in \{0, 1, \ldots, 11\}$$

- ω^0 (blue)
- ω^2 (magenta)
- ω^5 (olive)

Edges aligned to a few basic directions



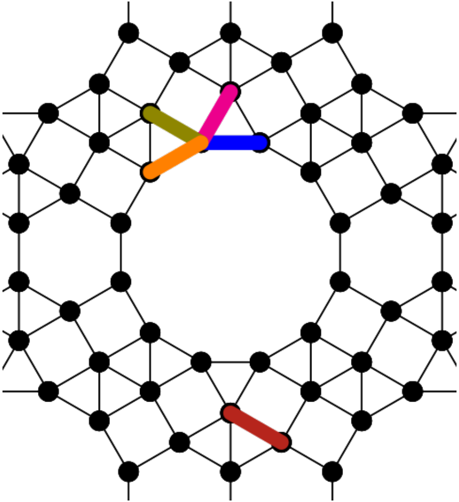
roots of unity

$$\omega^{12} = 1, \quad \omega = e^{\frac{2\pi i}{12}}$$

$$\omega^{n} = e^{\frac{2\pi i}{12} n}, \quad n \in \{0, 1, \ldots, 11\}$$

- ω^0
- ω^2
- ω^5
- ω^7

Edges aligned to a few basic directions



roots of unity

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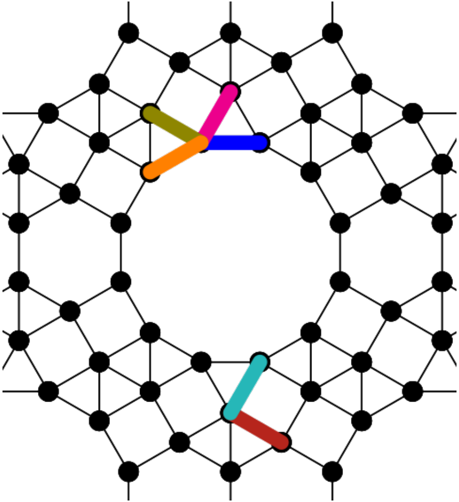
$$\omega^n = e^{\frac{2\pi i}{12} n}, \quad n \in \{0, 1, \ldots, 11\}$$

- ω^0
- ω^2
- ω^5
- ω^7
- ω^{11}

Edges aligned to a few basic directions



roots of unity

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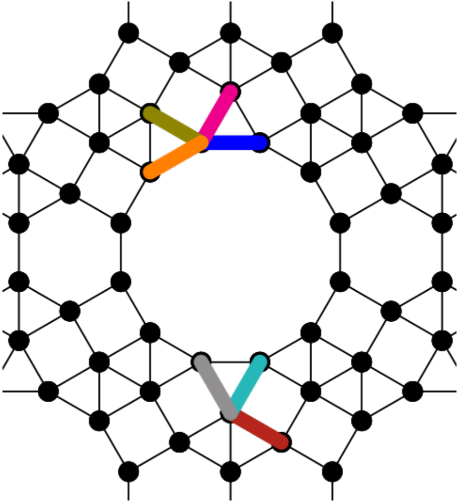
- ω^0
- ω^2
- ω^5
- ω^7
- ω^{11}
- ω^2

Edges aligned to a few basic directions



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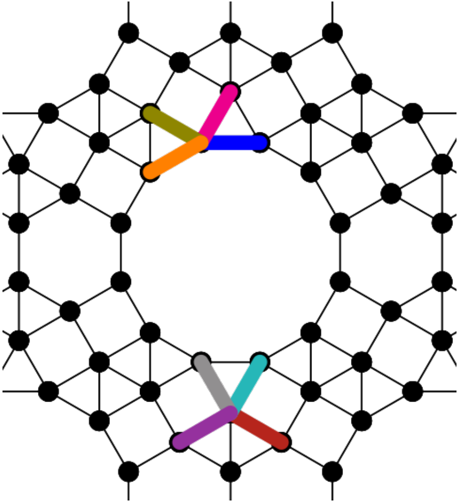
$$\omega^n = e^{\frac{2\pi i}{12} n}, \quad n \in \{0, 1, \ldots, 11\}$$

ω^0	ω^{11}
ω^2	ω^2
ω^5	ω^4
ω^7	

Edges aligned to a few basic directions



roots of unity

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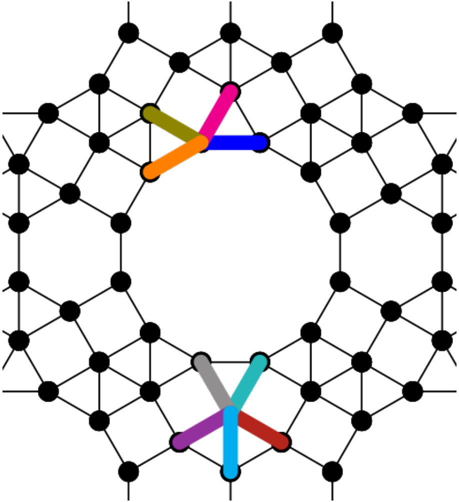
- ω^0
- ω^2
- ω^5
- ω^7
- ω^{11}
- ω^2
- ω^4
- ω_7

Edges aligned to a few basic directions



roots of unity

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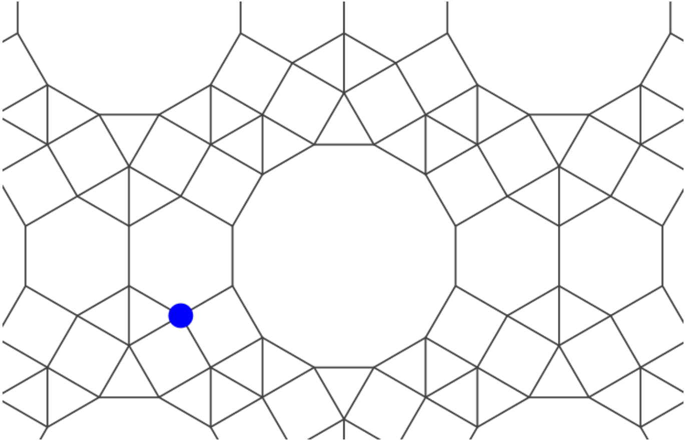
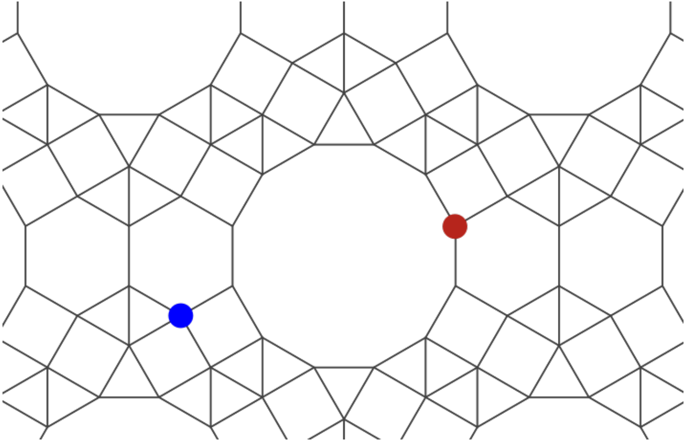
- ω^0
- ω^2
- ω^5
- ω^7

- ω^{11}
- ω^2
- ω^4
- ω_7
- ω^9

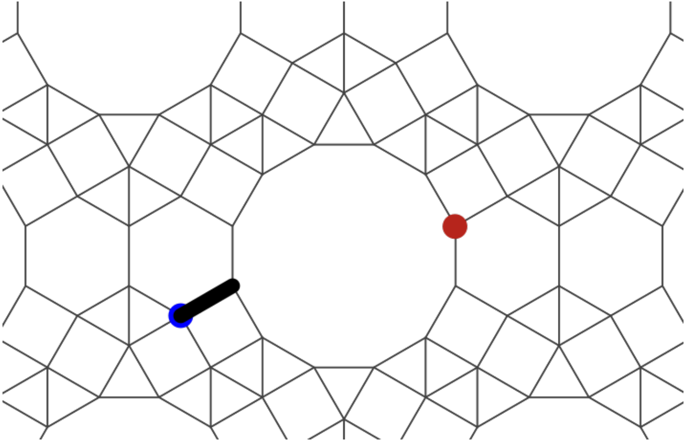
Vertices as integer linear combinations of basic directions



Vertices as integer linear combinations of basic directions

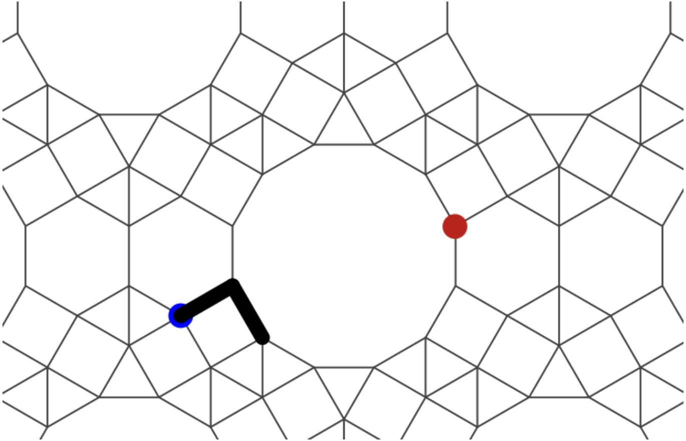


Vertices as integer linear combinations of basic directions



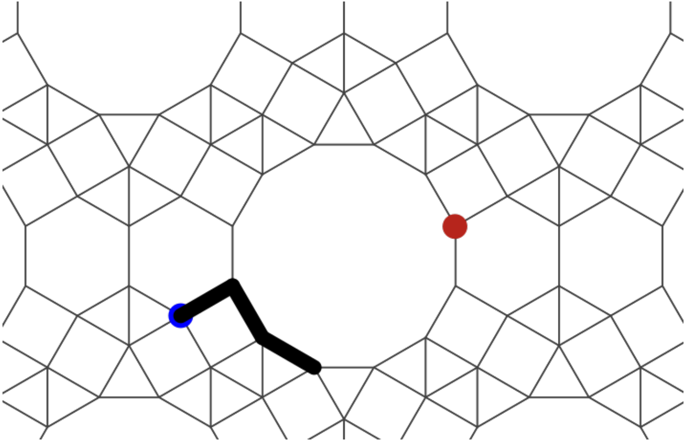
ω

Vertices as integer linear combinations of basic directions



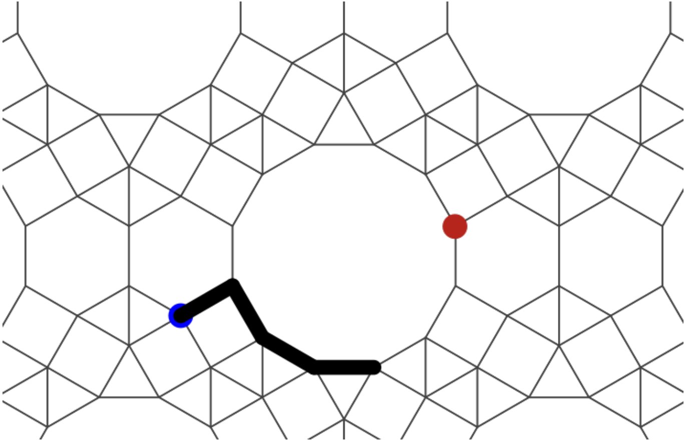
$\omega + \omega^{10}$

Vertices as integer linear combinations of basic directions



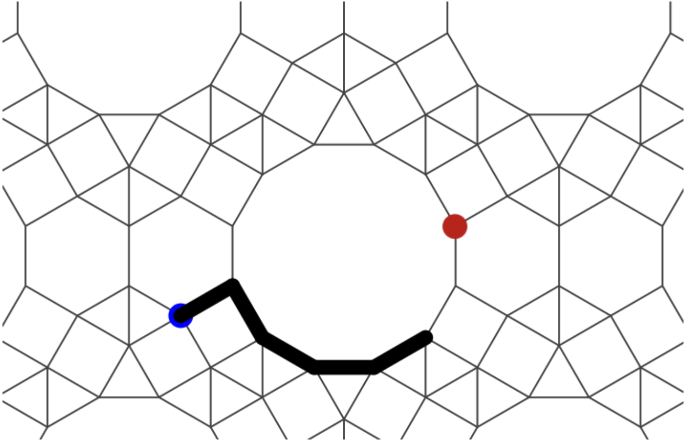
$\omega + \omega^{10} + \omega^{11}$

Vertices as integer linear combinations of basic directions



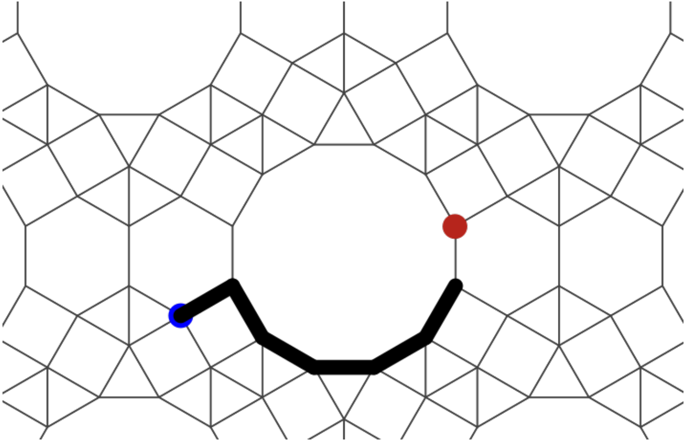
$\omega + \omega^{10} + \omega^{11} + \omega^{0}$

Vertices as integer linear combinations of basic directions



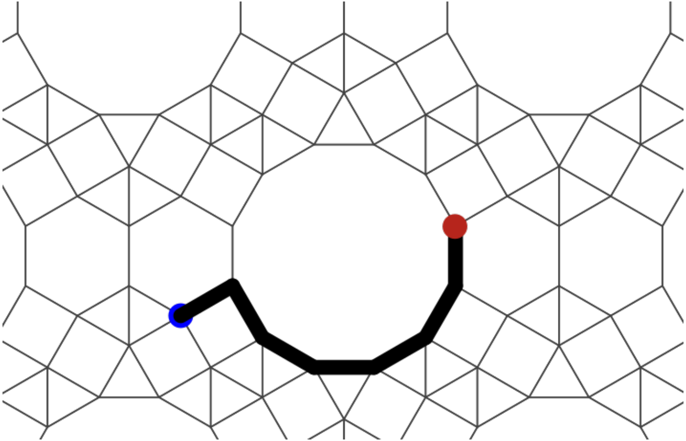
$$\omega + \omega^{10} + \omega^{11} + \omega^{0} + \omega$$

Vertices as integer linear combinations of basic directions



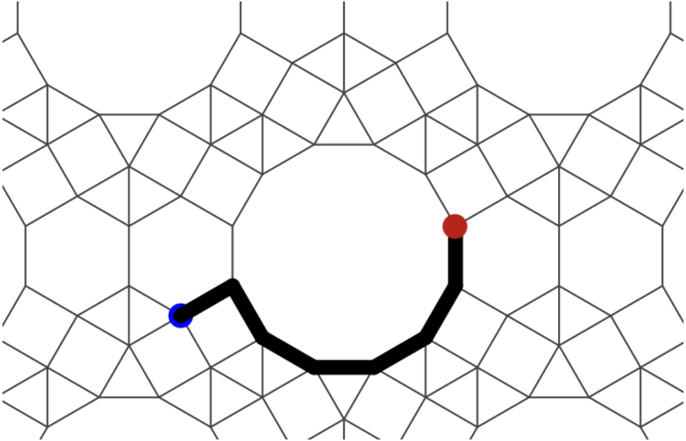
$\omega + \omega^{10} + \omega^{11} + \omega^{0} + \omega + \omega^{2}$

Vertices as integer linear combinations of basic directions



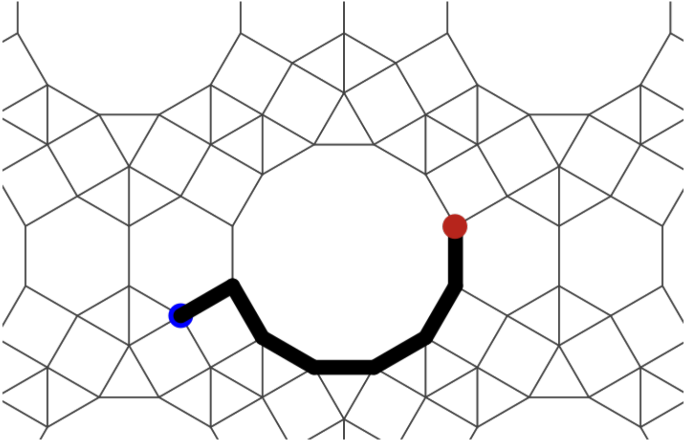
$\omega + \omega^{10} + \omega^{11} + \omega^{0} + \omega + \omega^{2} + \omega^{3}$

Vertices as integer linear combinations of basic directions



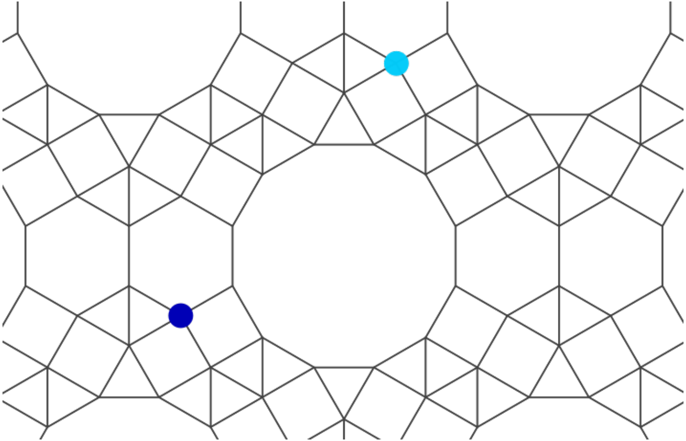
$$\omega + \omega^{10} + \omega^{11} + \omega^{0} + \omega + \omega^{2} + \omega^{3} = \omega^{11} + \omega^{10} + \omega^{3} + \omega^{2} + 2\omega + 1$$

Vertices as integer linear combinations of basic directions



$$\omega + \omega^{10} + \omega^{11} + \omega^{0} + \omega + \omega^{2} + \omega^{3} = \omega^{11} + \omega^{10} + \omega^{3} + \omega^{2} + 2\omega + 1 = V - O$$

Translations as integer linear combinations of basic directions



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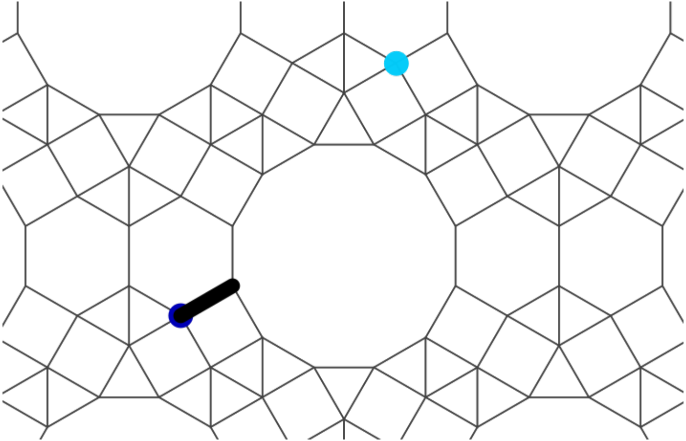
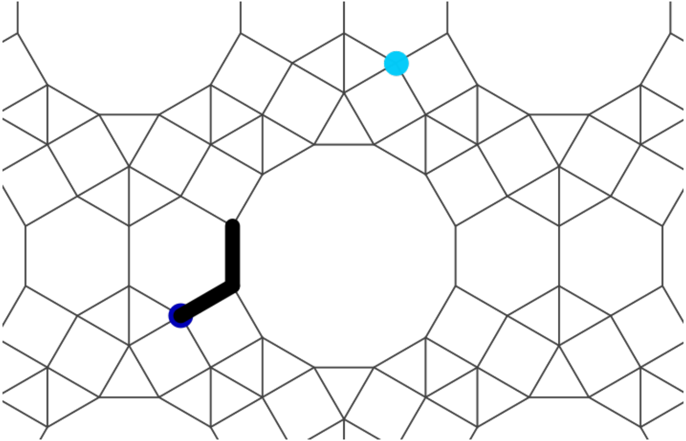


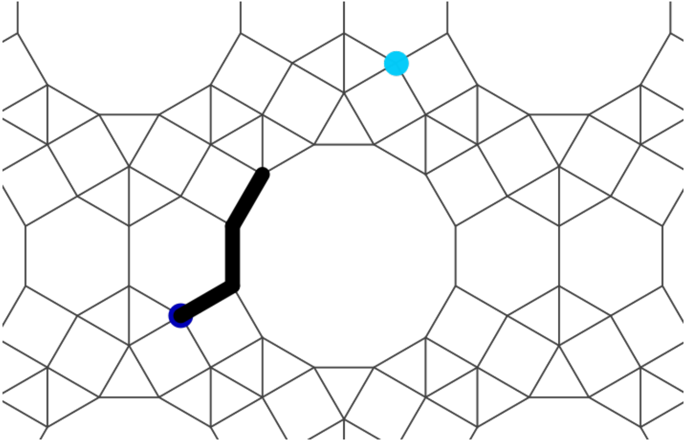
ω

Translations as integer linear combinations of basic directions



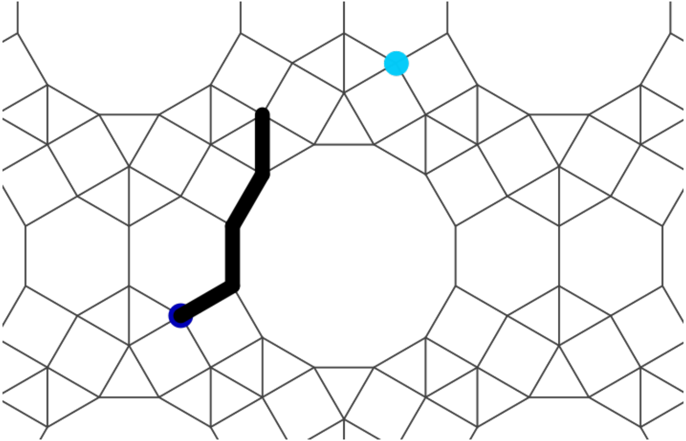
$\omega + \omega^3$

Translations as integer linear combinations of basic directions



$\omega + \omega^3 + \omega^2$

Translations as integer linear combinations of basic directions



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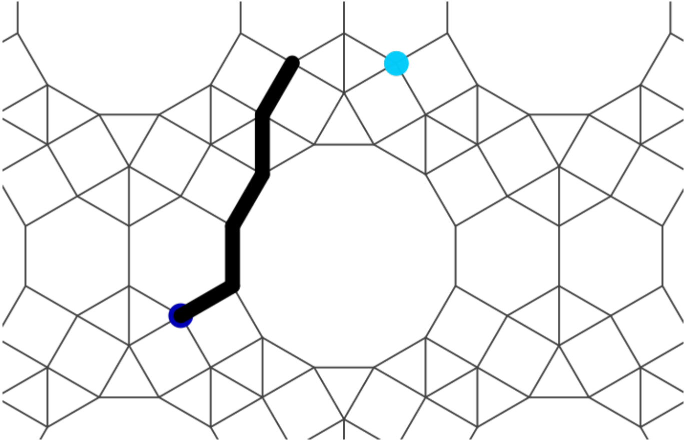

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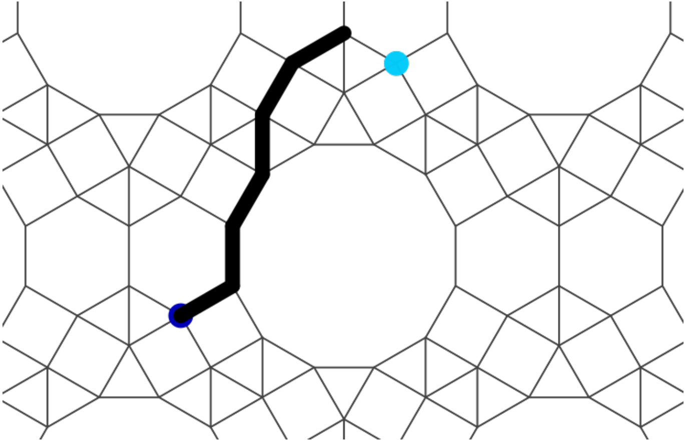
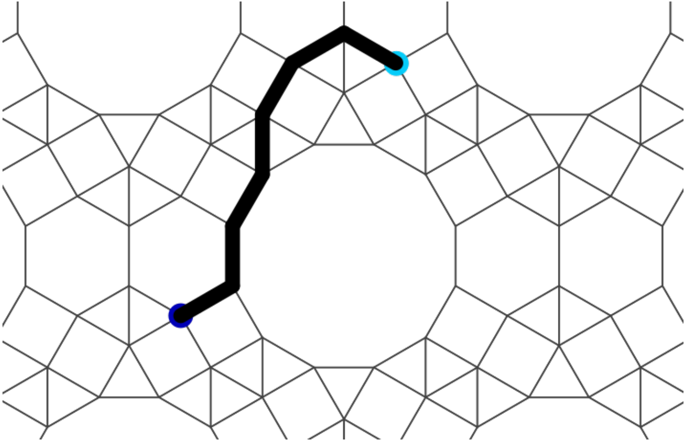
Translations as integer linear combinations of basic directions



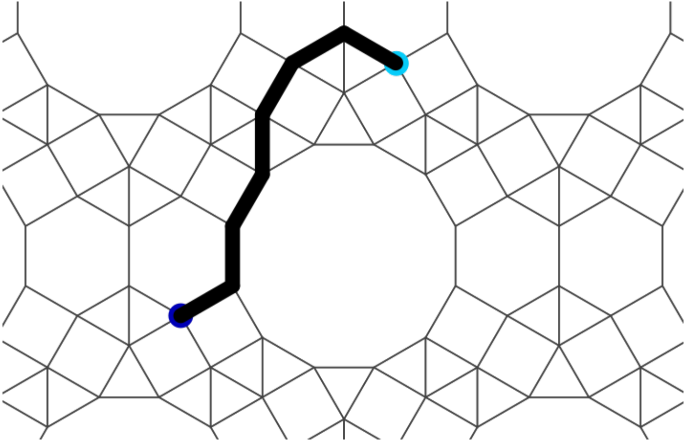
$\omega + \omega^3 + \omega^2 + \omega^3 + \omega^2 + \omega$

Translations as integer linear combinations of basic directions



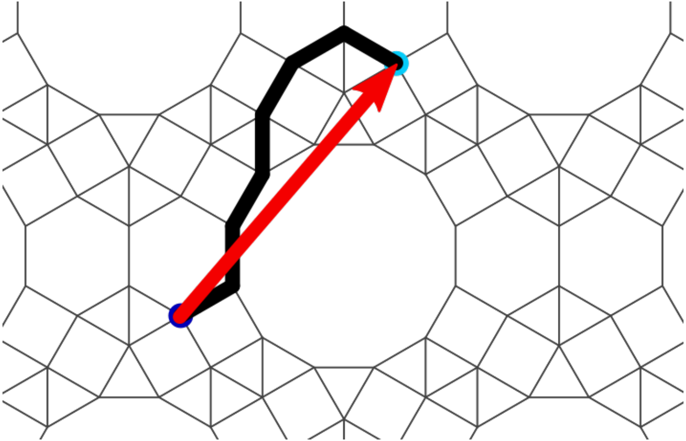
$\omega + \omega^3 + \omega^2 + \omega^3 + \omega^2 + \omega + \omega^{11}$

Translations as integer linear combinations of basic directions



$$\omega + \omega^3 + \omega^2 + \omega^3 + \omega^2 + \omega + \omega^{11} = \omega^{11} + 2\omega^3 + 2\omega^2 + 2\omega$$

Translations as integer linear combinations of basic directions



$$\omega + \omega^3 + \omega^2 + \omega^3 + \omega^2 + \omega + \omega^{11} = \omega^{11} + 2\omega^3 + 2\omega^2 + 2\omega = T - O$$

Tiling symbols

Vertices and translation vectors are expressed in
$\mathbb{Z}[\omega]$ = polynomials in ω

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Polynomials in ω reduced mod $\omega^4 - \omega^2 + 1$,
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give a unique representation!

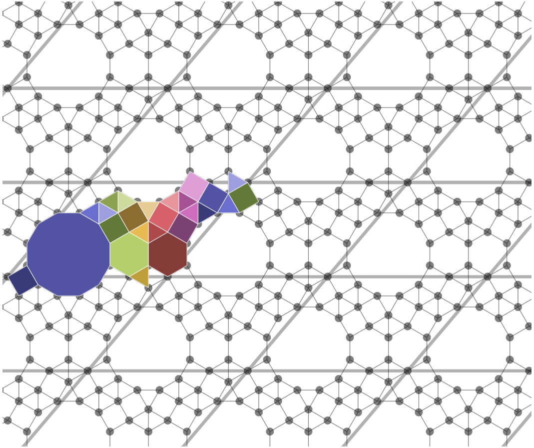
$$\begin{aligned}
\omega^4 &= -1 + \omega^2 &&= [-1, 0, 1, 0] \\
\omega^5 &= -\omega + \omega^3 &&= [0, -1, 0, 1] \\
\omega^6 &= -1 &&= [-1, 0, 0, 0] \\
\omega^7 &= -\omega &&= [0, -1, 0, 0] \\
\omega^8 &= -\omega^2 &&= [0, 0, -1, 0] \\
\omega^9 &= -\omega^3 &&= [0, 0, 0, -1] \\
\omega^{10} &= 1 - \omega^2 &&= [1, 0, -1, 0] \\
\omega^{11} &= \omega - \omega^3 &&= [0, 1, 0, -1]
\end{aligned}$$

Tiling symbols

Each tiling is represented by:

- two translation vectors
  define the fundamental region
- set of seeds
  vertices inside fundamental region
- translation vectors and seeds expressed as
  integer linear combinations of basic directions

Tiling symbols



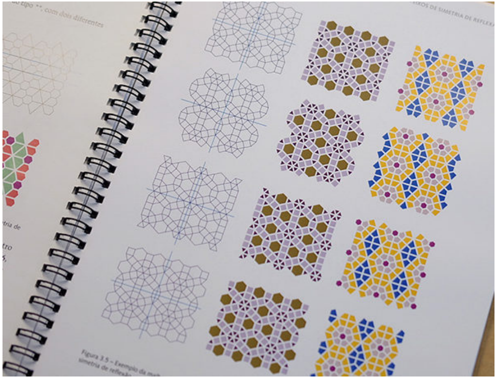
translation vectors

$$T_1 = [0, 2, 3, 1]$$
$$T_2 = [2, 6, 0, -3]$$

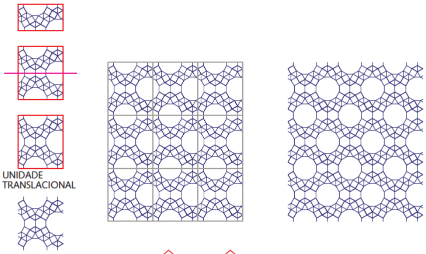
seeds

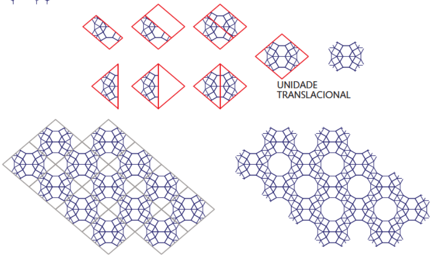
$$S_1 = [0, 0, 0, 0]$$
$$S_2 = [0, 2, 1, 0]$$
$$S_3 = [0, 3, 1, 0]$$
$$S_4 = [1, 1, 0, 0]$$
$$\vdots$$
$$S_{25} = [2, 1, 1, 3]$$

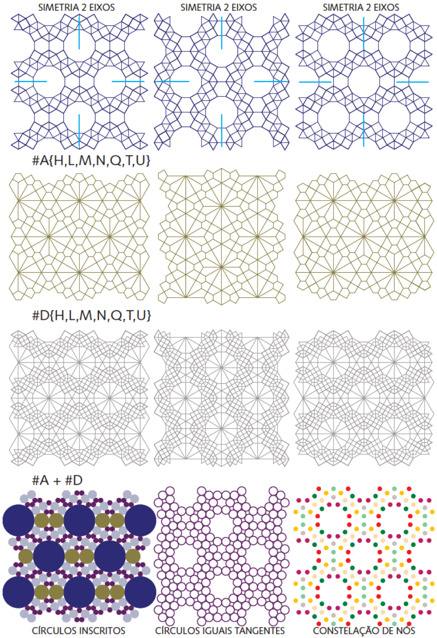
Book & catalogue: 200+ Arquimedean tilings



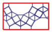
“Sobre malhas arquimedianas”, Ricardo Sá e Asla Medeiros e Sá, 2017





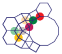


#A{H,L,M,N,Q,T,U}

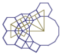


FRAGMENTO
FUNDAMENTAL

- H = 4-6-12
- L = 3-3-4-12
- M = 3-4-3-12
- N = 3-4-46



- Q = 3-3-6-6
- T = 3-3-3-4-4
- U = 3-3-4-3-4



POLÍGONOS DUAIS

Web catalogue: 1248 tilings

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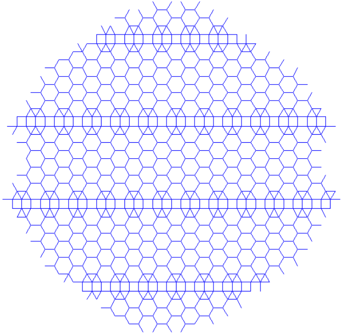
Brian Galebach

6-Uniform Tiling 1 of 673



Numbers of Tilings

		m-Archimedean															
		1	2	3	4	5	6	7	8	9	10	11	12	13	14	>14	Total
n-Uniform	1	11	0	0	0	0	0	0	0	0	0	0	0	0	0	0	11
	2	0	20	0	0	0	0	0	0	0	0	0	0	0	0	0	20
	3	0	22	39	0	0	0	0	0	0	0	0	0	0	0	0	61
	4	0	33	85	33	0	0	0	0	0	0	0	0	0	0	0	151
	5	0	74	149	94	15	0	0	0	0	0	0	0	0	0	0	332
	6	0	100	284	187	92	10	0	0	0	0	0	0	0	0	0	673
	7	0	?	?	?	?	?	7	0	0	0	0	0	0	0	0	?
	8	0	?	?	?	?	?	20	0	0	0	0	0	0	0	0	?
	9	0	?	?	?	?	?	?	8	0	0	0	0	0	0	0	?
	10	0	?	?	?	?	?	?	27	0	0	0	0	0	0	0	?
	11	0	?	?	?	?	?	?	?	1	0	0	0	0	0	0	?
	12	0	?	?	?	?	?	?	?	?	0	0	0	0	0	0	?
	13	0	?	?	?	?	?	?	?	?	?	?	?	0	0	0	?
	14	0	?	?	?	?	?	?	?	?	?	?	?	?	0	0	?
	>14	0	?	?	?	?	?	?	?	?	?	?	?	?	?	0	?
	Total	11	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	0	∞

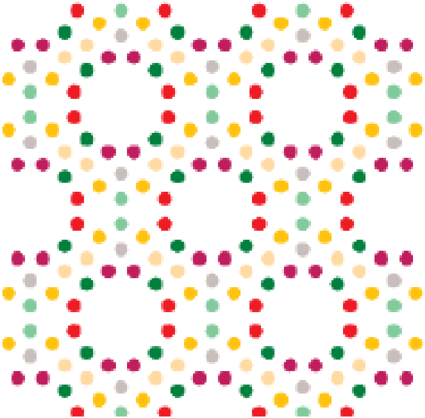
SVG samples in Wikipedia for $n \leq 5$

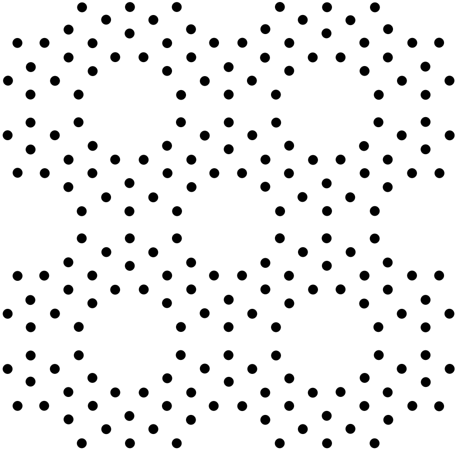
THE ON-LINE ENCYCLOPEDIA OF INTEGER SEQUENCES®

founded in 1964 by N. J. A. Sloane

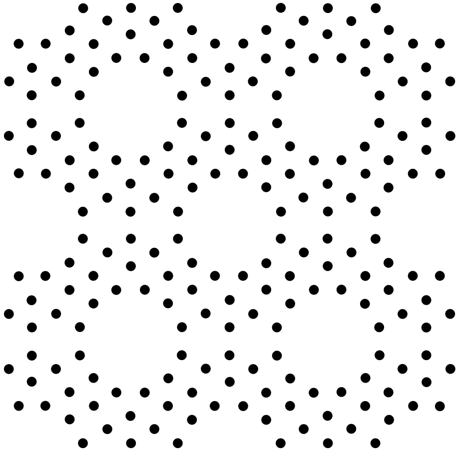
probabilitysports.com/tilings.html – Brian Galebach, 2002

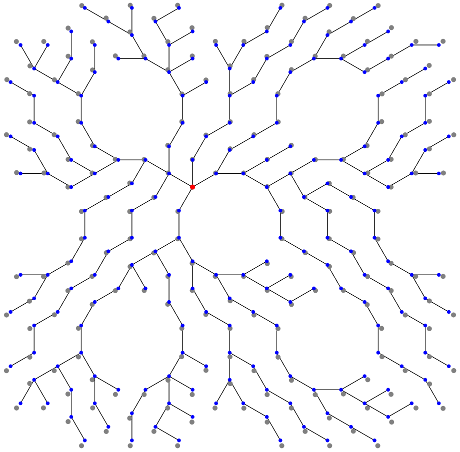
1. Find approximate coordinates for the vertices



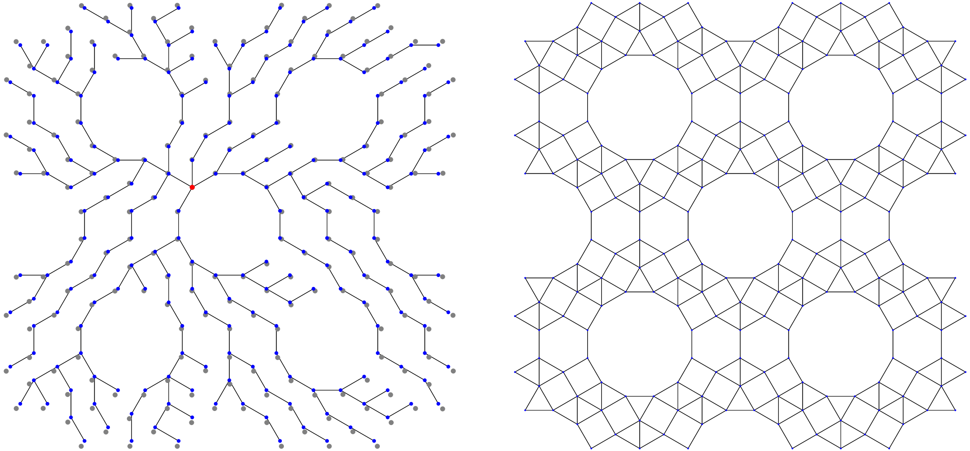


2. Correct the vertices: basic directions + unit length $\to \mathbb{Z}[\omega]$

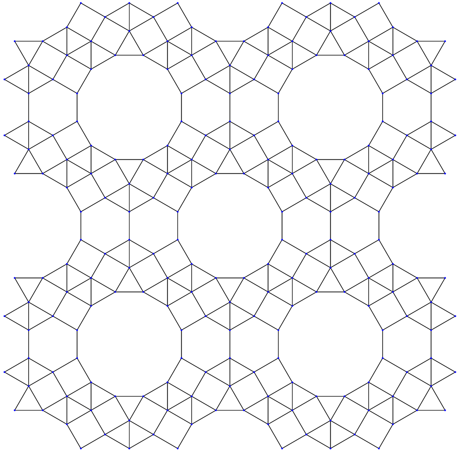


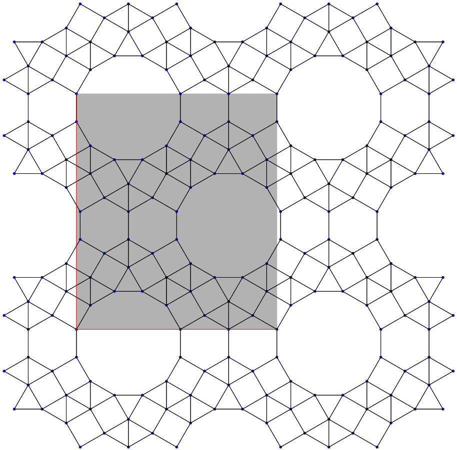


3. Find the edges: stars

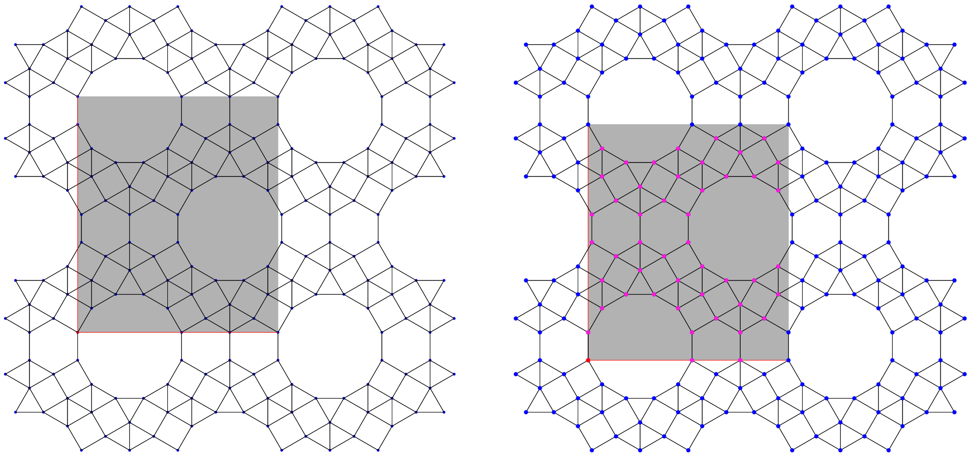


4. Find the translations: transitive equivalence + score

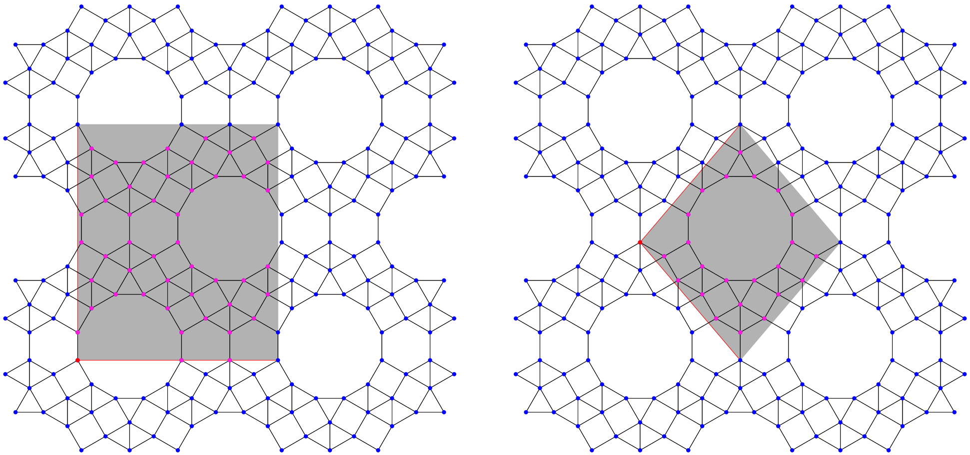




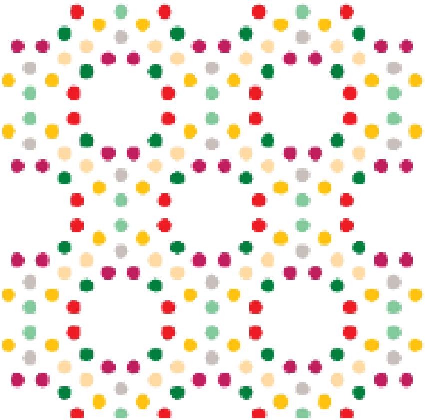
5. Find the seeds

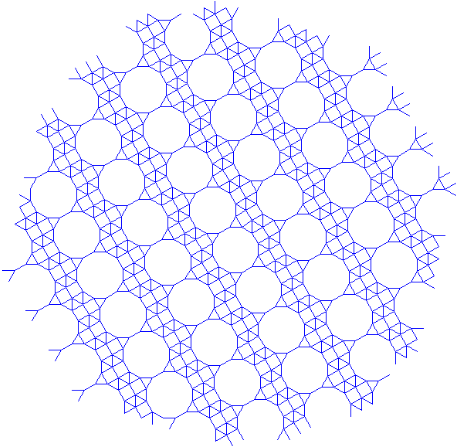


6. Minimize translation vectors



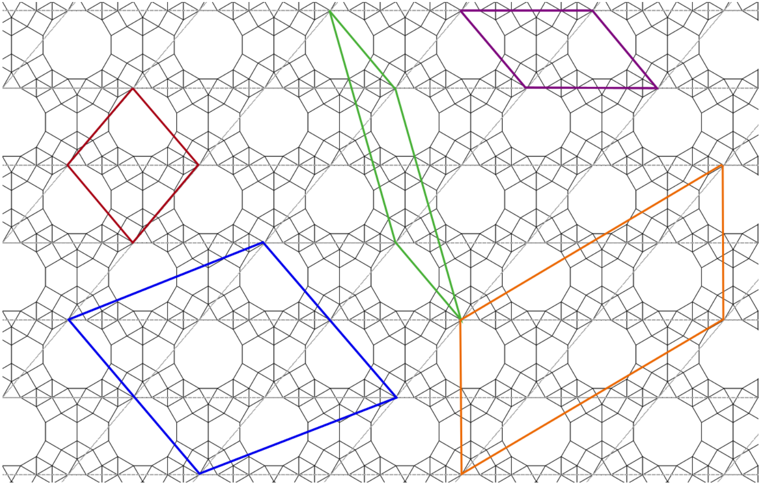
Match equivalent tilings





Equivalent representations

- many choices for translation vectors given a translation grid



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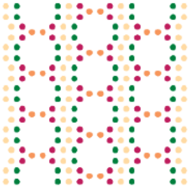
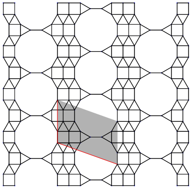
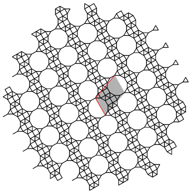
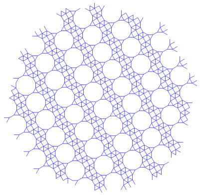

Equivalent representations

Translation vectors can be written as $T = AW$:

$$\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix} \begin{pmatrix} 1 \\ \omega \\ \omega^2 \\ \omega^3 \end{pmatrix}$$

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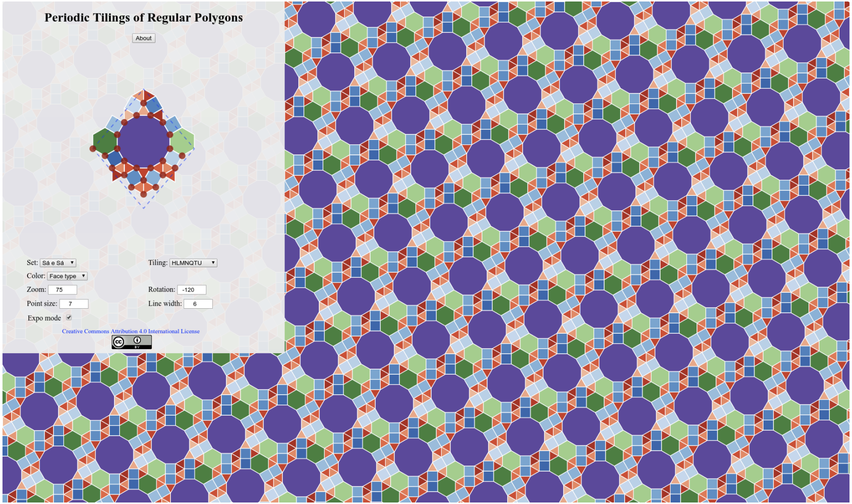
All rotations and origin choices are tested

Web interface to catalogue



Results and future work

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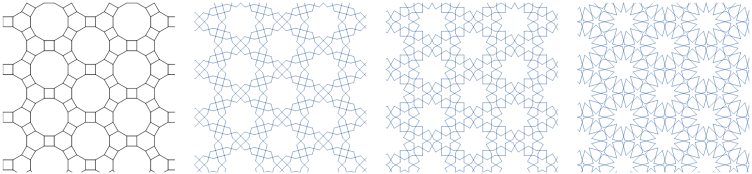
- Nice image synthesis applications



(4.6.12) $\theta = 45^\circ$ $\theta = 60^\circ$ $\theta = 75^\circ$

image by C. Kaplan

Web interface to catalogue

www.impa.br/∼cheque/tiling/

Acquiring Periodic Tilings of Regular Polygons from Images

José Ezequiel Soto Sánchez · IMPA
Asla Medeiros e Sá · FGV
Luiz Henrique de Figueiredo · IMPA

Visgraf Vision and Graphics Laboratory

impa
Instituto de Matemática Pura e Aplicada