Yves Meyer

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Wavelets

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IMPA

Contributions

Yves Meyer was awarded the 2010 Gauss Prize for fundamental contributions to number theory, operator theory and harmonic analysis, and his pivotal role in the development of wavelets and multiresolution analysis.

He also received the 2017 Abel Prize “for his pivotal role in the development of the mathematical theory of wavelets.”

The Scientist

Stéphane Mallat calls him a “visionary” whose work cannot be labelled either pure or applied mathematics, nor computer science either, but simply “amazing”.

Wavelets

From the mid-1980s, in what he called a “second scientific life”, Meyer, together with Daubechies and Coifman, brought together earlier work on wavelets into a unified picture.

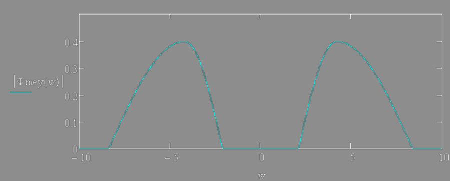
In 1986 Meyer and Pierre Gilles Lemarié-Rieusset showed that wavelets may form mutually independent sets of mathematical objects called orthogonal bases. Coifman, Daubechies and Stéphane Mallat went on to develop applications to many problems in signal and image processing.

The Meyer Wavelet

(1985)

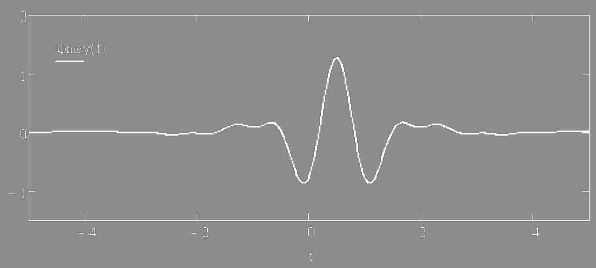
- First Non-Trivial Orthogonal Wavelet Basis
- C^∞ Continuously Differentiable
- Non-Compact Support
- Defined in Frequency Domain
- Continuous Wavelet Transform

Definition



$$\Psi_{mey}(w) = \begin{cases} \frac{1}{\sqrt{2\pi}} \sin\left(\frac{\pi}{2}\upsilon\left(\frac{3|w|}{2\pi}-1\right)\right) e^{\frac{jw}{2}} & if \quad \frac{2\pi}{3} \le |w| \le \frac{4\pi}{3} \\ \frac{1}{\sqrt{2\pi}} \cos\left(\frac{\pi}{2}\upsilon\left(\frac{3|w|}{4\pi}-1\right)\right) e^{\frac{jw}{2}} & if \quad \frac{4\pi}{3} \le |w| \le \frac{8\pi}{3} \\ 0 & otherwise \end{cases}$$

The Function



- Space Domain

The Next Frontier

- "Big Questions":
  - How to Discretize the Wavelet Transform?
  - How to Systematically Construct Wavelet Basis?
- The Answer
  - Multiresolution Analysis

Multiresolution Analysis

(1988)

- Yves Meyer and Stephane Mallat
- Foundations of Modern Signal Processing
- Many Applications in Science and Engineering

Multiresolution Analysis and Filter Banks

Outline

- Scale Spaces
- Multiresolution Analysis
- Dilation Equations
- Fast Wavelet Transform
- Two-Channel Filter Banks
- Matrix Implementation
- Examples

Scale

Natural Concept

- Physical

  Measurements, Data Acquisition
- Perception

  Focus, Features
- Applications

  Units, Computation

Scale Spaces

V_s : Space of Scale s

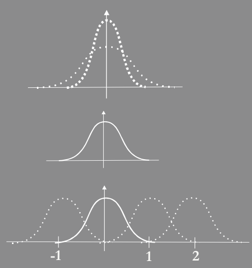
- Scaling Function

$$\phi_s(x) = \frac{1}{\sqrt{|s|}}\phi\left(\frac{x}{s}\right)$$

$$\int \phi_s(x)dx = 1$$

- Basis of V_s

$$\{\phi_{s,k} \mid \phi_s(x-k)\}$$



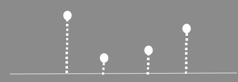
Representation of a Function in V_S

- Orthogonal Projection

$$\mathrm{Proj}_{V_S}(f) = \sum_k \langle f, \phi_{s,k} \rangle \phi_{s,k}(x)$$

- Representation Operator

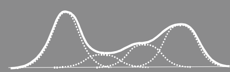
$$c_k^s = \langle f, \phi_{s,k} \rangle = f * \phi$$



(discrete sequence)

- Reconstruction Operator

$$f_s(x) = \sum_k c_k^s \phi_{s,k}(x)$$



(continuous function)

Resolution and Multiresolution

Scale ⇔ *Resolution*

- Data Representation
  - Computation - Discrete Elements (*Samples*)
  - Resolution - *Samples* / Unit of Scale
- Need for Representation at Multiple Scales
  - Different Features - Different Scales
  - Efficient Computation - Coarse to Fine Scale

Multiresolution Analysis

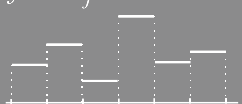
Sequence of Scale Spaces $(V_j),\ j \in Z$

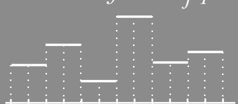
1. Inclusion: $V_j \subset V_{j-1}$
2. Scaling: $f(x) \in V_j \iff f(2^{-j}x) \in V_0$
3. Density: $\mathrm{closure}\{\bigcup_j V_j\} = L^2(\mathbf{R})$
4. Maximality: $\bigcap_j V_j = \{0\}$
5. Scale Basis: $\exists\ \phi(x)$ s.t. $\{\phi(x-k)\}$ basis of V_0

Two Scale Relation

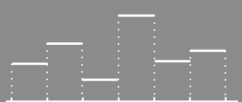
Axioms 1. and 2. (Nested Spaces and Dyadic Scaling)

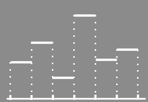
$f \in V_j \iff f \in V_{j-1}$





$f(x) \in V_j \iff f(2x) \in V_{j-1}$



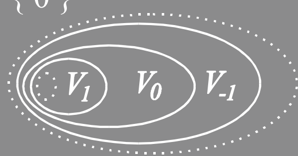


Completeness

Axioms 3. and 4. (complete covering of function space)

$$\mathrm{closure}\{\bigcup_j V_j\} = L^2(\mathbf{R})$$

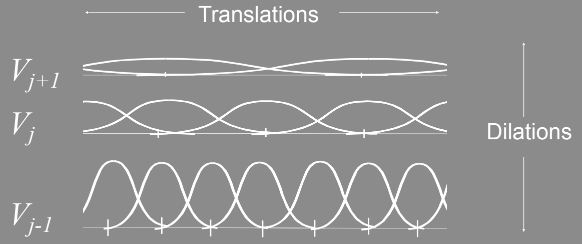
$$\bigcap_j V_j = \{0\}$$



Multiresolution Basis

Axioms 5. and 2. (basis family generated from a single ϕ)

$$\phi_{j,k}(x) = 2^{-j/2}\,\phi(2^{-j}x - k)$$



From Multiresolution to Wavelets

Multiresolution Ladder

- Coarse to Fine

$$f \in V_j \;\Rightarrow\; f \in V_{j-1}$$

- Fine to Coarse (*information loss)*

$$f \in V_{j-1} \qquad f \notin V_j$$

Wavelets: Complementary Subspaces

Intuition: difference between two consecutive levels

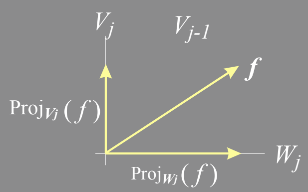
W_j : Details in V_{j-1} that cannot be represented in V_j

Wavelet Spaces

Definition: Orthogonal Complement of V_j in V_{j-1}

therefore: $V_{j-1} = V_j \oplus W_j$

$$\mathrm{Proj}_{V_{j-1}}(f) = \mathrm{Proj}_{V_j}(f) + \mathrm{Proj}_{W_j}(f)$$



Wavelet Basis

Orthogonality of Spaces

- Decomposition of L^2

$$W_j \perp W_k \qquad j \neq k$$

- Basis of W_j

$$L^2(\mathbb{R}) = \bigoplus_{j \in Z} W_j$$

$$\psi_{j,k}(x) = 2^{-j/2}\,\psi(2^{-j}x - k)$$

Multiresolution Decomposition

- Scale and Wavelet Spaces
- Function Decomposition

$$\{0\} \cdots \subset V_1 \subset V_0 \subset V_{-1} \subset \cdots L^2(\mathbf{R})$$

$$\cdots \quad W_1 \quad W_0 \quad W_{-1} \quad \cdots\cdots$$

$$f \in V_0$$

$$f = \mathrm{Proj}_{W0}(f) + \cdots + \mathrm{Proj}_{Wn}(f) + \mathrm{Proj}_{Vn}(f)$$

Two-Scale Operators

- Transform Representation Sequences

  Discrete Operators

  Move Between Consecutive Levels j and $j+1$

- Act on Multiresolution Hierarchy

  Fine $\leftrightarrow$ Coarse

  Fine $\leftrightarrow$ Detail

Key to Efficient Computation

Dilation Equations

- Scaling Basis: $\phi_0 \in V_0 \subset V_{-1}$

$$\phi_0 = \sum_k \langle \phi_0, \phi_{-1,k} \rangle \phi_{-1,k}(x) = \sum_k h_k \phi_{-1,k}(x)$$

- Wavelet Basis: $\psi_0 \in W_0 \subset V_{-1}$

$$\psi_0 = \sum_k \langle \psi_0, \phi_{-1,k} \rangle \phi_{-1,k}(x) = \sum_k g_k \phi_{-1,k}(x)$$

Expression of ϕ and ψ

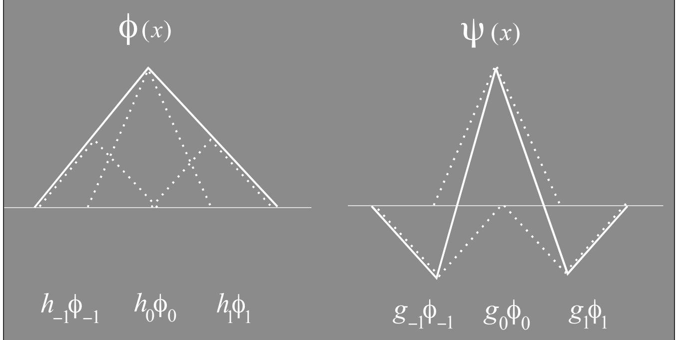
- Reflexive Definition

$$\phi(x) = \sqrt{2} \sum_k h_k \phi(2x-k)$$

- ψ defined in terms of ϕ

$$\psi(x) = \sqrt{2} \sum_k g_k \phi(2x-k)$$

Shape of ϕ and ψ



Fine to Coarse: Operator *H*

- Recursive Computation of Inner Products with ϕ

$$\langle f, \phi_{j,k} \rangle = \left\langle f, \sum_n \bar{h}_{n-2k} \phi_{j-1,k} \right\rangle = \sum_n \bar{h}_{n-2k} \langle f, \phi_{j-1,k} \rangle$$

- Discrete Convolution and Decimation

$$c_n^j = (\downarrow 2) \bar{H} * c_n^{j-1}$$

- Change of Representation $V_j \subset V_{j-1}$

$$c_n^j \xleftarrow{\bar{H}} c_n^{j-1}$$

Fine to Detail: Operator *G*

- Recursive Computation of Inner Products with ψ

$$\langle f, \psi_{j,k} \rangle = \left\langle f, \sum_n \bar{g}_{n-2k} \phi_{j-1,k} \right\rangle = \sum_n \bar{g}_{n-2k} \langle f, \phi_{j-1,k} \rangle$$

- Discrete Convolution and Decimation

$$d_n^j = (\downarrow 2) \bar{G} * c_n^{j-1}$$

- Change of Representation $W_j \subset V_{j-1}$

$$d_n^j \xleftarrow{\bar{G}} c_n^{j-1}$$

Wavelet Decomposition

- Assume: $f^j \in V^j$
- Start with: $c_n^j = \langle f, \phi_{j,k} \rangle$
- Decomposition Scheme

$$c_n^j \xrightarrow{\bar{H}} c_n^{j+1} \xrightarrow{\bar{H}} c_n^{j+2} \quad \ldots$$

$$c_n^j \xrightarrow{\bar{G}} d_n^{j+1}, \quad c_n^{j+1} \xrightarrow{\bar{G}} d_n^{j+2} \quad \ldots$$

Coarse and Detail to Fine

- Recover c^{j-1} from c^{j} and d^{j}

$$\begin{aligned} c_n^{j-1} &= \left\langle f^{j-1}, \phi_{j-1,n} \right\rangle \\ &= \left\langle \sum c_k^j \phi_{j,k} + \sum d_k^j \psi_{j,k}, \phi_{j-1,n} \right\rangle \\ &= \sum c_k^j \left\langle \phi_{j,k}, \phi_{j-1,n} \right\rangle + \sum d_k^j \left\langle \psi_{j,k}, \phi_{j-1,n} \right\rangle \\ &= \sum h_{n-2k} c_k^j + \sum g_{n-2k} d_k^j \end{aligned}$$

- Reconstruction Operator (*Expansion and Convolution*)

$$c_n^{j-1} = H * (\uparrow 2)\, c_n^j + G * (\uparrow 2)\, d_n^j$$

Wavelet Reconstruction

- Start with: $\left(c^{j+m}, d^{j+m}, \ldots, d^{j+1}\right)$

- Decomposition Scheme

$$\begin{array}{ccccc} c_n^{j+m} & \xrightarrow{H} & \cdots\ c_n^{j+1} & \xrightarrow{H} & c_n^j \\ & \overset{G}{\nearrow} & & \overset{G}{\nearrow} & \\ d_n^{j+m} & & \cdots\ d_n^{j+1} & & \end{array}$$

Wavelets and Filters

A Look in Frequency Domain

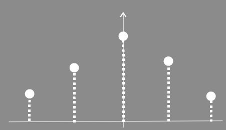
- Operators H and G
  - Discrete Filters
- Fast Wavelet Transform
  - Sub-band Filtering
- Signal Processing Framework
  - Filter Design Techniques
  - Implementation Scheme

Low-Pass Filter

Discrete Operator $\boldsymbol{H}$

- Impulse Response

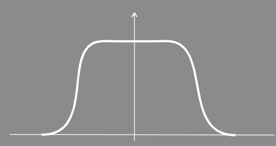
$$(h_k) = (\ldots, h_{-1}, h_0, h_1, \ldots)$$



- Transfer Function

$$H(\omega) = \sum_k h_k e^{-ik\omega}$$

H is 2π -Periodic

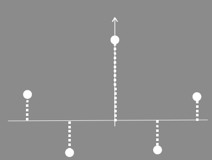


High-Pass Filter

Discrete Operator $\boldsymbol{G}$

- Impulse Response

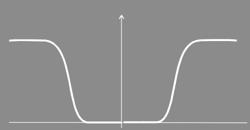
$$(g_k) = (\ldots, g_{-1}, g_0, g_1, \ldots)$$



- Transfer Function

$$G(\omega) = \sum_k g_k e^{-ik\omega}$$

G is 2π -Periodic

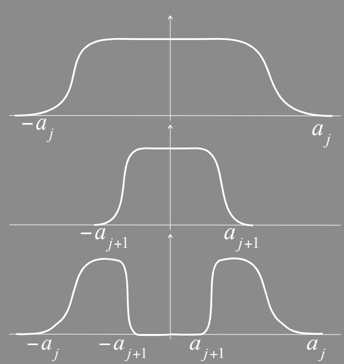


Multiresolution in Frequency Domain

$f_j \in V_j \Rightarrow \hat{f}_j \in [-a_j, a_j]$

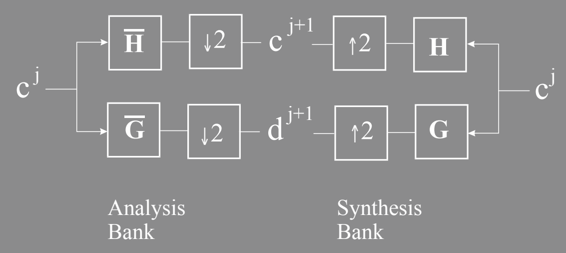
$\hat{f}_{j+1} \in [-a_{j+1}, a_{j+1}]$

$\hat{o}_{j+1} \in [-a_j, -a_{j+1}] \cup [a_{j+1}, a_j]$



Two Channel Filter Banks

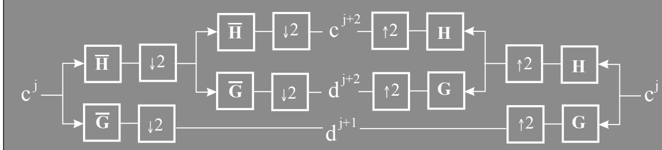
- One-Level Transform: *Sub-band Filtering*
  Perfect Reconstruction



Tree-Structured Filter Bank

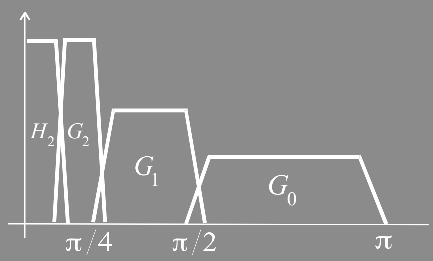
Wavelet Transform

- Recursive Filtering
- Same Filters (*invariance: dyadic scale, integer shift*)



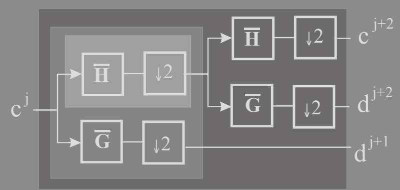
Frequency Splitting

- Recursive Filtering
  Dyadic Bands
  Energy Preservation



Matrix Implementation of FWT

- Filtering with Subsampling
- Two Channel Filter Bank
- Tree-Structured Filter Banks



Filtering with Subsampling

- Convolution and Decimation: $y = (\downarrow 2) F x$

$$\begin{pmatrix} y_0 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} f_0 & & \cdots & f_1 \\ f_1 & f_0 & & \cdots \\ \cdots & f_1 & f_0 & \\ & \cdots & f_1 & f_0 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

- Filter Matrix : $y = \downarrow F x$

$$\begin{pmatrix} y_0 \\ y_1 \end{pmatrix} = \begin{pmatrix} f_0 & & \cdots & f_1 \\ \cdots & f_1 & f_0 & \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Two-Channel Filter Bank

- Analysis Bank

$$M = \overline{\begin{pmatrix} (\downarrow 2) H \\ (\downarrow 2) G \end{pmatrix}} = \overline{\begin{pmatrix} \downarrow H \\ \downarrow G \end{pmatrix}}$$

- Analysis Matrix

$$\begin{pmatrix} c_0^1 \\ c_1^1 \\ d_0^1 \\ d_1^1 \end{pmatrix} = \overline{\begin{pmatrix} h_0 & & \cdots & h_1 \\ \cdots & h_1 & h_0 & \\ g_0 & & \cdots & g_1 \\ \cdots & g_1 & g_0 & \end{pmatrix}} \begin{pmatrix} c_0^0 \\ c_1^0 \\ c_2^0 \\ c_3^0 \end{pmatrix}$$

- Synthesis Bank

$$S = M^{-1} = M^{\mathrm{T}}$$

* *Critical Sampling*

Tree-Structured Filter Bank

- Decomposition and Reconstruction Matrices for Level j

$$D_j = \begin{pmatrix} M_k & 0 \\ 0 & I_{n-k} \end{pmatrix} \qquad R_j = D_j^{-1} = \begin{pmatrix} M_k^{\mathrm{T}} & 0 \\ 0 & I_{n-k} \end{pmatrix}$$

with $k = N/2^j$

- Example of Decomposition Sequence

$$\begin{pmatrix} c_0^3 \\ d_0^3 \\ d_0^2 \\ d_1^2 \\ d_0^1 \\ d_1^1 \\ d_2^1 \\ d_3^1 \end{pmatrix} = \begin{pmatrix} M_2 & \\ & I_6 \end{pmatrix} \begin{pmatrix} M_4 & \\ & I_4 \end{pmatrix} \left(M_8 \right) \begin{pmatrix} c_0^0 \\ c_1^0 \\ c_2^0 \\ c_3^0 \\ c_4^0 \\ c_5^0 \\ c_6^0 \\ c_7^0 \end{pmatrix}$$

FWT Matrix Computation

- Data Vector

$$v^j = \left((c^j)(d^j) \cdots (d^{j-m}) \right)$$

- Direct Transform (Analysis Bank)

$$v^{j+1} = D_j v^j$$

$$v^{m+1} = D_m \cdots D_1 D_0 c^0$$

- Inverse Transform (Synthesis Bank)

$$v^j = R_j v^{j+1}$$

$$c^0 = R_0 R_1 \cdots R_m v^{m+1}$$

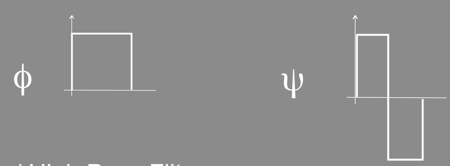
Example

Haar: Simplest Case

- Scaling Function and Wavelet
- Transform Scheme
- Matrix Computation
- Function Decomposition
- Change of Basis

Haar Wavelets

- Scaling Function and Wavelet

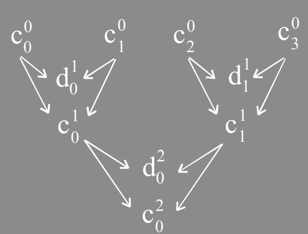


- Low / High Pass Filters

$$H = \frac{1}{\sqrt{2}}(1,\ 1) \qquad G = \frac{1}{\sqrt{2}}(1,\ -1)$$

Haar Transform

- Input: $f = (c_0^0, c_1^0, c_2^0, c_3^0)$



- output: $\bar{f} = (c_0^2, d_0^2, d_0^1, d_1^1)$

Haar Decomposition

Representation | *Projection*

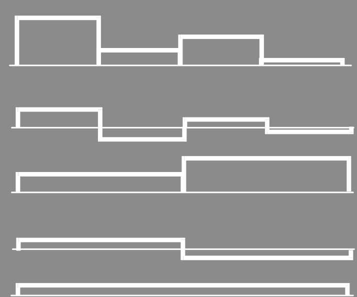
$f^0 = (c_0^0, c_1^0, c_2^0, c_3^0)$

$o^1 = (d_0^1, d_1^1)$

$f^1 = (c_0^1, c_1^1)$

$o^2 = (d_0^2)$

$f^2 = (c_0^2)$



Haar Matrix Computation

- Wavelet Matrix: (*H* and *G*)

Second Level — First Level

$$\begin{pmatrix} r & r & & \\ r & -r & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} r & r & & \\ & & r & r \\ r & -r & & \\ & & r & -r \end{pmatrix} \begin{pmatrix} 9 \\ 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 4 \\ 1 \end{pmatrix}$$

- Normalization: $r = 1/\sqrt{2}$

Change of Basis Interpretation

- Scale Representation

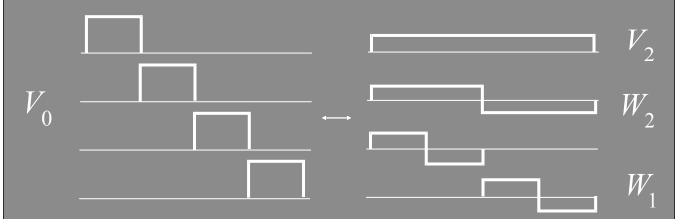
$$f = \begin{pmatrix} 9 \\ 1 \\ 2 \\ 0 \end{pmatrix} = 9\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

- Multiresolution Representation

$$f = \begin{pmatrix} 9 \\ 1 \\ 2 \\ 0 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} + 4\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

Representation in the Two Bases

Scale Basis — *Multiresolution Basis*



Bi-Orthogonal Wavelets

Bi-Orthonormal Basis

- Dual Basis

$$\langle e_i, f_j \rangle = 0, i \neq j$$

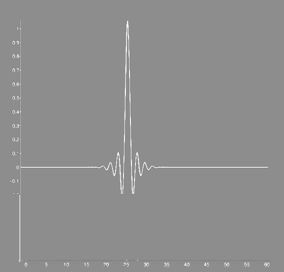
$$\langle e_i, f_i \rangle = 1$$

- Represent with one Basis, Reconstruct with the Other

$$v = \sum_{j=1}^{n} \langle v, e_j \rangle f_j = \sum_{j=1}^{n} \langle v, f_j \rangle e_j$$

- Computationally Similar to Orthonormal Basis
- More Degrees of Freedom to Construct Basis

Meyer Bi-orthogonal Wavelets



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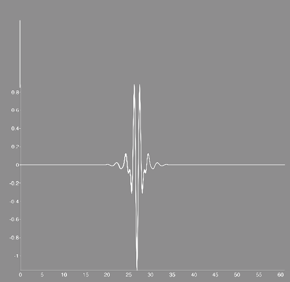

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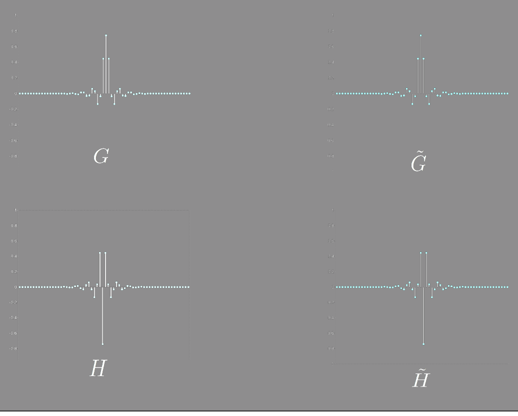
Discrete Filter Pairs



Conclusions

Computational Scheme for Wavelets

- Multiresolution Analysis
- Non-Redundant Wavelets
- Two-Channel Filter Banks
- Function Decompostion
- Wavelet Transform as a Basis Change

- Bi-Orthogonal Wavelets