Towards a Generalization of Dupire's Equation for Several Assets

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We pose the problem of generalizing Dupire's equation for the price of call options on a basket of underlying assets. We present an analogue of Dupire's equation that holds in the case of several underlying assets provided the volatility is time dependent but not asset-price dependent. We deduce it from a relation that seems to be of interest on its own.

1 Introduction

A fundamental problem in Financial Mathematics is that of calibrating the underlying model from market data. This is crucial, for example, in hedging and portfolio optimization. Such data may consist of underlying asset prices, or, as in many applications, derivative prices on such assets. An example, of central importance herein is an *European call option*. It gives the bearer the right, but not the obligation, of buying an asset B for a given strike price K at a certain maturity date T.

In the present work we are concerned with the problem of determining the model's volatility based on the quoted prices of a basket option for arbitrary values of the strike, the weights, and the maturity. Although, this is a highly idealized situation, it already poses some very interesting mathematical challenges, as we shall see in the sequel. The results presented here should be valuable for the development of effective methods to estimate the local volatility in multi-asset markets where a sufficiently large set of basket options is traded.

In the standard Black-Scholes [2] model for option pricing, the underlying asset is assumed to follow a dynamics described by the stochastic differential equation

$$\frac{dS}{S} = \mu dt + \sigma dW,$$

where W is a Brownian motion, μ is a drift coefficient, and σ is the volatility of the underlying asset. In the classical Black-Scholes theory, σ is assumed to be constant.

Despite the enormous success of such model, it is known that in practice it cannot consistently price options with different strike prices and maturity dates, as the volatility empirically appears not to be constant over time. Furthermore, if one computes the *implied volatility* from the quoted price one verifies empirically that different strikes and maturities lead to different implied volatilities for options on a given asset. This is known as the *smile effect* and was discussed in a pioneering paper by B. Dupire [5].

Due to the smile effect, volatility estimates based on historical data are considered not to be reliable. Another approach consists in trying to determine the volatility from the option prices in the market. This leads to a challenging inverse problem. See, for example, [3, 6, 8].

In [5], Dupire considered a model for the dynamics of the underlying asset in which the volatility depends both on the time t and on the stock price S. More precisely,

$$\frac{dS}{S} = \mu dt + \sigma(S,t)\, dW \ . \tag{1.1}$$

This type of model is known as a local volatility model. Other approaches have been proposed in which the volatility follows another stochastic process.

Dupire has shown that in the local volatility model, the volatility can, in principle, be recovered from market data if the price of European options on the underlying asset were known for all the strike prices K and maturity dates T.

The celebrated Dupire equation for the case of a single asset reads as follows

$$\frac{\partial C}{\partial T} = \frac{\sigma^2(K,T)K^2}{2}\frac{\partial^2 C}{\partial K^2} + (r(T) - D(T))\left(C - K\frac{\partial C}{\partial K}\right),$$

or in other words

$$\sigma = \sqrt{\frac{\frac{\partial C}{\partial T} - (r(t) - D(t))\left(C - K\frac{\partial C}{\partial K}\right)}{\frac{K^2}{2}\frac{\partial^2 C}{\partial K^2}}}\ .$$

Here, $C(t, S_t, T, K)$ is the undiscounted European call option price, $r(t)$ is the risk-free interest rate and $D(t)$ is the dividend rate. The price C satisfies, under the usual assumptions of liquidity, absence of arbitrage, and transaction costs (perfect market), the Black-Scholes equation

$$\begin{cases} \frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2(S,t)S^2\frac{\partial^2 C}{\partial S^2} + r(S\frac{\partial C}{\partial S} - C) = 0, & S > 0,\ t < T \\ C(S,T) = (S-K)^+. \end{cases} \tag{1.2}$$

In practice, however, the option prices are known only for a few maturity dates and strike prices and some interpolation is needed. The computed volatility depends strongly on the interpolation used. Due to the ill-posed character of this inverse problem, some regularization strategy has to be used to ensure the numerical stability of the reconstruction. See [3, 6]. In any case, Dupire's formula plays a fundamental role in several methods that have been proposed to tackle this problem.

Let us now consider, the multi-asset situation, which is very important in practice. In particular, it could be applied to index options.

Here, the dynamics is given by

$$\frac{dS_i}{S_i} = \mu_i dt + \sum_{j=1}^{N} \sigma_{ij} dW_j \ , \tag{1.3}$$

where W denotes the N-dimensional Brownian motion with respect to the risk-neutral measure. Here $\sigma_{ij} = \sigma_{ij}(S,t)$ is the volatility matrix, $\mu_i = \mu_i(t)$ is the risk-neutral drift, with $\mu_i(t) = r(t) - D_i(t)$ where D_i is the dividend rate of the i-th asset, and $W = (W_1, \ldots, W_N)$ is a standard N -dimensional Brownian motion.

For technical reasons, we shall assume throughout this paper that the volatility matrix $((\sigma_{ij}(t,S)))$ and the drift vector $\mu_j(t,S)$ are smooth and bounded, i.e.,

$$|\mu_j(t,S)| \le C \quad \text{and} \quad |\sigma_{ij}(t,S)| \le C \ . \tag{1.4}$$

Furthermore, we shall assume that the matrix $A = (a^{ij}) = \frac{1}{2}\sigma\sigma^t$ satisfies the uniform ellipticity condition: there exist constants $\lambda, \Lambda > 0$ such that

$$\lambda |y|^2 \le \sum_{i,j}^{n} a_{ij}(t,S) y_i y_j \le \Lambda |y|^2 \ . \tag{1.5}$$

Given a vector of weights $w = (w_1, w_2, \ldots, w_N)$ with $w_i \ge 0$, we consider an *European basket option*, that is, a contract giving the holder the right to buy a basket composed of w_i units of the i-th asset at a maturity date T upon paying a strike price K.

Here, the value

$$B = \sum_{j=1}^{N} w_j S_j$$

is called the *basket price (or index) composed of the stocks* S_i.

The fair price of such an option is

$$P(S_t, t, K, T) = e^{-\int_t^T \mu_i(\tau) d\tau} E_t^*[(\sum_{i=1}^{N} w_i S_{i,T} - K)^+]$$

where E_t^* denotes the expected value at time t under the so-called risk-neutral probability. It turns out to be simpler to work with the undiscounted call-price

$$C_w = e^{\int_t^T \mu_i(\tau) d\tau} P = E_t^*[(\sum_{i=1}^{N} w_i S_{i,T} - K)^+].$$

Our goal is to address the following natural question:

Is there a generalization of Dupire's equation for the multi-asset context?

We have a partial answer to this question, under additional assumptions, the most restrictive of all being that of having an asset-price independent volatility. More precisely, our main result reads as follows:

Theorem 1.1 *Assume that the volatility matrix σ_{ij} is a deterministic* locally integrable *function of time, then the fair price C_w of the European basket call option satisfies*

$$\frac{\partial C_w}{\partial T} = \sum_{i=1}^{N} \mu_i w_i \frac{\partial C_w}{\partial w_i} + \sum_{i,j=1}^{N} a_{ij} w_i w_j \frac{\partial C_w^2}{\partial w_i \partial w_j}, \tag{1.6}$$

where $A = (a_{ij})$ denotes the matrix given by $A = \frac{1}{2}\sigma\sigma^t$.

The proof of this result will be the subject of Section 3 as well as that of Appendix A.

Let p denote the transition probability density corresponding to the stochastic process defined by Equation (1.3), and let $\mathbf{s}$ denote the surface measure in the set

$$L_w \stackrel{\text{def}}{=} \left\{ (S_1, \ldots, S_N) \Big| \sum_{j=1}^{N} w_j S_j = K, S_j \geq 0 \right\} . \tag{1.7}$$

Theorem 1.1 relies on the following remarkable relation, that seems to be of interest in its own:

$$\sum_{i=1}^{N} \mu_i w_i \frac{\partial C_w}{\partial w_i} = \frac{\partial C_w}{\partial T} - \sum_{i,j=1}^{N} \int_{L_w} a_{ij} S_{i,T} S_{j,T} p(S_t, t, S_T, T) \frac{w_i w_j}{|w|} d\mathbf{s} . \tag{1.8}$$

Remark 1.2 *If no dividends are paid then $\mu_i = r$ for all i, and using the Euler's equation (3.3) we can re-write (1.6) as*

$$\frac{\partial C_w}{\partial T} = r(C_w - K \frac{\partial C_w}{\partial K}) + \sum_{i,j=1}^{N} a_{ij} w_i w_j \frac{\partial C_w^2}{\partial w_i \partial w_j}$$

2 Review of Dupire's Equation and Related Facts

A key point in the derivation of the one-dimensional Dupire's equation is that one may express the price of an European call option as

$$C(t, S_t, T, K) = \int_{-\infty}^{\infty} p(S_t, t, S_T, T)(S - K)^+ dS_T$$

where $p(t, S_t, \tilde{t}, S_{\tilde{t}})$ is the transition probability density corresponding to the stochastic process defined by Equation (1.1). From the PDE viewpoint, p is fundamental solution associated to the N-dimensional Black-Scholes equation (1.2). Using the fundamental theorem of calculus we deduce that

$$\frac{\partial C}{\partial K} = - \int_{K}^{\infty} p(S_t, t, S_T, T) dS_T$$

Hence, we may recover the transition probability by computing the second derivative of the call price with respect to K

$$\frac{\partial^2 C}{\partial K^2} = p. \tag{2.1}$$

For comparison with the multi-dimensional case, it is convenient to consider a more general (discounted) call option C_w for buying w units of the stock with strike price K. Then,

$$C_w = E^*_{t_0}[(wS_T - K)^+].$$

Thus, C_w is plainly a homogeneous function of degree one, with respect to the variables K and w. Hence, it satisfies Euler's equation, namely

$$K\frac{\partial C_w}{\partial K} + w\frac{\partial C_w}{\partial w} = C_w.$$

Differentiating this equation with respect to K and w we get

$$K\frac{\partial^2 C_w}{\partial K^2} + w\frac{\partial^2 C_w}{\partial w \partial K} = 0,$$

and

$$K\frac{\partial^2 C_w}{\partial K \partial w} + w\frac{\partial^2 C_w}{\partial w^2} = 0.$$

Hence,

$$K^2\frac{\partial C_w}{\partial K^2} = w^2\frac{\partial^2 C_w}{\partial w^2},$$

and we conclude that Dupire's equation can be written in an equivalent form as

$$\frac{\partial C_w}{\partial T} = \mu w\frac{\partial C_w}{\partial w} + \frac{1}{2}\sigma^2 w^2\frac{\partial^2 C_w}{\partial w^2}.$$

3 The Multi-Asset Case

We now present a proof of Theorem 1.1. As before, the price of the basket option can be written as

$$C_w(S_t, t, K, T) = \int_{\mathbb{R}^N_+} p(S_t, t, S_T, T)\,(\sum_{i=1}^N w_i S_{i,T} - K)^+ dS_T$$

where $p(t, S_t, \tilde{t}, S_{\tilde{t}})$ is now the transition probability density associated to the stochastic process defined by (1.3), or from the PDE's viewpoint the fundamental solutions to the multidimensional Black-Scholes equation:

$$\frac{\partial C}{\partial T} + \sum_{i}^{N} \mu_i(t,S) S_i \frac{\partial C}{\partial S_i} + \sum_{i,j=1}^{N} a^{ij}(t,S) S_i S_j \frac{\partial^2 S}{\partial S_i \partial S_j} = 0\ . \tag{3.1}$$

The standard theory of parabolic equations does not apply directly to (3.1). However, under the usual change of variables $\tau = T - t$ and $X_i = \log S_i$, Equation (3.1) transforms into a non-degenerate parabolic equation.

Under the technical conditions (1.4) and (1.5), it can be proved that that (3.1) admits a fundamental solution p that is at least of class $C^{1,2}$ and decays exponentially when $\|S\| \to \infty$, together with its first and second order derivatives. This fact will be crucial in the following computations, since this ensures that all the boundary terms at infinity vanish.

The proof of the existence of the fundamental solutions under these assumptions can be done by using the so-called *parametrix method*, introduced by E. Levi [7] in 1907. We remark that our technical conditions (1.4) and (1.5), and the smoothness requirement on the coefficients could be certainly relaxed. See, for example, [4] for a construction of the fundamental solution in the unbounded coefficient case, using Levi's method. However, as our main interest in this paper is the financial significance of our results, we do not intend to state the most general conditions under which our computations are still valid.

We introduce the region

$$H_w \overset{\text{def}}{=} \left\{ S \in \mathbb{R}_+^N \middle| \sum_{i=1}^{N} w_i S_i \geq K \right\}.$$

Thus,

$$C_w(S_t, t, K, T) = \int_{H_w} p(S_t, t, S_T, T)(\sum_{i=1}^{N} w_i S_{i,T} - K)\, dS_T \tag{3.2}$$

We note that C_w is homogeneous of degree one in the variables $(w_1, w_2, \ldots, w_n, K)$. Hence, it satisfies Euler equation

$$\sum_{i=1}^{N} w_i \frac{\partial C_w}{\partial w_i} + K \frac{\partial C_w}{\partial K} = C_w \ . \tag{3.3}$$

In order to be able to compute the derivatives of C_w, it is convenient to re-write Equation (3.2) as an integral over a region independent of w. For this purpose, we introduce the change of variables

$$B = \sum_{i=1}^{N} w_i S_{i,T},$$

and

$$Q_i = \frac{w_i S_{i,T}}{\sum_{i=1}^{N} w_i S_{i,T}}$$

for $i = 1, \ldots, N-1$. Therefore, $Q \in \Delta_N$ where

$$\Delta_N = \{Q = (Q_1, Q_2, \ldots, Q_{N-1}) : Q_i \geq 0, \sum_{i=1}^{N-1} Q_i \leq 1\}$$

is the $N-1$ dimensional simplex. Thus,

$$S_T := S(Q, B) = \left(\frac{Q_1 B}{w_1}, \ldots, \frac{Q_{N-1} B}{w_{N-1}}, \frac{(1 - \sum_{i=1}^{N-1} Q_i) B}{w_N} \right).$$

The Jacobian of the change of variables

$$(S_{1,T}, \ldots, S_{N,T}) \longmapsto (Q_1, \ldots, Q_{N-1}, B)$$

is given by

$$J = \frac{\partial(S_{1,T}, \ldots, S_{N,T})}{\partial(Q_1, \ldots Q_{N-1}, B)} = \frac{B^{N-1}}{w_1 w_2 \ldots w_N}.$$

Thus, we obtain:

$$C_w(S_t, t, K, T) = \int_K^{\infty} \int_{\Delta_N} p\left(S_t, t, S(Q, B), T\right)(B - K) \frac{B^{N-1}}{w_1 w_2 \ldots w_N} dQ dB$$

Hence

$$\frac{\partial C_w}{\partial K} = \int_{\Delta_N} \left[p\left(S_t, t, S(Q, B), T\right)(B - K) \frac{B^{N-1}}{w_1 w_2 \ldots w_N} \right]_{B=K} dQ$$

$$- \int_K^\infty \int_{\Delta_N} p\left(S_t, t, S(Q,B), T\right) \frac{B^{N-1}}{w_1 w_2 \dots w_N} dQ dB$$

$$= - \int_K^\infty \int_{\Delta_N} p\left(S_t, t, S(Q,B), T\right) \frac{B^{N-1}}{w_1 w_2 \dots w_N} dQ dB,$$

and

$$\frac{\partial^2 C_w}{\partial K^2} = \int_{\Delta_N} p\left(S_t, t, S(Q,K), T\right) \frac{K^{N-1}}{w_1 w_2 \dots w_N} dQ\ .$$

Going back to the S_T-coordinates we easily obtain the following identity:

$$\frac{\partial^2 C_w}{\partial K^2} = \frac{1}{|w|} \int_{L_w} p(S_t, t, S_T, T) d\mathbf{s}, \tag{3.4}$$

where L_w is defined as in the introduction. This identity relates the second derivative of the call price C_w with respect to strike price K, to the integral of the probability density p over the set L_w.

Equation (3.4) is the multi-dimensional analogue of Equation (2.1); in probabilistic terms, the integral term expresses the probability that the basket B has a price K at the maturity date T, given that the price vector has the value S_t at time t, namely

$$\frac{\partial^2 C_w}{\partial K^2} = \frac{1}{|w|} P\left[B_T = K \middle| S_t \right].$$

However, this relationship does not seem to yield a suitable multidimensional generalization of Dupire's equation. For this reason, we also compute the derivatives $\frac{\partial C_w}{\partial w_i}$ to get

$$w_i \frac{\partial C_w}{\partial w_i} = - \int_K^\infty \int_{\Delta_N} \frac{\partial p}{\partial S_{i,T}} \left(S_t, t, S(Q,B), T\right)(B-K) \frac{Q_i B}{w_i} \frac{B^{N-1}}{w_1 w_2 \dots w_N} dQ dB \tag{3.5}$$

$$- \int_K^\infty \int_{\Delta_N} p\left(S_t, t, S(Q,B), T\right)(B-K) \frac{B^{N-1}}{w_1 w_2 \dots w_N} dQ dB$$

for $i = 1, \dots N-1$. It is straightforward to notice that upon extending the above notation so that $Q_N = 1 - \sum_{i=1}^{N-1} Q_i$, relation (3.5) also holds for $i = N$. Then,

$$\sum_{i=1}^N \mu_i w_i \frac{\partial C_w}{\partial w_i} = - \int_{H_w} \sum_{i=1}^N \mu_i [S_{i,T} \frac{\partial p}{\partial S_{i,T}} + p] (\sum_{i=1}^N w_i S_{i,T} - K)\, dS_T$$

$$= - \int_{H_w} \sum_{i=1}^N \frac{\partial}{\partial S_{i,T}} [\mu_i S_i p]\, (\sum_{i=1}^N w_i S_{i,T} - K)\, dS_T.$$

Now, we use the fact that p satisfies the multi-dimensional Fokker-Planck equation (see e.g. [9]):

$$\frac{\partial p}{\partial T} + \sum_{i=1}^N \frac{\partial}{\partial S_i} [\mu_i S_i p] - \sum_{i,j=1}^N \frac{\partial^2}{\partial S_i \partial S_j} [a_{ij} S_i S_j p] = 0.$$

Thus we obtain

$$\sum_{i=1}^N \mu_i w_i \frac{\partial C_w}{\partial w_i} = \int_{H_w} \left\{ \frac{\partial p}{\partial T} + \sum_{i,j=1}^N \frac{\partial^2}{\partial S_{i,T} \partial S_{j,T}} [a_{ij} S_{i,T} S_{j,T} p] \right\} (\sum_{i=1}^N w_i S_{i,T} - K)\, dS_T\ .$$

On the other hand, we compute the derivative of C_w with respect to the maturity date

$$\frac{\partial C_w}{\partial T} = \int_{H_w} \frac{\partial p}{\partial T}(S_t, t, S_T, T)(\sum_{i=1}^{N} w_i S_{i,T} - K)\, dS_T,$$

and then

$$\sum_{i=1}^{N} \mu_i w_i \frac{\partial C_w}{\partial w_i} = \frac{\partial C_w}{\partial T} + \int_{H_w} \sum_{i,j=1}^{N} \frac{\partial^2}{\partial S_{i,T} \partial S_{j,T}} [a_{ij} S_{i,T} S_{j,T} p]\, (\sum_{i=1}^{N} w_i S_{i,T} - K)\, dS_T.$$

Upon applying the divergence theorem, and using the fact that the boundary integral over ∂H_w vanishes, we get

$$\sum_{i=1}^{N} \mu_i w_i \frac{\partial C_w}{\partial w_i} = \frac{\partial C_w}{\partial T} - \int_{H_w} \sum_{i,j=1}^{N} \frac{\partial}{\partial S_{j,T}} [a_{ij} S_{i,T} S_{j,T} p] w_i dS_T \ .$$

As the exterior normal vector to L_w is given by $\frac{-w}{|w|}$, we obtain:

$$\sum_{i=1}^{N} \mu_i w_i \frac{\partial C_w}{\partial w_i} = \frac{\partial C_w}{\partial T} - \sum_{i,j=1}^{N} \int_{L_w} a_{ij} S_{i,T} S_{j,T} p(S_t, t, S_T, T) \frac{w_i w_j}{|w|} d\mathbf{s}. \tag{3.6}$$

On the other hand, after changing variables and integrating by parts identity (3.5) we also deduce that

$$\frac{\partial C_w}{\partial w_i} = \int_{H_w} p(S_t, t, S_T, T) S_{i,T} dS_T.$$

Then

$$w_i w_j \frac{\partial^2 C_w}{\partial w_j \partial w_i} = -\int_K^\infty \int_{\Delta_N} \frac{\partial p}{\partial S_{j,T}}((S_t, t, S(Q,B), T) Q_i B \frac{Q_j B}{w_j} \frac{B^{N-1}}{w_1 \dots w_N} dQ dB$$

$$-(1+\delta_{ij}) \int_K^\infty \int_{\Delta_N} p(S_t, t, S(Q,B), T) Q_i B \frac{B^{N-1}}{w_1 \dots w_N} dQ dB$$

where δ_{ij} is the Kronecker's delta. As before, using the fact that $\frac{\partial}{\partial S_{j,T}}(p S_{i,T} S_{j,T}) = \frac{\partial p}{\partial S_{j,T}} S_{i,T} S_{j,T} + p(1+\delta_{ij}) S_{i,T}$, we deduce that

$$\frac{\partial^2 C_w}{\partial w_i \partial w_j} = \frac{1}{|w|} \int_{L_w} p S_{i,T} S_{j,T} d\mathbf{s}. \tag{3.7}$$

Thus, if a_{ij} are time-dependent only , we obtain:

$$\frac{\partial C_w}{\partial T} = \sum_{i=1}^{N} \mu_i w_i \frac{\partial C_w}{\partial w_i} + \sum_{i,j=1}^{N} a_{ij} w_i w_j \frac{\partial^2 C_w}{\partial w_i \partial w_j}.$$

This concludes the proof of Theorem 1.1. □

4 Conclusions

Basket options play an important role in financial markets. One reason being that many indices could be considered a basket of different assets. We considered properties of option prices on baskets and posed the natural question of whether an analogue of Dupire's now classical formula exists. In this paper we presented a first step towards such formula. More precisely, we presented an equation that holds under the extra assumption that the volatility matrix σ is asset-price independent. A natural continuation of the present work would be to extend the result presented herein to a situation where σ depends also on the underlying asset prices. Although at this moment we do not have such generalization, we believe that it should somehow rely on Equations (3.6) and (3.7). One might even speculate it would involve a non-local operator.

Yet another natural continuation of the present work would be to use the results obtained herein to develop effective numerical methods to compute the matrix $A = \frac{1}{2}\sigma\sigma^t$.

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Appendix A An Alternative Derivation

In this appendix we present yet another derivation of the main result. We believe that the techniques employed herein provide a complementary view of the problem. For simplicity, throughout this section we shall write S to denote the stock price at time t.

Consider as before a basket option, with a pay-off function given by:

$$f = (\sum_{j=1}^{N} w_j S_j - K)^+ \ .$$

Ito-Tanaka formula [10] reads as

$$df = \sum_{i=1}^{N} \frac{\partial f}{\partial S_i} dS_i + \sum_{i,j=1}^{N} a_{ij} S_i S_j \frac{\partial^2 f}{\partial S_i S_j} dt$$

with $A = (a_{ij})$ as before. Note that

$$\frac{\partial f}{\partial S_i} = \mathsf{H}(\sum_{j=1}^{N} w_j S_j - K) w_i \ ,$$

where H denotes the Heaviside function given by $\mathsf{H}(s) = 1$ if $s > 0$ and zero otherwise. Furthermore,

$$\frac{\partial^2 f}{\partial S_i \partial S_j} = \delta(\sum_{j=1}^{N} w_j S_j - K) w_i w_j$$

Hence,

$$f(T) = f(t_0) + \sum_{i=1}^{N} \int_{t_0}^{T} \mathsf{H}(\sum_{j=1}^{N} w_j S_j - K) w_i S_i \mu_i dt$$

$$+\sum_{i,j=1}^{N}\int_{t_0}^{T}\mathsf{H}(\sum_{j=1}^{N}w_jS_j-K)w_i\sigma_{ij}dW_j+\sum_{i,j=1}^{N}\int_{t_0}^{T}\delta(\sum_{j=1}^{N}w_jS_j-K)a_{ij}S_iS_jw_iw_jdt$$

Now we take the expected value $E^*_{t_0}$ at time t_0 to get

$$C_w(t_0)=f(t_0)+\sum_{i=1}^{N}\int_{t_0}^{T}E^*_{t_0}\left[\mathsf{H}(\sum_{j=1}^{N}w_jS_j-K)w_iS_i\mu_i\right]dt \tag{A 1}$$

$$+\sum_{i,j=1}^{N}w_iw_j\int_{t_0}^{T}E^*_{t_0}\left[\delta(\sum_{j=1}^{N}w_jS_j-K)a_{ij}S_iS_j\right]dt.$$

In the sequel, we make use of the following

Lemma A.1 *Let $g:\mathbb{R}^N_+\to\mathbb{R}$. Then,*

$$\int_{\mathbb{R}^N_+}g(S)\delta(\sum_{j=1}^{N}w_jS_j-K)p(S_{t_0},t_0,S,t)dS=\frac{1}{|w|}\int_{L_w}g(S)p(S_{t_0},t_0,S,t)d\mathbf{s}.$$

Proof Let us define, in a similar way to that of Section 3, B and Q by

$$B=\sum_{i=1}^{N}w_iS_i,$$

and

$$Q_i=\frac{w_iS_i}{\sum_{i=1}^{N}w_iS_i}$$

for $i=1,\dots,N-1$. Then

$$\int_{\mathbb{R}^N_+}g(S)\delta(\sum_{j=1}^{N}w_jS_j-K)p(S_{t_0},t_0,S,t)dS=$$

$$=\int_K^\infty\int_{\Delta_N}g\left(S(Q,B)\right)\delta(B-K)\frac{B^{N-1}}{w_1w_2\dots w_N}dBdQ$$

$$=\int_{\Delta_N}g\left(S(Q,K)\right)p(S_{t_0},t_0,S,t)\frac{K^{N-1}}{w_1w_2\dots w_N}dQ$$

$$=\frac{1}{|w|}\int_{L_n}g(S)p(S_{t_0},t_0,S,t)dS$$

□

Back to Equation (A 1), we get

$$E^*_{t_0}\left[\delta(\sum_{j=1}^{N}w_jS_j-K)a_{ij}S_iS_j\right]=$$

$$= \int_{\mathbb{R}_+^N} a_{ij} S_i S_j \delta(\sum_{j=1}^N w_j S_j - K) p(S_{t_0}, t_0, S, t) dS$$

where, as before, p denotes the transition probability density. From the previous lemma,

$$E_{t_0}^*[\delta(\sum_{j=1}^N w_j S_j - K) a_{ij} S_i S_j] = \frac{1}{|w|} \int_{L_w} a_{ij} S_i S_j p(S_{t_0}, t_0, S, t) dS.$$

Furthermore,

$$E_{t_0}^*[\mathsf{H}(\sum_{j=1}^N w_j S_j - K) w_i S_i \mu_i] =$$

$$\int_{\mathbb{R}_+^N} \mu_i w S_{i,t} \mathsf{H}(\sum_{j=1}^N w_j S_j - K) p(S_{t_0}, t_0, S, t) dS.$$

On the other hand, upon computing the derivatives

$$\frac{\partial C_w}{\partial w_i} = \int_{\mathbb{R}_+^N} \mathsf{H}(\sum_{j=1}^N w_j S_j - K) S_i p(S_{t_0}, t_0, S, t) dS,$$

we deduce that

$$E_{t_0}^*[\mathsf{H}(\sum_{j=1}^N w_j S_j - K) w_i S_i \mu_i] = \mu_i w_i \frac{\partial C_w}{\partial w_i}.$$

Finally, from the identity

$$C_w(t_0) = f(t_0) + \sum_{i=1}^N \mu_i w_i \int_{t_0}^T \frac{\partial C_w}{\partial w_i} dt + \sum_{i,j=1}^N \frac{w_i w_j}{|w|} \int_{t_0}^T \int_{L_w} a_{ij} S_i S_j p(S_{t_0}, t_0, S, t) d\mathbf{s}\, dt,$$

we get

$$\frac{\partial C_w}{\partial T} = \sum_{i=1}^N \mu_i w_i \frac{\partial C_w}{\partial w_i} + \sum_{i,j=1}^N \frac{w_i w_j}{|w|} \int_{L_w} a_{ij} S_i S_j p(S_{t_0}, t_0, S, t) d\mathbf{s}.$$

Now, since

$$\frac{\partial^2 C_w}{\partial w_i \partial w_j} = \int_{\mathbb{R}_+^N} \delta(\sum_{j=1}^N w_j S_j - K) S_i S_j p(S_{t_0}, t_0, S, t) dS$$

$$= \frac{1}{|w|} \int_{L_w} S_i S_j p(S_{t_0}, t_0, S, t) d\mathbf{s},$$

we deduce once again that if the diffusion coefficients a_{ij} are deterministic functions depending only on time, then the generalized Dupire's equation (1.6) holds.

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