Lectures on Algebraic Quantum Field Theory and Operator Algebras

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Abstract. In this series of lectures directed towards a mainly mathematically oriented audience I try to motivate the use of operator algebra methods in quantum field theory. Therefore a title as "why mathematicians are/should be interested in algebraic quantum field theory" would be equally fitting.

Besides a presentation of the framework and the main results of local quantum physics these notes may serve as a guide to frontier research problems in mathematical physics with applications in particle and condensed matter physics for whose solution operator algebraic methods seem indispensable. The ultraviolet problems of the standard approach and the recent holographic aspects belong to this kind of problems.

Contents

1. Quantum Theoretical and Mathematical Background

The fact that quantum field theory came into being almost at the same time as quantum mechanics often lead people to believe that it is "just a relativistic version of quantum mechanics". Whereas it is true that both theories incorporated the general principles of quantum theory, the additional underlying structures, concepts and mathematical methods are remarkably different and this contrast manifests itself most visibly in the operator algebra formulation of local quantum physics (LQP) [1] whereas their use in quantum mechanics would be an unbalanced formalistic exaggeration. This distinction is less evident if one employs the standard quantization formulation which has close links with differential geometry.

Mathematicians who were exposed to the mathematical aspects of some of the more speculative ideas in contemporary high-energy/particle theory (supersymmetry, string theory, QFT on noncommutative spacetime), which despite their mathematical attraction were unable to make contact with physical reality (in some cases this worrisome situation already prevails for a very long time), often are not aware that quantum field theory (QFT) stands on extremely solid rocks of experimental agreements. To give one showroom example of quantum electrodynamics i. e. the quantum field theory of electrons/positrons and photons, the experimental and theoretical values of the anomalous magnetic moment of the electron relative to the Bohr magneton (a natural constant) μ_0 are

$$\begin{aligned} \left(\frac{\mu}{\mu_0}\right)_{exper} &= 1,001159652200(10) \\ \left(\frac{\mu}{\mu_0}\right)_{theor} &= 1,001159652460(127)(75) \end{aligned} \tag{1}$$

where the larger theoretical error refers to the uncertainty in the knowledge of the value for the fine-structure constant and only the second uncertainty is related to calculational errors in higher order perturbative calculations. The precision list can be continued to quantum field theoretic effects in atomic physics as the Lamb shift, and with somewhat lesser accuracy in the agreement with experiments may be extended to the electroweak generalization of quantum electrodynamics and remains qualitatively acceptable even upon the inclusions of the strong interaction of quantum chromodynamics.

Mathematician may even be less aware of the fact that only a few quantum field theoretician who have had their experience with the mathematical intricacies and conceptual shortcomings of the standard approach still believe that the present quantization approach (which uses classical Lagrangian and formal functional integrals as the definition of quantum electrodynamics (QED) or the standard electroweak model) has a mathematical existence outside perturbation theory[1], inspite of the mentioned amazing experimental agreement with perturbation theory. In fact there is hardly any theoretician who would be willing to take a bet about the mathematical existence of these Lagrangian models. Arguments to that extend are often presented in the physics literature by stating that "QED does not exist". Of course there is a theory involving electrons and photons (even if we presently do not know its correct mathematical description) and the critical arguments only go against the Lagrangian/functional quantization definition of the theory and not against the underlying principles of LQP which in fact developed to a large degree from ideas about QED.

[1] Perturbative QFT does not have the mathematical meaning of a well-defined object which is being perturbatively expanded, but is rather a formal deformation theory whose consistency does not imply anything about the existence of a possible associated QFT.

The cause of this critical attitude inspite of the overwhelming numerical success is twofold. On the one hand it is known that renormalized perturbation theory does not lead to convergent series in the coupling strength; rather the series is at best asmptotically convergent i.e. the agreement with experiments would worsen if one goes to sufficiently high perturbative orders (assuming that one could seperate out the contributions coming from interactions outside of QED). But there is also another more theoretical reason. To introduce interactions via polynomial pointlike coupling of free fields is pretty much ad hoc, i.e. if this recipe would not have worked, hardly anybody would have been surprised. In fact a sufficient intrinsic understanding of what constitutes interaction is the still missing cornerstone, even after 70 years of QFT. Only in recent years there have been serious attempts and partial successes on which we will comment in sections 3 and 4 of these notes.

Whereas in low spacetime dimensions (d=1+1, d=1 chiral models) the mathematical existence of interacting quantum field theories has been demonstrated by the presentation of certain controllable models, 4-dimensional local quantum physics beyond free systems has largely resisted attempts at demonstrating existence via construction of nontrivial models or otherwise.

This situation of having a perturbatively extremely successful description of particle physics whose existence as a bona fide QFT on the other hand has remained outside mathematical control is quite unique and in fact without parallel in the history of physics. But it should not be viewed as something embarrassing to be suppressed or covered by excuses ("there has to be a cutoff at the Plank length anyhow") because this situation is also the source of fascination and a great challenge; its enigmatic power should not be squandered. In the history of physics each conceptual framework (classical mechanics, classical field theory, statistical mechanics, quantum mechanics) was eventually shown to be mathematically consistent (usually by finding nontrivial models), i.e. the necessity of finding an incorporation into a more general framework was almost never coming from mathematical inconsistencies, but either from new experimental facts or from the theoretical merger of two different frameworks (example: relativity). If LQP build on Einstein causality and quantum principles should really turn out to be mathematically inconsistent, this would constitute a remarkable and enigmatic piece of insight which should be made visible and not covered behind cut-offs[2]. There is hardly any contrast in fundamental physics comparable to that between the verbal ease with which the word "nonlocal" is injected into discussions and on the other hand the conceptual problems faced in implementing nonlocality without destroying the whole fabric of an intrinsic interpretation of the formalism including the derivation of the all important scattering theory. For example string theory in its present formulation does not permit an intrinsic derivation of time-dependent scattering theory (rather the S-matrix is imposed by the Veneziano prescription).

Despite 50 years of attempts to render short distance properties more mild by ad hoc nonlocal/noncausal modific]ations each proposal has proven to cause more problems then it set out to solve [2] and it remains to be seen whether the proposals of achieving short distance improvements via noncommutativity of the spacetime localization pass the acid test of a complete physical interpretation which includes in particular the derivation of time dependent scattering theory.

The general message in the many failed attempts is that principles as causality and locality can not be overcome

[2]Of course the phenomenological parametrzation in terms of cut-off integrals is not effected by these remarks of how to deal with physcal principles.

by pedestrian ad hoc mathematical modifications, but rather require the discovery of more general physical principles of which they are limiting cases. The suggestion of the present LQP approach with respect to short distance problems is conservative or revolutionary depending on where one wants to put the emphasis; conservative in that it does not temper with the causality principles and revolutionary in that it views the short distance problems as an aspect of the limitation of the quantization method using "pointlike field coordinatizations" (akin to singular coordinates) and not part of the intrinsic frontiers set by the principles but only of their implementation.

In pursuit of this challenge there have been new and deep conceptual and mathematical inroads and investments over the last two decades; some of the older ones were described in [1]. The characteristic feature of those achievements obtained with operator algebra methods is, as already previously indicated, that they combine a revolutionary approach with respect to concepts and mathematical formalism with a conservative attitude concerning physical principles. In view of the fact that the very difficult and expensive high energy experiments did not reveal any indication of incompatibility with the general principles, this is a very reasonable procedure indeed.

Of course physicists need sometimes to move into the (following Feynman's saying) "blue yonder". But at times of poor experimental guidance, taking off without solid theoretical grounds under one's feet, such a trip may like that of the legendary flying dutchman end without finding a physical landing place over many generations.

I think that the framework of algebraic quantum field theory on which these lectures are based offers such a firm soil. In particular it provides a profound mathematical anchor to the concept of Einstein causality (and the closely related Haag duality) in the form of the Tomita-Takesaki modular theory of operator algebras. This is of course welcome because it maintains the radical nature of such important future projects as the approach to quantum gravity by elevating it to its deserved conceptual distinguished position.

In my contacts with mathematicians I often encountered a strong curiosity about the motivations and history of the physical concepts behind the various formalisms. In these lectures I will try to pay attention to this legitimate desire.

In the following I will give an exposition of some particle physics aspects of the operator algebra approach, but before I start to emphasize the differences to quantum mechanics, it is useful to present some concepts which actually originated there. We will freely use such acronyms as QM, QT, QFT, AQFT (algebraic QFT), LQP (local quantum physics) and OA (operator algebras).

1.1. Restrictions of the Superposition Principle: Superselection. It is helpful to recall briefly von Neumann world of QM and the changes it suffered subsequently. Von Neumann form of QM was basically the quantum mechanics of a single particle and its mathematical formulation which was that of Weyl's reformulation of Heisenberg's commutation relation in the unitary exponential form (for one degree of freedom)

$$\begin{aligned} [q,p] &= i\hbar \\ W(\alpha)W(\beta) &= e^{-i(\alpha,\beta)}W(\alpha+\beta) \\ W(\alpha) &= e^{i(\alpha_1 q+\alpha_2 p)},\ (\alpha,\beta)=\alpha_1\beta_2-\alpha_2\beta_1 \end{aligned} \tag{2}$$

This algebraic structure defines a unique C^*-algebra and the Stone-von Neumann theorem says that there is only one regular irreducible representation. As a consequence there is no loss of generality in the use of the Schroedinger representation where pure states are represented by vectors (modulo constant phase factors) in the Hilbert space of L^2-integrable wave functions[3] and mixed state with density matrices (positive trace-class operators). This simple situation leads to the standard Hilbert space setting of QT. As von Neumann pointed out, the irreducibility of the representation of the Heisenberg-Weyl algebra in which the observables correspond to Hermitian operators leads to the unrestricted superposition principle: with two vectors describing physically realizable states also their linear combinations are physically realizable (although in most cases, different from classical wave theory, one does not know from what source such a superposed state is produced). It was von Neumann who emphasized this pivotal conceptual difference of quantum mechanics from classical wave optics, a difference which even in modern textbooks is often squandered for some superficial calculational gains which this conceptually incorrect analogy offers.

Here it we should also recall another of von Neumann's contributions to quantum theory, namely his famous relation of commuting operators and commutant algebras to commensurability of measurements. The totality of operators which commute with a given set of observables forms a *weakly closed operator algebra* which in von Neumann's honor carries his name. They belong to the larger class of operator algebras which are closed with respect to the hermitian conjugation operation and the operator norm, whose abstract version (forgetting the Hilbert space) is called C^*-algebra. Although von Neumann algebras are special C^*-algebras it would not be appropriate to subsume them under the heading of C^*-algebras.

The von Neumann algebra generated by an irreducible representation of the Weyl algebra (the C^*-algebra of quantum mechanics) is $B(H)$, the algebra of all bounded operators in the Hilbert space H.

It gradually became clear that von Neumann's mathematical framework of quantum mechanics, which admitted unrestricted coherent superposition of state vectors (or equivalently gave the status of an observable to each selfadjoint operator), had to be amended in the presence of particles with different spin and of multiparticle states. There were certain superpositions of state vectors which cannot be physically realized for geometric reasons.

A historically famous example is that of Wick Wightman and Wigner [1]. They pointed out that if ψ_1 and ψ_2 are wave function of a particle with halfinteger respectively integer spin, their coherent superposition

$$\psi = \alpha_1\psi_1 + \alpha_2\psi_2 \tag{3}$$

which under the action of a 2π-rotation changes to

$$\psi' = -\alpha_1\psi_1 + \alpha_2\psi_2 \tag{4}$$

cannot carry a direct physical meaning since the expectation values of unrestricted observables from the algebra of all bounded operators in Hilbert space $B(H)$ are different in these two wave function. The observability of the relative phase in the change $\psi_1 \to e^{i\varphi}\psi_1$ of ψ, which is one of the most characteristic aspects of quantum theory, is prevented by the existence of a superselection rule: there is no observable A which can connect halfinteger and integer spins

$$\begin{aligned} &(\psi_1, A\psi_2) = 0\,\forall\, observables\, A \\ &\Longleftrightarrow E_\psi(A) \equiv (\psi, A\psi) = (\psi', A\psi') \equiv E_{\psi'}(A) \end{aligned} \tag{5}$$

[3] More precisely states in QM are identified with unit rays since the mutiplication if a vector with a phase factor does not change the physical expectation values.

In contrast to *selection rules* for e.g. electromagnetic transitions in atoms (relating final with initial spin of the atomic state) which suffer corrections in higher orders, these superselection rules are exact and therefore the prefix *super* has a sound physical meaning. In their presence the Hilbert space $\mathcal{H}$ becomes the direct sum of Hilbert $\mathcal{H}_i$ spaces inside which the unrestricted superposition principle holds

$$\mathcal{H} = \sum_i \mathcal{H}_i \tag{6}$$

The modern terminology is to call the labels i summarily (eigenvalues of) superselection charges (in the above example they are the $\pm$ univalence of spin). The algebra generated by the observables consists of a direct sum of algebras with no connecting operators between the different subspaces. The sum could in principle also be a continuous integral in which case the similarity with von Neumann's central decomposition into factors is not accidental.

Another related important example comes from the notion of identical particles in multiparticle state vectors ψ_N containing N particles. Permutations of particle labels in multiparticle states of identical particles can only change the representing state vector but not the associated physical state (the expectation values of observables) since they commute with the algebra $\mathcal{A}$ generated by the observables

$$[U(\sigma), \mathcal{A}] = 0,\, \sigma \in S_N \tag{7}$$
$$U(\sigma)\psi_N = \psi_N^\sigma,\, E_{\psi_N}(A) = E_{\psi_N^\sigma}(A)$$

The different irreducible representations of the permutation group S_N are conveniently depicted in terms of Young tableaus. The above commutation relations of permutations with observables which express the indistinguishability of identical particles impose a superselection rule between inequivalent representations belonging to different Young tableaus. The standard argument in the QM literature to explain that *only abelian permutation group representations* are realized in nature is based on a fallacious tautology: it uses tacitly Schur's lemma which is of course synonymous with the triviality of the commutant and hence with the claimed abelianess of the representations of permutations. A well-known mathematical counterexample is obtained by imagining a quantum mechanics in which the spin is not accessible to measurements[4]. In such a world of hidden spin degrees of freedom the spatial wave functions belong to different symmetry-types with orbital Young tableaus which are conjugate to the hidden N-particle spin Young tableaus (so that the antisymmetry refers to the tensor product space: orbitals$\otimes$spin). Since $s = \frac{1}{2}$ states N-particle belong to hight 2 tableaus, it is easy to see that the possible spin tableaus (and therefore also the conjugate orbital ones) are uniquely determined by the various values of the total spin. The energy eigenvalues of the different symmetry types are generally different even without a spin dependent interaction. In this example the nonabelian "parastatics" of the different contributing permutation group representations can of course be reprocessed back into ordinary Fermion statistics by reintroducing the spin multiplicity (which was lost by assuming nonobservability i.e. averaging over spin degrees of freedom). But the general statement, that it is always possible to convert parastatistics into Fermi/Bose statistics (plus multiplicities for an internal symmetry group to act on), is one of the most nontrivial theorems in particle physics. For its proof one needs the full power of the superselection theory in local quantum physics as well as some more recent group theoretical tools [3].

[4]Spin is of course a spacetime symmetry directly accessible to (Stern-Gerlach) experiments, whereas inner symmetries (isospin and generalizations) are not.

In local quantum physics[5] (LQP) superselected charges, despite their global aspects have a local origin which complies with the (Einstein-) causality and spectral stability principles; as in the classical theory of Maxwell and Einstein the Global originates from the Local. Therefore "Topological Field Theories" are not directly physical since they originate from LQP by ignoring localization aspects. The only way to interpret them physically is to remember the physical "flesh" of localization and transportability which was separated from these topological "bones". A good example is provided by the reference endomorphisms and the intertwiner formalism of the next section (and the ensuing Markov traces on the infinite braid group which have a natural extension to the mapping class group and 3-manifold invariants). In fact topological field theories are only topological from a differential geometric viewpoint, whereas in LQP the terminology "combinatorial" would be more appropriate.

In fact one could define LQP as being the theory of spacetime dynamics of local densities of superselected charges. It turns out that the localized version of these charges constitute the backbone of the observable algebra which, as we will see in more detail in the next section, is described by a map (a net) of spacetime regions into C^*-algebras. Nothing turns out to be lost if we define this algebra in terms of its vacuum representation i.e. as a map into concrete operator algebras in a common Hilbert space which contains a distinguished vacuum vector. On the present level of understanding of observable nets there is also no loss if we assume that the individual spacetime-indexed operator algebras are weakly closed i.e. are von Neumann algebras. In fact the physically admissable representations of this algebra are just the localized representations of the observable vacuum net.

Apart from the above quantum mechanical superselection rules of multiparticle statistics and the before mentioned Wick-Wightman-Wigner univalent spin superselection rules, the only other mechanism for encountering interesting superselection sectors in QM is by leaving the setting of Schroedinger quantum mechanics and admitting topologically nontrivial configuration spaces. Take for example quantum mechanics on a circle (instead of a line) or in more physical terms the Aharonov-Bohm effect i.e. the quantum physics in the vector potential associated with a stringlike idealized solenoid generating a δ-function-like magnetic flux of strength θ through the x-y plane in the z-direction of the solenoid (a situation which shares the same nontrivial topology with the circle). In this case the quantum mechanics is not unique but rather depends on an angle θ which in the A-B case has the physical interpretation of the magnetic strength. The intuitive reason (which can be made mathematically rigorous) is that if one realizes the maximal abelian subalgebra generated by the multiplication operator $x\,mod\,2\pi$ (the angular coordinate) on the space of periodic wave functions $L^2(S^1)$ then the canonical conjugate p is equal to $-i\partial_x$ plus an operator which commutes with an irreducible system x and, p and hence is central-valued and therefore a numerical constant say θ in each irreducible representation, so that $p = -i\partial_x + \theta$. If the x-space would be the real axis, the θ can be transformed away by a nonsingular unitary "gauge transformation" but the topology prevents its elimination in this case. An equivalent formulation would be to keep the x and p as in the Schroedinger representation and to encode the θ as a quasiperiodicity angle into wave functions (which hereby turn into trivializing wave sections in a complex line bundle on the circle). But the way which would suit our present purpose (which was to find illustrations of superselection rules in QM) best would be to encode the quantum mechanics on the circle into

[5]It should be clear by now that we prefer to use the terminology LQP instead of QFT whenever we want the reader not to think primarily about the standard text-book formalism (Lagrangian quantization, functional integrals) but rather about the underlying physical principles and alternative conceptually more satisfactory implementations [1].

an abstract (θ-independent) C^*-algebra and have the θ appear as a representation label for the various irreducible Hilbert space representations of this algebra. This can be done, but we will not pursue his matter, since our main interest is QFT which offers a quite different mechanism for obtaining superselection-sectors as equivalence classes of irreducible representations of C^*algebras, namely the mechanism of infinite degrees of freedoms which will be illustrated in the sequel.

Let us look at a standard example (following [9]) which requires only a modest amount of concepts from physics namely spins on a linear lattice, or mathematically a tensor product of an arbitrary large number N of two-dimensional matrix-algebras $Mat_2(\mathbb{C}) = alg\left\{\sigma_i, \mathbf{1}|i = 1,2,3\right\}$; here the right hand side is the physicists's notation for this complex algebra in terms of the three hermitian Pauli matrices which together with the identity form a linear basis of the space $Mat_2(\mathbb{C})$. The pure states on this algebra are described in terms of the unit rays associated with Hilbert space vectors in a two-dimensional Hilbert space H_2. In this simple case one even has an explicit parametrization of all density matrices (mixed states) in terms of a 3-dim. unit ball $\vec{n}^2 \leq 1$

$$\omega(A) = \frac{1}{2}Tr(\mathbf{1} + \vec{n}\vec{\sigma})A,\ A \in Mat_2(\mathbb{C}) \tag{8}$$

where the pure states reside on the surface of the unit ball. The object of our interest is the tensor product algebra

$$\mathcal{A}_N = \otimes^N Mat_2(\mathbb{C}) = Mat_{2^N}(\mathbb{C}) \tag{9}$$

acting irreducibly on the tensor Hilbert space $H_{2^N} = \otimes^N H_2$. Since we are interested in the "thermodynamic" (inductive) limit, we first define the infinite dimensional Hilbert space in which the limiting algebra can act as an operator algebra

$$H = \left\{ \sum_s c(s)\left|s\right\rangle \mid \sum_s \left|c(s)\right|^2 < \infty,\, s : \mathbb{Z} \to \{\pm 1\} \right\} \tag{10}$$

i.e. we choose a (nonseparable) Hilbert space spanned by a basis of binary sequences with the generators of the algebra being Pauli-matrices labeled by the points of the linear chain $x \in \mathbb{Z}$

$$\begin{aligned} \sigma_3(x)\left|s\right\rangle &= s(x)\left|s\right\rangle \\ \sigma_1(x)\left|s\right\rangle &= \left|s'\right\rangle \\ \sigma_2(x)\left|s\right\rangle &= is(x)\left|s'\right\rangle \end{aligned} \tag{11}$$

with $s'(y) = s(y)\forall y$ except for $y = x$ where $s'(x) = -s(x)$. Although the limiting Hilbert space is nonseparable, the inductive limit algebra $\mathcal{A}$ remains separable and simple (no ideals) even after its uniform closure (this is a rather general property of inductively defined limiting C^*-algebras); a basis is given by

$$\sigma_{k_1}(x_1)...\sigma_{k_N}(x_j),\ x_1 < ... < x_k \tag{12}$$

i.e. a product of Pauli-matrices at finitely many chain pointswhich acts on H according to (11). The big nonseparable Hilbert space decomposes into a noncountable sum of separable Hilbert spaces that are invariant under the $\mathcal{A}$-action

$$\begin{aligned} H &= \oplus_{[s]} H_{[s]} \\ H_{[s]} &= \overline{\mathcal{A}\left|s\right\rangle} \end{aligned} \tag{13}$$

where one equivalence class evidently consists of binary sequences which have common two-sided "tails" $s \sim s'$ if $s(x) = s'(x)$ for sufficiently large $x \in \mathbb{Z}$. By applying $\mathcal{A}$ to a vector in $H_{[s]}$ one cannot change the two-sided "tail" i.e.

$$\langle s' \left| A \right| s \rangle = 0, \ \forall A \in \mathcal{A}, \ [s] \neq [s'] \tag{14}$$

A physicist would use the magnetization to show the presence of superselection rules, e.g. he would consider the sequence of averaged magnetization in a piece of the chain of size 2n

$$M_n = \frac{1}{2n+1} \sum_{x=-n}^{n} \sigma_3(x) \tag{15}$$

As a result of the decreasing pre-factor in the magnetization M_n commutes for n$\rightarrow \infty$ with any basis element of $\mathcal{A}$. Since vectors $|\psi_\pm\rangle \in H_{[s_\pm]}$ $s_\pm(x) = \pm 1, \forall x$ are eigenvectors of M_n with eigenvalues ± 1 we have

$$\begin{aligned} &\langle \psi_+ \left| A \right| \psi_- \rangle = lim_{n\rightarrow\infty} \langle \psi_+ \left| M_n A \right| \psi_- \rangle = \\ &lim_{n\rightarrow\infty} \langle \psi_+ \left| A M_n \right| \psi_- \rangle = - \langle \psi_+ \left| A \right| \psi_- \rangle \\ &\curvearrowright \langle \psi_+ \left| A \right| \psi_- \rangle = 0 \end{aligned} \tag{16}$$

This illustration shows clearly the mechanism by which infinitely many degree's of freedom generate superselection rules: there a too many different configurations at infinity which cannot be connected by operators from the quasilocal $\mathcal{A}$ (the uniform limit of local operations); whereas the $\pm$ magnetization (and certain others as alternating anti-ferromagnetic states) sectors with respect to a chosen spin quantization direction (in our case the 3-direction) have a clearcut physical meaning and can take on the role of ground state vectors of suitably chosen dynamical systems with (anti)ferromagnetic interactions, this is not the case for most of the myriads of other sectors.

This also makes clear that infinite degrees of freedom in quantum systems do not only lead to inequivalent representations, but also that there are far too many of them in order to be physically relevant. One needs a selection principle as to what states are of physical interest. In particle physics Einstein causality of the observable algebra and a suitable definition of localization of states relative to a reference state (the vacuum) constitute the cornerstone of what is often appropriately referred to as "Local Quantum Physics" (LQP). How to prove theorems and derive significant results in such a framework will be explained in the second lecture. Whereas the topological superselections in QM which we illustrated by the Aharonov-Bohm model (mathematically the QM on a circle) are the result of an over-idealization (infinitely thin stringlike solenoids, in order to reach mathematical simplicity[6]), the so-called vacuum polarization nature of the all-pervading vacuum reference state makes the infinite degree of freedom aspect of LQP an immutable physical reality. In the scaling limit of LQP which leads to conformal quantum field theory, there is also a topological aspect which enters through the compactification of Minkowski spacetime and the ensuing structure of the algebra (see the third section). But this mechanism is quite different and more fundamental than the over-idealizations of the Aharonov-Bohm solenoid. The typical situation is that the C^*-algebra $\mathcal{A}$ describing the observables of a system in LQP has a denumerable set of superselection sectors

[6]The problem with a "physical" solenoid of finite extension (which creates a less than perfect homogeneous magnetic field with a small contribution outside the solenoid) is a complicated boundary value problem *within the Schroedinger theory* and hence subject to the Stone- von Neumann uniqueness. In the idealized limit the field strength becomes encoded into the topological θ-angle.

and that the full Hilbert space which unites all representations is a direct sum

$$H = \bigoplus H_i \tag{17}$$

This decomposition is reminiscent of the decomposition theory of group algebras of compact groups apart from the fact that for the superselection sectors in LQP there exist "natural" intertwining operators which transfer superselection charges and connect the component spaces H_i (this has no analog in group theoretic representation theory). They are analogous to creation and annihilation operators in Fock space which intertwine the different N-particle subspaces.

Even though they themselves are not observables, they are nevertheless extremely useful. The best situation one can hope for is to have a full set of such operators which create a field algebra $\mathcal{F}$ in with no further inequivalent representations i.e. in which all representation labels ("charges") have become inner (charges within the field algebra). Note that the word "field" in this context does not necessarily refer to pointlike operator-valued distributions [11] but rather to the charge-transfer aspect of charge carrying operators which intertwine between different superselection sectors wheras the observables by definition stay in one sector.

Doplicher and Roberts [3] proved that observable algebras in four spacetime dimensions indeed allow a unique construction (after imposing the conventions of "normal commutation" relations between operators carrying different superselected charges [11]) of such a field algebra $\mathcal{F}$ from its observable "shadow" $\mathcal{A}$. This (re)construction of $\mathcal{F}$ from $\mathcal{A}$ is therefore reminiscent of Marc Kac's famous aphorism about a mathematical inversion problem which goes back to Hermann Weyl namely: "how to hear the shape of a drum?" The most startling aspect of their result is that the inclusion[7] of the two nets $\mathcal{A} \subset \mathcal{F}$ (apart from low-dimensional QFT) is completely characterized by the category of compact groups and that for each compact topological group there is a pair $(\mathcal{A} \subset \mathcal{F})$ within the setting of LQP which has the given group G as the fixed point group of $\mathcal{F}$

$$\mathcal{A} = \mathcal{F}^G \tag{18}$$

This equality also holds for each algebra in the net i.e. $\mathcal{A}(\mathcal{O}) = \mathcal{F}^G(\mathcal{O})$.

This observation goes a long way to *de-mystify the concept of inner symmetries* ("inner" in the physicists sense refers to symmetries related to superselected charges which commute with spacetime symmetries, whereas operator algebraist use inner/outer for unitarily implementable/nonimplementable auto- or endo- morphisms) which originated from Heisenberg's phenomenological introduction of isospin in nuclear physics and played a pivotal role through its group action on multiplicity indices of multicomponent Lagrangian fields. What results is an (presumably even for mathematicians) unexpected new road to group theory and group representations [3] via a "group dual" which is very different from that of the well-known Tanaka-Krein theory. The DR form of the group dual in turn emerges from the DHR superselection sector analysis, i.e. from an input which consists only of Einstein causality (known to mathematicians through its classical manifestation from relativistic wave propagation in the context of partial differential equation) and spectral stability (energy positivity).

In LQP of lower spacetime dimensions the rigid separation between spacetime and inner symmetries looses

[7] Both algebras consist of spacetime indexed subalgebras (nets) and for each there is an inclusion $\mathcal{A}(\mathcal{O}) \subset \mathcal{F}(\mathcal{O})$.

its meaning[8] and one is entering the realm of subfactor theory of V. Jones (which in a certain sense constitutes an extension of group theory), with braid group statistics being the main physical manifestation. Again a short interlude concerning the physicists use of the word "statistics" may be helpful. Historically the main physical manifestation of the difference between spacelike commuting Boson and anticommuting Fermi fields (or between (anti)symmetrized multiparticle tensor product spaces) was observed in the thermodynamical behavior resulting from statistical ensembles of such particles i.e. from their statistical mechanics. Even in situations in which there is no statistical mechanics involved, physicist continue to use the word "statistics" for the characterization of commutation relations of the fields which describe those particles.

Here it is helpful to remember a bit of history on the mathematical side. Group theory originated from Galois studies of inclusion of a (commutative) number field into an extended field (extended by the roots of a polynomial equation). The new subfactor theory in some way generalizes this idea to inclusions of particular families of nonabelian algebras. At the threefold junction between abstract quantum principles, the geometry of spacetime and inner symmetries stands one of the most startling and impressive mathematical theories: the Tomita-Takesaki modular theory, which in the more limited context of thermal aspects of open quantum systems was independently discovered by physicists [4]. It is, according to the best of my knowledge, the only theory capable to convert abstract domains of quantum operators and ranges of operator algebras within the algebra of all bounded operators $B(H)$ of the underlying Hilbert space into geometry and spacetime localization; and although this is not always obvious, this is also behind the geometrical aspects of subfactor theory.

The standard Hilbert space setting which one learns in a course of QM is only a sufficient tool if the algebraic structure of observables allows for only one (regular) representation as the p's and q's (encoded into the Weyl C*-algebra of QM). Whereas it is possible to present the standard particle theory in terms of computational recipes in this restricted setting (as it is in fact done in most textbooks), the extension of LQP into yet unexplored directions of particle physics requires a somewhat broader basis, including additional mathematical concepts.

In passing we mention that superselection rules also play a role in an apparently quite different area of fundamental quantum physics. It is commonly accepted that the Schrödinger cat paradox of QM (which is a dramatic setting of the von Neumann "reduction of the wave packet" dictum) becomes more palatable through the idea of a decoherence process in time which is driven by an environment i.e. there is a transition process in time from a pure quantum state of the object to a mixed (in a classical sense) state, and that for all practical purposes the coherence in the superposition $\psi_{live} + \psi_{dead}$ will have been lost in the limit of infinite time as a result of interactions with the infinite degrees of freedom of the environment. Ignoring the environment and the decoherence time, this takes on the form of the von Neumann wave packet collapse

$$\psi \overset{collapse}{\longrightarrow} \left|P\psi\right\rangle \left\langle P\psi\right| + \left|(1-P)\,\psi\right\rangle \left\langle (1-P)\,\psi\right| \underset{reading}{\overset{pointer}{\longrightarrow}} \begin{cases} P\psi \, with \; prob. \; \|P\psi\|^2 \; or \\ (1-P)\,\psi \, with \; prob. \; \|(1-P)\,\psi\|^2 \end{cases} \tag{19}$$

where for simplicity we assumed that the observable is a projection operator P. The mixed state on the right hand side has been represented as a mixture of two orthogonal components, but unlike a pure state a mixture has myriads of other (nonorthogonal) representations and it is not very plausible that in addition to the collapse into

[8]Whereas in higher dimensions ($d \geq 1+3$) with B/F statistics inner (compact group) and outer (spacetime) symmetries cannot be nontrivially "married" [1], in low dimensions they cannot be unequivocally "divorced".

a mixed state, the measurement also selects that orthogonal representation of the mixture among the myriads of other possibilities. Rather one believes that the measurement causes an interaction with the infinite degrees of freedom of the environment (the finite degrees of freedom of a quantum mechanical object may itself be the result of an idealization) and that what is observed at the end is a mixture of superselected states of the big system. Since the central decomposition in contrast to the aforementioned orthogonal mixture has an unambiguous classical meaning, this is the more satisfactory explanation of the decoherence process at infinite times [5]. Whereas it is quite easy to write down unitary time propagators which become isometric in the limit (or positive maps which generate mixtures already in finite time), it is a much more difficult task to create a realistic infinite degree of freedom model which describes the above features of a measurement process in a mathematically controllable way.

There is as yet no agreement about whether the full system (including the spacetime of the part of the universe which is spacelike separated from the laboratory during the finite duration of the measurement) remains in a pure state or undergoes a "reduction" into a mixture; it is not even clear whether this question belongs still to physics or is of a more philosophical nature. One idea which however resisted up to now a good conceptual and mathematical understanding is the possible existence of a more complete form of quantum theory which in addition to the standard dynamics also contains a process of "factualization" of events i.e. an interface between the potentiality of the standard (Copenhagen) interpretation and the *factuality of observed spacetime localized events* on a very basic level [1]. It seems that some of the oldest problems in quantum theory related to the quantum mechanical measurement process is still very much alive and that as a result of the importance of an infinite degree of freedom environment and locality, the theory of local quantum physics (which automatically generates the omnipresent environment of the spacelike separated infinite degree of freedom via the vacuum polarization property) may yet play an important role in future investigations.

It should be clear from this birds eye view of motivation and content of LQP that the properly adapted Einstein causality concept, which in its classical version originated at the beginning of the last century, still remains the pillar of the present approach to particle physics. With its inexorably related vacuum polarization structure and the associated infinite number of degrees of freedom it has given LQP its distinct fundamental character which separates it from QM. Despite its startling experimentally verified predictions and despite theoretical failures of attempts at its nonlocal modifications, its conceptual foundations for 4-dimensional interacting particles are presently still outside complete mathematical control.

1.2. Appendix A: The Superselection Sectors of CG. As a mathematical illustration of superselection rules we are going to explain the representation theory of (finite) group algebras using the setting of superselection sectors. In this way the reader becomes acquainted with the present notation and mode of thinking for a situation he may have already encountered in a different way.

Let G be a (not necessarily commutative) finite group. We affiliate a natural $\mathbf{C}^*$-algebra, the group-algebra $\mathbf{C}G$ with G in the following way:

- (i) The group elements g$\in G$ including the unit e form the basis of a linear vectorspace over $\mathbf{C}$:

$$x \in \mathbf{C}G, \quad x = \sum_g x(g)g\,, \quad with\ x(g) \in \mathbf{C} \tag{20}$$

- (ii) This finite dimensional vector space $\mathbf{C}G$ inherits a natural convolution product structure from G:

$$\left(\sum_{g\in G} x(g)g\right)\cdot\left(\sum_{h\in G} y(h)h\right)=\sum_{g,h\in G} x(g)y(h)g\cdot h=\sum_{k\in G} z(k)k \tag{21}$$

$$with\ z(k)=\sum_{h\in G} x(kh^{-1})y(h)=\sum_{g\in G} x(g)y(g^{-1}k)$$

- (iii) A *-structure, i.e. an antilinear involution:

$$x\rightarrow x^*=\sum_{g\in G} x(g)^* g^{-1}\ ,\quad i.e.x^*(g)=x(g^{-1})^* \tag{22}$$

Since :

$$\left(x^*x\right)(e)=\sum_{g\in G}\left|x(g)\right|^2\geq 0,\quad (=iff\quad x=0) \tag{23}$$

this *- structure is nondegenerate and defines a positive definite inner product:

$$(y,x)\equiv(y^*x)(e)$$

- (iv) The last formula converts $\mathbf{C}G$ into a Hilbert space and hence, as a result of its natural action on itself, it also gives a C^* norm (as any operator algebra):

$$||x||=\sup_{\|y\|=1}\|xy\|\,,\qquad C^*-condition:\ ||x^*x||=||x^*||\,||x|| \tag{24}$$

A $\mathbf{C}^*-$norm on a *-algebra is necessarily unique (if it exists at all). It can be introduced through the notion of spectrum.

It is worthwhile to note that (iii) also serves to introduce a tracial state on $\mathbf{C}G$ i.e. a positive linear functional φ with the trace property:

$$\varphi\left(x\right):=x(e),\quad \varphi(x^*x)\geq 0,\ \ \varphi(xy)=\varphi(yx) \tag{25}$$

This state (again as a result of (iii)) is even faithful, i.e. the scalar product defined by:

$$(\hat{x},\hat{y}):=\varphi(x^*y) \tag{26}$$

is nondegenerate. On the left hand side the elements of $\mathbf{C}G$ are considered as members of a vector space. The nondegeneracy and the completeness of the algebra with respect to this inner product (a result of the finite dimensionality of $\mathbf{C}G$) give a natural representation (the regular representation of $\mathbf{C}G$) on this Hilbert space:

$$x\hat{y}:=\widehat{xy} \tag{27}$$

The norm of these operators is identical to the previous one.

This construction of this "regular" representation λ_{reg} from the tracial state on the $\mathbf{C}^*$-group-algebra is a special case of the general Gelfand-Neumark-Segal (GNS-)construction presented in a later section.

Returning to the group theoretical structure, we define the conjugacy classes K_g and study their composition properties.

$$K_g := \left\{hgh^{-1}, h \in G\right\} \tag{28}$$

In particular we have K_e=$\{e\}$. These sets form disjoint classes and hence:

$$G = \cup_i K_i, \;\; |G| = \sum_{i=0}^{r-1} |K_i|\,, \quad K_e = K_0, \;\; K_{1,}....K_{r-1}, \quad r = \#classes \tag{29}$$

We now define central "charges":

$$Q_i := \sum_{g \in K_i} g \,\in \mathcal{Z}(\mathbf{C}G) := \{z,\, [z,x] = 0 \quad \forall x \in CG\} \tag{30}$$

It is easy to see that the center $\mathcal{Z}(\mathbf{C}G)$ consists precisely of those elements whose coefficient functions $z(g)$ are constant on conjugacy classes i.e. $z(g) = z(hgh^{-1})$ for all h. The coefficient functions of Q_i:

$$Q_i(g) = \begin{cases} 1 \;\; if \;\; g \in K_i \\ 0 \;\; otherwise \end{cases} \tag{31}$$

evidently form a complete set of central functions. The composition of two such charges is therefore a linear combination of the r independent $Q_i's$ with positive integer-valued coefficients (as a result of the previous formula (30)):

$$Q_i Q_j = \sum_l N_{ij}^l Q_l \tag{32}$$

The fusion coefficients N can be arranged in terms of r commuting matrices

$$\mathbf{N}_j, \;\; with \;\; (\mathbf{N}_j)_i^l = N_{ij}^l \tag{33}$$

The associativity of the 3-fold product QQQ is the reason for this commutativity, whereas the the N_j would by symmetric matrices iff the group itself is abelian.

Functions on conjugacy classes also arise naturally from characters χ of representations π

$$\chi^\pi(g) = Tr\pi(g)\,, \quad \chi^\pi(g) = \chi^\pi(hgh^{-1}) \tag{34}$$

This applies in particular to the previously defined left regular representation λ with $(\lambda_g x)(h) = x(g^{-1}h)$. Its decomposition in terms of irreducible characters goes hand in hand with the central decomposition of $\mathbf{C}G$:

$$\mathbf{C}G = \sum_l P_l \mathbf{C}G, \qquad Q_i = \sum_l Q_i^l P_l \tag{35}$$

The central projectors P_l are obtained from the algebraic spectral decomposition theory of the $Q_i's$ by inverting the above formula. The "physical" interpretation of the coefficients is: $Q_i^l = \pi_l(Q_i)$ i.e. the value of the i^{th} charge in the l^{th} irreducible representation. The P_l are simply the projectors on the irreducible components contained in the left regular representation. Since any representation of G is also a representation of the group algebra, every irreducible representation must occur in $\lambda_{reg}(\mathbf{C}G)$. One therefore is supplied with a complete set of irreducible representations, or in more intrinsic terms, with a complete set of r equivalence classes of irreducible representations. As we met the intrinsic (independent of any basis choices) fusion rules of the charges, we now encounter the intrinsic fusion laws for equivalence classes of irreducible representations.

$$\pi_k \otimes \pi_l \simeq \sum_m \tilde{N}_{kl}^m \pi_m \tag{36}$$

Whereas the matrix indices of the $N's$ label conjugacy classes, those of $\tilde{N}$ refer to irreducible representation equivalence classes. The difference of these two fusions is typical for nonabelian groups and corresponds to the unsymmetry of the character table: although the number of irreducible representations equals the number of central charges (= # conjugacy classes), the two indices in $\pi_l(Q_j)$ have a different meaning. With an appropriate renormalization this mixed matrix which measures the value of the j^{th} charge in the l^{th} representation we obtain the unitary *character matrix* $S_{lj} \equiv \sqrt{\frac{|K_j|}{|G|}} Tr\pi_l(g_j)$ (Tr is the normalized trace) which diagonalizes the commuting system of $N's$ as well as $\tilde{N}'s$:

$$\frac{S_{kj}}{S_{0j}}\frac{S_{lj}}{S_{0j}} = \sum_m \tilde{N}_{kl}^m \frac{S_{mj}}{S_{0j}}, \tag{37}$$

$$\frac{\sqrt{|K_j|}S_{ki}}{S_{k0}}\frac{\sqrt{|K_b|}S_{kj}}{S_{k0}} = \sum_c N_{ij}^c \frac{\sqrt{|K_c|}S_{kc}}{S_{k0}}$$

The surprise is that S shows up in two guises, once as the unitary which diagonalizes this $\tilde{N}_l(N_b)$-system, and then also as the system of eigenvalues $\frac{S_{la}}{S_{0a}}$ $(\frac{S_{kb}}{S_{k0}})$ which can be arranged in matrix form. We will not elaborate on this point. In section 3.2 we will meet an analogous situation outside of group theory which is symmetric in charge- and representation labels i.e. $N = \tilde{N}$ and S is a symmetric matrix,yet the composition of representations is not commutative.

In passing we mention that closely related to the group algebra $\mathbf{C}G$ is the so-called "double" of the group (Drinfeld):

$$D(G) = C(G) \bowtie_{ad} G \tag{38}$$

In this crossed product designated by $\bowtie$, the group acts on the functions on the group $C(G)$ via the adjoint action:

$$\alpha_h(f)(g) = f(h^{-1}gh) \tag{39}$$

The dimension of this algebra is $|G|^2$ as compared to dim$\mathbf{C}G = |G|$. Its irreducible representations are labeled by pairs $([\pi_{irr}], K)$ of irreducible representation and conjugacy class and therefore their matrices N and S are *selfdual.* In this sense group doubles are "more symmetric" than groups. In chapter 7 we will meet selfdual matrices S which cannot be interpreted as a double of a group and which resemble the S of abelian groups.

Returning to the regular representation we notice that the equivalence classes of irreducible representations appear with the natural multiplicity:

$$mult(\pi_l \ in \ \lambda_{reg}) = \ dim\pi_l \tag{40}$$

The results may easily be generalized to compact groups where they are known under the name of Peter-Weyl theory.

Since group algebras are very special, some remarks on general finite dimensional algebras are in order.

Any finite dimensional C^*-algebra $\mathcal{R}$ may be decomposed into irreducible components, and any finite dimensional irreducible C*-algebra is isomorphic to a matrix algebra $Mat_n(\mathbf{C})$. If the irreducible component $Mat_{n_i}(\mathbf{C})$ occurs with the multiplicity m_i , the algebra $\mathcal{R}$ has the form is isomorphic to the following matrix algebra:

$$\mathcal{R} = \bigoplus_i Mat_{n_i}(\mathbf{C}) \otimes 1_{m_i} \quad in \;\; \mathcal{H} = \oplus_i \mathcal{H}_{n_i} \otimes \mathcal{H}_{m_i} \tag{41}$$

and the multiplicities are unrelated to the dimensionalities of the components. The commutant of $\mathcal{R}$ in $\mathcal{H}$ is:

$$\mathcal{R}' = \oplus_i 1_{n_i} \otimes Mat_{m_i}(\mathbf{C}), \qquad Z := R \cap R' = \oplus_i \mathbf{C}{\cdot}1_{n_i} \otimes 1_{m_i} \tag{42}$$

The last formula defines the center.

1.3. Appendix B: Some Operator Algebra Concepts. Since operator algebras still do not quite belong to mainstream mathematics, it would be unrealistic for me to assume that a mainly mathematical audience is familiar with allthe mathematical concepts which I will use in these lectures. Whereas mathematicians usually have a stock of very profound knowledge which covers a rather small specialized region, the mathematical physicists knowledge tends to be better described by the Fouriertransform of the former. This is because a theoretical physicist cannot indulge in the luxury of being highly mathematically selective. Contrary to mathematicians he has to live at least part of his life with half-truths (however without ever loosing the urge to convert them into full truths). If one wants to understand the physical nature one has to be prepared for the unexpected and to create a large supply of mathematical knowledge for all potential future physical applications would be totally unrealistic. Unsolved problems in mathematical physics often cannot be that clearly formulated as e.g. Hilbert formulated the important mathematical problems at the beginning of last century. Only if a promising new theoretical Ansatz has been found, a physicist is willing to learn and invest in depth into the appropriate mathematics. For quantum theory this has occurred a long time ago and therefore the majority of mathematical physicist know Hilbert space theory and even a bit about operator algebras.

The fact that in recent decades Fields medals have been twice awarded to operator-algebra related work (Alain Connes, Vaughn Jones) shows however that this area is receiving an increased attention and recognition within mathematics.

The following collection of definitions and theorems are not to be confused with a mini-course on operator algebras. Their only purpose is a reminder of the kinds of objects I will use and to urge the uninitiated reader to consult some of the existent literature on the subject.

The objects to be represented are C*-algebras $\mathcal{A}$ i.e. normed (Banach) algebras with an antilinear involutive *-operation with the following consistency relations between them

$$\begin{aligned} \|A^*\| &= \|A\|\,,\;\; A \in \mathcal{A} \\ \|A^*A\| &= \|A\|^2 \end{aligned} \tag{43}$$

It is a remarkable fact that the norm of a *-algebra (which is already complete in that norm) is unique and solely determined by its algebraic structure namely through the formula

$$\|A\| = inf\left\{\rho \in \mathbb{R}_+,\; A^*A - r^2\mathbf{1} \text{ is invertible in } \mathcal{A}\;\forall r < \rho\right\} \tag{44}$$

Since only the subalgebra generated by A^*A and 1 is used in this definition, embedding of C^*-algebras into larger ones are automatically isometric and homomorphisms ϕ are automatically contracting i.e. $\|\phi(A)\| \leqslant \|A\|$.

A mathematical physicist often prefers the more concrete illustration of a C*-algebra as a norm-closed operator algebra in a Hilbert space. The following two theorems show that this is no restriction of generality

Theorem 1. *Every commutative unital C*-algebra is isomorphic to the multiplication algebra of continuous complex-valued functions on an* $L^2(\mathcal{M},\mu)$ *where* $\mathcal{M}$ *is an appropriately chosen measure space with measure* μ.

Theorem 2. *Every C*-algebra is isomorphic to a norm-closed *-algebra of operators in a Hilbert space*

The space $\mathcal{M}$ in the first theorem is the space of maximal ideals (closely related to the notion of spectrum which is related to the complement of the above notion of invertibility) of $\mathcal{A}$.

For the second theorem the concept of a state ω on $\mathcal{A}$ is important.

Definition 3. *A state* ω *on* $\mathcal{A}$ *is a linear functional on* $\mathcal{A}$ *with the following properties*

(1) $\omega(A^*A) \geqslant 0 \ \ \forall A \in \mathcal{A}$ *(positivity)*

(2) $\omega(\mathbf{1}) = 1$ *(normalization)*

The relation $|\omega(A)| \leq |A|$ and hence continuity is an easy consequence of this definition; this inequality also allows to introduce a norm $\|\omega\|$ on the state space of a C^*-algebra.

Each state has a representation associated with it and the canonical construction which establishes this relation is called the GNS construction (after Gelfand, Neumark and Segal). It basically consists in using the positivity for defining an inner product in a linear space (defined by the algebra)

$$(A,B)_\omega \equiv \omega(A^*B) \tag{45}$$

It is obvious that for a faithful state (no null ideal) this would be a positive definite inner product and on its Hilbert space closure one can define a faithful representation with $\mathbf{1}$ a cyclic vector $|\mathbf{1}\rangle$ for the representation

$$\pi(A)\,|B\rangle = |AB\rangle \tag{46}$$

The presence of a null ideal $\mathcal{N}$ requires to construct the Hilbert space from equivalence classes A mod $\mathcal{N}$ which forces the positive semidefinite inner product to become positive definite. Vice versa a cyclic representation with a distinguished cyclic vector (in physics usually the vacuum, a ground- or a KMS thermal- state vector) ψ_0 defines via $\omega_{\psi_0}(A) = \langle\psi_0\,|A|\,\psi_0\rangle$ a state.

The convex set $S(\mathcal{A})$ of states on $\mathcal{A}$ is a subset of the C* dual $\mathcal{A}^*$ and

$$\begin{aligned} &S_\pi(\mathcal{A}) \equiv \{\omega \in S(\mathcal{A})|\,\exists\rho \in B(H), \omega(A) = Tr\rho\pi(A)\} \\ &\rho \ density\ matrix\ \ i.e.\, \rho \geq 0,\ Tr\rho = 1 \end{aligned} \tag{47}$$

is a norm-closed subset of $S(\mathcal{A})$ called the folium associated with π. Evidently $S(\mathcal{A})$ decomposes into disjoint folia.

Whereas the notions of irreducibility/factoriality of a representation π as well as unitary equivalence $\pi_1 \simeq \pi_2$, and disjointedness $\pi_1 \overset{|}{\circ} \pi_2$ of two representations in terms of spaces of intertwiners (π_1, π_2) from $H_1 \rightarrow H_2$ are mostly familiar to mathematicians and mathematical physicist,

$$
\begin{aligned}
irred. &: (\pi,\pi) = \mathbb{C}\mathbf{1}\,,\ \ i.e.\, Schur's\ Lemma \\
factorial &: center\ of\ (\pi,\pi) = \mathbb{C}\mathbf{1} \\
unit.\,equiv. &: \exists\, unitary\, U \in (\pi_1,\pi_2) \\
disjoint &: (\pi_1,\pi_2) = \{\mathbf{0}\}
\end{aligned}
$$

quasiequivalence $\approx$ is less well known

$$\pi_1 \approx \pi_2 \iff S_{\pi_1}(\mathcal{A}) = S_{\pi_2}(\mathcal{A}) \tag{48}$$

For a factorial representations π is equivalent to all its subrepresentation (think of an allegory to Greek mythology: Laokoon and the multi-headed snake with heads growing immediately again after their beheading) i.e. $S_\pi(\mathcal{A})$ does not contain a closed subfolium. Two factorial representations are either quasiequivalent or disjoint (an extension of the situation presented by two irreducible representations).

Von Neumann algebras originate in QFT typically through the representation theory of C^*-algebras. Their Hilbert space representation $\pi(\mathcal{A})$ in H allows to take the weak closure which according to von Neumann's famous commutant theorem is equal to the double commutant $\pi(\mathcal{A})''$ of $\pi(\mathcal{A})$ in H. The so obtained von Neumann algebras are special weakly closed C^*-algebras which have no interesting representation theory since all representations which maintain a natural continuity property (normal representations) turn out to be quasiequivalent and have only one "folium of states". The physically relevant spacetime-indexed local algebras $\mathcal{A}(\mathcal{O})$ are in typical (presumably even in all) cases hyperfinite type III_1 von Neumann factors and hence are even unitarily equivalent, in other words there is up to unitary equivalence only one such algebra. The importance of factors i.e. von Neumann algebras with trivial center results from the fact that any von Neumann algebra allows a (generally continuous) central decomposition into factors and the latter can be classified in terms of equivalences between their projectors and will be explained in the sequel.

The system of all projectors $\mathcal{P}(M)$ in a von Neumann algebra M obeys the mathematical structure of a lattice. It is clear that unitarily equivalent projectors should be considered as part of an equivalence class and the first aim would be to understand the class structure. In order to have coherence of this equivalence notion with additivity of orthogonal projectors, one need to follow Murray and von Neumann and enlarge the class of equivalent projectors in the following way [6]

Definition 4. *Let $e, f \in \mathcal{P}(M)$, then*

1. *the two projectors are equivalent $e \sim f$ if there exists an partial isometry such that e and f are the source and range projectors: $u^*u = e,\ uu^* = f$*

2. *e is subequivalent to f, denoted as $e \preceq f$ if $\exists g \in \mathcal{P}(M)$ such that g is dominated by f and equivalent to $e : e \sim g \leq f$*

One easily checks that this definition indeed gives a bona fide equivalence relation in $\mathcal{P}(M)$. Via the relation between projectors and subspaces, these definitions and the theorems of the Murray von Neumann classification theory can be translated into relations between subspaces. The main advantage to restrict to factors is the recognition that *any two projectors are then subequivalent.* One calls a von Neumann algebra finite if a projector is never equivalent to a proper subprojector. Example: in $B(H)$ infinite dimensional spaces allow a partially isometric mapping on infinite dimensional subspaces and therefore this factor is infinite. $Mat_n(\mathbb{C})$ is of course a finite factor. It was a great discovery of Murray and von Neumann, that there exist infinite dimensional finite factors. In fact they defined:

Definition 5. *A factor M is said to be one of the following three types:*

1. I, if it possesses pure normal states (or minimal projectors).

2. II, if it not of type I and has nontrivial finite projectors.

3. III, if there are no nontrivial finite projectors.

Murray and von Neumann were able to refine their classification with the help of the trace. In more recent terminology a trace without an additional specification is a weight Tr with $Trxx^* = Trx^*x \; \forall x \in M$. A tracial state is a special case of a tracial weight.

The use of tracial weights gives the following refinement:

Definition 6. *Using normal tracial weight one defines the following refinement for factors:*

1. typeI_n if $ranTr\mathcal{P}(M) = \{0, 1, ..., n\}$, the only infinite type I factor is typeI_∞. Here the tracial weight has been normalized in the minimal projectors (for finite n this weight is in fact a tracial state).

2. typeII_1 if the Tr is a tracial state with $ranTr\mathcal{P}(M) = [0, 1]$; type II_∞ if $ranTr\mathcal{P}(M) = [0, \infty]$.

3. type III if no tracial weight exists i.e. if $ranTr\mathcal{P}(M) = \{0, \infty\}$.

In particular all nontrivial projectors (including $\mathbf{1}$) are Murray-von Neumann equivalent.

The classification matter rested there, up to the path-breaking work of Connes in the 70ies which in particular led to an important gain in understanding and complete classification of all hyperfinite type III factors.

Remark 7. *In LQP only the type I factor and the hyperfinite type III_1 factor are directly used. Besides the representation of the global algebra the type I factor features in the local split property (see next section). Local (wedge-, double cone-) algebras are of hyperfinite type III_1. For the formulation of the intertwiner formalism (topological field theory) of the superselection theory one also employs tracial hyperfinite type II_1 factors as an auxiliary tool.*

Although von Neumann algebras have no interesting representation theory, they are the ideal objects for the study of properties related to the relative positions of several of them in one common Hilbert space in particular inclusions of one into another. A baby version of an inclusion is as follows. Suppose that $Mat_2(\mathbb{C})$ acts not on its natural irreducible space $\mathbb{C}^2$ but by left action on the 4-dim Hilbert space $\mathcal{H}(Mat_2(\mathbb{C}), \frac{1}{2}Tr)$ where the inner product is defined in terms of the usual trace. In that space the commutant is of equal size and consists of $Mat_2(\mathbb{C})$

acting in the opposite order from the right which will be shortly denoted as $Mat_2(\mathbb{C})^{opp}$. Explicitly the realization of $\mathcal{H}$ as $\mathbf{C}^4$ may be defined as

$$\begin{pmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{pmatrix} \rightarrow \begin{pmatrix} \xi_{11} \\ \xi_{21} \\ \xi_{12} \\ \xi_{22} \end{pmatrix} \tag{49}$$

and the action of $\mathcal{A} = Mat_2(\mathbb{C})$ takes the following form:

$$a = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & a_{11} & a_{12} \\ 0 & 0 & a_{21} & a_{22} \end{pmatrix} \simeq \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \otimes \underline{1} \tag{50}$$

The most general matrix in the commutant $a' \in \mathcal{A}'$ has evidently the form:

$$a' = \begin{pmatrix} a'_{11} & 0 & a'_{12} & 0 \\ 0 & a'_{11} & 0 & a'_{12} \\ a'_{21} & 0 & a'_{22} & 0 \\ 0 & a'_{21} & 0 & a'_{22} \end{pmatrix} \simeq \underline{1} \otimes \begin{pmatrix} a'_{11} & a'_{12} \\ a'_{21} & a'_{22} \end{pmatrix}$$

The norm $\|\xi\| = \left(\frac{1}{2} Tr \xi^* \xi\right)^{\frac{1}{2}}$ is invariant under the involution $\xi \rightarrow \xi^*$ which in the $\mathbb{C}^4$ representation is given by the isometry:

$$J = \begin{pmatrix} K & 0 & 0 & 0 \\ 0 & 0 & K & 0 \\ 0 & K & 0 & 0 \\ 0 & 0 & 0 & K \end{pmatrix}, \quad K : \text{natural conjugation in } \mathbb{C} \tag{51}$$

We have:

$$j(\mathcal{A}) := J\mathcal{A}J = \mathcal{A}', \quad \text{antilin. map } \mathcal{A} \rightarrow \mathcal{A}' \tag{52}$$

which may be rewritten in terms of a linear anti-isomorphism:

$$a \rightarrow Ja^*J, \quad \mathcal{A} \rightarrow \mathcal{A}' \tag{53}$$

Consider now the trivial algebra $\mathcal{B} = \mathbb{C} \cdot \mathbf{1}_2$ as a subalgebra of $\mathcal{A} = Mat_2(\mathbb{C})$. In the $\mathbb{C}^4$ representation the B-algebra corresponds to the subspace:

$$\mathcal{H}_B = \left\{ \begin{pmatrix} \xi \\ 0 \\ 0 \\ \xi \end{pmatrix}, \xi \in \mathbb{C} \right\}, \quad \mathcal{H}_B = e_B \mathcal{H}, \quad e_B = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix} \tag{54}$$

The projector e_B commutes clearly with $\mathcal{B}$ i.e. $e_B \in \mathcal{B}'$. We now define a measure for the relative size of $\mathcal{B} \subset \mathcal{A}$ the Jones index:

$$[A : B] = \tau_{B'}(e_B)^{-1}, \quad \tau : \text{ normalized trace in } \mathcal{B}'$$

In our example $\tau(e_B) = \frac{1}{4}(\frac{1}{2} + 0 + 0 + \frac{1}{2}) = \frac{1}{4}$ i.e. the satisfying result that the Jones index is 4. The same method applied to the inclusion:

$$Mat_4(\mathbf{C}) \supset Mat_2(\mathbf{C}) \otimes \mathbf{1}_2 = \left\{ \begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix}, X \in Mat_2(\mathbf{C}) \right\} \tag{55}$$

also gives the expected result:

$$[A : B] = \frac{\dim Mat_4(\mathbf{C})}{\dim Mat_2(\mathbf{C})} = 4 \tag{56}$$

If, as in the previous cases B is a finite dimensional subfactor (i.e. a full matrix algebra) of A, the Jones index is the square of a natural number. For inclusions of finite dimensional semisimple algebras the index takes on more general values. For example:

$$Mat_2(\mathbf{C}) \oplus \mathbf{C} = \begin{pmatrix} X & & \\ & X & \\ & & x \end{pmatrix} \subset Mat_2(\mathbf{C}) \oplus Mat_3(\mathbf{C}) \tag{57}$$

$$X \in Mat_2(\mathbf{C}), \quad x \in \mathbf{C}1$$

Here the index is 3. It is easy to see that instead of the projector formula one may also use the incidence matrix formula:

$$[\mathcal{A} : \mathcal{B}] = \sum_{n,m} (\Lambda_n^m)^2 \tag{58}$$

The incidence matrix Λ is describable in terms if a bipartite graph. The number of say white vertices correspond to the number of full matrix component algebras for the smaller algebra and the black vertices labelled by the size of the components to the analogously labelled irreducible components of the bigger algebra. A connecting line between the two sets of vertices indicates that one irreducible component of the smaller is included into one of the bigger algebra. In our case:

$$\Lambda = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad ||\Lambda||^2 = 3 \tag{59}$$

¿From a sequence of ascending graphs one obtains important infinite graphs (Bratteli diagrams) which are very useful in the "subfactor theory" [7] which will appear in the mathematical appendix. In the infinite dimensional case the inclusion of full matrix algebras corresponds to the inclusion of von Neumann factors i.e. the "subfactor problem". In that case the spectrum of inclusions shows a fascinating and unexpected quantization phenomenon, the Vaughn Jones quantization formula for index ≤ 4. AFD (almost finite dimensional) C*-algebras obtained by sequences of ascending Bratteli diagrams equipped with tracial states enter LQP via the intertwiner algebra of charge transporters. A special case are the combinatorial theories which result from Markov-traces on selfintertwining transporters which contain the braid group and mapping class group (see subsection 2.4).

The finite dimensional inclusion theory has a very interesting infinite dimensional generalization through the subfactor theory which was initiated by Vaughan Jones. Whereas as in the above example inclusions of finite dimensional full matrix algebras have Jones indices which are squares of integers, infinite dimensional subfactors have a more interesting spectrum of Jones indices. For a presentation of subfactor theory using concepts and techniques of AQFT as well as its use for studying superselection sectors we refer to [10]

2. Superselections and Locality in Quantum Physics

In this second section I will explain the assumptions underlying LQP, its relation to more standard formulations of QFT and some of its important achievements.

Let us first list some assumptions which include the main properties of (Haag-Kastler) nets of observables.

- (i) There is an inclusion preserving map of compact regions $\mathcal{O}$ in Minkowski space into von Neumann operator algebras $\mathcal{A}(\mathcal{O})$ which are subalgebras of all operators $\mathcal{B}(\mathcal{H})$ in some common Hilbert space $\mathcal{H}$:

$$\begin{aligned} &\mathcal{A}:\mathcal{O}\rightarrow\mathcal{A}(\mathcal{O}) \\ &\mathcal{A}(\mathcal{O})\subset\mathcal{A}(\hat{O})\ if\ \mathcal{O}\subset\hat{\mathcal{O}} \end{aligned} \tag{60}$$

It is sufficient to fix the map on the Poincaré invariant family $\mathcal{K}$ of double cone regions ($V_{\pm}$:forward/backward lightcone)

$$\mathcal{O}=(V_{+}+x)\cap(V_{-}+y)\,,\ y-x\in V_{+} \tag{61}$$

Since the family of algebras $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O}\in\mathcal{K}}$ forms a net directed towards infinity of Minkowski space (two double cones can always be encloded into a larger one), one can naturally globalize the net by forming its inductive limit whose C^{*}-completion defines the quasilocal C^{*}-algebra $\mathcal{A}_{qua}$:

$$\mathcal{A}_{qua}=\overline{\bigcup_{\mathcal{O}\in\mathcal{M}}\mathcal{A}(\mathcal{O})}^{\|\cdot\|} \tag{62}$$

were the superscript indicates the uniform operator norm in terms of which the closure is taken. It is called "quasilocal" because its operators can still be uniformly approximated by those of the net (which excludes truly global operators as global charges). The C^{*}-algebras for noncompact regions are analogously defined by inner approximation with double cones $\mathcal{O}$[9]. Since they are concrete operator algebras in a common Hilbert space they have a natural von Neumann closure $\mathcal{M}=\mathcal{M}''$. The closely related (but independent) assumption (for double cones $\mathcal{O}$ of arbitray size)

$$\left\{\bigcup_{a}\mathcal{M}(\mathcal{O}+a)\right\}''=\mathcal{M}(M),\ M=Minkowski\ spacetime \tag{63}$$

is called *weak additivity* and expresses the fact that the *Global* can be constructed from the *Local*.

[9] There are also outer approximations approximations by intersections of wedges. Even if the result is geometrically identical, the associated algebras may lead to a genuine inclusion containing interesting physical information.

- (ii) *Einstein causality* and its strengthened form called Haag duality

$$Einstein\ causality : \mathcal{A}(\mathcal{O}) \subset \mathcal{A}(\mathcal{O}')' \tag{64}$$
$$Haag\ duality : \mathcal{A}(\mathcal{O}) = \mathcal{A}(\mathcal{O}')' \tag{65}$$
$$\mathcal{O}' = causal\ disjoint$$

- (iii) *Covariance and stability* (positive energy condition) with respect to the Poincaré group $\mathcal{P}$. For observable nets:

$$\begin{aligned} \alpha_{(a,\Lambda)}(\mathcal{A}(\mathcal{O})) &= \mathcal{A}(\Lambda\mathcal{O} + a) \\ &= AdU(a, \tilde{\Lambda})\mathcal{A}(\mathcal{O}) \\ U(a,1) &= e^{iPa},\ specP \in V_+, \\ \exists\, vacuum\, vector\ |0\rangle &\in H,\ P\,|0\rangle = 0 \end{aligned} \tag{66}$$

where the unitaries represent the covering group $\tilde{\mathcal{P}}$ in H. A particular case is that the P-spectrum contains the vacuum state $P = 0$. We will call the cyclically generated subspace $H_{vac} \equiv \overline{\mathcal{A}\,|0\rangle} \subset H$ the space of the vacuum sector. It is customary in LQP to assume that the common Hilbert space H in which the net is defined is the vacuum space H_{vac} and that the other physical representations are to be computed from the vacuum data by the methods of LQP explained in the next section.

- (iv) *Causal time slice property* (causal shadow property): Let $\mathcal{O}$ be the double cone like causal shadow region associated with a subregion $\mathcal{C}(\mathcal{O})$ of a Cauchy surface $\mathcal{C}$ and let U be a (timeslice) neighborhood of $\mathcal{C}(\mathcal{O})$ in $\mathcal{O}$, then

$$\mathcal{A}(\mathcal{O}) = \mathcal{A}(U) \tag{67}$$

- (v) *Phase space structure of LQP*

$$the\, map\, \Theta : \mathcal{A}(\mathcal{O}) \to e^{-\beta P_0}\mathcal{A}(\mathcal{O})\Omega\ is\ nuclear \tag{68}$$

i.e. the range of Θ is a "small" set of vectors contained in the image of a traceclass operator in $B(H)$.

Some additional comments on the physical ideas and some easy consequences.

In QM the physical interpretation of the commutant (of a collection of observable Hermitian operators) is that of a von Neumann algebra which is generated by all those observable operators whose measurements are commensurable relative to the given set of observables. The general setting of quantum theory offers no specific physical characterization of such algebras; however in LQP the Einstein causality property (ii) tells us that if the original collection generates all observables which can be measured in a given spacetime region $\mathcal{O}$, then the commensurable measurements are associated with observables in the spacelike complement $\mathcal{O}'$. Whereas Einstein causality limits the region of future influence of data contained in localized observable algebras $\mathcal{A}(\mathcal{O})$ to the forward closed light cone subtended by the localization region $\mathcal{O}$, the causal shadow property (iv) prevents the appearance of new degrees of freedom in the causal shadow $\mathcal{O}''$ from outside $\mathcal{O}$. In fact the idea that all physical properties can be extracted from an affiliation of an observable to a (possibly multiple-connected) spacetime region and its refinements is the most important and successful working hypothesis of LQP, whereas the standard formulation is ill suited to formalize this important aspect.

Haag duality is the equality in (64) i.e. the totality of all measurements relatively commensurable with respect to the observables in $\mathcal{A}(\mathcal{O})$ is exhausted by the spacelike disjoint localized observables $\mathcal{A}(\mathcal{O}')$. One can show (if necessary by suitably enlarging the local net within the same vacuum Hilbert space) that Haag duality can always be achieved in an Einstein causal net. It turns out that the inclusion of the original net in the Haag dualized net contains intrinsic (independent of the use of particular poitlike fields) information on "spontaneous symmetry breaking", an issue which will not be treated in this survey [8]. The magnitude of violation of Haag duality in other non-vacuum sectors is related to properties of their nontrivial superselection charge whose mathematical description is in terms of endomorphisms of the net (the Jones index of the inclusion $\rho(\mathcal{A}) \subset \mathcal{A}$ is a quantitative measure).

Neither Einstein causality nor Haag duality guaranty causal disjointness in the form of "statistical independence" for spacelike seperations i.e. a tensor product structure between two spacelike separated algebras (analogous to the factorization for the inside/outside region associated with a quantum mechanical quantization box). This kind of strengthening of causality is best formulated in terms of properties of states on the local algebras. It turns out that the nuclearity of the QFT phase space in (v) is a sufficient condition for the derivation of statistical independence [1]. This is done with the help of the so-called split property, a consequence of the nuclearity assumption which is interesting in its own right. It states the tensor factorization can be achieved if one leaves between the inside of a double cone $\mathcal{O}$ and its spacelike disjoint a "collar" region (physically for the vacuum fluctuations to settle down) i.e. if one takes instead the spacelike outside of a slightly bigger double $\hat{\mathcal{O}}$ cone which properly contains the original one. In that case there exists an intermediate type I factor $\mathcal{N}$ between the two double cone algebras $\mathcal{A}(\mathcal{O}) \subset \mathcal{N} \subset \mathcal{A}(\hat{\mathcal{O}})$ (there is even a canonically distinguished one) whose localization is "fuzzy" i.e. cannot be described in sharp geometrical terms beyond this inclusion. A type I factor is synonymous with tensor factorization

$$\begin{aligned} H &= H_{\mathcal{N}} \bar{\otimes} H_{\mathcal{N}'} \\ B(H) &= \mathcal{N} \bar{\otimes} \mathcal{N}' \end{aligned} \tag{69}$$

The positivity of energy is a specific formulation of stability adapted to particle physics which deals with local excitations of a Poincaré invariant vacuum. It goes back to Dirac's observation that if one does not "fill the negative energy sea" associated with the formal energy-momentum spectrum of the Dirac equation, the switching on of an external electromagnetic interaction will create a chaotic instability. In case of thermal states it is the so called KMS condition which secures stability[10].

The energy positivity leads via analytic properties of vacuum expectation values to the cyclicity of the vacuum with respect to the action of $\mathcal{A}(\mathcal{O})$ i.e. $\overline{\mathcal{A}(\mathcal{O})\Omega} = H$ and for $\mathcal{O}$'s with a nontrivial causal complement the use of causality also yields the absence of local annihilators i.e. $A\Omega = 0$, $A \in \mathcal{A}(\mathcal{O}) \curvearrowright A = 0$. This latter property is called separability of $(\mathcal{A}(\mathcal{O}), \Omega)$ and follows from cyclicity and causality. Both aspects together are known under the name of Reeh-Schlieder property and in operator algebra theory such pairs are called "standard". This property

[10] The KMS condition is a generalization of the Gibbs formula to open systems [1].

very different from what one is accustomed to in QM since it permits a creation of a particle "behind the moon" (together with an antipartcle in some other far remote region) by only executing local operations of small duration in an earthly laboratory. Mathematically this is the starting point for the Tomita-Takesaki modular theory which we will return to below. On the physical side the attempts to make this exotic mathematical presence of a dense set of state vectors by local operations physically more palatable has led to insights into the profound role of the phase space structure (v) [1].

2.1. Connection with pointlike formulation. Before we give an account of structural theorems, in particular the superselection structure following from these assumed properties, it is helpful to make a relation to the traditional formulation in terms of fields to which QFT owes its name. This arose from the canonical quantization of classical field theories which eventually found its more covariant but still formal formulation in terms of functional integrals using classical actions. The first successful attempt to overcome the "artistic" aspects[11] and to characterize the conceptual and mathematical properties of what one expects to lie behind the formal manipulations was given by Wightman [11] in terms of axioms about (not necessarily observable) pointlike fields. These axioms separate into two groups.

- $\mathcal{H}$-space and $\mathcal{P}$-group

1. Unitary representation $U(a,\alpha)$ in H of the covering group $\widetilde{\mathcal{P}}$ of $\mathcal{P}$, $\alpha \in SL(2,C)$
2. Uniqueness of the vacuum Ω, $U(a,\alpha)\Omega = \Omega$
3. Spectrum condition: $specU \in \bar{V}_+,$ the forward light cone.

- Fields

1. Operator-valued distributions: $A(f){=}\int A(x)f(x)d^4x,\ \ f \in \mathcal{S}$ (the Schwartz space of "tempered" testfunctions) is an unbounded operator with a dense domain $\mathcal{D}$ such that the function $\langle\psi_2\left|A(x)\right|\psi_1\rangle$ exists as a sesquilinear form for $\psi_i \subset \mathcal{D}$

2. Hermiticity: with A, also A^* belongs to the family of fields and the affiliated sesquilinear forms are as follows related: $\langle\psi_2\left|A^*(x)\right|\psi_1\rangle = \langle\psi_1\left|A(x)\right|\psi_2\rangle$

3. $\widetilde{\mathcal{P}}$-covariance of fields: $U(a,\alpha)A(x)U^*(a,\alpha) = D(\alpha^{-1})A(\Lambda(\alpha)x+a)$. For observable fields only integer spin representations (i.e. representations of $\mathcal{P}$) occur.

4. Locality: $\left[A^{\#}(f),A^{\#}(g)\right]_{\mp} = 0$ for suppf$\times$suppg ($\times$:supports are spacelike separated).

Haag duality, statistical independence and primitive causality (the causal shadow property) allow no natural formulation in terms of individual pointlike field coordinates, they are rather relations between algebras. The process of Haag dualization of a net affects the relation between fields and local algebras, there is no extension of pointlike fields involved.

[11] By this we mean properties which serve to start the calculation as e.g. the validity of a functional integral representation which the physical correlation functions (after renormalization) obtained at the end do not obey. The Wightman approach avoids this mathematical imbalance.

Formally the fields obey a dynamical law describing their causal propagation in timelike/lightlike direction. The relation between the two approaches is very close as far as the intuitive content of the physical principles is concerned. In fact the best way to relate them is to think about fields as being akin to coordinates in geometry and the local algebras as representing the coordinate-free intrinsic approach[12]. The smeared fields $A(f)$ with $suppf \subset \mathcal{O}$ play the role of formal (affiliated) generators of $\mathcal{A}(\mathcal{O})$, but since the latter are unbounded operators with a dense domain, the relation involves domain problems which are similar (but more difficult) than the connection between Lie algebras and Lie groups in the noncompact case. When the first monographs on this axiomatic approach were written [11], these domain properties were thought of as technicalities. However as a result of recent developments in "modular localization" one now knows that these domains contain physical information in particular information about the geometric localization of the operators.

In the opposite direction from nets of algebras to fields one does not expect in general that all degrees of freedom of a local theory can be described in terms of pointlike covariant fields at least if the theories are not scale invariant. In such a case the Lagrangian framework is too narrow and one must use the LQP framework. This is in particular the case if one "takes a holographic image, transplants or scans" a theory which was generated by pointlike fields as will be explained in the last section.

The pointlike Wightman approach leads, as a result of its analytic formalism, to fairly easy proofs of the existence of an antiunitary TCP operator Θ which implements the total spacetime reflection symmetry $x \rightarrow -x$ which simultaneously involves a conjugation of the superselection charge [11]. In the algebraic setting the proof requires presently a mild additional assumptions. [13].

2.2. Localization and Superselection. In order to study localized states and the associated representations it is convenient to have a global algebra which contains all $\mathcal{A}(\mathcal{O})$. For this purpose we use $\mathcal{A} \equiv \mathcal{A}_{qua}$ as defined in (62)

The strongest form of localization of states is that of Doplicher Haag and Roberts (DHR) [1]:

Definition 8. *A positive energy state is DHR-localized (relative to the vacuum) if the associated GNS representation $\pi(\mathcal{A})$ is unitarily equivalent to the vacuum representation $\pi_0(\mathcal{A})$ in the spacelike complement $\mathcal{O}'$ of any preassigned compact region (double cone) $\mathcal{O}$*

$$\pi|_{\mathcal{A}(\mathcal{O}')} \simeq \pi_0|_{\mathcal{A}(\mathcal{O}')} \ \forall \mathcal{O} \tag{70}$$

The definition in particular implies that a state which is strictly localized in $\mathcal{O}$, i.e. $\omega(A) = \omega_0(A) \ \forall A \in \mathcal{A}(\mathcal{O}')$, is also localized in the DHR sense [1]. This localization underlies the standard formulation of QFT which is based on covariant pointlike fields and covers in particular the fields featuring in the Lagrangian quantization formalism. The important step which converts the localization of states/representations into the localized superselected charges uses the Haag duality

[12] However the differences between the intrinsic algebraic approach as compared to that with pointlike fields appears greater than that on the geometric side, since in the coordinate-free geometric approach coordinates one still uses coordinate patches in the definition of a manifold.

Proposition 9. *Localized DHR representations can be expressed in terms of "localized charges" which are described in terms of localized endomorphism ρ of $\mathcal{A}$*

$$\pi \simeq \pi_0 \circ \rho \tag{71}$$

Proof. Pick any region $\mathcal{O}$ and use the existence of a unitary partial intertwiner $V(\mathcal{O})$ following from the above definition

$$V(\mathcal{O})\pi_0(A) = \pi(A)V(\mathcal{O}),\ A \in \mathcal{A}(\mathcal{O}') \tag{72}$$
$$\hat{\pi}(A) = V(\mathcal{O})^{-1}\pi(A)V(\mathcal{O}),\ \forall A \in \mathcal{A}_{qua}$$

where the second line is the definition of a representation which is equivalent to π and is identical to π_0 in its restriction to $\mathcal{A}(\mathcal{O}')$. Therefore *for all regions* $\mathcal{O}_1 \supset \mathcal{O}$ the range of $\hat{\pi}$ is according to Haag duality contained in that of $\pi_0 : \hat{\pi}(\mathcal{A}(\mathcal{O}_1)) \subset \pi_0(\mathcal{A}(\mathcal{O}_1))$. This is so because $[\pi_0(A'), \hat{\pi}(A)] = \pi([A', A]) = 0$ for $A' \in \mathcal{A}(\mathcal{O}_1'), A \in \mathcal{A}(\mathcal{O}_1)$. This relation together with Haag duality then tells us that $\hat{\pi}(\mathcal{A}(\mathcal{O}_1)) \subset \pi_0(\mathcal{A}(\mathcal{O}_1'))' \overset{HD}{=} \pi_0(\mathcal{A}(\mathcal{O}_1))$, from which one concludes that

$$\rho := \pi_0^{-1}\hat{\pi},\ \mathcal{A} \to \mathcal{A}$$

defines an endomorphism of $\mathcal{A}$.

Endomorphisms are generalizations of automorphism; they are not required to be morphisms of the algebra onto itself but may have a subalgebra as an image. The endomorphisms in LQP are faithful. They are called localized in $\mathcal{O}$ if $\rho(A) = A$, $\forall A \in \mathcal{A}(\mathcal{O})$, and they are said to be transportable if for any given $\tilde{\mathcal{O}}$ there exists an equivalent endomorphism $\tilde{\rho} \in [\rho]$ with $loc\tilde{\rho} \subset \tilde{\mathcal{O}}$. Since the latter region contains no limitation of a minimal size, there is no fundamental length in the DHR setting. It is often convenient to identify the algebra $\mathcal{A}$ with its faithful vacuum representation and write $\rho(A)\psi$ instead of $\pi_0 \circ \rho(A)\psi$. A neat way to remember that we are using the action on the vacuum Hilbert space but mediated through the endomorphism ρ

$$A : (\rho, \psi) \mapsto (\rho, \rho(A)\psi) \tag{73}$$

is to denote this different use of the vacuum Hilbert space H_0 as a representation space for ρ in form of a pair $H_\rho \equiv (\rho, H_0)$. This notation will later be allowed to develop a life of its own; it suggests the introduction of a C^*-algebra with bimodule properties called the *reduced field bundle.*

The marvelous achievement of converting localized transportable representations π into endomorphisms with the same properties of the observable algebra $\mathcal{A}$ is that now one may define a product structure of endomorphisms simply by acting successively, i.e. $(\rho_2 \circ \rho_1)(A) \equiv \rho_2(\rho_1(A)) \quad \forall A \in \mathcal{A}$. With this composition we have achieved a natural definition for the product of two representations

$$\pi_1 \circ \pi_1 \equiv \pi_0 \circ \rho_2\rho_1 \tag{74}$$

It is now appropriate to define in more precise terms what we mean by superselection sectors. Since unitary equivalent DHR representations correspond precisely to inner equivalent (i.e. by unitaries in $\mathcal{A}$) $\rho's$, we call a (superselection) sector a class of inner equivalent endomorphisms $[\rho]$ associated with a given ρ. It immediately follows, that whereas the individual endomorphisms compose in a noncommutative manner, the composition of sectors is abelian

$$[\rho_2]\,[\rho_1] := [\rho_2\rho_1] = [\rho_1\rho_2] = [\rho_1]\,[\rho_2] \tag{75}$$

This is a consequence of causality and the localizability and transportability of the endomorphisms which results in commutativity in case of their spacelike separation

$$\rho_2\rho_1 = \rho_1\rho_2,\ if\ loc\rho_1 \times loc\rho_2 \tag{76}$$

The proof is very simple: if both sides are applied to an $A \in \mathcal{A}(\mathcal{O}) \subset \mathcal{A}$ with $\mathcal{O}$ spacelike to both locρ_i, the relation obviously holds. But this standard situation can always be achieved by suitably transporting the ρ_i into two "spectators" ρ_i' with locρ_i' causally disjoint from the $\mathcal{O}$ by using suitably localized charge transporters U_i in such away that the localization of these unitaries does not destroy the commutativity in the process of changing back to the original $\rho_i's$. In these arguments one uses the (also easily proven) fact that the localization of the composite $\rho_2\rho_1$ is $O_{12} := \mathcal{O}_1 \vee \mathcal{O}_2$ i.e. the smallest double cone containing both $\mathcal{O}_i$.

An important step in the development of an intertwiner calculus is the realization that projectors E which project onto subrepresentations π on $H_\pi = E_\pi H_0$ commute with $\pi_0 \circ \rho_2\rho_1(\mathcal{A}(\mathcal{O}_{12}')) = \pi_0(\mathcal{A}(\mathcal{O}_{12}'))$ and hence belongs to the algebra $\pi_0(\mathcal{A}(\mathcal{O}_{12}))$. This permits the introduction of isometric intertwiners T which map H_0 onto the subspace $E_\pi H_0$ i.e. their source space is H_0 and their range space H_π. These isometries intertwine the endomorphisms

$$T\rho(A) = \rho_2\rho_1(A)T,\ \ T^*T = 1,\ TT^* = E \tag{77}$$
$$\pi = \pi_0 \circ \rho$$

The mathematical basis of this is the "property B" (due to Borchers [1]) stating that a projection operator E which is localized in $\mathcal{O}$ allows a factorization into intertwiners T with $locT = \tilde{\mathcal{O}}$ for any $\tilde{\mathcal{O}} \supset\supset \mathcal{O}$ (proper inclusion, i.e. no touching of boundaries so that a full neighborhood of the trivial translation in the translation group can act on $\mathcal{A}(\mathcal{O})$). This is a well-known property of properly infinite von Neumann algebras of type III in which case the factorization does not need the $\tilde{O}$ enlargement. The property B follows from additivity whereas proving that the local von Neumann algebras are hyperfinite of type III$_1$ (for which the corresponding von Neumann factors are unique) is more involved. In our special case (77) where $E \in \mathcal{A}(\mathcal{O}_{12})$ and hence according to property B $locT$ is a double cone which properly contains O_{12}.

The intertwiners (77) form a (Banach)space in $\mathcal{A}$, for which one sometimes uses the notation $(\rho_2\rho_1, \rho)$ i.e. $T : \rho \to \sigma$ forms the space $(\sigma|\rho)$ and their composition in $\mathcal{A}$ again belongs to an intertwiner space $ST = S \circ T \in (\tau|\rho)$ if $S \in (\tau|\sigma)$. Evidently for ρ irreducible (i.e. $\rho(\mathcal{A})' \cap \mathcal{A} = \mathbb{C}\mathbf{1}$), the T's are isometric and form a Hilbert space within the C^*-algebra $\mathcal{A}_{qua}$ i.e. $S^*T \in \mathbb{C}\mathbf{1}$.

Our main interest in the following will be the intertwiner calculus within the set Δ_0 of localized, transportable endomorphisms which have conjugates and have finite statistics. We call two irreducible localized endomorphisms ρ and $\bar{\rho}$ conjugate to each other if the sector $[\rho\bar{\rho}]$ contains the vacuum sector i.e. if there exist isometries $R \in (\bar{\rho}\rho|id)$, $\bar{R} \in (\rho\bar{\rho}|id)$; finiteness of statistics then leads to their uniqueness in case of irreducible endomorphisms. Their unique existence is closely linked to the existence of a unique "left inverse" ϕ and conditional expectation

$E : \mathcal{A} \to \rho(\mathcal{A})$

$$\phi(A) := R^* \overline{\rho}(A) R,\ A \in \mathcal{A} \tag{78}$$
$$E(A) := \rho \circ \phi(A)$$

This left inverse draws its name from the relation $\phi(A\rho(B)) = \phi(A)B$ and E has the properties of a projection of $\mathcal{A}$ onto $\rho(\mathcal{A})$; both properties are immediately read off from the definition of R. With an additional minimality requirement the uniqueness continues to hold in the case of general $\rho' s$ (on which the ϕ and E depend) [10].

This still leaves us with explaining statistics. The statistics operator of a pair ρ_1, ρ_2 is a distinguished flip operator $\varepsilon(\rho_1, \rho_2) \in (\rho_1\rho_2|\rho_2\rho_1)$ which is explicitly defined in terms of charge transporters $U_i \in (\hat{\rho}_i|\rho_i)$ which shift the localization into spacelike separated regions $\hat{\mathcal{O}}_1$, $\hat{\mathcal{O}}_2$

$$\varepsilon(\rho_1, \rho_2) = \rho_2(U_1^*) U_2^* U_1 \rho_1(U_2),\ if\ \hat{\mathcal{O}}_2 < \hat{\mathcal{O}}_1 \tag{79}$$

If the spacelike ordering is inverted, the same formula represents $\varepsilon(\rho_2, \rho_1)^*$ instead of $\varepsilon(\rho_1, \rho_2)$. It is easy to see that the definition (79) does not depend on the choice of the "spectator" endomorphisms $\hat{\rho}_i$ as long as one does not change their topological relation. Whenever there is no topological separation between the two orders (which in the case of double cones occurs for spacetime dimension $d \geq 2+1$) one has the additional relation of their equality. It is easy to show (see the second appendix to this section) that $\varepsilon \equiv \varepsilon(\rho, \rho)$ generates the Artin braid group via $g_k \equiv \rho^{k-1}(\varepsilon)$ and that the action of the $\rho_i' s$ on the $\varepsilon(\rho_1, \rho_2)$ define some colored groupoid version (see appendix of [19]). Since for spacetime dimension $d \geq 2+1$ with compact (double cone) localization there is only one localization class, we obtain with $\varepsilon^2 = 1$ the permutation group statistics of standard QFT.

These intertwiner spaces are empty precisely if there are no common subrepresentations. Similar to the use of creation and annihilation operators for Fock spaces, these charge intertwiners can be used to intertwine between the different charge subspaces of one "master space". This is done in the following way. From the Δ_0 one chooses one representative per irreducible sector and defines (with $\nabla_0 \subset \Delta_0$ being the reference set)

$$H = \bigoplus_{\rho_i \in \nabla_0} H_{\rho_i} \tag{80}$$
$$H_{\rho_i} = (\rho_i, H_0)$$
$$((\rho_i, \psi), (\rho_j, \varphi)) = \delta_{\iota j}(\psi, \varphi)$$

We will call this space the reduced field bundle space and our aim is to define on H with help of the system of reference intertwiners a kind of bimodule C^*-algebra, the so called reduced field bundle $\mathcal{F}$.

We start with defining isometric intertwiners $T_e^{(i)} \in (\rho_\alpha \rho|\rho_\beta)$ associated with $\rho' s$ from ∇_0 where the edge e stands for the superselection channel of the three irreducible endomorphisms with charge $c(e) = \rho$, source $s(e) = \rho_a$ and range $r(e) = \rho_\beta$. The finiteness assumption insures that these algebraic Hilbert spaces in $\mathcal{A}_{qua}$ are finite dimensional $dim(\rho_\alpha \rho|\rho_\beta) = (N_\rho)_\alpha^\beta < \infty$. Assume that the T_e have been chosen orthonormal

$$T_e^* T_{e'} = \delta_{ee'} 1_e \tag{81}$$
$$\sum T_e T_e^* = \mathbf{1}$$

In the completeness relation the sum extends over $r(e)$ as well as over (here suppressed) possible degeneracy index i. For the special case that ρ or ρ_α we choose $T_e = 1$ whereas for $\rho_\beta = id$ (and hence $\rho_\alpha = \bar{\rho}$) we take $T_e = R_\rho$. The definition of the reduced field bundle is now

$$\mathcal{F} = \bigoplus_e (e, \mathcal{A}) \tag{82}$$
$$F(e, A)(\rho_\alpha, \psi) = \delta_{\rho_\alpha s(e)}(r(e), T_e^* \rho_\alpha(A)\psi)$$

The successive application of the last line leads to a product formula for the operators in $\mathcal{F}$.

$$F(e_2, A_2)F(e_1, A_1) = \delta_{s(e_2)r(e_1)} \sum_{e,f} D_{f,e}^{e_2 \circ e_1} F(e, A_f) \in \mathcal{F} \tag{83}$$

with the fusion coefficients D being analogs of the group theoretic Clebsch-Gordan coefficients. With the above rules which allow to re-express everything in terms of the ∇_0 basis and the associated $T_e's$ one calculates straightforwardly

$$A_f = T_f^* \rho_1(A_2) A_1 \tag{84}$$
$$D_{f,e}^{e_2 \circ e_1} = T_{e_2}^* T_{e_1}^* \rho_\alpha(T_f) T_e \in (r(e)|r(e)) = \mathbb{C}$$
$$\rho_a(T_f) = \sum_{e_i, e} D_{f,e}^{e_2 \circ e_1} T_{e_1} T_{e_2} T_e^*$$

$\mathcal{F} \supset F(\mathbf{1}, \mathcal{A}) \simeq \mathcal{A}$ (where the edge denoted by $\mathbf{1}$ corresponds to the sum over all edges with $c(e) = 0 \curvearrowright s(e) = r(e)$) becomes a Banach subalgebra of $B(H)$. A more detailed analysis [19] shows that there is a *-operation which renders it a C^*algebra such that the observable subalgebra $\mathcal{A} \simeq F(\mathbf{1}, \mathcal{A})$ acts on $\mathcal{F}$ in a *bimodule manner*

$$F(\mathbf{1}, A)F(e, B) = F(e, \rho_\alpha(A)B) \tag{85}$$
$$F(e, B)F(\mathbf{1}, A) = F(e, BA), \ \ A, B \in \mathcal{A}$$

Similarly to the composition law (84) $\mathcal{F}$ has a commutation structure related to the localization of its operators. We define $locF = \mathcal{O}$, $F \in \mathcal{F}$ to be that region $\mathcal{O}$ for which F commutes with $\mathcal{A}(\mathcal{O}')$ i.e.

$$F(\mathbf{1}, \mathcal{A}(\mathcal{O}'))F(e, A) = F(e, A)F(\mathbf{1}, \mathcal{A}(\mathcal{O}')) \tag{86}$$

The definition is in fact independent of the source and range projection and can be solely expressed in terms of $\rho = c(e)$ in form of the existence of a charge transporter U with $locAdU\rho \subseteq \mathcal{O}$ and $locUA \subseteq \mathcal{O}$. The F's are in general nonlocal relative to each other and obey

$$F(e_2, A_2)F(e_1, A_1) = \sum_{f_1 \circ f_2} R_{f_1 \circ f_2}^{e_2 \circ e_1}(+/-) F(f_1, A_1) F(f_2, A_2) \tag{87}$$
$$locF_1 \lessgtr locF_2$$
$$R_{f_1 \circ f_2}^{e_2 \circ e_1}(+/-) = T_{e_1}^* T_{e_2}^* \rho_a \begin{pmatrix} \varepsilon(\rho_1, \rho_2) \\ \varepsilon(\rho_2, \rho_1)^* \end{pmatrix} T_{f_1} T_{f_2}, \ \ \rho_a = s(e_1) \tag{88}$$

where similarly to the previous D-matrix case the numerical R-matrices result from expanding the flip operator $\varepsilon(\rho_1, \rho_2)$ in the complete T_e intertwiner basis

$$\rho_a \begin{pmatrix} \varepsilon(\rho_1, \rho_2) \\ \varepsilon(\rho_2, \rho_1)^* \end{pmatrix} T_{f_2} T_{f_1} = \sum_{f_2 \circ f_1} R_{f_2 \circ f_1}^{e_2 \circ e_1}(+/-) T_{e_1} T_{e_2} \tag{89}$$

Up to unitary equivalence, the R-matrices are determined by the Markov traces on the B_∞ braid group (see appendix to this section). As already mentioned the DHR analysis for $d \geq 2+1$ gives permutation group statistics $B_\infty \to S_\infty$. But it allows in addition to the complete antisymmetric/symmetric Fermi/Bose permutation group representations also "parastatistics", i.e. irreducible S_n representations with mixed Young tableaus of size n. There was a suspicion since the beginning of the 70ies that behind the reduced field bundle structure with its source and range dependent operators, there may be a more natural description in terms of a field algebra $\mathcal{F}$ where the basic degrees of freedom were Fermions/Bosons but with multiplicities on which an "inner" symmetry group can act. At that time it was already known that all standard QFTs (including all models which were associated with Lagrangian quantization) with an internal symmetry group allow a descend to an observable algebra $\mathcal{A}$ consisting of the fixed points of the action of the compact symmetry group on the field algebra $\mathcal{A} = \mathcal{F}^G$ in such a way that the latter can be reconstructed from its "observable shadow" as a kind of cross product of the observable algebra with a group dual $\hat{G}$,i.e. $\mathcal{F} = \mathcal{A} \bowtie \hat{G}$. Doplicher and Roberts finished this extensive program of proving the (unique after imposing a natural physical convention) existence of $\mathcal{F}$ from only using the priciples of observable algebras around 1990 [3]. For their solution the discovery of a new duality theory different from the Tannaka-Krein theory was essential. In this way the mathematically efficient but conceptually somewhat mysterious internal symmetry concept (which historically entered particle physics with Heisenbergs introduction of isospin) was finally demystified: the inner group symmetry resulted from the unfolding of causality and superselection rules encoded in the observable algebras; it is part of "how to hear (reconstruct) the shape of a drum (the field algebra)" using again the famous metaphor of Marc Kac. We will not explain any of the additional concepts and theorems which led to those amazing results and refer the interested reader to the literature [3].

In the case of braid group statistics such an encoding into an extended algebra is not known and one has to be content with the field bundle (in the context of braid group statistics also refereed to as "exchange algebras") whose objects even in the pointlike limit turn out not to be ordinary fields whose closed source and range space is the full Hilbert space but rather field bundle (vertex) operators with a partial source and range space (in the context of conformal quantum field theories often referred to as "vertex operators).

For the case of chiral field theories i.e. nets indexed by the intervals on a circle one may also develop the superselection theory by studying 2-interval algebras. In that case one does not have to leave the vacuum representation because the information about the superselection sectors is contained in the violation of Haag duality i.e. in the nontrivial inclusion [14]

$$\mathcal{A}(E) \subset \mathcal{A}(E')' \tag{90}$$
$$E = I_1 \cup I_2,\ I_i\ disjoint$$
$$E' = S^1 \backslash E$$

The content of this double interval inclusion may be be canonically reprocessed into an endomorphism. The latter in turn is isomorphic to the so-called Longo-Rehren (LR) endomorphism which for rational (finite number of DHR sectors) theories has the form

$$\rho_{LR} = \sum_i \rho_i \otimes \rho_i^{opp} on\ \mathcal{A} \otimes \mathcal{A}^{opp} \tag{91}$$

This mathematical structure corresponds to the physical picture that although in the vacuum representation the global charge vanishes, the two interval situation allows for a charge/conjugate charge (or particle/antiparticle) split in which all existing local superselection charges participate in a democratic fashion [1]. This is yet another manifestation of the doctrin that all physical information resides in the vacuum representation of observable nets which has no analog in QM.

It is interesting to note that simple particle physics ideas which date back to the beginnings of AQFT, as the working hypothesis that the vacuum state on nets of algebras contains all of particle physics and only needs the right mathematical tools in order to explicitly reveal its complete content, are now being vindicated by beautiful operator algebraic methods.

2.3. Example: Modular Construction of Interaction-Free Nets. In this section I will briefly sketch how one obtains the interaction-free local net operator algebras directly from the Wigner particle theory without passing through pointlike fields.

For the purpose of explanatory simplicity we start from the complex wave function (momentum) space of the $(m, s = 0)$ representation for a neutral (selfconjugate) scalar particle

$$H_{Wig} = \left\{ \varphi(p) | \int |\varphi(p)|^2 \frac{d^3p}{2\sqrt{p^2+m^2}} < \infty \right\} \tag{92}$$
$$(u(\Lambda, a)\varphi)(p) = e^{ipa}\varphi(\Lambda^{-1}p)$$

The first step consists in defining a real subspace which describes wedge-localized wave functions. For the construction of the standard t-x wedge $W_{st} = (x > |t|, y, z$ arbitrary) we use the x-t Lorentz boost $\Lambda_{x-t}(\chi)$ and the t-x reflection $r : (x, t) \to (-x, -t)$ which according to well-known theorems is represented antiunitarily in the Wigner theory[13]. One then starts from the unitary boost group $u(\Lambda(\chi)$ and forms (by the standard functional calculus) the unbounded "analytic continuation" in the rapidity χ. Using a notation which harmonizes with that used in the later Tomita-Takesaki modular theory, we define the following operators in H_{Wig}

$$\mathfrak{s} = \mathfrak{j}\delta^{\frac{1}{2}} \tag{93}$$
$$\mathfrak{j} = u(r)$$
$$\delta^{it} = u(\Lambda(-2\pi t))$$

where $u(\Lambda(\chi)$ and $u(r)$ are the unitary/antiunitary representations of these geometric transformations in the (doubled, if required by antiparticles of opposite charge) Wigner theory. Note that $u(r)$ is apart from a π-rotation around the x-axis the one-particle version of the TCP operator.

Since the antiunitary t-x reflection commutes with the t-x boost δ^{it}, it inverts the unbounded δ i.e. $j\delta = \delta^{-1}j$ which is formally the analytically continued boost at the imaginary value $t = -i$. As a result of this commutation relation the unbounded antilinear operator $\mathfrak{s}$ is involutive on its domain of definition i.e. $\mathfrak{s}^2 \subset 1$ so that it may be used to define a real subspace (closed in the real sense i.e. its complexification is not closed)

$$H_R(W) = \{\varphi \in H_{Wig} | \mathfrak{s}\varphi = \varphi\} \tag{94}$$

[13] In case of charged particles the Wigner theory needs a particle/antiparticle doubling.

These unusual properties, which are not met anywhere else in QT, encodes geometric localization properties within abstract operator domains [12] [15]. They also preempt the relativistic locality properties of QFT which Wigner looked for in his representation approach but without finding the correct one (he found instead the Newton-Wigner localization [1] which is not covariant[14]). The localization in the opposite wedge i.e. the $H_R(W^{opp})$ subspace turns out to correspond to the symplectic (or real orthogonal) complement of $H_R(W)$ in H i.e. $\mathrm{Im}(\psi, H_R(W)) = 0 \curvearrowright \psi \in H_R(W^{opp})$. One furthermore finds the following properties for the subspaces called "standardness"

$$\begin{aligned} &H_R(W) + iH_R(W) \textit{ is dense in } H \\ &H_R(W) \cap iH_R(W) = \{0\} \end{aligned} \tag{95}$$

The subspaces have instead the following covariance properties

$$u(a, \Lambda)H_R(W) = H_R(\Lambda W + a) \tag{96}$$

The last line expresses the covariance of this family of wedge-localized real subspaces and follows from the covariance of the operator $\mathfrak{s}$. Having arrived at the wedge localization spaces, one may construct localization spaces for smaller spacetime regions by forming intersections over all wedges which contain this region $\mathcal{O}$

$$H_R(\mathcal{O}) = \bigcap_{W \supset \mathcal{O}} H_R(W) \tag{97}$$

These spaces are again standard and covariant. They have their own "pre-modular" (the true Tomita modular operators appear below) object $\mathfrak{s}_{\mathcal{O}}$ and the radial and angular part $\delta_{\mathcal{O}}$ and $j_{\mathcal{O}}$ in their polar decomposition (93), but this time their action cannot be described in terms of spacetime diffeomorphisms since for massive particles the action is not implemented by a geometric transformation in Minkowski space. To be more precise, the action of $\delta_{\mathcal{O}}^{it}$ is only local in the sense that $H_R(\mathcal{O})$ and its symplectic complement $H_R(\mathcal{O})' = H_R(\mathcal{O}')$ are transformed onto themselves (whereas j interchanges the original subspace with its symplectic complement), but for massive Wigner particles there is no geometric modular transformation (in the massless case there is a modular diffeomorphism of the compactified Minkowski space). Nevertheless the modular transformations $\delta_{\mathcal{O}}^{it}$ for $\mathcal{O}$ running through all double cones and wedges (which are double cones "at infinity") generate the action of an infinite dimensional Lie group. Except for the finite parametric Poincaré group (or conformal group in the case of zero mass particles) the action is "fuzzy" i.e. not implementable by a diffeomorphism on Minkowski spacetime. The emergence of these *fuzzy acting Lie groups is a pure quantum phenomenon*; there is no analog for the classical mechanics of a particle. They describe hidden symmetries [41][39]) which the Lagrangian formalism does not expose.

Note also that the modular formalism characterizes the localization of subspaces, but (in agreement with particle localization measurements through counters) is not able to distinguish individual elements in that subspace. There is a good physical reason for that, because as soon as one tries to do that, one is forced to leave the unique Wigner (m, s) representation framework and pick a particular covariant wave functions by selecting one specific intertwiner

[14] The inappropriateness of the the Born probability interpretation for the definition of covariant spacetime localization is related to several problems (viz. the Klein paradox) one faces if one imposes QM concepts onto relativistic wave equations. In some way the radical difference between the local algebras $\mathcal{A}(\mathcal{O})$ (hyperfinite type III_1 von Neumann algebras) and quantum mechanical algebras is already preempted in the Wigner theory.

among the *infinite set* of u and v intertwiners which link the unique Wigner (m, s) representation to the countably infinite covariant possibilities [12]. In this way one would then pass to the framework of covariant fields explained and presented in the first volume of Weinberg's book on QFT[16]. The description of an individual wave function in $H_R(W)$ or $H_R(\mathcal{O})$ in the standard setting depends on the choice of covariant intertwiners[15]. A selection by e.g. invoking Euler equations and the existence of a Lagrangian formalism may be convenient for doing particular perturbative computations or as a mnemotechnical device for classifying polynomial interaction densities, but is not demanded as an intrinsic attribute of local quantum physics.

The way to avoid the highly nonunique covariant fields is to pass from real subspaces *directly* to von Neumann subalgebras of the algebra of all operators in Fock space $B(H_{Fock})$. This step is well-known. For integral spin s one defines with the help of the Weyl (or CAR in case of Fermions) functor $Weyl(\cdot)$ the local von Neumann algebras [12][15] generated from the Weyl operators as

$$\mathcal{A}(W) := alg\left\{Weyl(f)|f \in H_R(W)\right\} \tag{98}$$

a process which is sometimes misleadingly called "second quantization". These Weyl generators have the formal appearance

$$\begin{aligned} Weyl(f) &= e^{ia(f)} \\ a(f) &= \sum_{s_3=-s}^{s} \int (a^*(p, s_3) f_{s_3}(p) + h.c.) \frac{d^3p}{2\omega} \end{aligned} \tag{99}$$

i.e. unlike the covariant fields they are independent of the nonunique u-v intertwiners which appear in the definition of (m, s) non-unique covariant fields (109) and depend solely on the unique Wigner data. An analogue statement holds for the halfinteger spin case for which the CAR functor maps the Wigner wave function into the fermionic generators of von Neumann subalgebras. The particle statistics turns out to be already preempted by the pre-modular theory on Wigner space[16] [12] (see also additional remarks further down). The close connection of the Wigner particle structure via modular theory with localization makes it easier to understand why in the standard framework of particle physics it never has been possible to find a nonlocal alternative associated with an elementary length. Recent attempts based on noncommutative geometry certainly are outside the Wigner particle framework and their main problem is to maintain consistency with observed particle physics and its underlying principles.

The local net $\mathcal{A}(\mathcal{O})$ may be obtained in two ways, either one first constructs the spaces $H_R(\mathcal{O})$ via (97) and then applies the Weyl functor, or one first constructs the net of wedge algebras (98) and then intersects the algebras according to

$$\mathcal{A}(\mathcal{O}) = \bigcap_{W \supset \mathcal{O}} A(W) \tag{100}$$

[15] The ambiguity in the intertwiners covers only the linear part of choices of field coordinatizations. The full ambiguity is related to the Borchers class of relatively local fields which in the free case consists of all Wick-polynomial composites.

[16] This preempting of multiparticle statistcs and spacelike field commutations in the structure of one-particle wave functions (via premodular properties) is one of several surprising phenomena which indicate that relativistic wave function are closer to LQP then they are to Schrödinger wave functions. Another well-known indication is the so called Klein paradox which occurs if one couples relativistic wave functions to external fields.

The functorial mapping Γ between the orthocomplemented lattice of real Wigner subspaces and subalgebras of $B(H_{Fock})$ maps the above pre-modular operators into those of the Tomita-Takesaki modular theory

$$J, \Delta, S = \Gamma(\mathfrak{j}, \delta, \mathfrak{s}) \tag{101}$$

(for the fermionic CAR-algebras there is an additional modification by a "twist" operator). Whereas the "pre-modular" operators denoted by small letters act on the Wigner space, the modular operators J, Δ have an Ad action ($AdUA \equiv UAU^*$) on von Neumann algebras in Fock space which makes them objects of the Tomita-Takesaki modular theory

$$SA\Omega = A^*\Omega,\ S = J\Delta^{\frac{1}{2}} \tag{102}$$

$$Ad\Delta^{it}\mathcal{A} = \mathcal{A} \tag{103}$$

$$AdJ\mathcal{A} = \mathcal{A}'$$

The operator S is that of Tomita i.e. the unbounded densely defined normal operator which relates the dense set $A\Omega$ to the dense set $A^*\Omega$ for $A \in \mathcal{A}$ and gives J and $\Delta^{\frac{1}{2}}$ by polar decomposition. The nontrivial miraculous properties of this decomposition are the existence of an automorphism $\sigma_\omega(t) = Ad\Delta^{it}$ which propagates operators within $\mathcal{A}$ and only depends on the state ω (and not on the implementing vector Ω) and a that of an antiunitary involution J which maps $\mathcal{A}$ onto its commutant $\mathcal{A}'$. The theorem of Tomita assures that these objects exist in general if Ω is a cyclic and separating vector with respect to $\mathcal{A}$. An important thermal aspect of the Tomita-Takesaki modular theory is the validity of the Kubo-Martin-Schwinger (KMS) boundary condition [1]

$$\omega(\sigma_{t-i}(A)B) = \omega(B\sigma_t(A)),\ \ A, B \in \mathcal{A} \tag{104}$$

i.e. the existence of an analytic function $F(z) \equiv \omega(\sigma_z(A)B)$ holomorphic in the strip $-1 < Imz < 0$ and continuous on the boundary with $F(t-i) = \omega(B\sigma_t(A))$. The fact that the modular theory applied to the wedge algebra has a geometric aspect (with J equal to the TCP operator times a spatial rotation and $\Delta^{it} = U(\Lambda_W(2\pi t)))$ is not limited to the interaction-free theory [1]. These formulas are identical to the standard thermal KMS property of a temperature state ω in the thermodynamic limit if one formally sets the inverse temperature $\beta = \frac{1}{kT}$ equal to $\beta = -1$. This thermal aspect is related to the Unruh-Hawking effect of quantum matter enclosed behind event/causal horizons.

Our special case at hand, in which the algebras and the modular objects are constructed functorially from the Wigner theory, suggest that the modular structure for wedge algebras may always have a geometrical significance with a fundamental physical interpretation in any QFT. This is indeed true, and within the Wightman framework this was established by Bisognano and Wichmann [1].

If we had taken the conventional route via interwiners and local fields as in [16], then we would have been forced to use the Borchers construction of equivalence classes[17] [11] in order to see that the different free fields associated

[17] The class of local covariant free fields belonging to the same (m,s)-Wigner representation is a linear subclass of the full equivalence class of relative local fields associated with a free field which comprises all Wick-polynomials. Each cyclic field in that class generates the same net of algebras. In the analogy with coordinates in differential geometry this subclass corresponds to linear coordinate transformations.

with the (m,s) representation with the same momentum space creation and annihilation operators in Fock space are just different generators of the same coherent families of local algebras i.e. yield the same net. This would be analogous to working with particular coordinates in differential geometry and then proving at the end that the important objects of interests as the physical S-matrix are independent of the interpolating fields (i.e. independent of the "field-coordinatizations").

The above method can be extended to all (m,s) positive energy Wigner representations. The boost transformation for $s \neq 0$ has a nontrivial matrix part whose analytic continuation for the construction of Δ requires some care. It is very interesting to note that the spin-statistics connection can be already seen on the level of the pre-modular structure of the Wigner representation before one arrives at the operator algebras in Fock space.

It is interesting to note that not all positive energy Wigner representations will lead to compactly localized algebras with pointlike generating fields. The two notable exceptions are:

1. Wigner's famous "continuous spin" zero mass representations in which the two-dimensional euclidean fixed point group of a lightlike vector $p = (1,0,0,1)$ is faithfully represented (which, different from the helicity of the photon-neutrino family, requires an infinite dimensional Hilbert space). The spaces $H_R(\mathcal{O})$ are trivial for compact $\mathcal{O}$ i.e. the intersection of the nontrivial wedge spaces (97) only contains the zero vector.

2. The Wigner representation theory for massive particles in d=2+1 admits any spin value ("any"-ons). For $s \neq (half)integer$ the spaces $H_R(\mathcal{O})$ are trivial if $\mathcal{O}$ is compact and nontrivial if $\mathcal{O}$ is a spacelike cone. For $s = (half)integer$ the double cone spaces $H_R(\mathcal{O})$ are nontrivial as in higher dimensions.

The general pre-modular theory for positive energy representations allows to prove [15] the standardness and nontriviality of $H_R(W)$ and $H_R(W_1 \cap W_2) \equiv H_R(W_1) \cap H_R(W_2)$ for two orthogonal $W_i's$, but the nontriviality of any smaller noncompact or compact region depends on the nature of the stability group of a physical (positive energy) momentum. The optimal noncompact localization properties of the famous Wigner continuous spin positive energy representations have not been investigated. Whether one can relate physically acceptable objects with these irreducible Wigner representations depends very much on the answer to the best possible localization properties.

It is easy to see that for any case s≠integer there is a mismatch between the geometrically opposite and the symplectic opposite i.e.

$$H_R(W') \neq H_R(W)^{'} \tag{105}$$

$$W' = W^{opp} = Rot(\pi)W$$

$$H_R(W)^{'} = TH_R(W') \tag{106}$$

One needs an additional "twist" T in order to transform one into the other. The distinction between the geometric and the symplectic opposite in H_{Wig} i.e. the appearance of T is also the reason why the Weyl functor is only appropriate for integer spin. For halfinteger spin for which T turns out to be multiplication by i the geometric complement suggest to look at the complement in the sense of the real bilinear form $Imi(f,g) \simeq Re(f,g)$. Without going into details we mention that this modification leads entails the necessity to use the CAR functor for fermions in case of halfinteger spin. The Fock space version of the multiplication with i turns out to be the twist operator appearing in the DHR work on Fermions [1]. But whereas for s=halfinteger this twist does not force the compact localization spaces to be trivial and only changes the multiparticle symmetrized tensor products into the antisymmetric ones, the twist for anyonic spin has quite different more dramatic consequences for the localization and the multiparticle structure. As we have already seen the localization cannot be better than semiinfinite string-like and as far as the multiparticle structure is concerned one can show that it cannot be described by a tensor product at all if one wants sharper than wedge localizations. This follows from a No-Go theorem by [17] who proved that spacelike cone localized anyonic fields which have nonvanishing matrix elements between the vacuum and the anyonic spin one-particle state and fulfill braid group commutation relations (which they are required to do by the general spin-statistics theorem) cannot be fields which applied to the vacuum create a pure one-particle state vector without the admixture of vacuum polarization components. In the terminology of the last section one may say there are no spacelike cone localized anyonic PFG's. Such one particle creating operators only exist for the larger wedge localization, a fact which strongly suggest to use the Wigner description only for the construction of the wedge algebras of "free" anyons and use those to descend to smaller localizations by the method of intersections. Normally the presence of these particle-antiparticle "clouds" in addition to one-particle components are thought of as a characteristic property of the interaction, but here they are caused by the braid group statistics even in the absence of genuine interactions. The distinction of *free versus interacting* based on Lagrangian quantization is clearly not very appropriate in such a situation.

It is of paramount importance to explicitly construct these free anyonic fields for a given spin-statistics structure. Even though they are like Boson/Fermion fields uniquely determined by their Wigner particle structure, some of the conceptual problems in their explicit construction are still open. Their physical importance results from the fact that besides braid group statistics and the related quantum symmetries there is nothing else which distinguishes LQP in low dimensions from Fermions/Bosons in d>1+2. The nonrelativistic limit does not eradicate the braid group statistics inasmuch as the Fermi/Bose alternative and the spin-statistics connection is not lost in this limit. Hence d=1+2 braid group statistics particles cannot be descibed by QM even if one is only interested in their nonrelativistic behavior. The complementary statement (which sounds more provocative) would be to say that QM of Bosons/Fermions owes its physical relevance to the fortuitous fact that there are relativistic fields (namely ordinary free fields) which create one-particle state vectors without any vacuum polarization admixture.

So if the new phenomena of high T_c-superconductivity and the fractional quantum Hall effect are characteristic for low (two spatial) dimensions they should be related to the braid group spin-statistics structure and hence outside the range of the standard Lagrangian quantization approach. The appearance of amplification factors from statistical dimension for plektons (of potential use for high T_c) and rational statistical phases (of potential use in the fractional Hall effect) are very encouraging, but there is still a long way to go before the quantum field theory of anyons/plektons and their electromagnetic couplings to external fields is understood on the level of say the understanding of the electromagnetic properties of Dirac Fermions. The present understanding of the fractional Hall effect has been obtained via the "edge current" approximation in which the fractional statistics effect enters via the simpler statistics structure of chiral theories.

In the presence of interactions, the structure of the wedge algebras is not only determined by the Wigner theory but the S-matrix also enters in the characterization of wedge-localized state vectors. There exist however wedge-localized operators which, if only applied once to the vacuum, create a one-particle state vector; whereas for any smaller localization region this would not be compatible with the presence of interactions unless there are in addition vacuum polarization clouds. In certain interacting cases in low spacetime dimensions the "polarization free generators" (PFG's) have nice (temperedness) analytic properties which keep them close to free systems; in fact their Fourier transforms obey a Faddeev-Zamolodchikov algebra. In the last section we will explain this situation in some more detail.

In passing we briefly remind the reader of the standard way of combining the Wigner particle picture with Einstein causality through the introduction of pointlike covariant "field coordinatizations".

The covariant field construction is synonymous with the introduction of intertwiners between the unique Wigner (m,s) representation and the multitude of Lorentz-covariant momentum-dependent spinorial (dotted and undotted) tensors, which under the homogenous L-group transform with the irreducible $D^{[A,B]}(\Lambda)$ matrices.

$$u(p)D^{(s)}(R(\Lambda,p)) = D^{[A,B]}(\Lambda)u(\Lambda^{-1}p) \tag{107}$$

The only restriction imposed by this intertwining is:

$$\mid A-B\mid \leq s\leq A+B \tag{108}$$

This leaves many A,B (half integer) *choices* for a given s. Here the $u(p)$ intertwiner is a rectangular matrix consisting of $2s+1$ column vectors $u(p,s_3),\ s_3=-s,...,+s$ of length $(2A+1)(2B+1)$. Its explicit construction using Clebsch-Gordan methods can be found in Weinberg's book [16]. Analogously there exist antiparticle (opposite charge) $v(p)$ intertwiners: $D^{(s)*}(R(\Lambda,p)\longrightarrow D^{[A,B]}(\Lambda)$. The covariant field is then of the form:

$$\begin{aligned}\psi^{[A,B]}(x) = \frac{1}{(2\pi)^{3/2}}\int\{e^{-ipx}\sum_{s_3}u(p_1,s_3)a(p_1,s_3) + \\ + e^{ipx}\sum_{s_s}v(p_1,s_3)b^*(p_1,s_3)\}\frac{d^3p}{2\omega}\end{aligned} \tag{109}$$

where $a^{\#}$ and $b^{\#}$ are the creation/annihilation operators for particles/antiparticles, i.e. the n-fold application of the particle/antiparticle creation operators generate the symmetrized (for integer spin) or antisymmetrized (for half-integer spin) tensor product subspaces of Fock space.

Since the range of the A and B (undotted/dotted) spinors is arbitrary apart from the fact that they must fulfil the inequality (108) with respect to the given physical spin s[18], the number of covariant fields is countably infinite. Fortunately it turns out that this loss of uniqueness does not cause any harm in particle physics. If one defines the polynomial *-algebras $\mathcal{P}(\mathcal{O})$ as the operator algebras generated from the smeared field with Schwartz test functions of support *supp* $f\in\mathcal{O}$ [11]

$$P(\mathcal{O}) =^* -alg\left\{\psi(f)\,|\,suppf\subset\mathcal{O}\right\}$$

one realizes that these localized algebras do not depend on the representative covariant field chosen from the (m,s) class. In fact all the different covariant fields which originate from the (m,s) representation share the same

[18] For the massless case the helicity inequalities with respect to the spinorial indices are more restrictive, but one Wigner representation still admits a countably infinite number of covariant representations.

creation/annihilation operators. This gave rise to the linear part of the Borchers equivalence classes of relatively local fields. The full Borchers class [11] generalized the family of Wick polynomials to the realm of interactions and gave a structural explanation of the insensitivity of the S-operator. Although the local operator algebras cannot be directly obtained from the fields, the polynomial algebras of the latter are (under some weak domain assumptions) affiliated to the von Neumann algebras $\mathcal{A}(\mathcal{O})$

An important property of free fields which fulfill an equation of motion is the validity of the quantum version of the Cauchy initial value problem. The algebraic counterpart is the causal shadow property (see beginning of this section) which for simple connected spacetime regions $\mathcal{O}$ reads

$$\mathcal{A}(\mathcal{O}) = \mathcal{A}(\mathcal{O}'') \tag{110}$$

where $\mathcal{O}'$ denotes the causal complement and the causal complement of the causal complement is the causal completion (or causal shadow) of $\mathcal{O}$. As stated previously the causal completion of a piece of timeslice or a piece of spacelike hypersurface is the double cone subtended by those regions. In order to derive this property one does not have to invoke the Cauchy initial value problem of pointlike fields; it is a functorial consequence of an analog property of localized real subspaces of H_W

$$H_R(\mathcal{O}) = H_R(\mathcal{O}'') \tag{111}$$

If a higher dimensional theory which fulfills the causal shadow property is restricted to a lower dimensional manifold containing the time direction (this is sometimes called a (mem)brane), then one obtains a physically unacceptable theory in which, as one moves upward in time, new degrees of freedom enter sideways. On the other hand if one tries to extend a free theory in a brane to the ambient space the resulting theory is only causally consistent if the objects are independent of the transversal directions in the ambient space. If the degrees of freedom in the brane are pointlike fields, the degrees of freedom in the ambient fields are the same, they just look like spacelike strands going into the transverse direction[19]. So the extension into an ambient world are not described by standard field degrees of freedom (for a more see the last section).

Another problem which even in the Wigner setting of noninteracting particles has not yet been solved is the pre-modular theory for disconnected or topologically nontrivial regions e.g. in the simplest case for disjoint double intervals of the massless s=$\frac{1}{2}$ model on the circle. This could be the first inroad into the terra incognita of nongeometric "quantum symmetries" of purely modular origin without a classical counterpart.

2.4. Appendix A: Coherence Relations involving Exchange Operators. With the help of the definition (79) of the exchange operator $\varepsilon(\rho_1, \rho_2) \in (\rho_2\rho_1|\rho_1\rho_2)$ in terms of charge transporters one can derive a set of consistency relations which are most easily remembered in form of their graphical representations. Irreducible endomorphisms are represented by vertical lines, the later acting ones to the right of the former acting. Intertwiners are represented by graphs, an intertwiner $T \in (\rho_1...\rho_n|\rho_1'...\rho_m')$ has m lines which enter from below and n lines which leave above. The multiplication $S \circ T$ is represented by juxtaposing the S-graph on top of the T-graphs (only defined for matching source lines of S with the range lines of T, however note that if necessary left lines may be

[19]These transversal strands are not the "dynamical strings" of string theory because the latter have more degrees of freedom than fields.

added for matching without changing the operator). The graph of T^* is the upside-down mirror image of that of T. The flip $\varepsilon(\rho_1,\rho_2)$ is represented by ρ_1-line which passes from right down to left above overneath a ρ_2-line which in turn starts from from left down to right above underneath the ρ_1 line (indicated by a breaking of the ρ_2 line around the point of crossing). The graphical representation of an action of ρ on an intertwiner T i.e. $\rho(T)$ is a ρ-line on the right of the T-graph. Since the basic T_e-vertices with $\rho' s$ from the reference set ∇_0 form a complete set, any intertwiner can in principle be written as a linear combination of products of $T_e' s$ and their Hermitian adjoints.

After these graphical rules have been justified one can immediately check that the composite exchange operators obey the following formulas ([18][19])

$$\begin{aligned} \varepsilon(\rho_3,\rho_1\rho_2) &= \rho_1(\varepsilon(\rho_3,\rho_2))\varepsilon(\rho_3,\rho_1) \\ \varepsilon(\rho_1\rho_2,\rho_3) &= \varepsilon(\rho_1,\rho_3)\rho_1(\varepsilon(\rho_2,\rho_3)) \end{aligned} \tag{112}$$

whereas the ε together fulfills the following coherence relations with intertwiners $T \in (\rho_2|\rho_1)$

$$\begin{aligned} \rho_3(T)\varepsilon(\rho_1,\rho_3) &= \varepsilon(\rho_2,\rho_3)T \\ \rho_3(T)\varepsilon(\rho_3,\rho_1)^* &= \varepsilon(\rho_3,\rho_2)^*T \end{aligned} \tag{113}$$

The proof is straightforward and uses in addition to the graphical rules of the T_e intertwiners and the action of the ρ the representation (79) in terms of a trivial crossing for the exchange of the spectator endomorphisms which again allows for a graphical representation. It is much simpler to remember these intertwiner relations in terms of their graphs.

There are some special cases which, because of their importance will be separately mentioned. One is the exchange-fusion (pentagon) relation which is the last formula for $T = T_e$ i.e. one of the basic intertwiners. For $T = R \in (\bar\rho\rho|id)$ and $\varepsilon(\rho_1,\rho_2) = 1$ if one of the $\rho_i' s$ is id, one gets

$$\rho_3(R) = \varepsilon(\bar\rho\rho,\rho_3)R = \varepsilon(\rho_3,\bar\rho\rho)^*R \tag{114}$$

Finally there is the famous Artin relation (adapted to colored braids)

$$\rho_3(\varepsilon(\rho_1,\rho_2))\varepsilon(\rho_1,\rho_3)\rho_1(\varepsilon(\rho_2,\rho_3)) = \varepsilon(\rho_2,\rho_3)\rho_2(\varepsilon(\rho_1,\rho_3))\varepsilon(\rho_1,\rho_2) \tag{115}$$

The usual braid group and the usual Artin relation results from specialization to one color $\rho_i \equiv \rho$, $i = 1, 2, 3$ in which case the Artin generators of the braid B_n group on n+1 strands are $g_k = \rho^{k-1}(\varepsilon)$ $k = 1...n$ with $\varepsilon \equiv \varepsilon(\rho,\rho)$ which fulfill the Artin braid relations

$$g_k g_{k+1} g_k = g_{k+1} g_k g_{k+1} \tag{116}$$

Note that this construction from a local net of observable algebras represents the Artin generators as composites of charge transporters and endomorphisms acting on them, so that the Artin relations are a consequence of the more basic relations between charge transporters. This is possible because the braid structure is embedded in a the ambient net of algebras which has a very rich algebraic structure. In particular the braid group structure comes equipped with a natural representation structure in terms of Markov traces on the B_∞ group algebra, which is the subject of the next appendix.

Remark 10. *The braid group B_n (and its special case the permutation group S_n) has a natural inclusive structure $B_n \subset B_{n+1}$ which permits to take the inductive limit B_∞. This property is related to its importance for particle statistics. Particle physics fulfills the so-called cluster property: the physics of n particles is contained in that of n+1 particles) and results from the latter by shifting one particle to spatial infinity, thus converting it into a "spectator". Particle statistics is the discrete structure which remains after one removes the localization aspect and the relic of the cluster property is reflected in the inclusive aspect of the statistics group and in the Markov property of the Markov trace on B_∞.*

Often the matrix representors of the braid group relations (116) are called Yang-Baxter relations but this is neither physically nor mathematically correct; physically, because the more complicated true Yang-Baxter relation belong to the concept of scattering theory and not to particle/field statistics and mathematically because whereas the representation theory of braids and knots is a well established area of the V. Jones subfactor theory, the Yang-Baxter relation have yet no firm position in mathematics (despite serious attempts to get to that structure by "Baxterization" of braid group representations).

2.5. Appendix B: Classification of admissable $\mathbf{B_\infty}$ Representations. The charge-carrying fields, which in the LQP setting are operators in the field bundle, form an exchange algebra in which R-matrices which represent the infinite braid group B_∞ appear. The admissable physical representations define a so called Markov trace on the braid group, a concept which was introduced by V. Jones but already had been used for the special case of the permutation group S_∞ in the famous 1971 work of Doplicher, Haag and Roberts [1]. Since this very physical method has remained largely unknown[20] outside a small circle of specialists, its renewed presentation in this appendix may be helpful

In this classification approach one starts with fusing and decomposing braided endomorphisms. The simplest case is a basic irreducible endomorphism ρ whose iteration leads to a "two channel" irreducible decomposition [18]

$$\rho^2 \simeq id \oplus \rho_1 \tag{117}$$
$$i.e.\ \left[\rho^2\right] = [id] \oplus [\rho_1]$$

where id denotes the identity endomorphism. This is the famous case leading to the Jones-Temperley-Lieb algebra, whereas the more general two-channel case

$$\rho^2 \simeq \rho_1 \oplus \rho_2 \tag{118}$$

gives rise to the Hecke algebra. Finally the special 3-channel fusion [21]

$$\rho^2 \simeq id \oplus \rho_1 \oplus \rho_2 \tag{119}$$

yield the so-called Birman-Wenzl algebra.

Each single case together with the Markov trace yields of a wealth of braid group representations. The first case comprises all the selfconjugate minimal models and is asymptotically associated (see below) with $SU(2)$ which is

[20] Particle physicists who are very familiar with group theory use a deformation theory known as the "quantum group" method. Although its final results are compatible with the structure of quantum theory, the intermediate steps are not (no Hilbert space and operator algebras, appearance of null-ideals). The present method is "quantum" throughout.

a pseudo self-conjugate group, whereas the second is similarly associated with $SU(n)$ for $n > 2$. Finally the third one has an assoiation with SO(n). There are of course also isolated exceptional fusion laws which do not produce families and whose basic fusion law cannot be viewed as arising from looking at higher composites of the previous families. In all such cases one finds a "quantization" from the positivity of the Markov-trace [7]; in the first case this is the famous Jones quantization. All cases have realizations in chiral QFT as exchange algebra (or reduced field bundle) operators associated with the current or W observable algebras.

The classification of the admissable braid group representation associated to the above fusion laws (and the associated knot- and 3-manifold- invariants) is a purely combinatorial problem of which a simpler permutation group version (for which only (118) occurs) was already solved in 1971 by DHR [1]. The method requires to study tracial states on the mentioned abstract C^*-algebras and the resulting concrete von Neumann algebras are factors of type II_1. These operator algebras which are too "small" in order to be able to carry continuous translations and allowing localization are often referred to as "topological field theories". In the present approach these combinatorial data are part of the superselection structure. If combined with the nature of the charge-carrying fields i.e. the information whether they form multiplets as in the case of current algebras or whether there are no such group theoretic multiplicities the have the same R-matrices and the same statistical dimensions (quantum dimensions) but their statistical phases and therefore their anomalous dimensions may be different. The numerical R-matrices determined from the Markov trace formalism fix the structure of the exchange algebras.

The DHR-Jones-Wenzl technique constructs the tracial states via iterated application of the left inverse of endomorphisms (or by the iteration of the related V. Jones basic construction in subfactor theory). Under the assumption of irreducibility of ρ (always assumed in the rest of this section) the previously introduced left inverse ϕ maps the commutant of $\rho^2(\mathcal{A})$ in $\mathcal{A}$ into the complex numbers:

$$\phi(A) = \varphi(A)\underline{1}, \quad A \in \rho^2(\mathcal{A})' \tag{120}$$

and by iteration a faithful tracial state φ on $\cup_n \rho^n(\mathcal{A})'$ with:

$$\phi^n(A) = \varphi(A)\underline{1}, \quad A \in \rho^{n+1}(\mathcal{A})'$$
$$\varphi(AB) = \varphi(BA), \quad \varphi(\underline{1}) = 1$$

Restricted to the $\mathbf{C}RB_n$ algebra generated by the ribbon braid-group which is a subalgebra of $\rho^n(\mathcal{A})'$ the φ becomes a tracial state, which can be naturally extended ($B_n \subset B_{n+1}$) to $\mathbf{C}RB_\infty$ in the above manner and fulfills the "Markov-property":

$$\varphi(a\sigma_{n+1}) = \lambda_\rho \varphi(a), \quad a \in \mathbf{C}RB_n \tag{121}$$

The terminology is that of V. Jones and refers to the famous Russian probabilist of the last century as well as to his son, who among other things constructed knot invariants from suitable functionals on the braid group. The "ribbon" aspect refers to an additional generator τ_i which represents the vertical 2π rotation of the cylinder braid group ($\simeq$ projective representation of B_n) [18][19].

It is interesting to note in passing that the Markov-property is the combinatorial relict of the physicist's cluster property which relates the n-point correlation function in local QFT to the n-1 point correlation (or in QM the physics of n particles to that of n-1 by converting one of the particles into a spectator by removing it to infinity. This Russian "matrushka" structure of inclusive relations requires to deal with the inductive limit B_∞ of the B_n braid groups. This picture is similar to that of cluster properties which was already used in our attempts to describe the QM statistics in the first section. The existence of a Markov trace on the braid group of (low dimensional) multi-particle statistics is the imprint of the cluster property on particle statistics. As such it is more basic than the notion of internal symmetry. It precedes the latter and according to the DR theory it may be viewed as the other side of the same coin on which one side is the old (compact group-) or new (quantum-) symmetry. With these remarks the notion of internal symmetry which historically started with Heisenberg's isospin in nuclear physics becomes significantly demystified.

Let us now return to the above 2-channel situation [18]. Clearly the exchange operator ε_ρ has maximally two different eigenprojectors since otherwise there would be more than two irreducible components of ρ^2. On the other hand ε_ρ cannot be a multiple of the identity because ρ^2 is not irreducible. Therefore ε_ρ has exactly two different eigenvalues λ_1, λ_2 i.e.

$$(\varepsilon_\rho - \lambda_1 \underline{1})(\varepsilon_\rho - \lambda_2 \underline{1}) = 0 \tag{122}$$

$$\Longleftrightarrow \varepsilon_\rho = \lambda_1 E_1 + \lambda_2 E_2 \ , \quad E_i = (\lambda_i - \lambda_j)^{-1} (\varepsilon_\rho - \lambda_j) \, , \quad i \neq j \tag{123}$$

which after the trivial re-normalization of the unitaries $g_k := -\lambda_2^{-1} \rho^{k-1}(\varepsilon_\rho)$ yields the generators of the Hecke algebra:

$$\begin{aligned} g_k g_{k+1} g_k &= g_{k+1} g_k g_{k+1} \\ g_k g_l &= g_l g_k \ , \quad |j-k| \geq 2 \\ g_k^2 &= (t-1) g_k + t \ , \quad t = -\frac{\lambda_1}{\lambda_2} \neq -1 \end{aligned} \tag{124}$$

The physical cluster property in the algebraic form of the existence of a tracial Markov state leads to a very interesting "quantization" [21] which was first pointed out by V. Jones [7]. Consider the sequence of projectors:

$$E_i^{(n)} := E_i \wedge \rho(E_i) \wedge ... \wedge \rho^{n-2}(E_i) \ , \quad i = 1, 2 \tag{125}$$

and the symbol $\wedge$ denotes the projection on the intersection of the corresponding subspaces. The notation is reminiscent of the totally antisymmetric spaces in the case of Fermions. The above relation $g_1 g_2 g_1 = g_2 g_1 g_2$ and $g_1 g_n = g_n g_1$, $n \geq 2$ in terms of the E_i reads:

$$\begin{aligned} E_i \rho(E_i) E_i - \tau E_i &= \rho(E_i) E_i \rho(E_i) - \tau \rho(E_i) \ , \quad \tau = \frac{t}{(1+t)^2} \\ E_i \rho^n(E_i) &= \rho^n(E_i) E_i \ , \quad n \geq 2 \end{aligned} \tag{126}$$

The derivation of these equations from the Hecke algebra structure is straightforward. The following recursion relation [20] of which a special case already appeared in the DHR work [1] is however tricky and will be given in the sequel

[21] In these notes we use this concept always in the original meaning of Planck as a spectral discretization, and not in its form as a deformation.

Proposition 11. *The projectors $E_i^{(n)}$ fulfill the following recursion relation ($t = e^{2\pi i\alpha}$, $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$) :*

$$E_i^{(n+1)} = \rho(E_i^{(n)}) - \frac{2\cos\alpha \sin n\alpha}{\sin(n+1)\alpha}\rho(E_i^{(n)})E_j\rho(E_i^{(n)}) \ , \quad i \neq j, \quad n+1 < q \tag{127}$$
$$E_i^{(q)} = \rho(E_i^{(q-1)}) \quad , \quad q = \inf\{n \in \mathbf{N}, n\,|\alpha| \geq \pi\} \;\; if \;\alpha \neq 0,\; q = \infty \; if \; \alpha = 0$$

The DHR recursion for the permutation group S_∞ is obtained for the special case t=0 i.e. $\alpha = 0$. In this case the numerical factor in front of product of three operators is $\frac{n}{n+1}$.

The proof is by induction. For n=1 the relation reduces to the completeness relation between the two spectral projectors of ε_ρ : $E_i = 1 - E_j$, $i \neq j$. For the induction we introduce the abbreviation $F = E_j\rho(E_i^{(n)}) = \rho(E_i^{(n)})E_j$ and compute F^2. We replace the first factor $\rho(E_i^{(n)})$ according to the induction hypothesis by:

$$\rho(E_i^{(n)}) = \rho^2(E_i^{(n-1)}) - \frac{2\cos\alpha\sin(n-1)\alpha}{\sin n\alpha}\rho^2(E_i^{(n-1)})\rho(E_j)\rho^2(E_i^{(n-1)}) \tag{128}$$

We use that the projector $\rho^2(E_i^{(n-1)})$ commutes with the algebra $\rho^2(\mathcal{A})'$ (and therefore with $E_j \in \rho^{(2)}(\mathcal{A})'$), and that its range contains that of $\rho(E_i^{(n)})$ i.e. $\rho^2(E_i^{(n-1)})\rho(E_i^{(n)}) = \rho(E_i^{(n)})$. Hence we find:

$$F^2 = E_j\rho(E_i^{(n)}) - \frac{2\cos\alpha\sin(n-1)\alpha}{\sin n\alpha}\rho^2(E_i^{(n-1)})E_j\rho(E_j)E_j\rho(E_i^{(n)}) \tag{129}$$

Application of (126) with $\tau = \frac{1}{2\cos\alpha}$ to the right-hand side yields:

$$F^2 = E_j\rho(E_i^{(n)}) - \frac{\sin(n-1)\alpha}{2\cos\alpha\sin\alpha}\rho^2(E_i^{(n-1)})E_j\rho(E_i^{(n)}) = \frac{\sin(n+1)\alpha}{2\cos\alpha\sin n\alpha}F \tag{130}$$

where we used again the above range property and a trigonometric identity. For $n = q-1$ the positivity of the numerical factor fails and by $F^2E_j = (FF^*)^2$ and $FE_j = FF^*$ the operator F must vanish and hence E_j is orthogonal to $\rho(E_j^{(q-1)})$ which is the second relation in (127). For $n < q-1$ the right-hand side of the first relation in (127) with the help of (130) turns out to be a projector which vanishes after multiplication from the right with $\rho^k(E_j), k = 1, ..., n-2$ as well as with E_j. The remaining argument uses the fact that this projector is the largest with this orthogonality property and therefore equal to $E_i^{(n+1)}$ by definition of $E_i^{(n+1)}$ Q.E.D.

The recursion relation (127) leads to the desired quantization after application of the left inverse ϕ :

$$\phi(E_i^{(n+1)}) = E_i^{(n)}\left(1 - \frac{2\cos\alpha\sin n\alpha}{\sin(n+1)\alpha}\eta_j\right), \quad i \neq j \tag{131}$$
$$\eta_j = \phi(E_j),\; 0 \leq \eta_j \leq 1,\; \eta_1 + \eta_2 = 1$$

¿From this formula one immediately recovers the permutation group DHR quantization in the limit $\alpha \to 0$. In that case positivity of the bracket restricts η_j to the values $\frac{1}{2}(1 \pm \frac{1}{d}),\; d \in \mathbf{N} \cup 0$ and the resulting permutation group representation is associated to the $SU(d)$-group. For $\alpha \neq 0$ one first notes that from the second equation (127) one obtains (application of ϕ):

$$\eta_j E_i^{(q-1)} = \phi(E_j\rho(E_i^{(q-1)})) = \phi(E_jE_i^{(q)}) = 0, \quad i \neq j \tag{132}$$

where the vanishing results from the orthogonality of the projectors. Since $\eta_1 + \eta_2 = 1$ we must have $E_i^{(q-1)} = 0$ for i=1,2, q$\geq$ 4, because $E_i^{(q-1)} \neq 0$ would imply $\eta_j = 0$ and $E_j^{(q-1)} = 0$. This in turn leads to $E_j \equiv E_j^{(2)} = 0$

which contradicts the assumption that ε_ρ possesses two different eigenvalues. This is obvious for $q=3$ and follows for $q>3$ from the positivity of ϕ (131) for n=2:

$$\phi(E_j^{(3)}) = -\frac{\sin\alpha}{\sin 3\alpha} E_j^{(2)} \quad \curvearrowright E_j^{(2)} = 0 \quad \curvearrowright E_i^{(q-1)} = 0,\ i=1,2,\ q\geq 4 \tag{133}$$

Using (131) iteratively in order to descend in n starting from $n=q-2$, positivity demands that there exists an $k_i \in \mathbf{N}, 2\leq k_i \leq q-2$, with:

$$\eta_i = \frac{\sin(k_i+1)\alpha}{2\cos\alpha\sin k_i\alpha},\ i=1,2 \quad \curvearrowright \sin(k_1+k_2)\alpha = 0 \tag{134}$$

where the relation results from summation over i. Since the only solutions are $\alpha = \pm\frac{\pi}{q}$, $k_1 = d$, $k_2 = q-d$, $d\in N$, $2\leq d\leq q-2$, one finds for the statistics parameters of the plektonic 2-channel family the value:

$$\lambda_\rho = \sum_{i=1}^{2} \lambda_i \eta_i = -\lambda_2\left[(t+1)\eta_1 - 1\right] = -\lambda_2 e^{\pm\pi i(d+1)/q} \frac{\sin\pi/q}{\sin d\pi/q} \tag{135}$$

a formula which allows for a nice graphical representation. We have established the following theorem:

Theorem 12. *Let ρ be an irreducible localized endomorphism such that ρ^2 has exactly two irreducible subrepresentations. Then [18]:*

- ε_ρ has two different eigenvalues λ_1, λ_2 with ratio

$$\frac{\lambda_1}{\lambda_2} = -e^{\pm 2\pi i/q}, \quad q\in\mathbf{N}\cup\{\infty\},\ q\geq 4 \tag{136}$$

- The modulus of the statistics parameter $\lambda_\rho = \phi(\varepsilon_\rho)$ has the possible values

$$|\lambda_\rho| = \begin{cases} \frac{\sin\pi/q}{\sin d\pi/q}, & q<\infty \\ \frac{1}{d}, 0 & q=\infty \end{cases} ,d\in N,\ \ 2\leq d\leq q-2 \tag{137}$$

- The representation $\varepsilon_\rho^{(n)}$ of the braid group B_n which is generated by $\rho^{(k-1)}(\varepsilon_\rho)$, $k=1,...,n-1$ in the vacuum Hilbert space is an infinite multiple of the Ocneanu-Wenzl representation tensored with a one dimensional (abelian) representation. The projectors $E_2^{(m)}$ and $E_1^{(m)}$ are "cutoff" (vanish) for $d<m\leq n$ and $q-d<m\leq n$ respectively

- The iterated left inverse $\varphi = \phi^n$ defines a Markov trace tr on B_n :

$$tr(b) = \varphi\circ\varepsilon_\rho(b) \tag{138}$$

The "elementary" representation which is characterized by two numbers d and q gives rise to a host of composite representation which appear if one fuses the ρ, ρ_1, ρ_2 and reduces and then iterates this process with the new irreducible ρ' etc. We will not present the associated composite braid formalism. With the same method one can determine the statistical phases up to an anyonic (abelian) phase. In order to have a unique determination, one needs (as in the original DHR work) an information on the lowest power of ρ which contains the identity endomorphism (the vacuum representation) for the first time. A special case of this is $\rho^2\supset id$ i.e. the selfconjugate Jones-Temperley-Lieb fusion. Here we will not present these computations of phases.

The problem of 3-channel braid group statistics [21] has also been solved with the projector method in case that one of the resulting channels is an automorphism τ:

$$\rho^2 = \rho_1 \oplus \rho_2 \oplus \tau \tag{139}$$

In that case ε_ρ has 3 eigenvalues μ_i which we assume to be different:

$$(\varepsilon_\rho - \mu_1)(\varepsilon_\rho - \mu_2)(\varepsilon_\rho - \mu_3) = 0 \tag{140}$$

The relation to the statistics phases ω_ρ, ω_i is the following: $\mu_i^2 = \frac{\omega_i}{\omega^2}$. In addition to the previous operators $G_i = \rho^{i-1}(\varepsilon_\rho) = (G_i^{-1})^*$ we define projectors:

$$E_i = \rho^{i-1}(TT^*) \tag{141}$$

where $T \in (\rho^2 \, |\tau)$ is an isometry and hence E_i are the projector onto the eigenvalue $\lambda_3 = \lambda_\tau$ of G_i. In fact one finds the following relations between the G_i and E_i :

$$\begin{aligned} E_i &= \frac{\mu_3}{(\mu_3 - \mu_1)(\mu_3 - \mu_2)}(G_i - (\mu_1 + \mu_2) + \mu_1\mu_2 G_i^{-1}) \\ E_i G_i &= \mu_3 E_i \end{aligned} \tag{142}$$

This together with the trilinear relations between the $G_i's$ and $E_i's$ as well as the commutativity of neighbors with distance ≥ 2 gives (upon a renormalization) the operators g_i and e_i which fulfill the defining relation of the Birman-Wenzl algebra which again depends on two parameters. The Markov tracial state classification again leads to a quantization of these parameters except for a continuous one-parameter solution with statistical dimension $d = 2$ which is realized in conformal QFT as sectors on the fixed point algebra of the $U(1)$ current algebra (which has a continuous one-parameter solution) under the action of the charge conjugation transformation (often called "orbifolds" by analogy to constructions in differential geometry).

Finally one may ask the question to what extend these families and their descendents and some known isolated exceptional cases exhaust the possibilities of plektonic exchange structures. Although there are some arguments in favor, the only rigorous mathematical statement is that of Rehren who proved that for exchange dimension $d < \sqrt{6}$ that this is indeed the case [22].

3. Conformally invariant Local Quantum Physics

There are two situations for which the algebraic methods go significantly beyond kinematics and reveal constructive aspects of dynamical properties, i.e. in the present context properties which are important in the explicit construction of interacting models. One such situation arises if the superselection sectors are related to time- or light-like distances rather then the DHR superselection structure which is associated with spacelike causality. This is the case for higher dimensional (timelike) and chiral (lightlike) conformal theories and will be subject of this section. The other situation is that of modular localization. It turns out that in theories with massive particles and an asymptotically complete scattering interpretation, the modular theory for the wedge localized algebras is directly related to the scattering S-operator; this will be explained in the next section.

The hope is that conformal theories stay sufficiently close to free theories so that they remain mathematically controllable but without becoming completely free. There is a mathematical theorem which supports this idea. It states that the interpolating field associated with a zero mass particle is necessarily a free field with canonical scale dimension [23]. The only way to evade this undesired free situation (totally "protected" in the parlance of recent perturbative supersymmetric gauge theory) is to have at least one field with anomalous dimension in the theory.

The observables of a conformal quantum field theory (CQFT) obey in addition to the spacelike commutativity (Einstein causality) also timelike commutativity (Huygens causality); the interaction is limited to lightlike distances. In fact Huygens causality is mathematically equivalent with a net on the compactification $\bar{M}$ of Minkowski space M. This at first sight seems to force us back into the realm of free zero mass theories and its integer-valued (for Bosons to which we will mostly limit ourselves) spectrum of scale dimensions but fortunately there is one saving grace namely the fact that charge-carrying fields[22] associated with such Huygens algebras live on the very rich covering space $\widetilde{M}$ which comes with a new global causality concept. With other words the existence of any field which does not live on $\bar{M}$ but rather requires $\widetilde{M}$, is the indicator of a conformal interaction.

The projections of these globalized-charge carrying operators turn out to be nonlocal, but they have physically and mathematically completely controllable noncausality which is due to their natural origin within the setting of LQP. Whereas in massive QFT the timelike region is the arena of interaction and remains the unknown dynamical "black box", conformal theories permit for the first time to expose this dynamical region for a systematic study resulting in timelike exchange algebra relations [26] and their classification. There is good reason to believe that this additional structure will lead to the first explicit construction of 4-dimensional QFTs and hence may successfully solve the more than 70 year old existence problem for QFTs in physical spacetime.

In the following let us recall the geometric aspects of $\bar{M}$ and $\widetilde{M}$ before we will adapt the superselection analysis of the previous section to the realm of conformal theories.

The simplest type of conformal QFT is obtained by realizing that zero mass Wigner representation of the Poincaré group with positive energy (and discrete helicity) allow for an extension the conformal symmetry group $SO(4,2)/Z_2$ without enlargement of the Hilbert space. Besides scale transformations, this larger symmetry also incorporates the fractional transformations (proper conformal transformations in 4-dim. vector notation)

$$x' = \frac{x - bx^2}{1 - 2bx + b^2x^2} \tag{143}$$

It is often convenient to view this formula as the action of the translation group $T(b)$ conjugated with a (hyperbolic) inversion I

$$I : x \rightarrow \frac{-x}{x^2} \tag{144}$$

$$x' = IT(b)Ix \tag{145}$$

which does not belong to the above conformal group, although it is unitarily represented and hence a Wigner symmetry in the Wigner representation space. For fixed x and small b the formula (143) is well defined, but globally it mixes finite spacetime points with infinity and hence requires a more precise definition; in particular in view of the positivity energy-momentum spectral properties in its action on quantum fields. Hence as preparatory

[22] The fact that conformal observables on $\bar{M}$ have an integer-valued scale spectrum does by itself not imply the absence of interactions since there is no reason why the observable correlations are identical to those of Wick polynomials of free fields or why such observables imply the existence of a zero mass particle state.

step for the quantum field theory concepts one has to achieve a geometric compactification. This starts most conveniently from a linear representation of the conformal group $SO(d,2)$ in d+2-dimensional auxiliary space $\mathbb{R}^{(d,2)}$ (i.e. without direct field theoretic significance) with two negative (time-like) signatures

$$G = \begin{pmatrix} g_{\mu\nu} & & \\ & -1 & \\ & & +1 \end{pmatrix} \tag{146}$$

and restricts this representation to the (d+1)-dimensional forward light cone

$$LC^{(d,2)} = \{\xi = (\xi, \xi_4, \xi_5); \xi^2 + \xi_d^2 - \xi_{d+1}^2 = 0\} \tag{147}$$

where $\xi^2 = \xi_0^2 - \vec{\xi}^2$ denotes the d-dimensional Minkowski length square. The compactified Minkowski space $\bar{M}_d$ is obtained by adopting a projective point of view (stereographic projection)

$$\bar{M}_d = \left\{ x = \frac{\xi}{\xi_d + \xi_{d+1}}; \xi \in LC^{(d,2)} \right\} \tag{148}$$

It is then easy to verify, that the linear transformations which keep the last two components invariant, consist of the Lorentz group and that those transformations which only transform the last two coordinates yield the scaling formula

$$\xi_d \pm \xi_{d+1} \rightarrow e^{\pm s}(\xi_d \pm \xi_{d+1}) \tag{149}$$

leading to $x \rightarrow \lambda x, \lambda = e^s$. The remaining transformations, namely the translations and the fractional proper conformal transformations, are obtained by composing rotations in the ξ_i-ξ_d and boosts in the ξ_i-ξ_{d+1} planes.

A convenient description of Minkowski spacetime M in terms of this d+2 dimensional auxiliary formalism is obtained in terms of a "conformal time" τ

$$M_d = (sin\tau, \mathbf{e}, cos\tau), \;\; e \in S^{d-1} \tag{150}$$

$$t = \frac{sin\tau}{e^d + cos\tau}, \;\; \vec{x} = \frac{\vec{e}}{e^d + cos\tau} \tag{151}$$

$$e^d + cos\tau > 0, \;\; -\pi < \tau < +\pi$$

so that the Minkowski spacetime is a piece of the d-dimensional wall of a cylinder in d+1 dimensional spacetime which becomes tiled with the closure of infinitely many Minkowski worlds. If one cuts the wall on the backside in the τ-direction, this carved out piece representing d-dimensional compactified Minkowski spacetime has the form of a d-dimensional double cone symmetrically around $\tau = 0, \mathbf{e} = (\mathbf{0}, e^d = 1)$ without its boundary[23]. The above directional compactification leads to an identification of boundary points at "infinity" and give e.g. for d=1+1 the compactified manifold the topology of a torus. The points which have been added at the infinity to M namely $\bar{M}\backslash M$ are best described in terms of the d-1 dimensional submanifold of points which are lightlike with respect to the (past infinity) apex at $m_{-\infty} = (0,0,0,0,1,\tau = -\pi)$. The cylinder walls form the universal covering $\widetilde{M}_d = S^{d-1} \times \mathbb{R}$ which is tiled in both τ-directions by infinitely many Minkowski spacetimes ("heavens and hells")

[23]The graphical representations are apart from the compactification (which involves identifications between past and future points at time/light-infinity) the famous Penrose pictures of M.

[24]. If the only interest would be the description of the compactification $\bar{M}$, then one may as well stay with the original x-coordinates and write the d+2 ξ-coordinates (following Dirac and Weyl) as

$$\begin{aligned} &\xi^\mu = x^\mu,\ \mu = 0, 1, 2, 3 \\ &\xi^4 = \frac{1}{2}(1 + x^2) \\ &\xi^5 = \frac{1}{2}(1 - x^2) \\ &i.e.\ \ (\xi - \xi')^2 = (x - x')^2 \end{aligned} \tag{152}$$

Since ξ is only defined up to a scale factor, we conclude that only lightlike differences retain an objective meaning in $\bar{M}$.

An example of a physical theory on $\bar{M}$ is provided by non-interacting photons. The impossibility of a distinction between space- and time- like finds its mathematical formulation in the optical Huygens principle which says that the lightlike separation is the only one where the physical fields propagate and hence where intuitively speaking an interaction can happen. In the terminology of LQP this means that the commutant of an observable algebra localized in a double cone consists apparently of a (Einstein causal) connected spacelike- as well as two disconnected (Huygens causality) timelike- pieces. But taking the compactification into consideration one realizes that all three pieces are connected and the space/time-like distinction is meaningless on $\bar{M}$. In terms of Wightman correlation functions this is equivalent to the rationality of the analytically continued Wightman functions of observable fields which includes an analytic extension into timelike Jost points [25][26].

Therefore in order to make contact with particle physics aspects, the use of either the covering $\widetilde{M}$ or of more general fields (see next section) on $\bar{M}$ is important since only in this way one can implement the pivotal property of causality together with the associated localization concepts. As first observed by I. Segal [27] and later elaborated and brought into the by now standard form in field theory by Lüscher and Mack [24], a global form of causality can be based on the sign of the invariant

$$\begin{aligned} &(\xi(\mathbf{e}, \tau) - \xi(\mathbf{e}', \tau'))^2 \gtrless 0,\ hence \\ &|\tau - \tau'| \gtrless 2\left| Arcsin\left(\frac{\mathbf{e} - \mathbf{e}'}{4}\right)^{\frac{1}{2}}\right| = |Arccos\,(\mathbf{e} \cdot \mathbf{e}')| \end{aligned} \tag{153a}$$

where the $<$ inequality characterizes global spacelike distances and $>$ corresponds to positive and negative global timelike separations. Whereas the globally spacelike region of a point is compact, the timelike region is not. The concept of global causality solves the so called Einstein causality paradox of CQFT [28] which consisted in the observation that there are massless QFT (example: the massless Thirring model) for which the unitary implementation of (143), which for sufficiently large b-parameters transforms spacelike into timelike distances (passing through lightlike infinity), would create a causality clash since the anomalous dimension fields are only Einstein but not Huygens local. The notion of global causality (64) in the sense of the covering space avoids the intermediate trespassing through lightlike separations and uses unitaries which implement the covering group acting on the covering space instead of (143).

For a particle physicist the use of covering space with its many heavens and hells above and below is not so attractive because the experimental hardware is not conformal covariant. Therefore it is helpful to know that there is a way of re-phrasing the physical content of globally causal conformal fields (which violate the Huygens principle and instead exhibit the phenomenon of “reverberation” [28] inside the forward light cone) in the setting of the ordinary Minkowski world M of particle physics without running into the trap of the causality paradox of the previous section. In this way the use of the above ξ- parametrization would loose some of its importance and the changed description may be considered as an alternative to the Lüscher-Mack approach on covering space.

This was achieved a long time ago in a joint paper involving the present authors [29]. The main point of this work was to point out that the global causality structure could be encoded into a global decomposition theory of fields (conformal block decomposition) with respect to the center of the conformal covering. Local fields, although they behave apparently irreducibly under infinitesimal conformal transformations, transform in general reducibly under the action of the global center of the covering $Z(\widetilde{SO(d,2)})$. This central reduction was the motivation in for the global decomposition theory of conformal fields in [29]. In the present setting it reads:

$$F(x) = \sum_{\alpha,\beta} F_{\alpha,\beta}(x),\ \ F_{\alpha,\beta}(x) \equiv P_\alpha F(x) P_\beta \tag{154}$$
$$Z = \sum_\alpha e^{2\pi i\theta_\alpha} P_\alpha$$

These component fields behave analogous to trivializing sections in a fibre bundle; the only memory of their origin from an operator on covering space is their quasiperiodicity

$$\begin{aligned} ZF_{\alpha,\beta}(x)Z^* &= e^{2\pi i(\theta_\alpha - \theta_\beta)} F_{\alpha,\beta}(x) \\ U(b)F(x)_{\alpha,\beta}U^{-1}(b) &= \frac{1}{[\sigma_+(b,x)]^{\delta_F-\zeta}\,[\sigma_-(b,x)]^{\zeta}} F(x)_{\alpha,\beta} \\ \zeta &= \frac{1}{2}(\delta_F + \theta_\beta - \theta_\alpha) \\ U(\lambda)F(x)_{\alpha,\beta}U^{-1}(\lambda) &= \lambda^{\delta_F} F(\lambda x)_{\alpha,\beta} \end{aligned} \tag{155}$$

where the second line is the transformation law of special conformal transformation of the components of an operator F with scale dimension δ_F sandwiched between superselected subspaces H_α and H_β associated with central phase factors $e^{2\pi i\theta_\alpha}$ and $e^{2\pi i\theta_\beta}$ and the last line is the scale transformation. Using the explicit form of the conformal 3-point function it is easy to see that phases are uniquely given in terms of the scaling dimensions δ which occur in the conformal model [29].

$$e^{2\pi i\theta} \in \begin{cases} \{e^{2\pi i\delta} |\, \delta \in scaling\,spectrum\} & Bosons \\ \{e^{2\pi i(\delta+\frac{1}{2})} |\, \delta \in scaling\,spectrum\} & Fermions \end{cases} \tag{156}$$

A central projector projects onto the subspace of all vectors which have the same scaling phase i.e. onto the so called conformal block associated with the center, so the labelling refers to (in case of Bosons) the anomalous dimensions $mod(1)$.

The prize one has to pay for this return to the realm of particle physics on M in terms of component fields (154) is that these projected fields are not pointlike Wightman fields and hence there is no chance to associate them with a Lagrangian or Euclidean action; the timelike decomposition theory transcends the standard QFT approach though not its underlying principles. Unlike ordinary pointlike fields the component fields depend on a source and range projector and if applied to a vector, the source projector has to match the Hilbert space i.e. $F_{\alpha,\beta}$ annihilates the vacuum if P_β is not the projector onto the vacuum sector. This is very different from the behavior of the original F which, in case it was localized in a region with a nontrivial spacelike complement, cannot annihilate the vacuum. This kind of projected fields are well known from the exchange algebra formalism of chiral QFT [30] and they appear in a rudimentary form already in [29].

The crucial question now is: what is the timelike/spacelike structure of the double-indexed component fields? Whereas it is easy to see that the $F_{\alpha,\beta}$ are genuinely nonlocal fields without any spacelike commutation relation, consistency considerations using analytic properties of F-correlation functions suggest the following ($\pm$) timelike relations

$$F_{\alpha,\beta}(x)G_{\beta,\gamma}(y) = \sum_{\beta'} R^{(\alpha,\gamma)}_{\beta,\beta'} G_{\alpha,\beta'}(y)F_{\beta',\gamma}(x),\ x > y \tag{157}$$
$$R \rightarrow R^{-1}\ for\ x < y$$

i.e. a commutation relation with R-matrices which generate a representation of the infinite braid group B_∞. The consisteny problems of the simultaneous validity of these timelike braid group structure with the spacelike permutation group (Bosons/Fermions) commutation relations have been analysed within the analytic framework of vacuum correlation functions and their analytic continuations [26]. This suggests [31] the presence of the following group G_∞ which generalizes the braid group on infinitely many strands B_∞

$$b_it_j = t_jb_i,\ |i-j| \geq 2 \tag{158}$$
$$b_it_jt_i = t_jt_ib_j,\ |i-j| = 1$$
$$b_ib_jt_i = t_jb_ib_j,\ |i-j| = 1$$

Here the b's are the generators of the braid group, i.e. they fulfill the Artin relations (116) among themselves, whereas the t's are the transpositions which generate the permutation group. The above "mixed" relations are the consistency relations between the timelike braiding and the spacelike permuting. There is yet no systematic study of the representation theory of G_∞ apart from the determination of a particular family of abelian representations [31]. Neither does there presently exist a derivation of the mixed group G_∞ within the DHR superselection formalism.

3.1. Chiral Conformal QFT. For d=1+1 one encounters a very special situation which leads to a significant simplification of the above formalism. Already for the classical wave equation the 2-dimensional situation is very different from the higher dimensional one. Whereas in higher dimension the characteristic initial value problem is uniquely defined by giving data on one lightfront only, for d=1+1 one needs the characteristic data on the right and left lightray in order to have a unique specification throughout spacetime[24]. This leads to the well-known doubling of degrees of freedom: the general wave solution in massless d=1+1 consists of right- and left- moving chiral contributions.

[24]For the propagation in the massive case the data on one light ray suffice.

The starting point of the chiral factorization of d=1+1 massless conformal QFT is the observation that the conformal group $PSO(2,2)$ factorizes into two $PSL(2) \simeq PSU(1,1)$. One then naturally expects that the subtheories which commute with the left/right Moebius group $PSL(2) = SL(2)/\mathbb{Z}_2$ are the two chiral components into which the theory tensor-factorizes, a fact which one can rigorously prove in the LOP setting [32]

$$H = H_+ \otimes H_- \tag{159}$$

$$\mathcal{A}(M^{(1,1)}) = \mathcal{A}(\mathbb{R}) \otimes \mathcal{A}(\mathbb{R})$$

$$\mathcal{A}(\bar{M}^{(1,1)}) = \mathcal{A}(S^1) \otimes \mathcal{A}(S^1) \tag{160}$$

where the last line is the factorization of the compactification on which the $PSU(1,1)$ acts independently on each factor. The Moebius group $PSL(2,R)$ is generated by the following transformations (translations, dilations, proper conformal transformations)

$$x \to x + a \tag{161}$$

$$x \to \lambda x$$

$$x \to \frac{x}{1 - bx}$$

which form a finite dimensional subgroup of the diffeomorphism group of the one-point compactified real line $\mathring{\mathrm{R}} \simeq S^1$. In most of the literature one finds the following formula for the action of the (global) diffeomorphism $x \to f(x)$ on the fields

$$A(x) \to (f'(x))^{d_A} A(f(x)) \tag{162}$$

This formula is incorrect for fields with anomalous dimension d_A. In fact these fields live on the covering space and cannot obey a transformation law as (162) in which the x-dependent prefactor is not operator- but only numerical-valued. Numerical valuedness is only possible for the transformation of the component fields $A_{\alpha,\beta}$ in the central decomposition (157). But even in that case the correct law depends on the source and range projectors through a frequency split e.g. under proper conformal transformations is (see also [29])

$$A_{\alpha,\beta}(x) \to \left(\frac{1}{(1-bx)^2}\right)_+^{d_A - \xi} \left(\frac{1}{(1-bx)^2}\right)_-^{\xi} A_{\alpha,\beta}\left(\frac{x}{1-bx}\right) \tag{163}$$

$$\xi = \frac{1}{2}(d_A + \theta_\beta - \theta_\alpha)$$

This frequency split maintains the spectrum property[25]. Only inside analytically continued correlation functions of products of observable fields (integer scale dimensions) the transformation law appears as if it would be coming from (162)

$$w(z_1, ..., z_n) \to \prod_i (f'(z_i))^{d_A} w(f(z_1), ..., f(z_n))$$

$$w(z_1, ..., z_n) = anal.cont.\, of \ \langle 0 | A(x_1) ... A(x_n) | 0 \rangle$$

[25] In one mathematical treatise [34] this inconsisteny of (162) was noticed but in order to maintain harmony with the physics literature its clarification was not pursued.

Diffeomorphisms beyond the Moebius group change the vacuum into another state and destroys the analytic properties of the vacuum expectations by generating singularities within the natural (BWH) analyticity domain of Wightman functions [11]. There is however no mathematical concept by which one can avoid the more complicated transformation laws of operators under those conformal transformations which change infinity. Fields which depend holomorphically on complex coordinates z, as they appear sometimes in the literature on chiral QFT and in string theory, are meaningless since analytic properties are not part of the operator algebra but enter through states and manifest themselves in state vectors and vacuum correlation functions[26]; in thermal representations of the same operator algebra they are completely different. Therefore the terminology of physicists of calling (local) chiral fields "holomorphic" is somewhat unfortunate. As exemplified in formula (163), the transformation properties of anomalous dimension component fields $A_{\alpha\beta}$ has subtle phases in case the diffeomorphisms change the infinity of Minkowski spacetime.

Unlike to the higher dimensional conformal theories where there is a consistency problem between space- and time-like causality of observables, the separations in the chiral theories are lightlike and "chiral causality" simply means disjointness of the lightlike intervals. In this case the previous decomposition theory of fields on the covering (157) only involves positive/negative lightlike distances and the compatibility problem between space- and time-like algebraic structure is absent. For chiral theories one knows many models in which a nontrivial braid group structure does occur.

The chiral case is also more easily susceptible to an algebraic analysis in terms of DHR localized endomorphisms on $\mathcal{A}(S^1)$, since one only has to pay attention to one kind of Haag duality. All the content of the previous chapter is applicable, one just has to replace the double cones by intervals on S^1. The observable algebra is best described as a pre-cosheaf which is a map

$$I \to \mathcal{A}(I),\ I \subset\subset S^1 \tag{164}$$

which is isotonic, local (i.e. operators commute for disjoint localization)and fulfills the positive energy condition as in section 2. Haag duality in the compactified description follows automatically from conformal invariance and locality. The ordering structure on a light ray does not only permit the more general braid group structure instead of the usual permutation group, but one can even proof that a chiral model with only (anti)commuting fields is associated to a free field theory in Fock space.

There are two important questions which go beyond the content of second chapter:

1) How does one systematically construct chiral theories from the superselection data?

2) How does one use these data for the construction of d=1+1 conformal field theories?

The attentive reader will notice that the first question implies a change of strategy. Whereas for the structural analysis of LQP it was advantageous to start with the observable algebra in order to classify the admissable superselection structures and in particular the admissible (braid group) statistics, for the actual construction of models it may turn out to be easier to look directly for the spacetime carriers of the superselection charges without constructing first the observable algebras. This is precisely the message from the interaction free case since the

[26]Whereas it is true that fields applied to the vacuum define normalizable state vectors if the imaginary parts of their analytical continued spacetime arguments are ordered, this ordering prescription prevents an encoding into universal operator domains which (as in the Wightman approach) are preserved under successive applications.

free fields themselves are more easily constructed (see the second section) than their associated local observables which usually are given in terms of local composites. For chiral theories in particular this would mean to look for an alternative method to the construction of anomalous dimensional fields than that via representation theory of current algebras.

Whereas the first problem is still at the beginning of its understanding (see remarks in next section), the extension of chiral theories to two-dimensional local theories is a well-studied subject [32]. It is part of a general extension problem of $\mathcal{A} \subset \mathcal{B}$, in this case of the extensions of endomorphisms on $\mathcal{A}$ to endomorphisms on $\mathcal{B}$.

3.2. Charge Transport around $\mathbf{S}^1$. Up to now we were not completely precise about in what mathematical way a global algebra with its localizable representations and endomorphisms should be related to the net of local algebras. This cavalier attitude did not cause any problem as long as the local algebras were indexed by double cones (more general compact regions) in Minkowski spacetime. In this case the system of local algebras is really a bona fide net in the strict mathematical sense since two double cones are always contained in a sufficiently large third double cone i.e. the local system is directed towards infinity. But then there is a natural globalization namely the C^*-algebra of the inductive limit $\mathcal{A}_{qua}$ in the operator norm topology introduced in section 2 (62). This C^*-algebra still remembers its local origin: its elements can be approximated uniformly by local elements. If we would have taken the von Neumann closure, the limit would have been too big for an interesting representation theory. It is then easy to show that a representation of the net of local algebras (see below) is uniquely associated with a representation (in the usual sense) of $\mathcal{A}_{qua}$, a fact which we have already used in section 2. In conformal theories however this procedure would be very clumsy and artificial since e.g. in the chiral case infinity in the compact S^1 description is a special point like any other point which is not left invariant by the conformal transformations. In that case the globalization is most efficiently done in terms of a universal C^*-algebra $\mathcal{A}_{univ}(S^1)$ which is different (it turns out to be bigger) from the "non-compact" DHR quasilocal algebra $\mathcal{A}_{qua}(\mathbb{R})$.

In order to understand its construction, we note that the net $\{\mathcal{A}(I)\}_{I\subset S^1}$ is not directed; in fact because of this it is not a net in the mathematical use of that word, but rather a "pre-cosheaf". The globalization of a pre-cosheaf is done relative to a class of distinguished representations and leads to the so-called universal C*-algebra $\mathcal{A}_{univ}$ as follows

Definition 13. *$\mathcal{A}_{univ}$ is the C^*algebra which is uniquely determined by the system (precosheaf[27]) of local algebras $(A(I))_{I\in\mathcal{T}}$, $\mathcal{T}$ = family of proper intervals $I \subset S^1$ and the following universality condition:*

(i) there are unital embeddings $i^I : \mathcal{A}(I) \rightarrow \mathcal{A}_{univ}$ s. t..

$$i^J \mid_{\mathcal{A}(I)} = i^I \quad if \quad I \subset J,\ I, J \in \mathcal{T} \tag{165}$$

and $\mathcal{A}_{univ}$ is generated by the algebras $i^I(\mathcal{A}(I)),\ I \in \mathcal{T}$;

(ii) for every coherent family of representations $\pi^I : \mathcal{A}(I) \rightarrow \mathcal{B}(H_\pi)$ there is a unique representation π of $\mathcal{A}_{univ}$ in H_π s. t..

$$\pi \circ i^I = \pi^I \tag{166}$$

[27]The terminology results from the fact that it is dual to the pre-sheaf of partial normal states associated with the $\mathcal{O}'s$. Often physicists by abuse of laguage continue to use "nets" even if the system of $\mathcal{O}'s$ is not directed as in case of compact spacetimes.

The universal algebra inherits the action of the Möbius group as well as the notion of positive energy representation through the embedding.

The universal algebra has more (global) elements than the quasilocal algebra of the DHR theory, in fact one obtains $\mathcal{A}_{qua}(\mathbb{R})$ by puncturing $\mathcal{A}_{univ}(S^1)$ at "infinity": $\mathcal{A}_{qua} \equiv \mathcal{A} \subset \mathcal{A}_{univ}$ with the consequence that the vacuum representation π_0 ceases to be faithful ($\mathcal{A}_{univ}$ includes annihilators of the vacuum) and the global superselection charge operators which are outer for $\mathcal{A}_{qua}$ become inner for $\mathcal{A}_{univ}$. From this observation emerges the algebra of Verlinde [33] (explained below) which originally was obtained by geometric (rather than local quantum physics) arguments in the limited context of chiral conformal theories. The removal of a point ξ from S^1 (this removal recreates the infinity of $\mathcal{A}_{qua}$) forces $\mathcal{A}_{univ}$ to shrink to $\mathcal{A}$.

Most of this new features can be seen by studying global intertwiners in $\mathcal{A}_{univ}$. Let $I, J \in \mathcal{T}$ and $\xi, \zeta \in I' \cap J'$ (i.e. two points removed from the complements) and choose ρ and σ s. t.. locρ, loc$\sigma \subset I$ and $\hat{\rho} \in [\rho]$ with loc$\hat{\rho} \subset J$. Then the statistics operators $\varepsilon(\rho, \sigma)$ and $\varepsilon(\sigma, \rho) \in \mathcal{A}(I) \subset A_\xi \cap A_\zeta$ are the same (i.e. they don't need a label ξ or ζ) independently of whether we use the quasilocal algebra $\mathcal{A}_\xi$ or $\mathcal{A}_\zeta$ for their definition. By Haag duality a charge transporter $V : \pi_0\rho \rightarrow \pi_0\hat{\rho}$ lies both in $\pi_0(\mathcal{A}_\xi)$ and $\pi_0(\mathcal{A}_\zeta)$. However its pre-images with respect to the embedding are different iff the monodromy operator is nontrivial i.e. iff the braidgroup representation does not reduce to that of the permutation group. In fact:

$$\begin{aligned} & V_\rho \equiv V_+^* V_- \text{ with } V_+ \in \mathcal{A}_\xi, \quad V_- \in \mathcal{A}_\zeta \\ & V_\rho \in (\rho, \rho) \end{aligned} \tag{167}$$

is a global selfintertwiner, which is easily shown to be independent of the choice of V and $\hat{\rho}$ (i.e. $\hat{\rho}$ is a "spectator" in the aforementioned sense). The representation of the statistics operators in terms of the charge transporters $\varepsilon(\rho, \sigma) = \sigma(V_+)^* V_+,\ \varepsilon(\sigma, \rho)^* = \sigma(V_-)^* V_-$ leads to:

$$\sigma(V_\rho) = \varepsilon(\rho, \sigma) V_\rho \varepsilon(\sigma, \rho) \ \curvearrowright \pi_0\sigma(V_\rho) = \pi_0 \left[\varepsilon(\rho, \sigma)\varepsilon(\sigma, \rho)\right] \tag{168}$$

The first identity is very different from the relation between $\varepsilon's$ due to local intertwiners. The global intertwiner V_ρ is trivial in the vacuum representation, thus showing its lack of faithfulness with respect to $\mathcal{A}_{univ}$. The global aspect of V_ρ is only activated in charged representations where it coalesces with monodromy operators. From its definition it is clear that it represents a charge transport once around the circle[28]. In the analytically continued vacuum expectation values the algebraic monodromy of charge transport aquires the monodromy around a branch cut in complex function, but in the vein of previous remarks, the multivaluedness of branchings in analytical continuations has no direct place in the operator algebra theory.

The left hand side of the first equation in (168) expresses a transport "around" in the presence of another charge σ, i.e. a kind of charge polarization. Let us look at the invariant version of V_ρ namely the global "Casimir" operators $W_\rho = R_\rho^* V_\rho R_\rho : id \rightarrow id$. This operator lies in the center $\mathcal{A}_{univ} \cap \mathcal{A}_{univ}{}'$ and depend only on the class (=sector) $[\rho]$ of ρ. By explicit computation[19] one shows that after the numerical renormalization $C_\rho := d_\rho W_\rho$ one

[28]Note that in A_{univ} which corresponds to a compact quantum world it is not possible to "dump" unwanted charges to "infinity", as in the case of A_{quasi}, but instead one encounters "polarization" effects upon charge transportation once around i.e. the round transport in the presence of charged endomorphism is different from that in the vacuum endomorphism.

encounters the fusion algebra:

$$
\begin{aligned}
&(i)\quad C_{\sigma\rho} = C_\sigma \cdot C_\rho \\
&(ii)\quad C_\rho^* = C_{\bar\rho} \\
&(iii)\quad C_\rho = \sum_\alpha N^\alpha C_\alpha \quad if \;\; \rho \simeq \oplus_\alpha N^\alpha \rho_\alpha
\end{aligned}
\tag{169}
$$

Verlinde's modular algebra emerges upon forming matrices with row index equal to the label of the central charge and the column index to that of the sector in which it is measured:

$$
S_{\rho\sigma} := \left| \sum_\gamma d_\gamma^2 \right|^{-\frac{1}{2}} d_\rho d_\sigma \cdot \pi_0 \sigma(W_\rho) \tag{170}
$$

In case of nondegeneracy of sectors, which expressed in terms of statistical dimensions and phases means $\left|\sum_\rho \kappa_\rho d_\rho^2\right|^2 = \sum_\rho d_\rho^2$, the above matrix S is equal to Verlinde's matrix S [33]which together with the diagonal matrix $T = \kappa^{-1} Diag(\kappa_\rho)$, with $\kappa^3 = (\sum_\rho \kappa_\rho d_\rho^2)/\left|\sum_\rho \kappa_\rho d_\rho^2\right|$ satisfies the defining equations of the generators of the genus 1 mapping class group which is $SL(2,\mathbb{Z})$

$$
\begin{aligned}
&SS^\dagger = 1 = TT^\dagger, \quad TSTST = S \\
&S^2 = C, \quad C_{\rho\sigma} \equiv \delta_{\bar\rho\sigma} \\
&TC = CT
\end{aligned}
\tag{171}
$$

It is remarkable that these properties are common to chiral conformal theories and to d=2+1 plektonic models even though the localization properties of the charge-carrying fields are quite different i.e. the S-T structure is not limited to conformal theories as the original Verlinde argument (which uses geometry properties ascribed to correlation functions instead of charge transporters of LQP) may suggest. One also obtains the general validity of the phase relation:

$$
\frac{\kappa}{|\kappa|} = e^{-2\pi i c/8} \tag{172}
$$

where c is a parameter which in the chiral conformal setting is known to measure the strength of the two-point function of the energy-momentum tensor (the Virasoro central term). This identification may be derived by studying the (modular) transformation properties of the Gibbs partition functions for the compact Hamiltonian L_0 of the conformal rotations under thermal duality transformations $\beta \to 1/\beta$. For d=2+1 plektons, no similar physical interpretation is known. It is interesting to confront these results with the structure of superselection rules of group algebras in the appendix of the first section.

Lemma 14. *The matrix S is similar to the character matrix in the appendix section 1.2. However in distinction to nonabelian finite groups (which also yield a finite set of charge sectors of the fixed point observable algebra) the present nonabelian sectors produce a symmetric "character" matrix S which signals a perfect self-duality between charge measurers $\{Q\}$ and charge creators $\{\rho\}$ i.e. as if the representation dimensions and the size of the conjugacy classes coalesce. The group theoretic case does not lead to the S-T modular group structure. Furthermore the algebra $\mathcal{Q}$ generated by the central charges and the action of the endomorphisms on those charges[29] do not contain the DR group theoretic structure since it only involves endomorphisms with nontrivial monodromy.*

[29] This action leads out of the center and generates a global subalgebra of $\mathcal{A}_{univ}$.

This strongly suggests to try to understand the new "quantum symmetry" property in terms of the structural properties of the algebra $\mathcal{Q}$. As a generalization of the Verlinde matrix S one finds for the $Q's$ in the presence of more than one polarization charges the entries of the higher genus mapping class group matrices. The reason is that in addition to the above process whose schematic description

$$vacuum \overset{split}{\longrightarrow} \rho\bar{\rho} \underset{selfintertw.}{\overset{global\ \rho}{\longrightarrow}} \rho\bar{\rho} \overset{fusion}{\longrightarrow} vacuum \tag{173}$$

led to the global intertwiner $W_\rho = R_\rho^* V_\rho R_\rho$, there are the more involved global intertwiners associated with processes in which the global selfintertwining occurs after a split of a *non-vacuum charge* σ and a later fusion to μ which appear in a $\rho\bar{\rho}$ reduction:

$$\sigma \overset{split}{\longrightarrow} \alpha\beta \underset{intertw.}{\overset{global\ \alpha}{\longrightarrow}} \alpha\beta \overset{fusion}{\longrightarrow} \mu, \quad \sigma,\mu \subset \rho\bar{\rho} \tag{174}$$

with the global intertwiner $V_\alpha \in (\alpha,\alpha)$ being used in $T_{e(\sigma)}^* V_\alpha T_{e(\mu)}$ where $T_{e(\mu)}$ is the $\alpha\beta \rightarrow \mu$ fusion intertwiner and the Hermitian adjoints represent the corresponding splitting intertwiner. As in the vacuum case, the selfintertwiners V_α become only activated after the application of another endomorphism say η, i.e. in the presence of another charge η (hence the name "polarization" mechanism). It can be shown that the following $T_{e(\sigma)}^* V_\alpha T_{e(\mu)}$ operators are the building blocks of the mapping class group matrices which have multicharge-"measurer" (Q-) and multicharge-creator (ρ-) column and row multiindices and are formed from repeated use of operators of the form

$$\phi_\lambda((T_{g(\eta)}\eta(T_{e(\mu)}^* V_\alpha T_{e(\sigma)})T_{f(\eta)}^*)) : \eta \rightarrow \eta \tag{175}$$

Here $T_{f(\eta)}$ and $T_{g(\eta)}$ are the intertwiners corresponding to the charge edges $f(\eta) : \lambda\sigma \rightarrow \eta$ and $g(\eta) : \lambda\mu \rightarrow \eta$, whereas ϕ_λ is the left inverse of the endomorphism λ. Besides the global intertwiners V, we only used the local splitting intertwiners and their Hermitian adjoints which represent the fusion intertwiners. The main question is: why do we organize the numerical data of the global charge-measurer and charge-creator algebra $\mathcal{Q}$ as entries in a multiindex matrix? This is ultimately a question about physical interpretation and the use of this algebra $\mathcal{Q}$ in LQP. The difficulty here is, that although in the present stage of development of LQP one understands the combinatorial properties of superselected charges including their braid group statistics, there is yet no understanding of the d=2+1 spacetime carriers of these properties which would be needed for applications to fractional quantum Hall- or High T_c-phenomena. One would expect the above matrix S and its multicharge mapping class group generalization to show up in scattering of "plektons". The formalism and its interpretation for charged fields with braid group statistics is expected to be quite different from standard Lagrangian physics and attempts to treat plektons within the standard setting by manipulating Chern-Simons Lagrangians have remained inconclusive. In the operator algebra setting this *natural non-commutativity* (i.e. without changing the classical spacetime indexing of nets into something noncommutative) caused by braid group statistics is more visible and suggests a constructive approach based on the (Tomita) modular wedge localization (see next section) which is presently under way, but there is still a long way to go.

Finally some additional remarks on the higher dimensional conformal case treated at the beginning of this section are called for. In that case we have stayed away from the formulation in terms of endomorphism of $\mathcal{A}_{univ}(\bar{M})$ and the ensuing charge transport around the compact Dirac-Weyl world $\bar{M}$. The reason is not that we have doubts about its validity, but rather that for the consistency with the spacelike Boson/Fermion DHR statistics we would have to understand how the $\bar{M}$ timelike Haag duality is related to the spacelike Haag dualization on M. Since dualizations in the pointlike field setting do not change the pointlike fields themselves but only the way in which algebras indexed by spacetime regions are generated by these fields, we found it safer to use the pointlike framework in the hope a future more rigorous treatment using endomorphisms and showing consistency of timelike/spacelike aspects will confirm our findings.

4. Constructive use of Modular Theory

In order to formulate the modular localization concept in the case of interacting particles, one must take note of the fact that the scattering matrix[30] S of local QFT is the product of the interacting TCP-operator Θ (mentioned in the second section) with the free (incoming) TCP operator Θ_0 and (since the rotation by which the Tomita reflection J differs from Θ is interaction-independent as all connected Poincaré transformations are interaction-independent) we have:

$$S = \Theta \cdot \Theta_0, \quad S = J \cdot J_0 \tag{176}$$

and as a result we obtain for the Tomita involution $\check{S}$ (to avoid confusion with the S-matrix we now write $\check{S}$ for the Tomita involution which was called S in the previous section)

$$\check{S} = J\Delta^{\frac{1}{2}} = SJ_0\Delta^{\frac{1}{2}} = S\check{S}_0 \tag{177}$$

Again we may use covariance in order to obtain $\check{S}(W)$ and the localization domain of $\check{S}(W)$ as $\mathcal{D}(\check{S}(W)) = \mathcal{H}_R(W) + i\mathcal{H}_R(W)$ i.e. in terms of a net of closed real subspaces $\mathcal{H}_R(W) \in \mathcal{H}_{Fock}$ of the incoming Fock space. However now the construction of an associated von Neumann algebra is not clear since an "interacting" functor from subspaces of the Fock space to von Neumann algebras is not known. In fact whereas the existence of a functor from the net of real localized *Wigner subspaces* $H_R(W) \subset H_{Wig}$ to a net of von Neumann algebras is equivalent to

$$H_R(W_1 \cap W_2) = H_R(W_1) \cap H_R(W_2) \tag{178}$$

The equality can be shown to become an inequality $\subset$ for the above localized subspaces $\mathcal{H}_R$ of *Fock space*.

As in the free case, the modular wedge localization does not use the full Einstein causality but only the so-called "weak locality", which is just a reformulation of the TCP invariance [11]. Weakly relatively local fields form an equivalence class which is much bigger than the local Borchers class, but they are still associated to the same S-matrix (or rather to the same TCP operator [11]). Actually the S in local quantum physics has two different interpretations: S with the standard scattering interpretation in terms of large time limits of suitably defined operators obtained from localized (compactly as in the sense of DHR or along spacelike cones [1]) operators, and on the other hand S in its role to provide modular localization in interacting theories as in the above formulas. There is *no parallel outside local quantum physics* to this two-fold role of S as a scattering operator and simultaneously as a relative modular invariant between an interacting- with its associated free- system. Whereas most concepts and properties which have been used in standard QFT as e.g. time ordering and interaction picture formalism are

[30] Mathematicians who are not familiar with the physically pivotal scattering theory which relates interacting to free theories may look up [1] or take (176) as a definition of S. The modular construction is independent of the scattering interpretation of this operator.

shared by nonrelativistic theories, modular localization is a new characteristic structural element in LQP which is closely related to the vacuum polarization property.

The so-called *inverse problem* in QFT is the question whether an admissable S (i.e. one which fulfills the general S-matrix properties as unitarity and the analytic crossing symmerty) has a unique associated QFT. Since the S-matrix has no unique attachment to a particular field coordinate but is rather affiliated with a local equivalence of field coordinatizations, the natural arena for this typ of question is the LQP algebraic setting. Indeed this modular localization setting allows to show that if an admissable S-operator has any associated LQP theory at all, it must be unique.

4.1. Polarization-Free Generators. The special significance of the wedge-localization in particle physics is due to the fact that it is the smallest localization region for which there exists operators G such that their one-fold application to the vacuum $G\Omega$ is a one-particle state vector without the admixture of particle-antiparticle vacuum polarization clouds[31]. We call such operators "polarization-free generators" (PFG) [12] [42][35]. Since they are necessarily unbounded, we present a more precise definition.

Definition 15. *A closed operator G is called a polarization-free generator (PFG) if (i) it is affiliated with a wedge algebra $\mathcal{A}(W)$, (ii) has the vacuum Ω in its domain and in that of its adjoint G^* and (iii) $G\Omega$ and $G^*\Omega$ are in the mass m one-particle subspace $\mathcal{H}^{(1)} = E_m\mathcal{H}$ (E_m=one-particle projector).*

The existence of these operators is a consequence of the following general theorem of modular theory:

Theorem 16. *Let Φ be any vector in the domain of the Tomita modular operator $\check{S}$ associated with the modular theory of ($\mathcal{A}$,Ω). Then there exists a closed operator F which is affiliated with $\mathcal{A}$ and together with its adjoint F^* contains Ω in its domain and satisfies*

$$F\Omega = \Phi,\ F^*\Omega = S^*\Phi$$

Proof. The previous theorem is then a consequence for $\mathcal{A} = \mathcal{A}(W)$ and the fact that there exists a dense set of one particle states in the domain of $\Delta_W^{\frac{1}{2}}$=$U(\Lambda_W(\chi = i\pi))$ (i.e. the analytically continued L-boost) which is identical to the domain of $\check{S}_W$.

Although the PFG's for wedge regions always exist, their use for the construction of the wedge algebras from the wedge localized subspace is presently limited to their "temperedness".

Definition 17. *A polarization-free generator G is called tempered if there exists a dense subspace $\mathcal{D}$ (domain of temperedness) of its domain which is stable under translations such that for any $\Psi \in \mathcal{D}$ the function $x \to GU(x)\Omega$ is strongly continuous and polynomial bounded in norm for large x, and the same holds also true for G^*.*

It turns out that tempered PFG's generate wedge algebras which stay close to interaction-free theories, in fact for $d \geqslant 1+2$ the S-matrix is trivial i.e. equal to the identity. In d=1+1 it is easy to exhibit large classes of nontrivial examples by modifying commutation relations in momentum space. Using the rapidity parametrization

[31] Here we assume the usual LSZ setting of scattering theory i.e. the existence of particle states. As previously mentioned this excludes interacting conformal theories.

for on-shell momentum $p = m(cosh\theta, sinh\theta)$, the commutation relation for free creation and annihilation operators reads ($\pm$ (anti)/commutator)

$$[a(\theta), a(\theta')]_{\pm} = 0 \tag{179}$$
$$[a(\theta), \alpha^*(\theta')]_{\pm} = \delta(\theta - \theta')$$

The iterative application of the creation operator defines a basis in Fock space. We start with the Fock space of free massive Bosons or Fermions. In order to save notation we will explain the main ideas first in the context of selfconjugate (neutral) scalar Bosons. Using the Bose statistics we use for our definitions the "natural" rapidity-ordered notation for n-particle state vectors

$$a^*(\theta_1)a^*(\theta_2)...a^*(\theta_n)\Omega, \;\; \theta_1 > \theta_2 > ... > \theta_n \tag{180}$$

and define new creation operators $Z^*(\theta)$ as follows: in case of $\theta_i > \theta > \theta_{i+1}$ and with the previous convention we set

$$Z^*(\theta)a^*(\theta_1)...a^*(\theta_i)...a^*(\theta_n)\Omega = \tag{181}$$
$$S(\theta - \theta_1)...S(\theta - \theta_i)a^*(\theta_1)...a^*(\theta_i)a^*(\theta)...a^*(\theta_n)\Omega$$

where $S(\theta)$ represents θ-dependent functions of modulus one. With $Z(\theta)$ as the formal adjoint one finds the following two-particle commutation relations

$$Z^*(\theta)Z^*(\theta') = S(\theta - \theta')Z^*(\theta')Z^*(\theta) \tag{182}$$
$$Z(\theta)Z^*(\theta') = S(\theta' - \theta)Z^*(\theta')Z(\theta) + \delta(\theta - \theta')$$

where the formal Z adjoint of Z^* is defined in the standard way. The $*$-algebra property requires $S(\theta)^* = S(\theta)^{-1} = S(-\theta)$. This structure leads in particular to

$$Z^*(\theta_1)...Z^*(\theta_n)\Omega = a^*(\theta_1)...a^*(\theta_n)\Omega \tag{183}$$
$$Z^*(\theta_n)...Z^*(\theta_1)\Omega = \prod_{i>j} S(\theta_i - \theta_j)a^*(\theta_1)...a^*(\theta_n)\Omega$$

for the *natural/opposite* order, all other orders giving partial products of S's. Note that for momentum space rapidities it is not necessary to worry about their coalescence since only the L^2 measure-theoretical sense (and no continuity) is relevant here. In fact the mathematical control of these operators i.e. the norm inequalities involving the number operator $\mathbf{N}$ hold as for the standard creation/annihilation operators

$$\left\| \mathbf{N}^{-\frac{1}{2}} \int Z^*(\theta)f(\theta)d\theta \right\| \leq (f, f)^{\frac{1}{2}}$$

Let us now imitate the free field construction and ask about the localization properties of these F-fields

$$F(x) = \frac{1}{\sqrt{2\pi}} \int (e^{-ipx}Z(\theta) + h.c.) \tag{184}$$

This field cannot be (pointlike) local if S depends on θ since the on-shell property together with locality leads to the free field formula. In fact it will turn out (see next section) that the smeared operators $F(f) = \int F(x)f(x)d^2x$ with

$$suppf \in W_0 = \{x; \; x^1 > |x^0|\} \tag{185}$$

have their localizations in the standard wedge W. But contrary to smeared pointlike localized fields, the wedge localization cannot be improved by improvements of the test function support inside W. Instead the only way to come to a local net of compactly localized algebras (and, if needed, to their possibly existing pointlike field generators) is by intersecting oppositely localized wedge algebras (see below). This improvement of localization by algebraic means instead of by sharpening the localization of test functions (quantum- versus classical- localization) is the most characteristic distinction from the standard formalism. *It takes care of noncommutative features of LQP without violating its principles unlike the use of noncommutative geometry (the introduction of noncommutative spacetime) in particle physics.*

It follows from modular theory that the wedge localization properties of the above PFG's $F(f)$ are most conveniently established via the KMS properties of their correlation functions.

Theorem 18. *The KMS-thermal property of the pair $(A(W),\Omega)$ is equivalent to the crossing symmetry of the S-coefficient in (182)*

For a sketch of the proof we consider the KMS property of the affiliated PFG's $F(f)$. For their 4-point function the claim is

$$
\begin{aligned}
(\Omega, F(f_{1'})F(f_{2'})F(f_2)F(f_1)\Omega) &\equiv \langle F(f_{1'})F(f_{2'})F(f_2)F(f_1)\rangle_{therm} \qquad (186)\\
&\overset{KMS}{=} \left\langle F(f_{2'})F(f_2)F(f_1)F(f_{1'}^{-2\pi i})\right\rangle_{therm}\\
&\Leftrightarrow S(\theta) = S(i\pi-\theta) \qquad (187)
\end{aligned}
$$

Here we only used the cyclic KMS property (the imaginary 2π-shift in the second line corresponds to the modular holomorphy (104) re-expressed in terms of the Lorentz boost parameter) for the four-point function. The relation is established by Fourier transformation and contour shift $\theta\rightarrow\theta-i\pi$. One computes

$$
\begin{aligned}
F(\hat{f}_2)F(\hat{f}_1)\Omega &= \int\int f_2(\theta_2-i\pi)f_1(\theta_1-i\pi)Z^*(\theta_1)Z^*(\theta_2)\Omega + c.t. \qquad (188)\\
&= \int\int f_2(\theta_2-i\pi)f_1(\theta_1-i\pi)\{\chi_{12}a^*(\theta_1)a^*(\theta_2)\Omega +\\
&+ \chi_{21}S(\theta_2-\theta_1)a^*(\theta_2)a^*(\theta_1)\Omega\} + c\Omega
\end{aligned}
$$

where the χ are the characteristic function for the differently permuted θ-orders. The analogous formula for the bra-vector is used to define the four-point function as an inner product. If S has no pole in the physical strip, the KMS property is obviously equivalent to the crossing symmetry of $S(\theta)$. If $S(\theta)$ has a (necessarily crossing symmetric) pole in the in the physical strip, the contour shift will produce an unwanted terms which wrecks the KMS relation. The only way out is to modify the previous relation by adding a compensating bound state contribution to the scattering term

$$
\begin{aligned}
F(\hat{f}_2)F(\hat{f}_1)\Omega &= (F(\hat{f}_2)F(\hat{f}_1)\Omega)_{scat} \qquad (189)\\
&+ \int d\theta f_1(\theta_1+i\theta_b)f_2(\theta_2-i\theta_b)\left|\theta,b\right\rangle \left\langle\theta,b\right|Z^*(\theta-i\theta_b)Z^*(\theta+i\theta_b)\left|\Omega\right\rangle
\end{aligned}
$$

The second contribution then compensates the pole contribution from the contour shift. In general the shift will produce an uncompensated term from a crossed pole whose position is obtained by reflecting in the imaginary axis around $i\frac{\pi}{2}$. which creates the analogous crossed bound state contribution. In our simplified selfconjugate model it is the same term as above. In the presence of one or several poles one has to look at poles in higher point functions.

Despite the different conceptual setting one obtains the same formulas as those for the S-matrix bootstrap of factorizing models and hence one is entitled to make use of the bootstrap technology in this modular program. In fact the involution J for the present model turns out to be (the S without the θ-dependent argument denotes the S-operator in Fock space)

$$J = SJ_0 \tag{190}$$
$$S^* a^*(\theta_1)...a^*(\theta_n)\Omega = \prod_{i>j} S(\theta_i - \theta_j) a^*(\theta_1)...a^*(\theta_n)\Omega$$

so even without invoking scattering theory we see that the operator S fulfills the modular definition of the S-matrix.

Using a suitable formalism it is easy to see that the PFG generators can be generalized to particle/antiparticle multiplets. In this case the coefficient functions S are matrix valued and the associativity of the Z-algebra is nothing else than the Yang-Baxter relation. Our notation using the letter Z is intended to indicate that the modular wedge localization (for those cases with tempered PFG's [35]) leads to a derivation and spacetime interpretation of the Zamolodchikov-Faddeev algebra [36]. The creation/annihilation generators of this algebra are simply the positive/negative energy contributions to the Fouriertransform of tempered PFG's and the crossing symmetry of the structure coefficients S is nothing else then the modular characterization of the wedge localization of the PFG's. This is a significant conceptual step which does not only equip the formally useful ZF algebra with a much needed physical interpretation, but also vindicates the old dream of the S-matrix bootstrap approach concerning the avoidance of ultraviolet problems. Neither the S-matrix nor the local algebras know about the short distance properties of the individual field coordinatizations. Different from the standard approach, the formfactors of fields are determined *before* their short distance properties can endanger their existence. The only presently known truely intrinsic ultraviolet behavior which one can associate with e.g. double cone algebras regardless of what generating pointlike field coordinate may have been used is the entropy which can be assigned to the "split inclusion" [1] which in physical terms consists of a double cone with a "collar" around it in form of a slightly larger double cone. This localization entropy is expected to have a divergence in terms of the inverse collar size and a Bekenstein area dependence as a result of the vacuum fluctuations near the surface of such a relativistic box [12].

Our wedge-localized PFG's do not only link the ZF algebra do the general principles of local quantum physics, but they also reduce the danger that the mathematically amazing results of the bootstrap-formfactor program for factorizing models[32] may end up to become a sectarian issue dissociated from the rest of particle physics.

After having constructed the generators which are affiliated with the wedge algebra $\mathcal{A}(W)$, one tries to construct the generators for double cone intersections

[32] For a short list of papers which have motivated and influenced my work on the PFG generators or are close to its underlying philosophy see [37] where the reader also finds a more detailed list of the many model contributions to the bootstrap-formfactor program.

$$\mathcal{A}(\mathcal{O}_a) := \mathcal{A}(W_a)' \cap \mathcal{A}(W) \tag{191}$$
$$\mathcal{O}_a = W_a^{opp} \cap W$$

where W_a is the a-translated wedge and the double cone $\mathcal{O}_a$ is defined in the last line. Making for an operators $A \in \mathcal{A}(\mathcal{O}_a)$ the formal Ansatz as a series in the $Z's$,

$$A = \sum \frac{1}{n!} \int_C ... \int_C a_n(\theta_1, ...\theta_n) : Z(\theta_1)...Z(\theta_n) :\in \mathcal{A}(W) \tag{192}$$

the relation which characterizes its affiliation with $\mathcal{A}(\mathcal{O}_a)$ has a simple form in terms of PFG's

$$\left[A, F_a(\hat{f})\right] = 0 \tag{193}$$

where $F_a(\hat{f})$ is the previously introduces PFG $F(\hat{f})$ translated by a. On can show [12] that this relation leads to the "kinematical pole relation" which is one of Smirnov's [37] formfactor axioms[33]. It relates the residuum of certain poles of a_n to the coefficient a_{n-2}. These meromorphic coefficient functions are related to the matrix elements of A between particle states which are called formfactors by the physicist. In the mathematical sense the collection of such matrix elements or a formal series as (192) define only a sesquilinear form. The control of associated operators has not yet been achieved and therefore the existence problem of the factorizing models, which in the LQP setting is the nontriviality

$$\mathcal{A}(\mathcal{O}_a) \neq \{\mathbb{C}\mathbf{1}\} \tag{194}$$

presently remains an open mathematical problem. But even though the construction of factorizing models remains incomplete, there is an important message concerning the ultraviolet problem in QFT. It is well known that in the standard approach the short distance behavior sets severe limits; it is impossible in $d = 1 + 3$ to associate a meaningful perturbation theory with interacting Lagrangian fields unless their short distance dimensions stay close to their canonical values (i.e. 1 for Bosons and $\frac{3}{2}$ for Fermions) since this would lead to nonrenormalizable situations. As a result of this restriction the number of different renormalizable coupling types is finite. The only fields which are allowed to have large values of short distance dimensions are the composites of the Lagrangian fields. The wedge localization approach, which avoids field coordinatizations at the outset, does not suffer from those Lagrangian short distance problems. Whereas the construction of pointlike fields in the causal perturbation theory is controlled by their short distance behavior, the existence of nontrivial theories in the modular localization approach is determined by the nontriviality of double cone algebras obtained by intersecting wedge algebras. Although a more detailed investigation may reveal a relation between these two structure, that the latter requirement appears less restrictive. Having constructed the double cone algebras, one may of course "coordinatize" them by pointlike fields; as long as one avoids such field coordinatizations in the process of construction no ultraviolet limitations have been introduced by the method of construction. In the Lagrangian approach we cannot even be sure that (apart from certain "superrenormalizable" models) the perturbatively renormalizable (i.e. ultraviolet-wise acceptable) models have rigorously existing models behind them.

[33] In the literature the kinematical pole relation is written for pointlike fields which formally corresponds to $a = 0$. The finite size leads to a better (Pailey-Wiener) large momentum behavior.

4.2. Modular Inclusions, Holography, Chiral Scanning and Transplantation. A modular inclusion in the general mathematical sense is an inclusion of two von Neumann algebras (in our case they are assumed to be factors) $\mathcal{N} \subset \mathcal{M}$ with a common cyclic and separating vector Ω and such that modular group $\Delta_{\mathcal{M}}^{it}$ for $t < 0$ transforms $\mathcal{N}$ into itself (compression of $\mathcal{N}$) i.e.

$$Ad\Delta_{\mathcal{M}}^{it}\mathcal{N} \subset \mathcal{N} \tag{195}$$

$$t \lesseqgtr 0, \pm half sided\ modular \tag{196}$$

(when we simply say modular, we mean $t < 0$) We assume that $\cup_t Ad\Delta_{\mathcal{M}}^{it}\mathcal{N}$ is dense in $\mathcal{M}$ (or that $\cap_t \Delta_{\mathcal{M}}^{it}\mathcal{N} = \mathbb{C}\cdot 1$). This modular inclusion situation may be viewed as a generalization of a situation studied by Takesaki [6] in which the modular group of $\mathcal{M}$ leaves $\mathcal{N}$ invariant. This then leads to the modular objects Δ^{it}, J of $\mathcal{N}$ being restrictions of those of $\mathcal{M}$ as well as the existence of a conditional expectation $E : \mathcal{M} \to \mathcal{N}$. Whereas for inclusions of abelian algebras conditional expectations always exist (physical application: the Kadanoff-Wilson renormalization group "decimation" or "integrating out" degrees of freedom in Euclidean QFT), the existence of noncommutative conditional expectations is tight to the shared modular group of the two algebras.

The above modular inclusion situation has in particular has the consequence that the two modular groups $\Delta_{\mathcal{M}}^{it}$ and $\Delta_{\mathcal{N}}^{it}$ generate a two parametric group of (translations, dilations) in which the translations have positive energy [38]. Let us now look at the relative commutant (see appendix of [39]). Let $(\mathcal{N} \subset \mathcal{M}, \Omega)$ be modular with nontrivial relative commutant. Then consider the subspace generated by relative commutant $H_{red} \equiv \overline{(\mathcal{N}' \cap \mathcal{M})\Omega} \subset H$. The modular unitary group of $\mathcal{M}$ leaves this subspace invariant since $\Delta_{\mathcal{M}}^{it}, t > 0$ maps $\mathcal{N}' \cap \mathcal{M}$ into itself by the inclusion being modular. Now consider the orthogonal complement of H_{red} in H. This orthogonal complement is mapped into itself by $\Delta_{\mathcal{M}}^{it}$ for positive t since for ψ be in that subspace, then

$$\left\langle \psi, \Delta_{\mathcal{M}}^{it}(\mathcal{N}' \cap \mathcal{M})\Omega \right\rangle = 0\ for\ t > 0. \tag{197}$$

Analyticity in t then gives the vanishing for all t, i.e. invariance of H_{red}.

Due to Takesaki's theorem [6], we can restrict $\mathcal{M}$ to H_{red} using a conditional expectation to this subspace defined in terms of the projector P onto H_{red}. Then

$$E(\mathcal{N}' \cap \mathcal{M}) \subset \mathcal{M}|_{\overline{(\mathcal{N}' \cap \mathcal{M})\Omega}} = E(\mathcal{M}) \tag{198}$$

$$E(\cdot) = P \cdot P \tag{199}$$

is a modular inclusion on the subspace H_{red}. $\mathcal{N}$ also restricts to that subspace, and this restriction $E(\mathcal{N})$ is obviously in the relative commutant of $E(\mathcal{N}' \cap \mathcal{M}) \subset E(\mathcal{M})$. Moreover using arguments as above it is easy to see that the restriction is cyclic with respect to Ω on this subspace. Therefore we arrive at a reduced modular "standard inclusion"

$$(E(\mathcal{N}) \subset E(\mathcal{M}), \Omega) \tag{200}$$

Standard modular inclusions are known to be isomorphic to chiral conformal field theories [40] i.e. they lead to the canonical construction of a net $\{\mathcal{A}(I)\}_{I \in \mathcal{K}}$ indexed by intervals on the circle with the Moebius group PL(2,R) acting in correct manner (including positive energy).

This theorem and its extension to modular intersections leads to a wealth of physical applications in QFT, in particular in connection with "hidden symmetries" symmetries which are of purely modular origin and have no interpretation in terms of quantized Noether currents [41][39].

The modular inclusion techniques unravel new structures which are not visible in terms of standard field coordinatizations. In order to provide a simple example, let us briefly return to d=1+1 massive theories. It is clear that in this case we should use the two modular inclusions which are obtained by sliding the (right hand) wedge into itself along the upper/lower light ray horizon. Hence we chose $\mathcal{M} = \mathcal{A}(W)$ and $\mathcal{N} = \mathcal{A}(W_{a_+})$ or $\mathcal{N} = \mathcal{A}(W_{a_-})$ where $W_{a\pm}$ denote the two upper/lower light-like translated wedges $W_{a\pm} \subset W$. As explained in section 2 following [43] (where cyclicity is shown for massive free fields), this case leads to a modular inclusion as above with the additional cyclicity $H_{red} = H$. In the case of the upper horizon of W we therefore have (omitting for simplicity the $\pm$ subscripts)

$$\begin{aligned} &\mathcal{A}(I(0,a)) \equiv A(W_a)' \cap \mathcal{A}(W) \\ &\overline{\mathcal{A}(I(0,a)\Omega} = H \end{aligned} \tag{201}$$

where the notation indicates that the localization of $\mathcal{A}(I(0,a))$ is thought of as the piece of the upper light ray interval between the origin and the endpoint a.

¿From the standardness of the inclusion one obtains according to the previous discussion an associated conformal net on the line, with the following formula for the chiral conformal algebra on the half line

$$\mathcal{A}(R_>) \equiv \bigcup_{t\geq 0} Ad\Delta_W^{it}\left(\mathcal{A}(I(0,a))\right) \subseteq \mathcal{A}(W), \tag{202}$$

with the equality sign

$$\mathcal{A}(R_>) = \mathcal{A}(W) \tag{203}$$

following from the cyclicity property

$$\overline{\mathcal{A}(R_>)\Omega} = \overline{\mathcal{A}(W)\Omega} \tag{204}$$

for the characteristic data on one light ray together with the before mentioned theorem of Takesaki (which gives $P = 1$ in this case). An entirely analogous argument applies to the lower horizon of W.

The argument is word for word the same in higher spacetime dimensions, since the appearance of transversal components do not modify the chain of reasoning. The cyclicity property (204) can be shown for for all free fields [43] except for massless d=1+1 fields which need both upper and lower characteristic data. One of course does not expect such modular properties to be effected by interactions. The following formal intuitive argument of the kind used by physicists suggests that the "characteristic shadow property" follows generally (apart from the mentioned d=1+1 massless exception) from the standard causal shadow property of QFT. Let us start from the special situation $\mathcal{A}(W) = \mathcal{A}(R_>^{(\alpha)})$ where $R_>^{(\alpha)}$ is a spacelike positive halfline with inclination α with respect to the x-axis. This is a consequence of the standard causal shadow property (the identity of $\mathcal{A}(\mathcal{O}) = \mathcal{A}(\mathcal{O}'')$ where $\mathcal{O}''$ is the causal completion of the convex spacelike region $\mathcal{O}$) in any spacetime dimension.The idea is that if this relation remains continuously valid for $R_{>}^{(\alpha)}$ approaching the light ray ($\alpha = 45\,^{\circ}$) which then leads to the desired equality. We will call the property (203) the "characteristic shadow property".

Physicists are accustomed to relate massless theories with PSL(2,$\mathbb{R}$)-invariant LQP. Although any d=1+1 massless theory is under quite general circumstances conformally invariant and factorizes into massless chiral theories, not every PSL(2,$\mathbb{R}$)-invariant chiral theory describes a massless situation. For the above construction of a chiral theory via modular inclusion this is obvious, since although the lightray momentum operators (generators of the lightray translations $U_{\pm}(a)$) $P_{\pm}$ have a gapless nonnegative continuous spectrum going down to zero, the physical mass spectrum of the original two-dimensional theory is given by $\mathbb{M}^2 = P_+P_-$ and starts with the zero value belonging to the vacuum being followed by a discrete one-particle state in the gap between zero and the start of the continuum.

The above holographic projection which via modular inclusion associates a chiral conformal theory with the originally two-dimensional massive theory did not change the algebras in the net but only their spacetime affiliation (and hence their physical interpretation). This becomes quite obvious if one asks how the lightlike $U_{-}(a)$ translation acts on the holographic projection i.e. on the chiral $\mathcal{A}_+(\mathbb{R})$ net. Using the natural indexing of $\mathcal{A}_+(\mathbb{R})$ in terms of intervals on $\mathbb{R}$, we see that $U_{-}(a)$ acts in a totally fuzzy way as an automorphism since it is not a member of the $PSL(2,\mathbb{R})$ Moebius group of $\mathcal{A}_+(\mathbb{R})$. With other words the holographic image is not a chiral conformal theory in the usual sense of zero mass physics with an associated characteristic energy-momentum tensor and an ensuing Virasoro-algebra (on which the opposite light cone translation would act trivially), but rather a $PSL(2,\mathbb{R})$ invariant theory with *additional automorphisms with "fuzzy" actions* in terms of the chiral net indexing. The holographic projection contains the same informations as the original two-dimensional net in particular their is no change of the number of degrees of freedom. The lowering of the spacetime dimension in the holographic process is accompanied by the conversion of some of the originally geometric automorphisms into fuzzy ones. These properties are very important if one uses lightray quantization (or the infinite momentum frame method). The original local information gets completely reprocessed and the reconstruction of the two-dimensional local quantum physics from its lightray quantization description is a nontrivial task.

This idea of holographic encoding also works in higher spacetime dimensions. In that case the formal analog of the lightray theory $\mathcal{A}_+(\mathbb{R})$ would be a $(d-1)$-dimensional lightfront net. One again starts with a modular inclusion of wedges by a lightlike translation. However this does not yet create a net on the lightfront horizon of the wedge. It turns out that the missing transversal net structure can be created by d-2 carefully selected additional L-boosts. The best way of describing the result is actually not in terms of a d-1 dimensional holographic lightfront net, but rather as a "scanning" of the original theory in terms of an abstract chiral theory which is brought into d-1 scanning positions in the same Hilbert space [44].

There is one very peculiar case of this holographic association of a theory with a lower dimensional one: the famous AdS_{d+1}-$CQFT_d$ relation (a correspondence between the so called anti deSitter spacetime manifold in d+1 and conformal field theory in d spacetime dimension). In this case the maximal symmetry groups namely SO(d,2) are the same and in particular the phenomenon of certain geometrically acting automorphisms turning fuzzy in the holographic image is absent. Furthermore the correspondence is not related to lightfront horizons and modular inclusions; it is the only known "sporadic" case of holography based on shared maximal symmetry which is intimately related to the d+2 dimensional linear formalism which one uses to conformally compactify the d-dimensional Minkowski space [45]. Although it is an isomorphism between two QFTs, its existence was first conjectured in the setting of string theory [46]. The rigorous proof in the setting of LQP [47] leads to a correspondence which deviates in one point from the way the original conjecture in the field theoretic setting was formulated in some publications. There the correspondence was thought to be one between two Lagrangian field theories i.e. two quantum field theories which have a presentation in terms of covariant pointlike fields. This is however not true, inasmuch as it is not the case in the previous modular inclusion-based holography (where this is obvious since the original geometric symmetry gets lost in the holographic projection). One really needs the field coordinate independent LQP setting of nets of operator algebras also in the AdS-CQFT relation [47]. Intuitively one of course expects this, since the content of an isomorphism between theories in different spacetime dimensions cannot be described in terms of pointlike maps. Using concepts from physics one realizes that a conformal theory which fullfils in addition to the already mentioned (see previous section) causality properties the requirement of primitive causality (the causal shadow property) which is the local quantum physical adaptation of a dynamical propagation law in time (i.e. of a classical hyperbolic differential equation), then the algebraic net of the canonically associated AdS theory cannot be generated by pointlike fields. Rather the best localized generators which the AdS theory possesses are “strands” which intersect in a pointlike manner the conformal boundary of AdS at infinity.

String theorist would perhaps say at this point that this is what they would have expected anyhow, but unfortunately these stringlike configurations extending into the bulk do not increase the number of degrees of freedom which the dynamical strings of string theory proper would do (this is the reason for calling tem “stands”). On the other hand these AdS strands are really strings in the sense of LQP localization, whereas localization in the target-space formalism of string theory has remained one of the most inconclusive and obscure issues.

Opposite to the discussed situation is the one where one starts with a pointlike field situation in AdS. Then the associated CQFT violates the causal shadow property: as one moves up into the causal shadow region of a piece of a (thin) timeslice there are more and more degrees of freedom entering from the AdS bulk. The d+1 dimensional AdS theory generated by pointlike fields contains too many degrees of freedom which destroy the equality between the algebra indexed by a connected spacetime region $\mathcal{O}$ and the algebra indexed by its causal completion (=causal shadow) $\mathcal{O}''$. Since the AdS spacetime from the point of view of particle physics is an auxiliary concept[34], it is more reasonable to relax particle physics requirement on the AdS side than on the conformal side; after all conformal theories are thought of as the scaling limit of massive physical theories.

This brings up the interesting question whether there are QFTs, which even after holographic unfolding (reprocessing into higher dimensions by changing the spacetime net indexing or by giving a spaetime meaning to a fuzzy subalgebra) do not allow an interpretation in terms of pointlike field coordinatization, i.e. *which are intrinsically operator-algebraic and not even formally obtainable by Lagrangian quantization.* The general answer to this question is not known, but for conformal theories a pointlike field coordinatization is always possible. For chiral theories proofs have been published [48] and there is no reason for thinking that these methods are limited to low dimensions. My personal opinion is that there may exist massive LQP theories which do not violate known physical

[34]It arises if one wants to reorganize the content of conformal theory in such a way that the so called conformal hamiltonian (which is the higher dimensional analog of the chiral rotation) becomes the true hamiltonian (in the sense of the Lagrangian formalism) [26].

principles and which do not allow a complete description in terms of pointlike fields.

It is interesting to note that there exist relations between QFTs on spacetimes with the same dimensionality which, although not requiring the use of modular inclusions, share the aspect of fuzzyness (hidden symmetries) of certain automorphisms. A nice illustration of such a "transplantation" was recently given in [49]. The authors start with a QFT on 4-dim. deSitter spacetime which is the submanifold $dS = \left\{ x \in \mathbb{R}^5 | \, x_0^2 - x_1^2 - ... - x_4^2 = -1 \right\}$. They then show that the so called Robertson-Walker spacetime (RW) in a certain parameter range has an isometric embedding into dS. Although this does not lead to transformation formulas for pointlike fields, it does allow to transplant the family of algebras of double-cone shaped regions on dS to corresponding (using the embedding) regions on RW. As one expects from a map which cannot be expressed in terms of pointlike fields, part of the geometrical SO(4,1) dS symmetry becomes fuzzy after the RW transplantation. These hidden symmetries are pure "quantum" and would not be there at all in the (semi)classical RW theory. Apparently the transplanted theory fulfills all the physical requirements presented in the second section (adapted to curved spacetime).

It is interesting to analyze some other ideas from string theory as "branes" and the Klein-Kaluza dimensional reduction within the algebraic setting of local quantum physics.

The (mem)brane idea consists in studying a theory which results from restricting a given theory to a spacetime submanifold containing the time axis by fixing one spatial coordinate. This is of course perfectly legitimate as an auxiliary mathematical device. If however one wants to attribute a physical reality to the restricted as well as to the ambient theory (which includes the requirement of the causal shadow property), one faces a similar problem as in the AdS-CQFT discussion above. The causal shadow property prevents to have pointlike fields in both cases; if the brane theory is generated by pointlike fields, these pointlike fields develop into transversal strands (which do not depend on the transversal coordinates) in the ambient theory and these stringlike configurations do not increase the degree of freedoms, they lack the dynamical aspects of string theory.

The problem with the Kaluza-Klein limit is more severe, since the attempt to reduce the number of spatial dimensions by making them compact and then letting their size go to zero will create uncontrollable vacuum fluctuations. Not only quantities as those which feature in the Casimir effect will diverge, but there is not even a good reason to believe that the vacuum state on the ambient algebra will stay finite on any local operator in the Kaluza-Klein limit. Operator methods are better suited to make this difficulty visible than Lagrangian quantization which especially in its functional action form is formally closer to semiclassical aspects. In the quantization approach it tends to be overlooked because the Kaluza-Klein reduction is made on the quantization level before the actual model calculations (and hence the memory of the ambient spacetime is lost), instead of first computing the correlation functions of the ambient theory and then taking the K-K limit.

Most of the problems touched upon in this section belong to the "unfinished business" of particle physics, whereas section 2 and part of section 3 (in particular the DHR theory) consists of material with a well-understood and firm mathematical and conceptual position. We included some "unfinished business" in these notes in order to counteract the widespread impression that operator algebra methods are limited to formulate already understood aspects of local quantum physics in a more rigorous fashion.

4.3. Concluding remarks. The aim of these lectures is to convince mathematicians that not only are operator algebra methods useful for their innovative power in problems of local quantum physics, but also that some of the concepts coming from LQP have left their mark on mathematics. This can be seen e.g. by the many contributions coming from LQP which preceeded discoveries in subfactor theory[35]. Although our main motivation and illustrations of modular inclusions were related to LQP, there have also been interesting recent applications of modular inclusions to noncommutative dynamical (Anosov- and K-) systems [50].

On the physical side the presently most promising ideas in my view are related to the ongoing development of modular theory. Concepts as modular inclusions are the first steps towards a seemingly very deep connection between the relative positioning of copies of one abstract operator algebra in one common Hilbert space and more geometrical spacetime properties and (finite or infinite dimensional) Lie groups which are generated by the modular groups of the various standard pairs (algebra, state vector) which can be associated with such situations. Apart from the action of Poincaré/conformal groups there are infinitely many modular actions which are "fuzzy" i.e. their local action on an algebra associated with a region cannot be encoded into a diffeomorphism. These actions are totally "quantum" i.e. they do not exist in classical theory and hence remain hidden in quantization procedures. Hitherto "fuzziness" of spacetime aspects was mostly noticed in connection with "noncommutative QFT" (the noncommutativity referring to spacetime), but here we met this behavior without changing any of the underlying principles as soon as we use methods which allow us to go beyond the confinements of the Lagrangian approach. With other words the use of OA-methods in QFT emulates some of the properties which result from noncommutative QFT.

Another problem which in my view can only be solved by operator algebra methods is to obtain an intrinsic understanding of *interactions* independent of field coordinatizations and their short distance behavior. In fact the physical Leitmotiv underlying these lectures is *how to adapt Wigner's ideas with the help of Tomita's modular theory to the realm of interactions.*

As mentioned, the modular methods tend to be more noncommutative than the standard methods although the noncommutative aspects are resulting from the same physical principles which underlie the standard approach. An example of this is the program of constructing the multiparticle spaces and interpolating fields for d=1+2 Wigner particles with braid group statistics using modular wedge localization as presented in section 2.2. In the standard setting the only Lagrangians which have a chance to have a relation with braid group statistics are those containing Chern-Simons terms, but the whole setup starting from Euclidean actions is too commutative to produce the real-time vacuum expectations of products of the spacetime carriers of these plektonic charges[36]. In order to counteract this there have been attempts to change the classical spacetime to something noncommutative [51]. The operator algebra approach would not change the classical spacetime structure which enters as the indexing of the net of algebras in agreement with the physical idea that the physics of d=1+2 plektons is laboratory physics and not quantum gravity. It would rather attempt to unravel noncommutative aspects of LQP by changing Lagrangian methods by modular constructions.

[35] Searching in the http://xxx.lanl.gov/ archives math-ph and math.OA under the names Doplicher, Evans, Longo, Mueger, Rehren, Roberts, Xu..., the reader finds many AQFT-induced contributions to operator algebras.

[36] In fact one Chern-Simons Lagrangian seems to describe only plektons without anti-plektons, but the question of discrete invariances as parity can only be settled in a formalism which has both (similar to neutrinos) and then it is a matter of how they interact.

On almost all issues considered in these notes the operator algebra framework offers a more conservative alternative than the more artistic (outside of perturbation theory) Lagrangian approach. In particular the underlying philosophy of LQP is not that of a search for better Lagrangians but rather that of unfolding and sharpening of physical principles. In that respect it is closer in spirit to condensed matter physics where one deemphasizes individual Hamiltonians in favor of the notion of universality classes (equivalence classes which share certain structural aspects, the most prominent being short distance class).

Since in the history of physics the times of greatest progress were those of contradictions and not of harmony, I believe that the importance of LQP will increase.

Last not least there are of course also strong inner-mathematical reasons for studying ideas about noncommutative geometry which are also related to the use of operator algebras [52]. It would be nice if there also would exist compelling physical reasons for a direct connection between particle physics and noncommutative geometry. The present problem with such a program is that the algebraic smoothness which distinguishes noncommutative geometry from C^*- or von Neumann algebras does not seem to be related to the smoothness/analyticity of physical correlation functions [1] (which is already accounted for by the von Neumann algebra nets of AQFT). Furthermore the use of noncommutative spacetime for the control of ultraviolet divergencies in laboratory (i.e. excluding quantum gravity) particle physics seems to be far-fetched in view of the fact that more conservative ideas as the modular localization structure contain the very strong message that the ultraviolet problem as we know it may well be a fake of singular field coordinatization which the Lagrangian quantization formalism enforces upon us.

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