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Inverse Problems of Partial Differential Equations in Industry and Engineering

A Tutorial with Examples and Techniques
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Contents

Introduction

Direct $\times$ Inverse Problems

Inverse problem theory is a fairly general phrase used to encompass a number of mathematical techniques associated to nonlinear problems that arise in many fields, especially in partial differential equations. Typically one is concerned with obtaining information about the coefficients of an equation by using data obtained from its solutions.

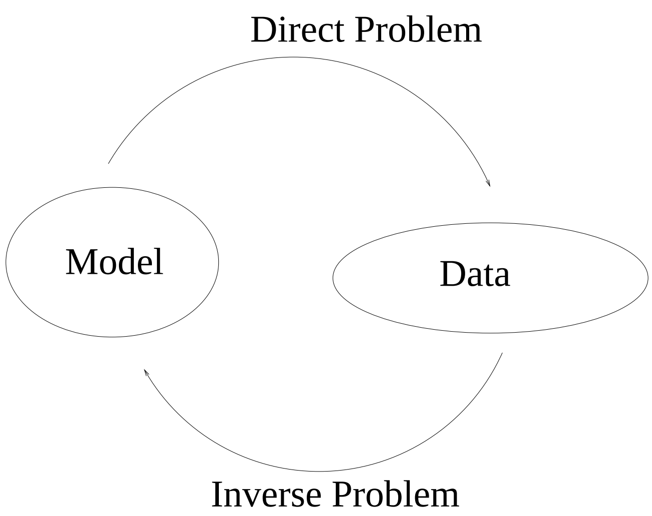

Fig. 0.1. Notion of Inverse Problem

For example, let's consider the linear wave equation with variable speed of propagation

$$\partial_t^2 \Phi - c^2(x)\Delta\Phi = g(x,t) \ . \tag{0.1}$$

Here, $\Phi(x,t)$ is the intensity of the perturbation at a given point x and time t. The term $g = g(x,t)$ corresponds to a forcing term. In many applications it is taken as Dirac's delta measure at a certain source location point $(x_s, 0)$ of space-time.

The *direct problem* consists in obtaining the solution Φ given the speed of propagation $c(x)$ and the forcing term g. Sometimes, only the asymptotic behavior of Φ can be obtained.

When studying the *inverse problem*, one tries to recover the speed of propagation of waves in the medium, by looking at the asymptotic behavior of the solutions, or the measured solution at certain chosen points in space.

Inverse problems are characterized by the fact that they are usually much harder than their direct counterparts. Furthermore, they are usually associated to ill-posed problems. Despite this, many of such problems can be treated in a very effective way, and yield practical results.

This ability to probe and map objects with radiation has been an intriguing concept in science during the last few decades. Research on non-invasive imaging has already had significant effects on the areas such as

- medical diagnosis,
- non-destructive evaluation,
- underwater imaging,
- microscopy, and
- seismic exploration.

The mathematical issues associated to inverse problems are extremely challenging. This is due to the fact that in most cases, one has to solve poorly conditioned equations for noisy and limited data. Therefore, they require special techniques to handle and understand the desired solutions.

1. Inverse Problems in Wave Propagation and Geophysics

1.1 Typical Physical Configurations and the Wave Equation

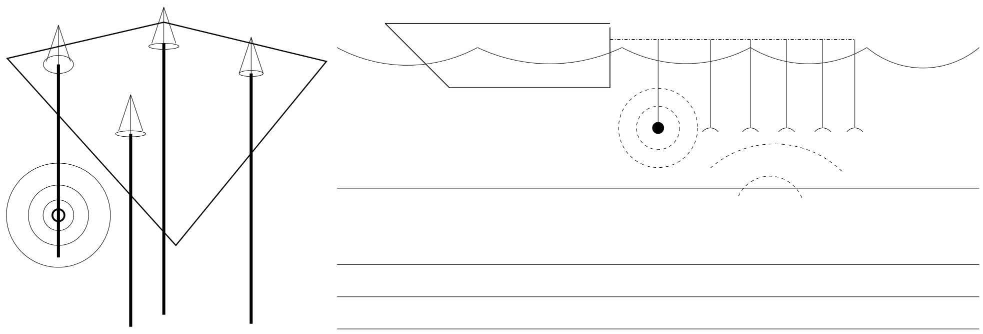

Fig. 1.1. A depiction of boreholes and of an off-shore survey.

The propagation of waves in a large class of seismic phenomena is governed by the linear acoustic wave equation

$$\frac{1}{\rho(x)v^2(x)}\frac{\partial^2}{\partial t^2}p(x,t) - \nabla \cdot \frac{1}{\rho(x)}\nabla p(x,t) = F(x,t)\ . \tag{1.1}$$

Here, $p = p(x,t)$ is the pressure at the point x and time t, $v(x)$ is the velocity of wave propagation at x and $\rho(x)$ is the density. Furthermore, the forcing term $F(x,t)$ corresponds to sources that are located at certain regions which in general are far from the region under scrutiny.

The above wave equation is derived under suitable small amplitude hypothesis for the pressure in a fluid. See for example Chapter 5 of [40].

For the direct-problem (1.1) to be well posed we assume also that the pulse $F(x,t)$ and the pressure $p(x,t)$ are zero for $t << 0$. Typically one would model $F(x,t) = f(t)\delta(x - x_s)$ where x_s is a source location.

The "ideal" inverse problem would be:

> Suppose we know the $p(x_r, t_r)$ for a large number of receiver positions x_r and times t_r and for different source signals $F(x,t)$. Find (or estimate) $\rho(x)$ and $v(x)$.

Obviously, this is a very ambitious and hard problem. Yet one can get a lot of information about $\rho(x)$ and $v(x)$ which is extremely useful for geophysical applications such as oil prospection, water reservoir location, geophysical surveys and so on. We refer the interested reader on practical applications to the webpage:
http:// sepwww.stanford.edu/sep/biondo/Lectures For the theoretical aspects an excellent reference is: http:// www.trip.caam.rice.edu

One avenue to try simplifying the above problem mathematically is to assume that the $\rho(x)$ is basically constant. In this case, the wave equation becomes

$$\frac{\partial^2}{\partial t^2}p(x,t) - v^2(x)\Delta p(x,t) = G(x,t) \ , \tag{1.2}$$

where now $G = v^2 f(t)\delta(x - x_s)/\rho$.

1.1.1 The Helmholtz Equation

As costumary in the study of signal and wave propagation we will study the decomposition of the solutions in terms of time harmonics. We will see the relation between the wave equation and the Helmholtz equation, which will reappear in a number of contexts. We follow closely lectures of G. Papanicoulaou at the 1999 Mathematical Geophysics Summer School.

Let us apply the Fourier transformation to equation (1.1). Physically this corresponds to considering the pressure as a superposition of time-harmonic sinusoidals of diferent phases. By writing[1]

$$p(x,t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u(x,\omega) e^{-i\omega t} \, d\omega \ ,$$

we get after a quick computation that each component $u(x,\omega)$ must satisfy the Helmholtz equation:

$$\left(\Delta + \frac{\omega^2}{v^2(x)} \right) u(x,\omega) = -\widehat{f}(\omega)\delta(x - x_s) \ , \tag{1.3}$$

where

$$\widehat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{i\omega t} \, dt \ .$$

We shall, from now on, write the velocity $v(x)$ as a perturbation (not necessarily small) of a *constant* velocity background c, so that

$$\frac{1}{v^2(x)} = \frac{1}{c^2}\left(1 + \alpha(x)\right) \ . \tag{1.4}$$

[1] Notice we are using $\exp(-i\omega t)$ for the Fourier superposition.

This constant velocity background c is usually the velocity of propagagation of the (acoustic) waves in the reference medium where the source are located. See Figure 1.2.

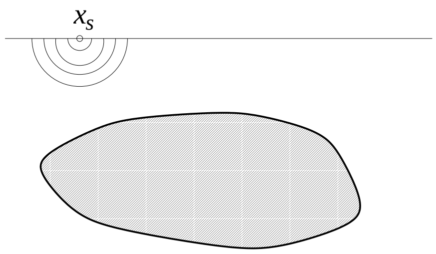

Fig. 1.2. The Source Location and the Hatched Region Under Scrutiny.

In order to guarantee well posedness for the Helmholtz equation above, one should impose radiation conditions. Physically this corresponds to the requirement that no waves are being generated at infinity. The so-called Sommerfeld radiation condition for (1.3) is

$$r\left(\frac{\partial u}{\partial r} - \frac{i\omega}{v}u\right) \to 0 \qquad \text{as} \qquad r \to \infty \ , \tag{1.5}$$

where $r = |x|$.

In the spirit of scattering theory, we are going to break the solution u as a superposition of two fields, one is the incident field u_I, and the other the scattered field u_S. So, that the total field

$$u = u_I + u_S \ .$$

We recall that the *support of a* function α is the closure of the set of all points where the function α does not vanish. Since we are assuming that the support of α does not contain the region where the sources are placed it follows that the incident field u_I is a solution of the free equation

$$\left(\Delta + \frac{\omega^2}{c^2}\right) u_I = -\widehat{f}(\omega)\delta(x - x_s) \ .$$

The sought scattered field u_S satisfies the equation

$$\left(\Delta + \frac{\omega^2}{c^2}\right) u_S = -\frac{\omega^2}{c^2}\alpha(x)\left(u_I + u_S\right) \ . \tag{1.6}$$

We recall now the notion of a Green's function $G(x, y; k)$ for the Helmholtz equation. It is a solution of

$$\left(\Delta + k^2\right) G = -\delta(x-y) \ , \tag{1.7}$$

where here $k = \omega/c$ is the wave number. In $\mathbb{R}^3$, the outgoing [2] Green's function is given by

$$G(x,y;k) = \frac{\exp(ik|x-y|)}{4\pi|x-y|} \ . \tag{1.8}$$

We now recall Green's identity with $\mathcal{L} \stackrel{\text{def}}{=} \Delta + k^2$ and D any sufficiently well behaved domain where the divergence theorem holds and not containing y. Then, we have

$$\int_D (G\mathcal{L}w - w\mathcal{L}G)dx = \int_{\partial D} (G\frac{\partial w}{\partial n} - w\frac{\partial G}{\partial n})\ dS(x) \tag{1.9}$$

where n is the exterior normal to the domain D at the boundary ∂D. We first use it to show the "sampling" property: If $y \in D$ then

$$\int_D w(x)\mathcal{L}G(x,y;k)dx = -w(y) \ ,$$

at least if w has compact support and is sufficiently smooth. This is essentially the meaning of the claim that G satisfies (1.7). Then, we apply the last identity to $w = u_s$ and $y = x_g$ the location of our gatherer, i.e., a geophone or a hydrophone. We leave as an exercise to the reader to check, using the above remarks and the fact that G and u_S satisfy the Sommerfeld radiation condition, that

$$u_S(x_g,\omega) = k^2 \int_\Omega \alpha(x)\left(u_I(x,\omega) + u_S(x,\omega)\right) G(x_g,x,k)\, dx \ , \tag{1.10}$$

where Ω is the support of $\alpha(x)$. Equation (1.10) is the well known Lipmann-Schwinger equation. Despite its simplicity it is crucial in the understanding of the scattering problem. It can be shown, under suitable assumptions on the function α, that it is equivalent to the existence of u_S satisfying the scattering problem posed above and endowed with the outgoing radiation condition.

If we define the operator

$$Kf = \int_\Omega \alpha(x)G(\cdot,x,k)f(x)\, dx \ ,$$

then the Lipmann-Schwinger takes the operational form

$$u_S - k^2 K u_S = k^2 K u_I \ . \tag{1.11}$$

[2] i.e., satisfying the Sommerfeld radiation condition.

1.1.2 The Born Approximation

We are now ready to define the so-called Born approximation. It consists in neglecting the scattered term u_S vis-a-vis the incoming term u_I. In this way we get u_B, which is an approximation for u_S given by:

$$u_B(x_g, \omega) = k^2 \int_\Omega \alpha(x) u_I(x, \omega) G(x_g, x, k)\, dx \ . \tag{1.12}$$

Physically, this means that we are neglecting the multiple scattering due to the perturbation α. Another way of looking at it is by saying that as a first order approximation, the scattered field is produced by superposing the contribution of each point x in the support of α as a scatterer of weight $k^2\alpha(x)$. See Figure 1.3.

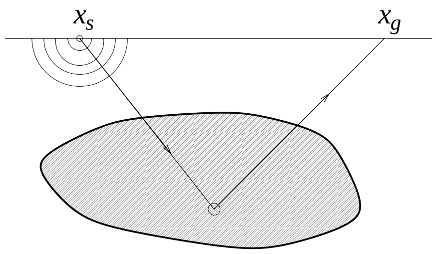

Fig. 1.3. The Single Scattering and the Born Approximation

Equation (1.12) is instrumental for wave inversion of *low contrast* media. It is interesting, however, to remark that the Born approximation is not necessarily associated with weak contrast $\alpha(x)$. It means that the scattered field is weak as compared to the incoming field. The latter is a consequence of the weak contrast media, as can be seen for example, by looking at the operational form of the Lipmann-Schwinger equation (1.11) . In the expression of the Neumann series we see that if the norm of α is sufficiently small, then we can bound the contribution of the terms K^n with $n \geq 2$ as $\mathcal{O}(||\alpha||^2)$.

We arrive now at one of the different equivalent versions of the inverse problem for acoustic wave inverstion within the context of the Born approximation: Let's assume that we have the measurements $u_B(x_g; \omega)$ for "all" values of ω and different detector (gatherer) and source positions. Determine the function α from equation (1.12).

1.1.3 One Analytic Approach

We shall now give a flavour of the procedure to solve (theoretically) this inverse problem. In order to do that we are going to concentrate in one single geometry, namely the one where the source and the detector are coincident. Thus $x_s = x_g$. This situation is refered in the geophysical literature as zero offset.

Let us denote by $x = (x^1, x^2, x^3)$ and place the x^3 axis as pointing down (in the good geophysical tradition). Our source and gatherer positions will be placed at $x_s = x_g = \xi = (\xi^1, \xi^2, 0)$. Equation (1.12) takes the form

$$u_B(\xi,\omega) = \frac{\omega^2}{c^2} \int_{x^3>0} \alpha(x) u_I(x,\omega) G(x_g, x, k)\, dx^1 dx^2 dx^3 \ . \tag{1.13}$$

If we take our incident wave as the result of a point source, then

$$u_I(x,\omega) = \widehat{f}(\omega) \frac{\exp(i\omega r/c)}{4\pi r} \ ,$$

where $r = |x - \xi|$. By making use of the form of the Green's function again we get

$$u_B(\xi,\omega) = \left(\frac{\omega}{4\pi c}\right)^2 \int_{x^3>0} \alpha(x) \frac{\exp(2i\omega r/c)}{r^2}\, dx^1 dx^2 dx^3 \ . \tag{1.14}$$

The task is to invert such integral. Indeed, we remark that at this point we have a mathematically precise direct and inverse problem. The direct problem takes the form: Given α determine u_B. The inverse problem is: Given u_B find α. We also note that we have some hope of success, at least theoretically, since the counting of independent variables is correct. By that we mean, provided we use all the frequency information and vary the source-gatherer position we have that both $\alpha(x^1, x^2, x^3)$ and $u_B(\xi^1, \xi^2, 0; \omega)$ depend on the same number of independent variables.

It turns out that from the practical point of view the determination of α from equation (1.14) is not a well posed problem. Indeed, in many reasonable spaces the integral operator acting on the right hand side is a compact operator acting on an infinite dimensional space. Its is inverse will be an unbounded operator. This, and many other "ill posed" problems have been the subject of intensive investigation, and were the subject of intense investigation. Some of the names that are quoted in this respect are Tikhonov and Morozov.

Let us now proceed with the teoretical inversion. We assume first that we are using an impulsive time source. Hence, $\widehat{f}(\omega) = 1$. Then,

$$u_s(\xi,\omega) = \left(\frac{\omega}{4\pi c}\right)^2 \int_{x^3>0} \alpha(x) \frac{\exp(2i\omega r/c)}{r^2}\, dx^1 dx^2 dx^3 \ .$$

Thus, upon multiplying by $(4\pi c/\omega)^2$ and differentiating with respect to ω we get:

$$\int_{\Omega} \alpha(x) \frac{\exp(2i\omega r/c)}{4\pi r} = -2\pi i c^3 \frac{\partial}{\partial \omega} \left(\frac{u_s}{\omega^2} \right) \ , \tag{1.15}$$

where $\Omega = \operatorname{supp}(\alpha)$ as above. Since $r = |x - \xi|$ we now recognize on the LHS of Equation (1.15) a convolution between $g_1(\xi - x, \omega) \stackrel{\text{def}}{=} \exp(2i\omega r/c)/(4\pi r)$, with $\alpha(x)$. Let us consider a two-dimensional Fourier Transform pair. We shall also denote it by a hat and will have (κ_1, κ_2) and (ρ_1, ρ_2) as conjugate variables:

$$\widehat{f}(\kappa) = \int\int e^{-2i\kappa\cdot\rho} f(\rho)\, d\rho$$
$$f(\rho) = \frac{1}{\pi^2} \int\int e^{2i\kappa\cdot\rho} \widehat{f}(\kappa)\, d\kappa \ .$$

We now use the two-dimensional Fourier Transform and the convolution theorem to get

$$\int_0^\infty dz\ \widehat{\alpha}(\kappa, z) \widehat{g_1}(\kappa, z, \omega) = -2\pi i c^3 \frac{\partial}{\partial \omega} \left(\frac{\widehat{u_s}}{\omega^2} \right) \ , \tag{1.16}$$

where we have taken the depth $z = x^3$ and the function

$$\widehat{g_1}(\kappa, z, \omega) = \int\int \frac{\exp(-2i\kappa \cdot \rho + 2i\omega\sqrt{\rho^2 + z^2}/c)}{4\pi\sqrt{\rho^2 + z^2}}\, d\rho \ .$$

We leave it as an exercise to the reader to verify that

$$\widehat{g_1}(\kappa, z, \omega) = -\frac{1}{4ik_3} \exp(2ik_3|z) \ ,$$

where $k_3 = \sqrt{k^2 - \kappa^2}$ with the appropriate choice of the branch of the square-root, i.e.:

$$k_3 = \begin{cases} \operatorname{sgn}(\omega)\sqrt{k^2 - \kappa^2} & , |\omega| \geq c\kappa \\ i\sqrt{\kappa^2 - k^2} & , |\omega| < c\kappa \end{cases} \tag{1.17}$$

Hence, we have transformed the equation into

$$\int_0^\infty dx^3 \widehat{\alpha}(\kappa, x^3) \exp(2ik_3|x^3|) = -8\pi k_3 c^3 \frac{\partial}{\partial \omega} \left(\frac{\widehat{u_s}(\kappa, \omega)}{\omega^2} \right) \ .$$

We now notice that the left hand side consists of an 1-dimensional Fourier transform. Hence

$$\widehat{\alpha}(\kappa, x^3) = -8c^3 \int_{-\infty}^{\infty} dk_3\ k_3 \frac{\partial}{\partial \omega} \left(\frac{\widehat{u_s}}{\omega^2} \right) exp(-2ik_3 x^3) \ .$$

Thus,

$$\alpha(x) = -\frac{8c^3}{\pi^2} \int_{\mathbb{R}^3} dk k_3 \frac{\partial}{\partial \omega} \left(\frac{\widehat{u_s}(\kappa, \omega)}{\omega^2} \right) \exp(2i(\kappa \cdot \rho - k_3 x_3) \ , \tag{1.18}$$

with k_3 given by (1.17) and $\omega = c\mathrm{sgn}(k_3)\sqrt{\kappa^2 + k_3^2}$. We now go to the physical time domain by Fourier transform

$$\alpha(x) = -\frac{8c^3}{\pi^2} \int_{\mathbb{R}^3} dk \int_{\mathbb{R}^2} d\xi \int_0^\infty dt \ k_3 e^{2i(\kappa\cdot(\rho-\xi)-\kappa_3 x^3)} \frac{\partial}{\partial \omega} (\frac{1}{\omega^2} e^{i\omega t}) \ .$$

The final result is

$$\alpha(x) = \frac{8c^3}{i\pi^2} \int_{\mathbb{R}^2} d\xi \int_{\mathbb{R}^3} dk \ k_3 e^{2i(\kappa\cdot(\rho-\xi)-\kappa_3 x^3)} \int_0^\infty dt \ t \ u_s(\xi, t) \exp(i\omega t) \left(1 + \frac{2i}{\omega t}\right) \tag{1.19}$$

This formula clearly presents difficulties for numerical implementations. It involves highly unstable numerical operations such as differentiation and division by quantities that could be close to zero (check!). It has similar characteristics to the classical Radon inversion formula, which was highly unstable numerically. Hence, its practical implementation “as is” is not a good idea.

1.1.4 Ray Theory

The Eikonal equation - geometric optics

We shall motivate the Eikonal equation from two perspectives, which are insteresting on their own right. First we are going to arrive at it from the WKBJ method applied to the Helmholtz equation. Then we are going to look at it directly from the perspective of the wave equation.

Let us revisit equation (1.3) assuming that $\widehat{f}(\omega) = 0$. Following the spirit of the WKBJ theory we seek asymptotic solutions as $\omega \to \infty$ to

$$\left(\Delta + \frac{\omega^2}{v^2(x)} \right) u(x, \omega) = 0 \ , \tag{1.20}$$

of the form

$$u(x, \omega) \approx e^{i\omega\tau(x)} \sum_{n\geq 0} \frac{A_n(x)}{(i\omega)^n} \ \text{, as } \omega \to \infty \ . \tag{1.21}$$

The interpretation of the ansatz (1.21) is that we have a phase term $\tau(x)$ and an amplitude $A(x, \omega)$ which corresponds to an asymptotic series in ω^{-1}. A straightforward computation gives us

$$\text{grad } u = \left(i\omega \text{grad } \tau \sum_{n\geq 0} \frac{A_n}{(i\omega)^n} + \sum_{n\geq 0} \frac{\text{grad } A_n}{(i\omega)^n} \right) e^{i\omega\tau} \ ,$$

and

$$\begin{aligned} \Delta u = \text{div grad } u = e^{i\omega\tau} \big\{ & -\omega^2 |\text{grad } \tau|^2 A_0 + \\ & i\omega \left(|\text{grad } \tau|^2 A_1 + 2\text{grad } A_0 \cdot \text{grad } \tau + (\Delta\tau) A_0 \right) + \\ \sum_{n\geq 0} & \left(-|\text{grad } \tau|^2 A_{n+2} + 2\text{grad } A_{n+1} \cdot \text{grad } \tau + \Delta A_n + A_n \Delta\tau \right) (i\omega)^{-n} \ \big\} \ . \end{aligned}$$

After substituting Δu into Equation (1.21) and balancing the terms of equal power in ω we get

$$|\nabla\tau|^2 = \frac{1}{v^2(x)} \tag{1.22}$$

$$2\nabla A_0 \cdot \nabla\tau + A_0 \Delta\tau = 0 \tag{1.23}$$

$$2\nabla A_{n+1} \cdot \nabla\tau + A_{n+1} \Delta\tau = -\Delta A_n \ , \ n > 0 \ . \tag{1.24}$$

Equation (1.22) is the Eikonal equation, which plays a fundamental role in wave propagation. It is the equation that describes the evolution of the phase τ of the time harmonic wave $u(x, \omega)$. The gradient of $\tau(x)$ indicates the direction of the rays of geometrical optics.

Equation (1.23) correponds to a homogeneous transport equation for the amplitude A_0 along the ray trajectory. Indeed, if $\tau(x)$ is known, then grad τ defines a vector field. Let us consider a unit speed integral line of such vector field. More precisely, a function $\sigma \mapsto x(\sigma)$ satisfying

$$\dot{x} = v^2(x) \text{grad } \tau(x) \ . \tag{1.25}$$

Notice that because of the Eikonal equation, this function $x(\cdot)$ indeed has unit speed and σ is an arclength along it. Then, Equation (1.23) is telling us that the amplitude changes along the above integral line according to

$$2\frac{\partial A_0}{\partial \sigma} = -A_0 \Delta\tau \ .$$

Equation (1.24) corresponds to an inhomogeneous transport equation for the correction of order $n + 1$ of the amplitude, which depends on the earlier contributions. See Figure 1.4.

Let us give yet another derivation of the Eikonal equation. Consider a fundamental solution of the wave equation

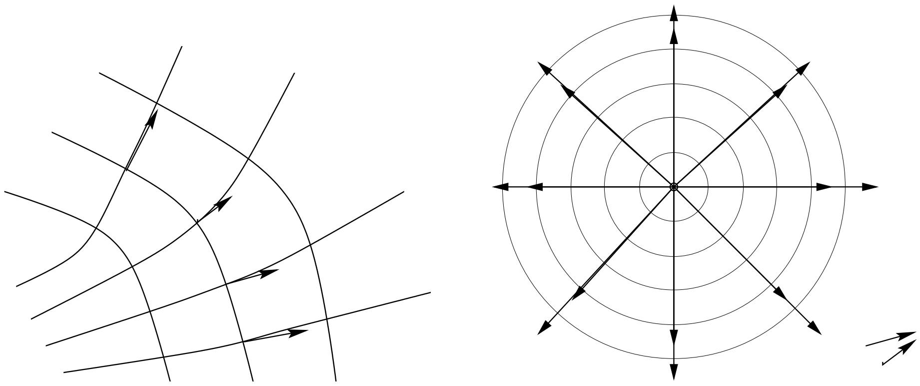

Fig. 1.4. The integral curves to grad $\tau(x)$ and the constant phase curves.

$$\begin{cases} \frac{1}{v^2(x)}\frac{\partial^2}{\partial t^2}G(x,t;x_s) - \Delta G(x,t;x_s) = \delta(t)\delta(x-x_s) \ . \\ G(x,t;x_s) \equiv 0 \quad , \quad t<0 \end{cases} \tag{1.26}$$

We seek a solution of the form:

$$u(x,t;x_s) = a(x,x_s)S(t-\tau(x;x_s)) + R(x,t;x_s) \tag{1.27}$$

where t and $|x-x_s|$ are small, and R a smoother function than S. This ansatz is based on the form for the fundamental solution of the wave equation in free space, which is given (for $c \equiv c_0$) by

$$G(x,t;x_s) = \frac{\delta(c_0 t - |x-x_s|)}{4\pi|x-x_s|} \ .$$

Substituting the ansatz (1.27) into the operator, we get

$$\begin{aligned} \left(\frac{1}{v^2(x)}\frac{\partial^2}{\partial t^2} - \Delta\right) u = {} & a\left(\frac{1}{v^2} - |\nabla\tau|^2\right) S''(t-\tau) \\ & + \left(2\nabla\tau\cdot\nabla a - a\Delta\tau\right) S'(t-\tau) \\ & + \Delta a \; S(t-\tau) \\ & + \frac{1}{v^2(x)}\frac{\partial^2}{\partial t^2}R - \Delta R \end{aligned}$$

The eikonal equation appears as the result of eliminating the highest order singularity term.

Two standard references for the Eikonal equation are: The classical masterpiece of Courant & Hilbert or F. John's [38].

Example: In free space the solutions of the Eikonal equation are given by $\tau(x,x_s) = |x-x_s|/c_0$. The constant level curves correspond to the location of the wave fronts. See Figure 1.4.

The solution of the Eikonal equation can be obtained by the method of characteristics, which plays an instrumental role in hyperbolic equations. We start by considering the vector field defined by equation (**??**) and take $x(\sigma)$ one of its solutions. Then a straight-forwards computation gives that

$$\frac{d}{d\sigma}\tau(x(\sigma)) = 1 \ .$$

Hence, $\tau(x(\sigma))$ can be identified with the parameter σ, except for possibly an additive constant. In other words, $\tau(x(\sigma)) = \tau(x(\sigma_0)) + \sigma - \sigma_0$. Now, the remark, which we are not going to prove in detail here, is that if we know the rays $x(\sigma)$ we can reconstruct τ. Such rays are constructed directly by defining:

$$\xi(\sigma) \stackrel{\text{def}}{=} \nabla\tau(x(\sigma)) \ .$$

Then,

$$\begin{aligned}
\dot{\xi}(\sigma) &= \nabla\dot{\nabla}\tau(x(\sigma))\dot{x}(\sigma) \\
&= v^2(x(\sigma))\nabla\dot{\nabla}\tau(x(\sigma))\nabla\tau(x(\sigma)) \\
&= v^2(x(\sigma))\nabla\frac{1}{2v^2} \\
&= -\frac{1}{2}|\nabla\tau|^2\nabla v^2(x)\Big|_{x=x(\sigma)}
\end{aligned}$$

We now define the function

$$H(x,\xi) = \frac{1}{2}v^2(x)|\xi|^2 \ ,$$

and consider the Hamilton's equations

$$\dot{x} = \nabla_\xi H(x,\xi) \tag{1.28}$$

$$\dot{\xi} = -\nabla_x H(x,\xi) \ . \tag{1.29}$$

For each $\xi(0) = \theta \in \mathbb{S}^2$ and $x(0) = x_s$ this system defines a curve in $\mathbb{R}^6$. The projection on the x space of such solutions of the system of equations (1.28) and (1.29) are the rays of classical geometric optics. The optical rays are crucial, since they are the paths along which information and singularities are transmited.

2. Mathematical Techniques

jointly with
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2.1 Distributions and Sobolev Spaces

The goal of this section is to clarify the notation concerning the main results of distribution theory and Sobolev spaces that are used throughout these notes. It is not intended to be a comprehensive introduction to the theory, but rather a "crash course" to make these notes self-contained. A number of excellent references can be found in [65].

Let Ω be an open set of $\mathbb{R}^n$, and denote by $\overline{\Omega}$ its closure. We use the multi-index notation: for $k = (k_1, \cdots, k_n) \in \mathbb{Z}^n_{\geq 0}$ and $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ we denote by

$$x^k \stackrel{\text{def}}{=} x_1^{k_1} \cdots x_n^{k_n} \ ,$$

$$\partial^k \stackrel{\text{def}}{=} \partial_{x_1}^{k_1} \cdots \partial_{x_n}^{k_n} \ ,$$

$$|k| \stackrel{\text{def}}{=} k_1 + \cdots + k_n \ .$$

Distribution theory grew as a need to overcome the technical difficulties associated with the process of differentiation. Very early, it was evident the need to introduce certain idealized elements which would correspond to derivatives of functions at points where the usual notion of derivative did not make sense. This leads naturally to the notion of weak derivatives and weak solutions of partial differential equations.

To study the notions of weak derivatives systematically, one introduces the notion of distribution spaces [60]. The basic remark is to think of functions by looking at their action on certain spaces of test functions. Physically, this would correspond to performing measurements against smooth "observables" to determine the behavior of functions close to a given point.

It is easy to see that a continuous function f on an open set Ω is uniquely determined if we know

$$\langle f, \phi \rangle \stackrel{\text{def}}{=} \int_{\Omega} f(x)\phi(x)dx \ ,$$

for any arbitrary given function $\phi \in C^{\infty}(\Omega)$. Furthermore, if the support of ϕ is compact, i.e. $\phi \in C_0^{\infty}(\Omega)$ then

$$\int_{\Omega} \partial^{\alpha} f(x)\phi(x)dx = \int_{\Omega} (-1)^{|\alpha|} f(x)\partial^{\alpha}\phi(x)dx \ .$$

However, the right hand side of this equality is defined even if the left hand side does not make sense. The first step to develop a model of a distribution space is to introduce the test functions. The second one is a topology. After that, one can define the continuous linear functionals which will correspond to the generalized functions of our space.

2.1.1 Test Spaces

In these notes $\mathcal{E}(\Omega)$ denotes the Fréchet space (see [58]) of C^{∞} functions on Ω endowed with the topology given by the semi-norms

$$p_{K,\alpha}(f) = \max_{x \in K} |\partial^{\alpha} f(x)| \ , \quad K \subset \Omega \text{ compact} \ .$$

By a Fréchet space we mean a locally convex, metrizable and complete topological space. In simples terms, $\mathcal{E}(\Omega)$ corresponds to the C^{∞} functions on Ω endowed with the convergence notion:

$$f_n \stackrel{n \to \infty}{\longrightarrow} f \text{ in } \mathcal{E}(\Omega)$$

iff for each compact K of Ω, and multi-index $\alpha \in \mathbb{Z}^n_{\geq 0}$, the sequence $\partial^{\alpha} f_n$ converges uniformly in K to $\partial^{\alpha} f$.

By $\mathcal{D}(\Omega)$ we will denote the space of C^{∞} functions on Ω of compact support. $\mathcal{D}(\Omega)$ will be endowed with topology that is more subtle and which we shall not describe here. The interested reader may consult [58]. We shall however give the "recipe" for convergence in this topology:

A sequence $\{\phi_n\}_{n=1}^{\infty}$ converges in $\mathcal{D}(\Omega)$ iff

(i) $\phi_n \stackrel{n \to \infty}{\longrightarrow} \phi$ in $\mathcal{E}(\Omega)$,
(ii) There exists $K \subset \Omega$ compact such that $\operatorname{supp} \phi_n \subset K$, $\forall n$.

The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ denotes the set of $C^{\infty}(\mathbb{R}^n)$ functions f such that:

$$p_{\alpha,\beta}(f) \stackrel{\text{def}}{=} \sup_{x \in \mathbb{R}^n} |x^{\alpha} \partial^{\beta} f(x)| < \infty, \qquad \forall \alpha, \beta \in \mathbb{Z}^n_{\geq 0} \ .$$

Here again we endow $\mathcal{S}(\mathbb{R}^n)$ with a topology that requires that $\phi_n \stackrel{n \to \infty}{\longrightarrow} \phi$ iff

$$p_{\alpha,\beta}(\phi_n - \phi) \stackrel{n\to\infty}{\longrightarrow} 0, \quad \forall \alpha, \beta \in \mathbb{Z}^n_{\geq 0} \ .$$

Once we have endowed the spaces $\mathcal{D}, \mathcal{E}, \mathcal{S}$ with very strong topologies, we expect that the corresponding topological duals $\mathcal{D}', \mathcal{E}', \mathcal{S}'$ are going to be very "permissive" spaces. Indeed, the notion of convergence in these dual spaces is going to be the weak-$*$ convergence.

It is a simple exercise to show that

$$\mathcal{D}(\mathbb{R}^n) \hookrightarrow \mathcal{S}(\mathbb{R}^n) \hookrightarrow \mathcal{E}(\mathbb{R}^n)$$

and

$$\mathcal{D}(\Omega) \hookrightarrow \mathcal{E}(\Omega) \ .$$

All the above inclusions with the symbol $\hookrightarrow$ are continuous w.r.t. the corresponding topologies. As a consequence, we get that

$$\mathcal{E}'(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n) \ ,$$

$$\mathcal{E}'(\Omega) \subset \mathcal{D}'(\Omega) \ .$$

We recall that $G \in \mathcal{D}'(\Omega)$ is zero in an open set $U \subset \Omega$ if for every $\varphi \in \mathcal{D}(\Omega)$ such that $\operatorname{supp} \varphi \subset U$, we have

$$\langle G, \varphi \rangle \stackrel{\text{def}}{=} G(\varphi) = 0.$$

The support of the distribution G, denoted by $\operatorname{supp} G$, is the complement to the largest open set where G is zero. It is obvious that $\operatorname{supp} G$ is closed.

The space $\mathcal{D}'(\Omega)$ is called the space of generalized functions on Ω, or *distributions* on Ω. The space $\mathcal{E}'(\Omega)$ is called the space of *compact* distributions on Ω. The reason for this being the fact that:

If $F \in \mathcal{D}'(\Omega)$ then,

$$F \in \mathcal{E}'(\Omega) \Leftrightarrow \operatorname{supp} F \subset K, \ \ \forall K \text{ compact in } \Omega \ .$$

2.1.2 Fourier Transform

One of the fundamental ingredients in the study of linear operator theory is the so-called Fourier transform. As a matter of fact, there exists a multitude of transformations that would qualify as Fourier transforms, each one according to the specific context. Let's start from a T-periodic function f on the line, i.e. f satisfies for a fixed $T \neq 0$

$$f(x) = f(x+T) \quad , \forall x \in \mathbb{R} \ .$$

Let's take

$$S \subset \frac{2\pi}{T}\mathbb{Z}$$

a finite set. It is clear that for any arbitrary linear combination of exponentials of the form

$$f(x) = \sum_{\nu \in S} c_\nu e^{i\nu x} \ , \tag{2.1}$$

the function $f(x)$ is periodic. The question is whether any T-periodic function could be written in the form (2.1), where now $S \subset (2\pi/T)\mathbb{Z}$ is not necessarily finite. It is clear that if we define the inner product

$$(f \mid g) = \frac{1}{T}\int_0^T f(x)\overline{g(x)}dx \ ,$$

then $c_\nu = (f \mid e^{i\nu x})$. The map $\widehat{f} : \nu \mapsto c_\nu = (f \mid e^{i\nu x})$ is called the (finite-interval) Fourier transform of f.

It is easy to see that $\{\exp(i\nu x)\}_{\nu \in (2\pi/T)\mathbb{Z}}$ is a complete orthonormal set in $L^2[0,T]$. In other words, any $f \in L^2[0,T]$ can be written in the form (2.1). If S is infinite, then such sum must be understood as an appropriate limit.

The next step is to view the (non-periodic) functions on the line as limits when $T \to \infty$ of periodic ones. In this case, we have that the set $(2\pi/T)\mathbb{Z}$ becomes dense on $\mathbb{R}$. We should look at integrals instead of sums, i.e. objects of the form

$$\int_{\mathbb{R}} c(\nu)e^{i\nu x}\, d\nu \ .$$

For a given $f \in \mathcal{S}(\mathbb{R}^n)$ the expression for the coefficients c_ν given above, motivates the definition of its Fourier transform $\widehat{f}$ as

$$\mathcal{F}[f](\xi) \overset{\text{def}}{=} \widehat{f}(\xi) \overset{\text{def}}{=} \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi} \ đx \ , \ f \in \mathcal{S}(\mathbb{R}^n)$$

where

$$đx \overset{\text{def}}{=} \frac{dx}{(2\pi)^{n/2}} \ .$$

We introduce the adjoint Fourier transform

$$\mathcal{F}^*[f](\xi) \overset{\text{def}}{=} f^\vee(\xi) \overset{\text{def}}{=} \int_{\mathbb{R}^n} f(x)e^{+ix\cdot\xi} \ đx \ .$$

The first crucial remark is the following proposition, whose proof can be found in most Fourier analysis textbooks [35]:

Proposition 2.1.1. *The maps $\mathcal{F}$ and $\mathcal{F}^*$ satisfy:*

(i) $\mathcal{F}$ *and* $\mathcal{F}^* : \mathcal{S} \longrightarrow \mathcal{S}$ *are 1-1 and onto.*
(ii) $\mathcal{F}$, $\mathcal{F}^*$ *are continuous w.r.t. the topology of* $\mathcal{S}$.
(iii) *For* $f \in \mathcal{S}(\mathbb{R}^n)$ *we have*

$$\mathcal{F}^*\left(\mathcal{F}f\right) = \mathcal{F}\left(\mathcal{F}^* f\right) = f \ .$$

(iv) *For* $f \in \mathcal{S}(\mathbb{R}^n)$ *we have*

$$(i\xi)^\alpha \, \partial^\beta \hat{f}(\xi) = (\partial^\alpha (-ix)^\beta f)\hat{\ }(\xi) \ ,$$

and

$$(-i\xi)^\alpha \, \partial^\beta f\check{\ }(\xi) = (\partial^\alpha (ix)^\beta f)\check{\ }(\xi) \ .$$

Proof. We shall mention only a few words about the proof of this proposition. The fourth item is a direct computation using integration by parts. This is then used to prove that $\mathcal{F}$ and $\mathcal{F}^*$ map $\mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ continuously. Once this is proven we can show that $\mathcal{F}$ and $\mathcal{F}^*$ are inverses of one another. This may be done in a number of different ways. See for example [57, 35]. We shall use a rather "non-traditional" method. For that we restrict ourselves to the case $n = 1$. The generalization to the n-dimensional case will be left as an exercise.

Consider the map:

$$\mathsf{G} : \mathcal{S}(\mathbb{R}) \ni f \mapsto \mathcal{F}^*\left(\mathcal{F}\left(f\right)\right) \in \mathcal{S}(\mathbb{R}) \tag{2.2}$$

First, notice that G commutes with the operator $\mathcal{M}_x$ defined by

$$\mathcal{M}_x f(x) \stackrel{\text{def}}{=} x f(x) \ .$$

Indeed,

$$(\mathcal{M}_x \, f)\hat{\ }\check{\ } = (i\partial_\xi \hat{f})\check{\ } = \mathcal{M}_x \, (\hat{f})\check{\ } \ .$$

Exercise 2.1.1. Show that if an operator $\mathsf{A} : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ commutes with the operator $\mathcal{M}_x$, then $\mathsf{A} = \mathcal{M}_{g(x)}$ for some function $g(x)$.

It is clear from the exercise that G is the operator of multiplication by a function $g(x)$. We now compute it using the function $f(x) = \exp(-\frac{x^2}{2})$.

Exercise 2.1.2. Check that

$$(e^{-\frac{x^2}{2}})\hat{\ }(\xi) = e^{-\frac{\xi^2}{2}} \ .$$

Hence, $g(x) = 1$, which concludes the proof of item (iii) of the proposition. Item (i) follows then immediately from the others. *Q. E. D.*

The notion of derivatives and other continuous operations on $\mathcal{S}$ can be easily extended to $\mathcal{S}'$. For example, if $T \in \mathcal{S}', \varphi \in \mathcal{S}$, then

$$\langle \partial^\alpha T, \varphi \rangle \stackrel{\text{def}}{=} \langle T, (-1)^{|\alpha|} \partial^\alpha \varphi \rangle$$

It is immediate to check that the claims of Proposition (2.1.1) all hold for the space $\mathcal{S}'$ (endowed with the weak-$*$ topology).

The Fourier transform extends naturally to many interesting spaces. In particular to $L^2(\mathbb{R}^n)$. The first step for the L^2 theory is the following:

Proposition 2.1.2. *For $f \in \mathcal{S}(\mathbb{R}^n)$ we have*

$$\int_{\mathbb{R}^n} |f(x)|^2 \, dx = \int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 \, d\xi \ .$$

Recall that the space $L^2(\mathbb{R}^n)$ is endowed with the usual inner product

$$(f \mid g) = \int_{\mathbb{R}^n} f(x)\overline{g(x)} \, dx \ .$$

Exercise 2.1.3. Verify that for $f, g \in \mathcal{S}(\mathbb{R}^n)$

$$(f \mid \mathcal{F}g) = (\mathcal{F}^* f \mid g) \ ,$$

which justifies the notation $\mathcal{F}^*$.

Since $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$ we have that $\mathcal{F}$ and $\mathcal{F}^*$ extend as isometries from $L^2(\mathbb{R}^n)$ into itself. We shall use the same notation for such extensions of the Fourier transform without any further remark.

As a consequence of Proposition 2.1.2 we get the so-called Plancherel theorem:

Theorem 2.1.1 (Plancherel). *The linear operators $\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ and $\mathcal{F}^* : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ satisfy*

$$\mathcal{F}\mathcal{F}^* = \mathcal{F}^*\mathcal{F} = I \ .$$

2.1.3 Sobolev Spaces

In many applications it is crucial to work with L^2 functions whose all derivatives up to a certain order belong to $L^2(\mathbb{R}^n)$. Because of Plancherel's theorem, and the extension of Proposition 2.1.1 to $\mathcal{S}'$, a function f whose all distributional derivatives up to order k belong to $L^2(\mathbb{R}^n)$ can be characterized by the requirement that

$$\int_{\mathbb{R}^n} (1 + |\xi|^2)^k \, |\widehat{f}(\xi)|^2 \, d\xi \quad < \infty \ .$$

This motivates considering for arbitrary $s \in \mathbb{R}$ the norm

$$\|f\|_s^2 \stackrel{\mathrm{def}}{=} \|f\|^2_{\mathcal{H}^s(\mathbb{R}^n)} \stackrel{\mathrm{def}}{=} \int_{\mathbb{R}^n} \langle \xi \rangle^{2s} |\widehat{f}(\xi)|^2 \, d\xi \ ,$$

where,

$$\langle \xi \rangle \stackrel{\mathrm{def}}{=} (1 + |\xi|^2)^{\frac{1}{2}} \ .$$

The Sobolev space $\mathcal{H}^s(\mathbb{R}^n)$ is defined by

$$\mathcal{H}^s(\mathbb{R}^n) = \{ f \in \mathcal{S}'(\mathbb{R}^n) \mid \ \|f\|_s^2 < \infty \ \} \ .$$

It is not difficult to see that $\mathcal{H}^s(\mathbb{R}^n)$ when endowed with the $\| \quad \|_s$ norm, is a Hilbert space. It is a natural question when a distribution in $\mathcal{H}^s$ corresponds to a C^k function. The general philosophy is that the faster the decay of the Fourier transform $\widehat{F}$, the smoother the distribution F . In fact we have:

Theorem 2.1.2 (Sobolev). *If* $s > \frac{n}{2} + k$ *, then* $\mathcal{H}^s(\mathbb{R}^n) \subset C^k_\infty(\mathbb{R}^n)$.

Here,

$$C^k_\infty(\mathbb{R}^n) = \{ f \in C^k(\mathbb{R}^n) \ \mid \ \lim_{|x| \to \infty} \partial^\alpha f(x) = 0 \ , \ \forall \alpha, \ \ |\alpha| \leq k \} \ .$$

For example, if $f \in \mathcal{H}^s(\mathbb{R})$ with $s > 1/2$ then $f \in C(\mathbb{R})$. Actually, more can be said about the Hölder properties of functions in $\mathcal{H}^s$. Recall that a function f is said to be (uniformly) Hölder continuous of order α if

$$|||f|||_\alpha \stackrel{\mathrm{def}}{=} \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x-y|^\alpha} \quad < \infty \ .$$

The class of bounded and Hölder continuous functions of order α, with the norm given by $||| \quad |||_\alpha + \| \quad \|_\infty$ forms a Banach space. It is denoted by C^α_b. We have the following embedding:

Theorem 2.1.3. *If* $s = \frac{n}{2} + \alpha$ *,* $0 < \alpha < 1$ *, then* $\mathcal{H}^s(\mathbb{R}^n) \subset C^\alpha_b(\mathbb{R}^n)$.

For a proof of these results we refer the reader to M. Taylor, Vol. 1 Chapter 4.

A key question that emerges when we are dealing with boundary value problems is the relation between the smoothness of the solutions to such problems in the interior of a domain, and their extension to the boundary. A first crucial step in this direction is one of the so-called *trace theorems*. Namely, let's consider the following situation:

Take $f \in \mathcal{S}(\mathbb{R}^n)$ and restrict it to $\mathbb{R}^{n-1}$ via

$$\mathsf{T} f(x_2, \cdots, x_n) \stackrel{\mathrm{def}}{=} f(0, x_2, \cdots, x_n) \ .$$

It is easy to see that this map extends continuously as a map from $\mathcal{H}^s(\mathbb{R}^n)$ to $\mathcal{H}^{s-1/2}(\mathbb{R}^{n-1})$ for $s > 1/2$. Indeed, if we write $f = \mathsf{T}u$ then

$$\widehat{f}(\xi') = \int_{\mathbb{R}} \widehat{u}(\xi) \quad d\!\!\!/\xi_1 \ ,$$

where $\xi = (\xi_1, \xi') \in \mathbb{R}^n$, and thus

$$|\widehat{f}(\xi')|^2 \le \left(\int_{\mathbb{R}} |\widehat{u}(\xi)|^2 \langle\xi\rangle^{2s} \quad d\!\!\!/\xi_1\right) \left(\int_{\mathbb{R}} \langle\xi\rangle^{-2s} \quad d\!\!\!/\xi_1\right) .$$

But if $s > 1/2$,

$$\begin{aligned} \int_{\mathbb{R}} \langle\xi\rangle^{-2s} \quad d\!\!\!/\xi_1 &= \int_{\mathbb{R}} (1 + |\xi'|^2 + \xi_1^2)^{-s} \quad d\!\!\!/\xi_1 \\ &= (1 + |\xi'|^2)^{-s+1/2} \int_{\mathbb{R}} \left(1 + \frac{\xi_1^2}{\langle\xi'\rangle^2}\right)^{-s} \frac{d\!\!\!/\xi_1}{\langle\xi'\rangle} \\ &= C \langle\xi'\rangle^{-2s+1} . \end{aligned}$$

Thus,

$$\int_{\mathbb{R}^{n-1}} \langle\xi'\rangle^{2(s-1/2)} \, |\widehat{f}(\xi')|^2 \, d\xi' \le C \int_{\mathbb{R}^n} |\widehat{u}(\xi)|^2 \langle\xi\rangle^{2s} \, d\xi .$$

Furthermore, the map T is onto for $s > 1/2$.

Interpolation Techniques

A vast number of situations in PDEs requires estimating norms of linear maps. Many times there is a "continuous" family of useful norms. For example the L^p or the $\mathcal{H}^s$ norms. In this section we consider a fundamental technique for such estimates, namely the so-called *interpolation techniques*. The setting is very general. Consider two Banach spaces, say E and F and let's assume that

$$F \hookrightarrow E \quad \text{(continuously embedded)} \ . \tag{2.3}$$

The idea now is to introduce a Banach space of "deformations" on the strip:

$$\Omega \stackrel{\text{def}}{=} \{z \in \mathbb{C} \ | \ 0 < \Re z < 1\}$$

and

$$\begin{aligned} \mathsf{H}_{E,F}(\Omega) = \big\{ & u : \overline{\Omega} \to E \ , \text{holomorphic on } \Omega \ , \\ & \text{bounded and continuous on } \overline{\Omega} \ , \text{s.t. } \sup_{y\in\mathbb{R}} \|u(1+iy)\|_F < \infty \ \big\} \ . \end{aligned}$$

The interpolation space $[E,F]_\theta$ is defined for a fixed $\theta \in [0,1]$ by

$$[E,F]_\theta = \{ u(\theta) \mid u \in \mathsf{H}_{E,F}(\Omega) \} .$$

The topology of $[E,F]_\theta$ is, by definition, the one that makes it isomorphic to

$$\mathsf{H}_{E,F}(\Omega) / \{ u \mid u(\theta) = 0 \} .$$

Proposition 2.1.3. *Let E, F and $\widetilde{E}, \widetilde{F}$ be Banach spaces as in (2.3). Suppose that $\mathsf{T} : E \to \widetilde{E}$ is a continuous linear map and the $\mathsf{T}(F) \subset \widetilde{F}$. Then,*

$$\mathsf{T}([E,F]_\theta) \subset [\widetilde{E}, \widetilde{F}]_\theta , \ \forall \theta \in [0,1] .$$

Proof. Take $v \in [E,F]_\theta$, hence $\exists u \in \mathsf{H}_{E,F}(\Omega)$ such that $u(\theta) = v$. It is not difficult to see that $\mathsf{T}u(z) \in \mathsf{H}_{\widetilde{E},\widetilde{F}}(\Omega)$, so $\mathsf{T}v = \mathsf{T}u(\theta) \in [\widetilde{E}, \widetilde{F}]_\theta$.

Q. E. D.

This proposition, though elementary, yields substantial simplification of the proof of many important results.

A direct consequence of the interpolation technique introduced above is the fact that we can characterize $\mathcal{H}^{k\theta}(\mathbb{R}^n)$ as

$$\mathcal{H}^{k\theta}(\mathbb{R}^n) = [L^2(\mathbb{R}^n) , \mathcal{H}^k(\mathbb{R}^n)]_\theta , \ \ \theta \in [0,1] .$$

We are now ready to discuss Sobolev spaces on arbitrary domains and on compact manifolds. The first remark is that if $\chi : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism which is identical to a linear map outside a compact set, then the induced map $\chi^* : \mathcal{S} \to \mathcal{S}$ preserves all the Sobolev spaces $\mathcal{H}^s(\mathbb{R}^n)$. It is an immediate application of the chain rule that

$$\chi^* : \mathcal{H}^k(\mathbb{R}^n) \to \mathcal{H}^k(\mathbb{R}^n) , \quad k \in \mathbb{Z}_{\geq 0} ,$$

is bounded. Indeed,

$$\begin{aligned} \|\chi^* f\|_k^2 &= \int_{\mathbb{R}^n} (1 + |\xi|^2)^k \, |(f \circ \chi)\hat{}(\xi)|^2 \, (\xi) \, d\xi \\ &= \int_{\mathbb{R}^n} \left(1 + \sum \xi_i^2\right)^k \, |(f \circ \chi)\hat{}(\xi)|^2 \, d\xi \\ &= \sum_{j \leq k} C_{j,k} \|\partial^j \, (f \circ \chi)\|_{L^2}^2 , \end{aligned}$$

where the constants $C_{j,k}$ can be expressed in terms of the binomial coefficients. If we write

$$\partial^{\alpha}\left(\chi^{*} f\right)=\partial^{\alpha}(f \circ \chi)=\sum c_{\alpha, \beta}(\chi)\left(\partial^{\beta} f\right) \circ \chi ,$$

then we can bound the terms $c_{\alpha,\beta}(\chi)$ in terms of the different derivatives of χ to get that

$$\|\chi^{*} f\|_{k}^{2} \leq C\,\|f\|_{k}^{2} .$$

One of the advantages of having interpolation techniques is that we can now use the integer estimates to get similar results in $\mathcal{H}^s$ for non-integer values of s. Another application of the interpolation techniques is the construction of Sobolev spaces with arbitrary index on manifolds. This will be discussed on the next paragraph.

Let M be a compact manifold. Therefore, it makes sense to say that a distribution belongs to $\mathcal{H}^s(M)$. More precisely, $u \in \mathcal{H}^s(M)$ iff

(i) $u \in \mathcal{D}'(M)$,
(ii) $\forall U$ a coordinate patch of M and $\Psi \in C_0^{\infty}(U)$, the distribution $\Psi u \in \mathcal{E}'(U)$ is also an element of $\mathcal{H}^s(U)$.

It is not hard to see that

$$[L^2(M), \mathcal{H}^k(M)]_\theta \;=\; \mathcal{H}^{k\theta}(M) ,$$

and that more generally, $\forall \alpha, \beta \in \mathbb{R}$

$$[\mathcal{H}^{\alpha}(M), \mathcal{H}^{\beta}(M)]_\theta \;=\; \mathcal{H}^{\alpha\theta+(1-\theta)\beta}(M) .$$

The natural isomorphism $\mathcal{H}^s(M)^* \;\cong\; \mathcal{H}^{-s}(M)$ is also preserved. The reader should convince him/herself that the definition presented herein coincides with the elementary ones (see for example [65]) for the Sobolev spaces

$$\mathcal{H}^s(\mathbb{T}) \;=\; \{u \in \mathcal{S}'(\mathbb{T}) \mid \sum_{n\in\mathbb{Z}} (1+n^2)^s\,|\widehat{u}(n)|^2 \;<\; \infty\} ,$$

where $\widehat{u}(n)$ here denotes the finite Fourier transform.

2.1.4 Sobolev Spaces on Bounded Domains

Let $\overline{\Omega}$ be a smooth, compact manifold with boundary $\partial\Omega$ and interior Ω. First the usual procedure to relate the interior Ω with $\overline{\Omega}$ is to look at a local chart close to the boundary. We use the notation $\mathbb{R}^n_+ = \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_1 > 0\}$ and the space

$$\mathcal{S}(\overline{\mathbb{R}^n_+}) = \{f \in \mathcal{S}(\mathbb{R}^n),\ \text{restricted to } \mathbb{R}^n_+ \;\} .$$

We introduce the space

$$\mathcal{H}^k(\mathbb{R}^n_+) = \{u \in L^2(\mathbb{R}^n_+) \,|\, \partial^\alpha u \in L^2(\mathbb{R}^n_+), \quad \forall \alpha, \quad |\alpha| \le k\} \quad .$$

Here the derivative ∂^α is taken in the sense of distributions in $\mathbb{R}^n_+$. The topology of $\mathcal{H}^k(\mathbb{R}^n_+)$ is the one given by the norm

$$\left(\sum_{|\alpha|\le k} \|\partial^\alpha u\|^2_{L^2(\mathbb{R}^n)} \right)^{1/2} \quad .$$

We claim that $\mathcal{S}(\overline{\mathbb{R}^n_+})$ is dense in $\mathcal{H}^k(\mathbb{R}^n_+)$. Indeed, use the fact that one can find constants $a_1, \cdots, a_N$ such that the operator

$$\mathsf{E}u(x) = \begin{cases} u(x) & \text{, if } x_1 \ge 0 \\ \sum_{j=1}^N a_j\, u(-jx_1, x') & \text{, if } x_1 < 0 \end{cases}$$

matches the derivatives of order less or equal to $N-1$ for any choice of function $u \in C^k(\mathbb{R}^n_{\ge 0})$, and $k \ge N-1$. For this to happen, the constraints on the coefficients a_j are given by

$$\sum_{j=1}^N (-j)^l\, a_j \;=\; 1, \quad \forall l = 0, \cdots, N-1 \;.$$

For the above choice of constants, it can be shown that E extends uniquely as a bounded map

$$\mathsf{E} \,:\, \mathcal{H}^k(\mathbb{R}^n_+) \longrightarrow \mathcal{H}^k(\mathbb{R}^n) \;.$$

As a corollary we get that the restriction of elements of $\mathcal{H}^k(\mathbb{R}^n)$ to $\mathcal{H}^k(\mathbb{R}^n_+)$ is *onto* . The space $\mathcal{H}^s(\mathbb{R}^n_+)$ for $s \in \mathbb{R}$ is defined by interpolation

$$\mathcal{H}^s(\mathbb{R}^n_+) = [L^2(\mathbb{R}^n_+), \mathcal{H}^k(\mathbb{R}^n)]_\theta$$

taking $k \ge s$ and $s = \theta k$.

Now we pass to the situation of a manifold with boundary. Let $\overline{\Omega}$ be a smooth, compact manifold with *smooth* boundary $\partial\Omega$. We shall also assume for simplicity [1] that $\overline{\Omega} \subset \mathbb{R}^n$, with $n = \dim \Omega$.

To define $\mathcal{H}^k(\Omega)$ we start by looking all linear differential operators P of order less or equal to k with coefficients in $C^\infty(\overline{\Omega})$ and require that $u \in \mathcal{H}^k(\Omega) \Leftrightarrow \mathsf{P}u \in L^2(\Omega)$, for every such differential operator. In a way analogous to what we did for $\mathcal{H}^k(\mathbb{R}^n_+)$, we show that $C^\infty(\overline{\Omega})$ is dense in $\mathcal{H}^k(\Omega)$ (recall that $\Omega = \overline{\Omega} \setminus \partial\Omega$). Also, as in the case of $\mathbb{R}^n_+$ by using coordinate patches we get an extension operator

[1] Otherwise one can use a trick as described on page 286 of M. Taylor's book.

$$\mathsf{E} : \mathcal{H}^k(\Omega) \longrightarrow \mathcal{H}^k(M) ,$$

where M is the manifold $\mathbb{T}^n$ in the case $\Omega \subset \mathbb{R}^n$, or more generally, a manifold without boundary containing Ω.

The interpolation technique gives for $s \geq 0$ a good definition for $\mathcal{H}^s(\Omega)$ as

$$\mathcal{H}^s(\Omega) = [L^2(\Omega), \mathcal{H}^k(\Omega)]_\theta , \quad k \geq s , \quad s = \theta k .$$

It is an exercise to check that for $s \geq 0$:

$$\mathcal{H}^s(\Omega) \cong \mathcal{H}^s(M)/\{u \in \mathcal{H}^s(M) \ \mid \ u\big|_\Omega = 0\} .$$

For negative values of s we use such characterization (it does not depend on the embedding $\Omega \to M$!). We can now summarize the main results about the newly introduced spaces:

Proposition 2.1.4. *Suppose that Ω is an n-dimensional compact manifold with smooth boundary. Then*

- $\mathcal{H}^s(\Omega) \subset C^k(\overline{\Omega})$, $s > n/2 + k$,
- $\mathcal{H}^s(\Omega) \subset C^\alpha(\overline{\Omega})$, $s = n/2 + \alpha$, $\alpha \in (0,1)$,
- $\iota : \mathcal{H}^s(\Omega) \hookrightarrow \mathcal{H}^k(\Omega)$ *is compact* [2] *for* $s > k \geq 0$.

We are now ready to state one of the crucial results about the restriction of elements of $\mathcal{H}^s(\Omega)$ to the boundary $\partial\Omega$.

Theorem 2.1.4. *Suppose that Ω is an n-dimensional compact manifold with smooth boundary. Let $\mathcal{T} : C^\infty(\overline{\Omega}) \longrightarrow C^\infty(\partial\Omega)$ be such that $\mathcal{T}u = u \restriction_{\partial\Omega}$. Then, for $s > 1/2$, $\mathcal{T}$ extends uniquely as a continuous map*

$$\mathcal{T} : \mathcal{H}^s(\Omega) \longrightarrow \mathcal{H}^{s-1/2}(\partial\Omega) .$$

Take Ω as before (compact manifold with boundary $\partial\Omega$). We consider the closure of $C_0^\infty(\Omega)$ in $\mathcal{H}^s(\Omega)$. This space is denoted by $\mathcal{H}_0^s(\Omega)$. Let M be a compact manifold with no boundary containing Ω. For $k \in \mathbb{Z}_{\geq 0}$ we can show that

$$\mathcal{H}_0^k(\Omega) \ = \ \{u \in \mathcal{H}^k(M) \ \mid \ \mathrm{supp}\ u \subset \overline{\Omega}\} .$$

This is an immediate consequence of the fact that if $k \in \mathbb{Z}_{\geq 0}$ is an integer, we can write the norm in $\mathcal{H}^k(\Omega)$ in terms of $\sum_{j=1}^k \|\mathsf{P}_j u\|^2_{L^2(\Omega)}$. Here for $\{\mathsf{P}_j | j = 1, \cdots , k\}$ is a suitable set of differential operators of order $\leq k$.

[2] We review the concept of compact operator in Appendix 2.2.

2.2 Compact Operators and SVD

The purpose of this section is to quickly review the proof of the so-called *singular value decomposition (S.V.D.)*. This result has a number of important application in numerical analysis, inverse problems, and numerical linear algebra. We follow B. Simon's [61]. For the numerical applications see [21].

Our approach and proof is directly in the infinite dimensional case making use of the spectral theorem for compact self-adjoint operators. Let's first recall some of these basic concepts.

Let $\mathcal{L}(X, Y)$ denote the space of bounded linear operators from X into Y. The so-called "uniform operator topology" is defined by the norm

$$||T||_{X,Y} \stackrel{\text{def}}{=} \sup_{x\neq 0} \frac{||Tx||_Y}{||x||_X} \ .$$

Remark 2.2.1. In these notes we use the convention that our *complex* Hilbert space inner-products are *linear* w. r. t. the first entry and *anti-linear* w. r. t. to the second one.

Definition 2.2.1. *Let X and Y be Banach spaces and $T : X \to Y$ a linear operator. T is called* compact *(or* completely continuous*) if it maps the unit ball B_X on a pre-compact set, i.e., $T(B_X)$ has compact closure. The set of compact operators from X to Y is denoted by $\mathcal{K}(X, Y)$.*

It can be easily shown that the above definition implies that T is a bounded operator. Furthermore, the set of compact operators is closed under limits in the uniform operator topology of $\mathcal{L}(X, Y)$.

Typical examples of compact operators are operators of finite dimensional range. Another important class of compact operators is given by integral operators.

Exercise 2.2.1. Let $k \in L^2(X \times Y)$, where (X, dx) and (Y, dy) are (separable) measure spaces and $X \times Y$ is given the measure $dx \otimes dy$. Show that the operator

$$Kf(x) = \int_Y k(x, y)f(y)dy$$

is a compact operator from $L^2(Y, dy)$ into $L^2(X, dx)$.

The next result shows that for Hilbert spaces the self-adjoint compact operators can be approximated by finite-rank operators.

Theorem 2.2.1 (Hilbert-Schmidt). *Let $\mathcal{H}$ be a separable* [3] *Hilbert space and $A \in \mathcal{L}(\mathcal{H})$ be a* compact self-adjoint *operator. Then, there exists an orthonormal basis of $\mathcal{H}$,*

[3] If $\mathcal{H}$ is non-separable a similar result still holds with the proviso that in this case $\ker A$ is going to be a non-separable space and the cardinality of the basis to its complement will be necessarily countable.

$$\{\phi_n\}_{n=1,\cdots}$$

such that each ϕ_n *is an eigenvector of* A*, i.e.*

$$A\phi_n = \lambda_n \, \phi_n \ .$$

Furthermore, we can reorder this basis so that

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n| \geq |\lambda_{n+1}| \geq \cdots \geq 0 \ ,$$

and

$$\lambda_k \neq 0 \implies \dim\ker(A - \lambda_k I) < \infty \ .$$

If $\dim \mathcal{H} = \infty$*, then necessarily* $\lim_{n\to\infty} \lambda_n = 0$.

As a corollary of the Hilbert-Schmidt theorem, it follows that:

Corollary 2.2.1. *If* $A = A^* \in \mathcal{K}(\mathcal{H})$ *then there exists an orthonormal set* $\{\phi_n\}_{n=1}^N$ *s.t.*

$$A = \sum_{n=1}^{N} \lambda_n \, \phi_n \, (\cdot \mid \phi_n) \ .$$

Furthermore, if $N = \infty$*, the limit takes place in the uniform topology of* $\mathcal{L}(\mathcal{H})$.

We are now ready to discuss the SVD result:

Theorem 2.2.2. *Let* $A \in \mathcal{K}(\mathcal{H})$*, then we can write*

$$A = \sum_{n=1}^{r} \sigma_n \, (\cdot \mid \psi_n) \, \phi_n, \tag{2.4}$$

where $r \in \mathbb{N} \cup \{\infty\}$ *and* $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ *; the sets* $\{\phi_n\}_{n=1}^r$ *and* $\{\psi_n\}_{n=1}^r$ *are orthonormal sets (not necessarily complete).*

Proof. Let $T = A^*A$, then $(Tx \mid x) \geq 0 \, , \forall x \in \mathcal{H}$. Also T is self-adjoint. By the Hilbert-Schmidt result, we can write

$$T = \sum_{n=1}^{r} \lambda_n \, \psi_n \, (\cdot \mid \psi_n) \ ,$$

where $r = \sup\{\, n \mid \lambda_n \neq 0 \,\}$. The set $\{\psi_n\}_{n=1}^r$ is orthonormal. Since $T \geq 0$, it follows that $\lambda_n > 0 \ , \quad \forall n < r + 1$.

Let

$$\sigma_n = \sqrt{\lambda_n} \ , \quad 1 \leq n < r + 1 \ ,$$

and

$$\phi_n \stackrel{\text{def}}{=} \frac{1}{\sigma_n} A\psi_n \ , \quad 1 \le n < r+1 \ .$$

Then,

$$T\psi_n = \sigma_n^2 \, \psi_n$$

implies that $A^* A \, \psi_n = \sigma_n^2 \, \psi_n$ and hence

$$\psi_n = \frac{1}{\sigma_n} A^* \, \phi_n \ .$$

Notice that $Ax = 0 \Leftrightarrow Tx = 0$. Thus, $\ker T = \ker A$.

Now, take $x \in \mathcal{H}$ and write it in the decomposition given by Corollary 2.2.1:

$$x = x_0 + \sum_{n=1}^{r} (x \mid \psi_n) \psi_n \ ,$$

where $x_0 \in \ker T = \ker A$. Set

$$y \stackrel{\text{def}}{=} \sum_{n=1}^{r} (x \mid \psi_n) \sigma_n \phi_n \ ,$$

which is well defined since $\sum_{n=1}^{r} |(x \mid \psi_n)|^2 < \infty$. Now, it is enough to notice that $A\psi_n = \sigma_n \phi_n$ to get that $y = Ax$, which completes the proof of equation (2.4). To show that $\{\phi_n\}$ is an orthonormal set, notice that

$$AA^* \, \phi_n = \frac{1}{\sigma_n} AA^* A\psi_n = \frac{1}{\sigma_n} \sigma_n^2 \, A\psi_n = \sigma_n^2 \, \phi_n \ .$$

Hence, each ϕ_n are eigenvalues of AA^*, and therefore mutually orthogonal, whenever two of such vectors belong to distinct eigenvalues. To show the normalization, we compute

$$\begin{aligned} 1 = (\psi_n \mid \psi_n) &= \left(\frac{1}{\sigma_n} A^* \phi_n \ \Big| \ \frac{1}{\sigma_n} A^* \phi_n \right) \\ &= \left(\frac{1}{\sigma_n^2} AA^* \phi_n \ \Big| \ \phi_n \right) \\ &= (\phi_n \mid \phi_n) \ . \end{aligned}$$

Q. E. D.

The SVD is a fundamental tool in the study of inverse problems. In Figure 2.1 we show the geometrical interpretation of the SVD. It tells us that each unit vector ψ_n is being sent by the map A into a vector of size σ_n along the direction ϕ_n.

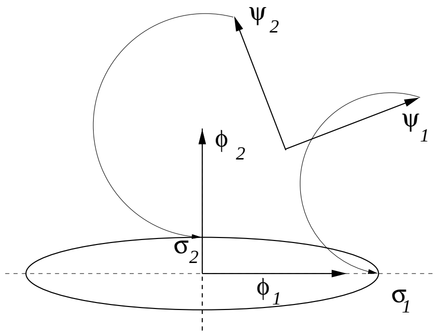

Fig. 2.1. Depiction of the SVD. Notice the unity vectors ψ_n being sent into vectors of length σ_n in the direction ϕ_n

As direct consequence of the SVD decomposition we get that the equation $Ax = y$ will have a solution if, and only if,

1. The vector $y \in \ker A^{*\perp}$, and
2. the sum
$$\sum_{n=1}^{r} \frac{1}{\sigma_n^2} |(y \mid \phi_n)|^2 < \infty \ .$$

In this case the solution of x will be given by

$$x = \sum_{n=1}^{r} \frac{1}{\sigma_n} (y \mid \phi_n) \psi_n \ .$$

In the finite dimensional case, the transformation will be invertible if, and only if, r is equal to the dimension of $\mathcal{H}$. In other words, the eigenvalues of A^*A are all nonzero. The presence of singular values close to zero indicates that the solution of the problem $Ax = y$ will be doomed to numerical instability. The condition number of a matrix $A \in \mathbb{C}^{n \times n}$ is the ratio σ_1/σ_r. See [21] for a comprehensive analysis of the numerical aspects related to the SVD and the concept of condition number.

In the infinite dimensional case, if $r = \infty$ then the sequence $\{\sigma_n\}$ must necessarily converge to zero. This shows the inherent instability of solving equations of the form $Ax = y$ when A is a compact operator in an infinite dimensional Hilbert space. For an excellent introduction to these and other methods see [5].

3. The Radon Transform

In this chapter we discuss a prototypical example of an inverse problem that has been instrumental in the development of computerized tomography.

The underlying physical model is that of a tissue, or more generally a medium, that absorbs the radiation according to the following law:

If we denote by $I(x_1)$ the intensity of radiation measured at a certain point x_1 due to a source located at x_0 with intensity $I(x_0)$ then

$$I(x_1) = I(x_0) \exp\left(- \int_{L(x_0,x_1)} \mu(s) ds \right) , \tag{3.1}$$

where $L(x_0, x_1)$ is the straight line joining x_0 to x_1, and $\mu = \mu(x)$ is the absorption.

One way of obtaining such a model is to start off from the (time independent) Boltzmann transport equation. If we denote by $\Psi = \Psi(x, \omega)$ the density of the radiation at a point $x \in \mathbb{R}^3$, traveling in the direction ω, and due to a source $q(x, \omega)$, then Ψ satisfies

$$\omega \cdot \nabla \Psi + \mu \Psi = q + \mu_s \int_{\mathbb{S}^2} \sigma(\omega, \omega') \Psi(\cdot, \omega') d\omega' , \tag{3.2}$$

where $\sigma(\omega, \omega')$ is the differential scattering cross section, which represents the probability of scattering from the direction ω' to the direction ω, $\mu = \mu(x)$ is the absorption coefficient, and $\mu_s = \mu_s(x)$ is a normalization factor (so that σ would correspond to a probability density).

For a number of situations (X-ray tomography in particular) one can neglect the contribution from the scattering to get

$$\omega \cdot \nabla \Psi + \mu \Psi = q . \tag{3.3}$$

Let us assume that this equation is true for a strictly convex body such as the one shown in Figure 3.1. Let's also assume that the source term has support outside the region Ω, say at a point x_0. Then, one can integrate equation along the line $L(x, x_0)$ joining x with x_0 to get

$$\Psi(x, \omega) = \Psi(x_0, \omega) \exp\left(- \int_{L(x_0,x)} \mu(s) ds \right) ,$$

where ω is the unit vector in the direction $x - x_0$. If we denote now by $I(x)$ the value of $\Psi(x, \omega)$ we get equation (3.1).

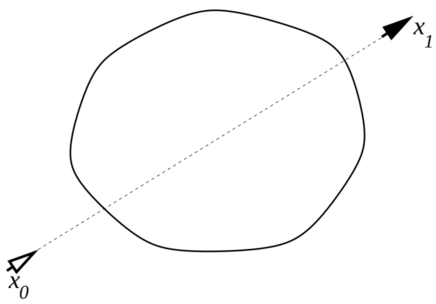

Fig. 3.1. An object being crossed by an X-ray

3.1 The Classical Radon Transform

Let Ω be an open domain of $\mathbb{R}^n$ and let $\mathcal{L}$ denote the set of all (straight) lines in Ω. In this section we shall be concerned with the following problem, which is directly motivated by the absorption of radiation in a body:

Given a function f defined on Ω, can one reconstruct f from

$$\mathcal{L} \ni L \longmapsto \int_L f(x)dx \ ?$$

This problem leads naturally to a more general one: Determining a function from the knowledge of its averages on arbitrary k-dimensional planes.

In order to approach the problem in a systematic way, we shall introduce the Radon transform. [1] It takes its name honoring Radon's original contribution (see its reprint in [56]). It played a key role in the development of computerized tomography, but its connections with other fields of Mathematics goes way beyond this application. For more information see also [14, 34, 33, 19, 67, 66, 20, 49].

We shall approach the study of the Radon transform coming from the point of view of the Fourier transform. [2] The latter is defined for $f \in \mathcal{S}(\mathbb{R}^n)$ by

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi} \ dx \ . \tag{3.4}$$

[1] We shall consider only the integrals on hyperplanes for simplicity. As mentioned above, it can be defined for arbitrary k-planes. See [33, 49] for generalizations

[2] An overview of the properties of the Fourier transform can be found in the Appendix 2.1

Now, the idea is to perform the integration first along planes perpendicular to ξ, and then introduce a phase factor. More precisely for $\xi \neq 0$, we define as in Fig. 3.2,

$$\omega \stackrel{\text{def}}{=} \frac{\xi}{|\xi|} ,$$

and $y \stackrel{\text{def}}{=} \omega \cdot x$.

Then, we may write the Fourier transform as

$$(2\pi)^{n/2}\widehat{f}(\xi) = \int_{\mathbb{R}} e^{-i|\xi|y} \left(\int_{x\cdot\omega=y} f(x)\, d^{n-1}x \right) dy ,$$

where here, we are using the symbol $d^{n-1}x$ to emphasize the integral in the $n-1$ dimensional space. This leads naturally to considering, for each fixed $\omega \in \mathbb{S}^{n-1}$, and $y \in \mathbb{R}$ the integral

$$\mathsf{R}\, f(\omega, y) \stackrel{\text{def}}{=} \int_{x\cdot\omega=y} f(x) \quad d^{n-1}x . \tag{3.5}$$

Equation (3.5) defines a map $\mathsf{R} \; : C_0^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{S}^{n-1}\times\mathbb{R})$, as well as a map from $\mathcal{S}(\mathbb{R}^n)$ into $C^\infty(\mathbb{S}^{n-1}\times\mathbb{R})$. If $f \in C_0^\infty(\mathbb{R}^n)$, then the support of $\mathsf{R}f(\omega,\cdot)$ is compact, whereas if f belongs to the Schwartz space so does $\mathsf{R}f(\omega,\cdot)$. Also, we have

$$\mathsf{R}f(\omega, y) = \mathsf{R}f(-\omega, -y) . \tag{3.6}$$

We also define, for further use,

$$\mathsf{R}_\omega f \stackrel{\text{def}}{=} \mathsf{R}f(\omega,\cdot) .$$

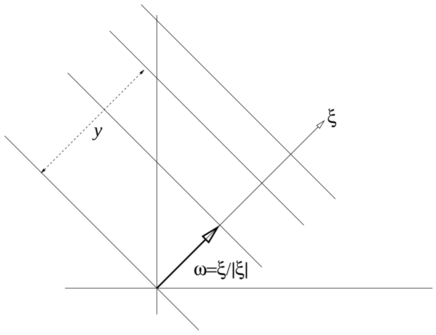

Fig. 3.2. Change of variables in Fourier's frequency space, illustrating the introduction of the Radon transform

This approach gives us immediately an important relation between the Radon transform and the 1-dimensional Fourier transform, which is known as the "Projection Theorem" or the "Fourier Slice theorem":

Theorem 3.1.1. *For $f \in \mathcal{S}(\mathbb{R}^n)$, $s \in \mathbb{R}$, and $\omega \in \mathbb{S}^{n-1}$ we have*

$$\mathcal{F}\left[\mathsf{R}_\omega f\right](s) = (2\pi)^{(n-1)/2}\widehat{f}(s\omega) \ , \tag{3.7}$$

where here $\mathcal{F}$ denotes the one-dimensional Fourier transform.

We are now in the position of answering the theoretical question: Does $\mathsf{R}f$ determine uniquely f? It is clear from the construction of R that the answer is positive. Indeed, $\mathsf{R}f$ yields the Fourier transform of f, which uniquely determines f.

3.2 The Inversion of the Radon Transform

Our next goal is to develop an inversion formula. It turns out that this inversion formula will depend on the parity of the dimension n. Such dependence is an interesting phenomenon, which we shall discuss later. In order to obtain the inversion, we shall first explore a few properties of R.

Let f be an element in $\mathcal{S}(\mathbb{R}^n)$. The Fourier inversion formula gives that

$$(2\pi)^n f(x) = \int_{\mathbb{R}^n} d^n\xi \;\; e^{i\xi\cdot x}\left(\int_{-\infty}^{+\infty} dy \;\; e^{-i|\xi|y}\,\mathsf{R}f(\frac{\xi}{|\xi|}, y)\right) \ .$$

We now perform the integration using spherical coordinates in the frequency domain. Denoting by ρ the distance from x to the origin and by $d\omega$ the volume element of the $(n-1)$-dimensional sphere $\mathbb{S}^{n-1}$ we have

$$d^n\xi = \rho^{n-1}d\rho \; d\omega \ .$$

Hence,

$$(2\pi)^n f(x) = \int_{\mathbb{S}^{n-1}} d\omega \int_0^\infty d\rho \; \rho^{n-1} \int_{-\infty}^{\infty} dy \;\; e^{i\rho(\omega\cdot x - y)}\mathsf{R}f(\omega, y) \ . \tag{3.8}$$

Our task reduces to computing the iterated inner integrals. But now we have to distinguish between the even and the odd dimensional case.

For n odd:

Here, using the symmetry (3.6) and the fact that $n-1$ is even, we can rewrite (3.8) as

$$(2\pi)^n f(x) = \frac{1}{2}\int_{\mathbb{S}^{n-1}} d\omega \int_{-\infty}^{\infty} d\rho \; \rho^{n-1} \int_{-\infty}^{\infty} dy \;\; e^{i\rho(\omega\cdot x - y)}\mathsf{R}f(\omega, y) \tag{3.9}$$

Now, in the sense of $\mathcal{S}'(\mathbb{R})$ we have

$$(2\pi)^{1/2}\widehat{\delta_{x_0}} = \exp(-ix_0 y)$$

and thus

$$\widehat{\delta_{x_0}^{(k)}}(y) = \frac{(iy)^k}{(2\pi)^{1/2}}\exp(-ix_0 y)\ .$$

Hence, for $g \in \mathcal{S}(\mathbb{R})$ we can write

$$\int_{-\infty}^{+\infty} d\rho\ \rho^{n-1}\int_{-\infty}^{+\infty} dy\ e^{i\rho(z-y)}g(y) = \frac{2\pi}{i^{n-1}}g^{(n-1)}(z)\ . \tag{3.10}$$

Bringing together equations (3.9) and (3.10) we get

$$f(x) = c_n \int_{\mathbb{S}^{n-1}} d\omega\ \left.\frac{\partial^{n-1}}{\partial y^{n-1}}\right|_{y=\omega\cdot x} \mathsf{R}f(\omega, y)\ ,$$

where $c_n = (2\pi)^{1-n}/(2i^{n-1})$.

For n even:

In this case, the expression gets more complicated, since we have to substitute ρ^{n-1} by an absolute value in equation (3.10). The expression will be given in terms of the Hilbert transform, i.e.,

$$\mathsf{H}f(x) \stackrel{\text{def}}{=} \frac{1}{\pi}\lim_{\epsilon\downarrow 0}\int_{|y-x|\geq\epsilon} dy\ \frac{f(y)}{x-y}\ , \quad f \in \mathcal{S}(\mathbb{R})\ .$$

In the sequel, we make use the following exercise:

Exercise 3.2.1. Verify that

$$\widehat{\mathsf{H}f}(\xi) = \frac{\operatorname{sgn}(\xi)}{i}\widehat{f}(\xi)$$

and conclude that for $g \in \mathcal{S}(\mathbb{R})$ and n even we have

$$\int_{-\infty}^{+\infty} d\rho\ |\rho|^{n-1} \int_{-\infty}^{+\infty} dy\ e^{i\rho(z-y)}g(y) = \frac{2\pi}{i^{n-2}}\mathsf{H}[g^{(n-1)}](z)\ . \tag{3.11}$$

As a consequence of the previous exercise, it follows from equation (3.8) that for *even* n we have

$$f(x) = \alpha_n \int_{\mathbb{S}^{n-1}} d\omega\ \left.\frac{\partial^{n-1}}{\partial y^{n-1}}\right|_{y=x\cdot\omega} \mathsf{H}[\mathsf{R}f(\omega,\cdot)](y)\ ,$$

where the constant $\alpha_n = -\pi/(2\pi i)^n$.

Remark 3.2.1. The formulae for n even and odd can be unified making use of the absolute value of the (unbounded) differential operator $\partial/\partial y$. It is however interesting to stress that for n even the result is highly nonlocal.

Remark 3.2.2. The Radon transform described above generalizes in many directions. First, it generalizes by considering integrals over k-planes, instead of hyperplanes. One particular case of which is the so-called X-ray transform [49]. It consists on the average of one dimensional lines. Another avenue is to consider the generalizations of the Radon transform to homogeneous spaces. The reader is invited to look at [33] for these matters.

3.3 Properties of the Radon Transform

In this section we shall explore some further properties of the Radon transform. Some of these properties will be used later in this chapter such as in Section 3.4, others will be illustrative of its power.

Let us consider the cylinder $\mathcal{Z} = \mathbb{S}^{n-1} \times \mathbb{R}$, and denote by $\mathcal{S}(\mathcal{Z})$ the Schwartz space on $\mathcal{Z}$, i.e. the set of functions $g \in C^\infty(\mathcal{Z})$ with the property that

$$\sup_{y,\omega} |y^m \partial^\alpha g(\omega, y)| < \infty \ ,$$

for every multi-index α and $m \in \mathbb{Z}_{\geq 0}$.

We now introduce two natural inner products[3]

$$(f \mid g)_{L^2(\mathcal{Z})} = \int_{\mathbb{S}^{n-1}} \int_{\mathbb{R}} d\omega \, dy \, f(\omega, y) \overline{g(\omega, y)}$$

and

$$(f \mid g)_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} dx \, f(x) \overline{g(x)}.$$

So far we have been working with R defined only for $\mathcal{S}(\mathbb{R}^n)$. Our goal now is to show that R extends in a natural way to other interesting spaces. It turns out that in order to do that, it is reasonable to obtain an adjoint for R. This adjoint is also useful in some reconstruction techniques.

Let's recall that $\mathcal{S}(\mathcal{Z})$ is dense in $L^2(\mathcal{Z})$. We now proceed formally, making use of the δ function. The energetic reader should have no problem in justifying this.

[3] We shall drop the spatial subscript whenever it is obvious which space we are using.

$$
\begin{aligned}
(\mathsf{R}f \mid g) &= \int_{\mathbb{R}}\int_{\mathbb{S}^{n-1}} dy\, d\omega \int_{\omega\cdot x = y} d^{n-1}x\, f(x)\overline{g(\omega,y)} \\
&= \int_{\mathbb{S}^{n-1}}\int_{\mathbb{R}}\int_{\mathbb{R}^n} d\omega\, dy\, d^n x\, \delta(y-\omega\cdot x) f(x)\overline{g(\omega,y)} \\
&= \int_{\mathbb{R}^n} d^n x \int_{\mathbb{S}^{n-1}} d\omega \int_{\mathbb{R}} dy\, \delta(y-\omega\cdot x) f(x)\overline{g(\omega,y)} \\
&= \int_{\mathbb{R}^n} d^n x\, f(x) \int_{\mathbb{S}^{n-1}} d\omega\, \overline{g(\omega,\omega\cdot x)} \\
&= (f \mid \mathsf{R}^* g).
\end{aligned}
$$

Therefore,

$$
(\mathsf{R}^* g)(x) = \int_{\mathbb{S}^{n-1}} d\omega\, g(\omega,\omega\cdot x).
$$

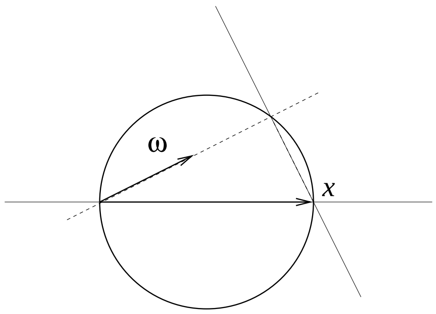

Fig. 3.3. The geometry of the adjoint Radon transform

The interpretation of this formula goes as follows: Whereas in the Radon transform for each hyperplane (indexed by (ω, y)) we compute an average of the given function f, in the adjoint Radon transform for each point we compute the average over all the hyperplanes passing through it. More precisely, if we identify the points of $\mathcal{Z}$ with (a double cover) of the set of all hyperplanes in $\mathbb{R}^n$ then, for a fixed $x \in \mathbb{R}^n$ the integral

$$
\int_{\mathbb{S}^{n-1}} d\omega\, g(\omega,\omega\cdot x)
$$

corresponds to summing over all planes passing through x.

Remark 3.3.1. If $g \in \mathcal{S}(\mathcal{Z})$, it is not hard to see that $\mathsf{R}^* g \in \mathcal{S}(\mathbb{R}^n)$. A minute's thought shows that $\mathcal{S}(\mathcal{Z})$ can be endowed with a locally convex topology by using the semi-norm

$$
p_{m,\alpha}(\varphi) \stackrel{\text{def}}{=} \sup_{\substack{y\,\in\,\mathbb{R}\\ \omega\,\in\,\mathbb{S}^{n-1}}} |y^m\, \partial^\alpha\, \varphi(\omega,y)|
$$

and that R and R^* are operators between $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}(\mathcal{Z})$.

We now have the following:

Proposition 3.3.1. *If $f \in \mathcal{S}(\mathbb{R}^n)$, then*

$$\mathsf{R}^*\,\mathsf{R}\,f = |\mathbb{S}^{n-2}|\,|x|^{-1} * f. \tag{3.12}$$

Proof. Just compute

$$\begin{aligned} \mathsf{R}^*\,\mathsf{R}\,f(x) &= \int_{\mathbb{S}^{n-1}} d\omega\,\mathsf{R}\,f(\omega, x\cdot\omega) \\ &= \int_{\mathbb{S}^{n-1}} d\omega \int_{\omega^\perp} d^{n-1}z\, f((x\cdot\omega)\omega + z) \\ &= \int_{\mathbb{S}^{n-1}} d\omega \int_{\omega^\perp} d^{n-1}z\, f(x+z)\ . \end{aligned} \tag{3.13}$$

Exercise 3.3.1. If $f \in \mathcal{S}(\mathbb{R}^n)$, then

$$\int_{\mathbb{S}^{n-1}} d\omega \int_{\omega^\perp} d^{n-1}z\, f(z) = |\mathbb{S}^{n-2}| \int_{\mathbb{R}^n} d^n z\, \frac{f(z)}{|z|}\,.$$

Since, $1/|z|$ is integrable at 0 in dimension $n \geq 2$, it is not hard to see that it defines an element of $\mathcal{S}'(\mathbb{R}^n)$. Thus, using (3.13) and the exercise we get the desired result. *Q. E. D.*

So far we have discussed the Radon transform for functions in $\mathcal{S}(\mathbb{R}^n)$. We shall now consider its extensions to other spaces. The first remark is that it naturally extends to $L^1(\mathbb{R}^n)$. Indeed, for $\omega \in \mathbb{S}^{n-1}$,

$$\int_{\mathbb{R}} ds |\mathsf{R}_\omega f(s)| \leq \int_{\mathbb{R}^n} dx |f(x)|$$

so,

$$||\mathsf{R} f||_{L^1(\mathcal{Z})} \leq |\mathbb{S}^{n-1}|\,||f||_{L^1(\mathbb{R}^n)}\,.$$

We shall now discuss the L^2 theory. But for that we have to restrict to functions with support in a compact set. Since our main application will be in this situation, this shall pose no problem. So, without loss of generality we consider that $f \in L^2(\mathbb{B}^n)$, where $\mathbb{B}^n$ is the open n-dimensional unit ball

$$\mathbb{B}^n = \{x \in \mathbb{R}^n \mid |x| < 1\}.$$

We follow closely Natterer's book [49] which contains many more useful results.

Notice that if $\operatorname{supp} f \subset \mathbb{B}^n$, then

$$\text{supp } \mathsf{R}_\omega f \subset (-1,1).$$

The space $L^2([-1,1],(1-s^2)^{(1-n)/2}ds)$ consists of the measurable functions on $[-1,1]$ such that

$$|||f|||^2 \stackrel{\text{def}}{=} \int_{-1}^{1} ds\,|f(s)|^2(1-s^2)^{(1-n)/2} < \infty.$$

We endow $L^2([-1,1],(1-s^2)^{(1-n)/2}\,ds)$ with the natural inner product associated to the above norm.

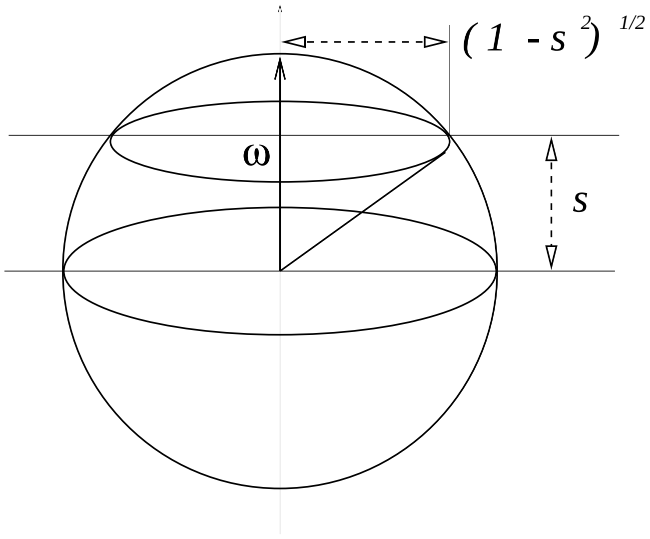

Fig. 3.4. The domain of integration for $\mathsf{R}_\omega\, f$ when supp $f \subset \mathbb{B}^n$

The first remark is that if $f \in \mathcal{S}(\mathbb{R}^n)$ with supp $f \subset \mathbb{B}^n$, then

$$|\mathsf{R}_\omega\, f(s)|^2 \leq (1-s^2)^{(n-1)/2}\, C_n \int_{y\in\omega^\perp} |f(s\omega+y)|^2\, d^{n-1}\, y. \tag{3.14}$$

where $C_n = |\mathbb{B}^{n-1}|$ is the volume of the $(n-1)$-dimensional ball. To see this, just notice that since supp $f \subset \mathbb{B}^n$ the domain of integration of $\mathsf{R}_\omega f(s)$, is a disk defined by the condition

$$\mathbb{D}_{\omega,s} \stackrel{\text{def}}{=} \left\{ s\omega + y \mid y \perp \omega \text{ and } |y| \leq (1-s^2)^{1/2} \right\} ,$$

as shown in Figure 3.4. Equation (3.14) follows from Cauchy-Schwarz inequality applied to $f \cdot \chi_{\mathbb{D}_{\omega,s}}$. Because of (3.14) we have that

$$\int_{-1}^{1} ds(1-s^2)^{(1-n)/2}\, |\mathsf{R}_\omega f(s)|^2 \leq C_n \int_{\mathbb{R}^n} |f(x)|^2\, dx\ . \tag{3.15}$$

To obtain an estimate for $\mathsf{R}f$ we just integrate (3.15) over the unit sphere in $\mathbb{R}^n$ to yield

$$\int_{\mathcal{Z}} |\mathsf{R}f(\omega,s)|^2\,(1-s^2)^{(1-n)/2}\,ds d\omega \le |\mathbb{S}^{n-1}| C_n \int_{\mathbb{R}^n} |f(x)|^2\,dx$$

We have thus shown that the operators R and R_ω, which were defined initially for functions in the corresponding Schwarz spaces, extend naturally to

$$\mathsf{R}_\omega \colon L^2(\mathbb{B}^n) \to L^2([-1,1],(1-s^2)^{(1-n)/2}\,ds)$$

and

$$\mathsf{R}\colon L^2(\mathbb{B}^n) \to L^2(\mathcal{Z},(1-s^2)^{(1-n)/2}\,ds d\omega)$$

as bounded operators w.r.t. the above mentioned L^2 norms.

Exercise 3.3.2. Show that for $f,g \in \mathcal{S}(\mathbb{R}^n)$ we have

$$\mathsf{R}_\omega(f*g) = \mathsf{R}_\omega f \,*\, \mathsf{R}_\omega g,$$

where the convolutions are defined in the appropriate spaces.

The next question we shall address is the continuity of the Radon transform in appropriate Sobolev spaces. Here again we follow closely [49].

First we introduce the Sobolev norms

$$||g||^2_{\mathcal{H}^\alpha(\mathcal{Z})} \stackrel{\text{def}}{=} \int_{\mathbb{S}^{n-1}} d\omega \int_{\mathbb{R}} d\eta (1+\eta^2)^\alpha |\widehat{g}(\omega,\eta)|^2$$

where for $g \in \mathcal{S}(\mathcal{Z})$, $\widehat{g}(\omega,\cdot)$ is the 1-dimensional Fourier transform of $g(\omega,\cdot)$. We denote by $\mathcal{H}^\alpha(\mathcal{Z})$ the closure of $\mathcal{S}(\mathcal{Z})$ with respect to the norm $||\quad||_{\mathcal{H}^\alpha(\mathcal{Z})}$ defined above. Recall that $\mathcal{H}^\alpha_0(\mathbb{B}^n)$ is the closure of the space $C^\infty_0(\mathbb{B}^n)$ with respect to the Sobolev norm in $\mathcal{H}^\alpha(\mathbb{R}^n)$. Our goal, is to show the heuristical claim that the Radon transform smoothes functions with compact support about $(n-1)/2$ derivatives.

The main estimate is given by the following:

Theorem 3.3.1. *Given α, there exist constants $c = c(\alpha,n)$ and $C = C(\alpha,n)$ such that $\forall\, f \in C^\infty_0(\mathbb{B}^n)$, we have*

$$c\,||f||_{\mathcal{H}^\alpha_0(\mathbb{B}^n)} \le ||\mathsf{R}f||_{\mathcal{H}^{\alpha+(n-1)/2}(\mathcal{Z})} \le C\,||f||_{\mathcal{H}^\alpha_0(\mathbb{B}^n)}\,.$$

Proof. To prove the lower bound we note that

$$\begin{aligned} ||f||_{\mathcal{H}^\alpha_0(\mathbb{B}^n)} &\le \int_{\mathbb{R}^n} |\xi|^{1-n}\,(1+|\xi|^2)^{\alpha+(n-1)/2}\,|\widehat{f}(\xi)|^2\,d\xi \\ &= \int_{\mathbb{S}^{n-1}} d\omega \int_0^\infty ds (1+s^2)^{\alpha+(n-1)/2}\,|\widehat{f}(s\omega)|^2, \end{aligned}$$

upon a change into spherical coordinates in Fourier space. Now, we make use of the Fourier Slice Theorem, which gives that

$$\mathcal{F}\mathsf{R}_\omega f(s) = (2\pi)^{(n-1)/2}\, \widehat{f}(s\omega).$$

Hence, after a quick computation

$$||\mathsf{R}\, f||^2_{\mathcal{H}^{\alpha+(n-1)/2}(\mathcal{Z})} = 2(2\pi)^{n-1} \int_{\mathbb{R}^n} |\xi|^{1-n}\,(1+|\xi|^2)^{\alpha+(n-1)/2}\,|\widehat{f}(\xi)|^2\,d\xi \qquad (3.16)$$

we get

$$||f||^2_{\mathcal{H}_0^\alpha(\mathbb{B}^n)} \le \frac{1}{c}\,||\mathsf{R}\, f||^2_{\mathcal{H}^{\alpha+(n-1)/2}(\mathcal{Z})}$$

with $c = 2(2\pi)^{n-1}$.

To get the upper bound we start from (3.16) breaking the integral into $|\xi| \le 1$ and $|\xi| \ge 1$. For $|\xi| \ge 1$ we have $|\xi|^2 \ge (1+|\xi|^2)/2$ and thus

$$\int_{|\xi|\ge 1} |\xi|^{1-n}(1+|\xi|^2)^{\alpha+(n-1)/2}\,|\widehat{f}(\xi)|^2\,d\xi \le 2^{(n-1)/2}\,||f||^2_{\mathcal{H}_0^\alpha(\mathbb{B}^n)}\,.$$

The part with $|\xi| \le 1$ is estimated by

$$\int_{|\xi|\le 1} |\xi|^{1-n}\,(1+|\xi|^2)^{\alpha+(n-1)/2}\,|\widehat{f}(\xi)|^2\,d\xi \le C_1(\alpha,n) \sup_{|\xi|\le 1} |\widehat{f}(\xi)|^2,$$

where

$$C_1(\alpha,n) \stackrel{\text{def}}{=} \int_{|\xi|\le 1} |\xi|^{1-n}\,(1+|\xi|^2)^{\alpha+(n-1)/2}\,d\xi.$$

The proof is now completed by showing that

$$\sup_{|\xi|\le 1} |\widehat{f}(\xi)| \le \beta ||f||_{\mathcal{H}_0^\alpha(\mathbb{B}^n)}\,,$$

for some $\beta = \beta(\alpha,n) < \infty$, which is left as an exercise to the reader. *Q. E. D.*

As a consequence of the previous theorem, it follows that the Radon transform extends as a bounded map from $\mathcal{H}_0^\alpha(\mathbb{B}^n)$ into $\mathcal{H}^{\alpha+(n-1)/2}\,(\mathcal{Z})$ with closed range. Furthermore the map

$$\mathsf{R}^{-1}\colon \mathrm{Ran}(\mathsf{R}) \to \mathcal{H}_0^\alpha(\mathbb{B}^n)$$

is bounded. At this point a very natural question arises: How to characterize the range of the Radon transform in the various domains mentioned above? This question is of practical importance, since we want to be able to say whether or not a set of measurements is indeed coming from $\mathsf{R}\, f$, for f in an appropriate space. The upshot of the discussion will be that the Radon transform is highly structured. Indeed, let's consider the moments

$$p_m(\omega) \stackrel{\text{def}}{=} \int_{\mathbb{R}} ds\, s^m\, \mathsf{R}_\omega f(s)$$

for $m = 0, 1, 2, \cdots$.

A simple computation gives that

$$\begin{aligned} p_m(\omega) &= \int_{\mathbb{R}} ds\, s^m \int_{\omega\cdot x = s} d^{n-1}x\, f(x) \\ &= \int_{\mathbb{R}} ds \int_{\omega\cdot x = s} d^{n-1}x\, (\omega\cdot x)^m\, f(x) \\ &= \int_{\mathbb{R}^n} d^n x\, (\omega\cdot x)^m\, f(x). \end{aligned}$$

Hence, if we write $\omega = (\omega_1, \cdots, \omega_n)$, then $p_m(\omega)$ is a *homogeneous* polynomial of degree m in the variables $\omega_1, \cdots, \omega_n$.
We have thus established the following:

Helgason-Ludwig Condition: The polynomials $p_m(\omega)$ are homogeneous of degree m, for $m = 0, 1, \cdots$.

It turns out that the Helgason-Ludwig condition is not only necessary but also a sufficient one, more precisely we have

Theorem 3.3.2. *Let $g \in \mathcal{S}(\mathcal{Z})$ be an even function. If the Helgason-Ludwig condition holds, then there exists $f \in \mathcal{S}(\mathbb{R}^n)$ such that $g = \mathsf{R}\, f$.*

We refer the interested reader to [49] for the proof of this elegant result.

3.4 Application to the Wave Equation

We shall now discuss an illustrative application of the Radon transform, which displays its power. The context is classical wave propagation. Namely, find a solution to the wave equation in n spatial dimensions:

$$\begin{cases} \partial_t^2\, u = \Delta u, \\ u\big|_{t=0} = f \\ \partial_t u\big|_{t=0} = g \end{cases} \tag{3.17}$$

Here, as usual, $u = u(t, x)$, where $t \in \mathbb{R}$ is the time variable, and $x \in \mathbb{R}^n$ is the spatial one. We have rescaled the time variable so that the speed of propagation is 1. We have asked the condition that $u \in C^2(\mathbb{R}\times\mathbb{R}^n)$ to keep matters as simple as possible. We shall also assume for simplicity that f and g belong to $\mathcal{S}(\mathbb{R}^n)$.

The key observation is the following formula, which has also many applications in the Radon transform theory:

$$\mathsf{R}_\omega[\Delta f](y) = \frac{\partial^2}{\partial y^2}\,\mathsf{R}_\omega f(y)\,, \qquad f \in \mathcal{S}(\mathbb{R}^n).$$

The proof of this comes directly from the Fourier Slice theorem and the fact that

$$\mathcal{F}[\Delta f](\xi) = -|\xi|^2\,\mathcal{F}[f].$$

Indeed, let for $\omega \in \mathbb{S}^{n-1}$,

$$g_\omega(y) \overset{\text{def}}{=} \frac{\partial^2}{\partial y^2}\,\mathsf{R}_\omega f(y)$$

and

$$\tilde{g}_\omega(y) \overset{\text{def}}{=} \mathsf{R}_\omega[\Delta f](y).$$

It is enough to show that $\mathcal{F}[g_\omega](\eta) = \mathcal{F}[\tilde{g}_\omega](\eta)$ for all $\eta \in \mathbb{R}$.
Now,

$$\begin{aligned}\mathcal{F}[\tilde{g}_\omega](\eta) &= (2\pi)^{(n-1)/2}\,\mathcal{F}[\Delta f](\eta\omega)\\ &= (2\pi)^{(n-1)/2}\,(-\eta^2)\mathcal{F}[f](\eta\omega).\end{aligned}$$

On the other hand, after integration by parts

$$\begin{aligned}\mathcal{F}[g_\omega](\eta) &= (2\pi)^{-1/2}\int_{\mathbb{R}} d\eta\, e^{-iy\eta}(-\eta^2)\mathsf{R}f_\omega(y)\\ &= (2\pi)^{(n-1)/2}(-\eta^2)\,\mathcal{F}[f](\eta\omega).\end{aligned}$$

Thus proving the claim. This fact has a deep meaning. As consequence of it, all the functional calculus based on Δ is translated by the Radon transform into the functional calculus based on $\partial^2/\partial y^2$.

We now apply this to the problem of solving the wave equation. If we apply the Radon transform to (3.17), and choose $\omega \in \mathbb{S}^{n-1}$ an arbitrary, but fixed, direction we are left with the equation:

$$\begin{cases}\partial_t^2\, u^\# = \partial_y^2\, u^\#\,,\\ u^\#\big|_{t=0} = f^\#\,,\\ \partial_t u^\#\big|_{t=0} = g^\#\,,\end{cases} \tag{3.18}$$

where the #-ed quantities are the ones obtained by applying R_ω with fixed ω.

We have thus transformed the n-dimensional wave equation into a family of (uncoupled) 1-dimensional wave equations indexed on the parameter ω. The solution is given by the classic d'Alembert formula:

$$u^{\#}(t,y) = \frac{1}{2}\left(f^{\#}(y+t) + f^{\#}(y-t)\right) + \frac{1}{2}\int_{y-t}^{y+t} g^{\#}(s)\,ds. \tag{3.19}$$

Let's focus our attention on the case $f = 0$, since one can easily show that if we know how to solve in this case, then we also know how to solve the general one by superposition.

Exercise: Show that the formula obtained using the inverse Radon transform on (3.19) is the same as the classical formula.

An interesting feature of formula (3.19) which is related to Huygens' classical construction is the following.

Let's consider that $f \equiv 0$, and focus on

$$u^{\#}(t,y) = \frac{1}{2}\int_{y-t}^{y+t} g^{\#}(s)\,ds.$$

If $n = 3$, then we get from Section 3.2

$$\begin{aligned} u(t,x) &= c_3 \int_{\mathbb{S}^2} d\omega\, \frac{\partial^2}{\partial y^2}\bigg|_{y=\omega\cdot x} \frac{1}{2}\int_{y-t}^{y+t} ds\, \mathsf{R}_\omega g(s) \\ &= c_3 \frac{1}{2}\int_{\mathbb{S}^2} d\omega\, \frac{\partial}{\partial t}\left(\mathsf{R}_\omega g(\omega\cdot x + t) + \mathsf{R}_\omega g(\omega\cdot x - t)\right) \\ &= c_3\, \frac{\partial}{\partial t}\int_{\mathbb{S}^2} d\omega\, \mathsf{R}_\omega g(\omega\cdot x + t), \end{aligned}$$

where in the last equality we used the fact that $\mathsf{R}_{-\omega}\, g(-\omega\cdot x - t) = \mathsf{R}_\omega\, g(\omega\cdot x + t)$.

As a consequence it is easy to see that the domain of influence of a point $x = x_0$ is at most a ball of radius $|t|$, and center x_0. What is very peculiar here, is that we know that the domain of influence of x_0 in the $n = 3$ case is precisely the sphere (Huygens' principle in its strong form!). Hence, somehow the contribution from all the distant points from x_0 cancels out. In fact a proof of this can be found in [33, 34].

3.5 Reconstruction Methods

In this section we shall describe a few techniques that are used in computerized tomography. We do not plan to provide a comprehensive survey, but rather to convey the flavor of the multi-disciplinary aspects of the subject.

We cover briefly the so-called Algebraic Reconstruction Techniques (ART), whose basic idea is to reduce the quest for $f\colon \Omega \subset \mathbb{R}^n \to \mathbb{R}$ such that

$$\int_L f\,dx = g_L\,, \qquad L = L_1, L_2, \ldots, L_p\,,$$

into an over-determined linear system of equations.

Here L denotes an arbitrary hyperplane intersecting the region Ω, and g_L denotes the corresponding value of the Radon transform.

One of the techniques used to solve such over-determined system is Kaczmarz method, which has an interest of its own.

We also discuss the method of filtered backprojections, which was instrumental in the development of computerized tomography, and still has lots of room for interesting questions. Our exposition follows very closely that of Natterer's [49].

3.5.1 Kaczmarz's Method

In many practical applications one is faced with the question of solving a large system of equations of the form

$$R_j\,f = g_j\,, \qquad j = 1, \ldots, p \tag{3.20}$$

where each $R_j\colon \mathcal{H} \to \mathcal{H}_j$ is a linear bounded map from a Hilbert space $\mathcal{H}$ onto one of its subspaces $\mathcal{H}_j$.
This is a typical problem for example in data fitting or geodesy.

For simplicity we assume that $\mathcal{H}$ is a *real* Hilbert space, but many of the remarks could be easily extended to complex ones.

The geometric idea behind Kaczmarz's method can be easily described with a system of linear equations on the plane. See Figure 3.5. Let's assume we have a system of p consistent equations, and denote by P_j the orthogonal projection onto the affine subspace of solutions to each of the equations

$$R_j\,f = g_j\,.$$

The idea now is to start from an initial guess, say f^0, and produce a sequence of iterates

$$f^k = P_p\,P_{p-1}\cdots P_1\,f^{k-1},$$

for $k = 1, 2, \cdots$. The point in the iteration is that the sequence $\{f^j\}_{j=1}^{\infty}$ converges to the solution of the system.

It turns out that for numerical reasons, it is interesting to introduce a relaxation parameter, similar to the one used in iterative methods such as the

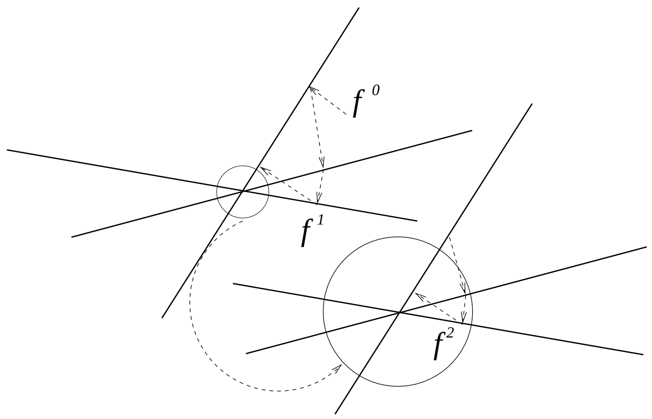

Fig. 3.5. Geometrical description of Kaczmarz method

successive over relaxation techniques (SOR). See [21, 32] for iterative methods. More precisely, let's define the map

$$P_j^\omega = (1-\omega)I + \omega\, P_j$$

and

$$P^\omega = P_p^\omega \cdots P_1^\omega\,.$$

Here, ω is the relaxation parameter, and the iteration is defined as

$$f^{k+1} = P^\omega\, f^k, \tag{3.21}$$

for $k = 0, 1, 2, \cdots$, and f^0 the initial guess to the solution. Typically f^0 could be taken as 0. When $\omega = 1$, the iteration defined above reduces to the geometric construction mentioned in Figure 3.5. Although in general P^ω is not a projection the idea is that we will obtain an iteration that converges to a suitable "solution" to the problem. Before making this statement more precise we shall make the method more explicit.

As usual, let R_j^* denote the adjoint of R_j , and let's assume that $R_j\colon \mathcal{H} \to \mathcal{H}_j$ is surjective. Then, R_j^* is injective and $R_j\, R_j^*$ possesses a left inverse.

Now to determine the orthogonal projection P_j on the affine space defined by

$$R_j\, f = g_j$$

we proceed as follows:

Since

$$P_j\, f - f \;\perp\; \ker(R_j)$$

there exists $u_j \in \mathcal{H}_j$ such that

$$P_j\, f = f + R_j^*\, u_j\,.$$

(Recall that $\ker R_j \perp \operatorname{Ran} R_j^*$).
But since P_j is a projection we must have that $R_j\, P_j\, f = g_j$, and thus

$$R_j\, R_j^*\, u_j = g_j - R_j\, f.$$

We now make use of the left inverse to $R_j\, R_j^*$, which as we mentioned above exists, to get

$$u_j = \left(R_j\, R_j^*\right)^{-1} (g_j - R_j\, f).$$

Therefore,

$$P_j\, f = f + R_j^* \left(R_j\, R_j^*\right)^{-1} (g_j - R_j\, f)$$

and the explicit form of the iteration then becomes [4] $f_0 = f^k$

$$f_j = f_{j-1} + \omega\, R_j^* \left(R_j\, R_j^*\right)^{-1} (g_j - R_j\, f_{j-1})$$

where $j = 1, \ldots, p$.
At the end of such loop we have $f^{k+1} = P^\omega f^k$.

For further use we define

$$R = \begin{bmatrix} R_1 \\ \vdots \\ R_p \end{bmatrix} .$$

The convergence is ensured by the following result, whose proof can be found in [49].

Theorem 3.5.1. *Let's assume that the system (3.20) has a solution. Then, if $0 < \omega < 2$ and the initial guess $f^0 \in \operatorname{Ran}(R^*)$, the sequence $\{f^k\}_{k=0}^\infty$ converges to the solution of (3.20) with minimal norm.*

Let's now restrict to the finite dimensional situation. It turns out that in practice, many times the presence of noise or numerical errors makes the system (3.20) incompatible. Then, the Kaczmarz iteration defined above still gives a meaningful result. Indeed, let's suppose that $\mathcal{H}$ is finite dimensional, then under the same hypothesis of Theorem 3.5.1, the Kaczmarz iteration converges to the unique solutions $f_\omega \in \operatorname{Ran}(R^*)$ of

$$R^* \left(D + \omega\, L\right)^{-1} (g - R\, f_\omega) = 0$$

where $D = \operatorname{diag}(R_1 R_1^*, \ldots, R_p R_p^*)$ and

$$L = \begin{bmatrix} 0 & 0 & & \cdots & \\ R_2 R_1^* & 0 & & & \\ \vdots & & \ddots & \ddots & \\ 0 & & & 0 & 0 \\ R_p R_1^* \ldots & & & R_p R_{p-1}^* & 0 \end{bmatrix} .$$

[4] The inner loop will be denoted by subscripts.

Furthermore, if (3.20) has a solution then f_ω is exactly this solution. Otherwise, it can be shown that

$$f_\omega = f_M + \mathcal{O}(\omega)$$

where f_M minimizes the form $(D^{-1}(g - Rf) \mid g - Rf)$ in $\mathcal{H}$.

3.5.2 Algebraic Reconstruction Technique

In this section we refine the Kaczmarz algorithm presented in the previous section to the case of ray tomography. We shall deal in particular with the two-dimensional situation.

Before proceeding, we mention that the approach described here is one of the several possible techniques. As before, we follow [49], where many other techniques are also described. Let's then turn our attention to the problem of obtaining a function $f\colon \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ from the integrals

$$\int_{L_j} f(x)\, dx = g_j\,, \quad j = 1, \ldots, N. \tag{3.22}$$

Now, the basic idea here is to divide the object into a certain number M of

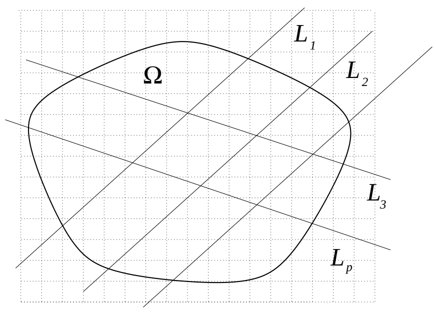

Fig. 3.6. The two dimensional Radon problem

picture elements, which are called pixels. One way of doing this is to superimpose onto Ω a grid of $M = N_1 \times N_2$ pixels, as shown in Figure 3.6. However, many other possibilities could be used, such as for example a grid in polar coordinates, which would be more suited for some data collection set up.

The function f will now be represented by its value F_m on each pixel P_m, $m = 1, \ldots, M$. The integral of equation (3.22) is now approximated by setting A_{jm} to be the length of the line L_j in the pixel P_m, as shown in Figure 3.7. The result is the discretization of equation (3.22) as

$$A_j^\top F = g_j\,, \quad j = 1, \ldots, N$$

where A_j is the column vector given by

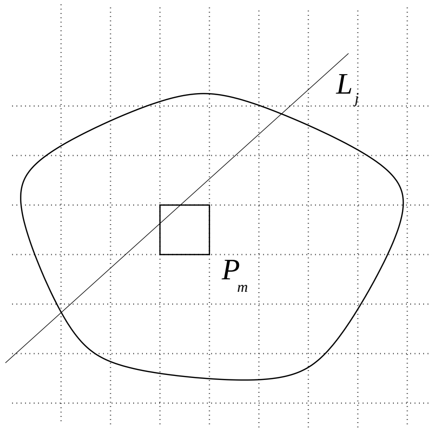

Fig. 3.7. The pixel P_m crossed by the line L_j

$$A_j^\top = [A_{j1}, \ldots, A_{jM}]$$

and F is the row vector whose m-th component is the value of f at P_m.

A minute's thought gives that the internal loop for the Kaczmarz iteration is given by

$$F_j = F_{j-1} + \frac{\omega}{|A_j|^2}\,(g_j - A_j^\top F_{j-1})A_j \tag{3.23}$$

for $j = 1, \ldots, N$, where ω is the relaxation parameter.

Remarks:

1. Typically the line L_j will intersect about $\sqrt{M}$ pixels (think for example of the case $M = N_1^2$, where N_1 is the number of pixels on the side of a square). Hence, each vector A_j will have only about $\sqrt{M}$ nonzero entries. This gives a rather sparse structure for the internal loop (3.23). More precisely, one needs about $\mathcal{O}(\sqrt{M})$ operations to compute (3.23), and a total of $\mathcal{O}(N\sqrt{M})$ operations to perform each of the Kaczmarz method iterations.

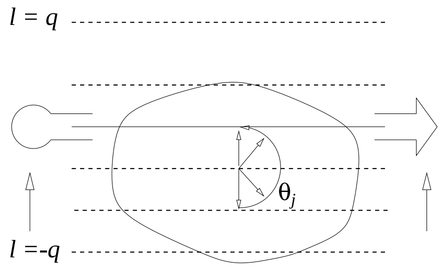

Fig. 3.8. Parallel scanning geometry

2. Let's consider the situation of a parallel scanning geometry, as described in Figure 3.8. Here the object undergoes $2q+1$ readings along lines parallel to one another, then it is rotated and another set of $2q+1$ readings is performed, and so on up to a maximum of say p directions.
   Then, in this situation we would have $M=(2q+1)^2$ and $N=(2q+1)p$.
   A practical choice for p is about $\pi\, q$. Then, the operation count is about $\mathcal{O}(p^3)$ operations. This is the same order of magnitude of the complete filtered back-projection algorithm.
3. As in many iterative numerical algorithms the Kaczmarz method is competitive only if convergence occurs in a relatively small number of iterations. On the other hand, the ART method is extremely versatile and easy to implement.
4. As a consequence of the general theory of the Kaczmarz method, which was described in the previous section it follows that the sequence $F^k,\ k=0,1,2,\cdots$, will converge to a generalized solution F_ω. Obviously in practice one does not expect that the over-determined system will be consistent. In this case, it can be shown (see [49]) that as $\omega \searrow 0,\ \ F_\omega$ converges to the minimizer of
$$\sum_{j=1}^{N} \frac{1}{|A_j|^2}\,(g_j - A_j^\top F)^2.$$

3.5.3 Filtered Back-projection Algorithm

No introductory text on the applications of the Radon transform would be complete without mentioning a few words about the interpretation of the inversion formula and the so-called filtered back-projection algorithm.

Our goal is to do that in a very condensed form, hoping this will entice the reader to look into the bibliography.

We start off from a side remark. Namely, the reconstruction formulae of (3.2) for n even and odd can be unified by looking at it from the viewpoint of the functional calculus for self-adjoint operators. Indeed, let's consider the skew-adjoint operator[5]

$$\begin{aligned} \frac{d}{dx} : \mathcal{H}^1(\mathbb{R}) \subset L^2(\mathbb{R}) &\to L^2(\mathbb{R}) \\ f &\mapsto f' \end{aligned}$$

Then, one can compute its absolute value by means of the Fourier transform

$$\left|\frac{d}{dx}\right| g = \mathcal{F}^{-1}\left[|\xi|\hat{g}(\xi)\right].$$

[5] by skew-adjoint we mean an operator which is i times a self-adjoint operator.

In this case, the Hilbert transform H allows us to write down the polar decomposition of d/dx as

$$\frac{d}{dx} = -\mathsf{H}\left|\frac{d}{dx}\right| .$$

The upshot is that

$$\left|\frac{d}{dx}\right|^{n-1} = \begin{cases} (-i)^{n-1}\dfrac{d^{n-1}}{dx^{n-1}}\,, & n \quad \text{odd} \\ (-i)^{n-1}\,\mathsf{H}\left(\dfrac{d}{dx}\right)^{n-1}, & n \quad \text{even.} \end{cases}$$

The unified formula takes the form

$$f(x) = c_n \int_{\mathbb{S}^{n-1}} d\omega\, |\mathsf{D}|^{n-1}\, \mathsf{R}_\omega\, f(\omega\cdot x)\ ,$$

where c_n is a numerical constant and $|\mathsf{D}|^{n-1}\,\mathsf{R}_\omega f$ corresponds to computing the $(n-1)$-st power of the absolute value of the derivative operator. In this case, we see that in general to obtain the inverse to the Radon transform one would undergo two steps: First some kind of filtering , which in this case is represented by the derivative and possibly (for n even) the Hilbert transform. Second, some kind of adjoint operation, which in this literature became known as a back-projection.

It turns out, however, that in order to make this process into a numerical method it is worthwhile looking into a "smeared" version of f. In other words, we seek an algorithm to compute $W_b * f$, where W_b is an approximation to the δ-function as $b \to \infty$.

Here, W_b is taken as a band-limited function, i.e. $\mathcal{F}[W_b](\xi) = 0$ if $|\xi| > b$. The reason being that the differentiation process is highly unstable numerically. Before we proceed, remark that for $f \in \mathcal{S}(\mathbb{R}^n)$ and $g \in \mathcal{S}(\mathcal{Z})$ we have that

$$(\mathsf{R}^*\, g) * f = \mathsf{R}^*(g\, *\, \mathsf{R}f), \tag{3.24}$$

where the convolution on the RHS is taken on the second argument over the real line, i.e.

$$(g * \mathsf{R}f)(\omega, y) = \int_{\mathbb{R}} ds\, g(\omega, s)\, \mathsf{R}f(\omega, y-s).$$

The proof of equation (3.24) is left as an exercise, as well as its extension to more general classes of functions g defined on $\mathcal{Z}$. The idea now is to take

$$W_b = \mathsf{R}^*\, w_b$$

and choose w_b judiciously so that by making use of (3.24) we get

$$W_b * f = \mathsf{R}^*(w_b * \mathsf{R}f). \tag{3.25}$$

A theoretical example would be the ideal low-pass filter, in which case one would have

$$W_b(x) = (2\pi)^{-n/2}\, b^n \, \frac{J_{n/2}(b|x|)}{(b|x|)^{n-2}}$$

where $J_{n/2}$ is the Bessel function of the first kind with exponent $n/2$ [44].

In equation (3.25) it becomes clear that the filtering process will be performed using the convolution with the function $W_b = \mathsf{R}^*\, w_b$.

Let's focus in the case of $\mathbb{R}^2$ and assume that our function f has support contained in the disc $\mathbb{D}$. Let's once again assume we are working in the parallel beam geometry, however these assumptions are not essential and the reader may consult [49] for more general situations. In this context the Radon transform

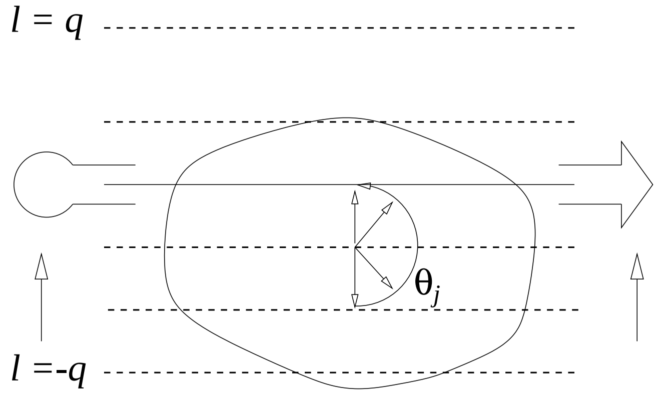

Fig. 3.9. The two dimensional case

$g = \mathsf{R}f$ will be presented as

$$\big\{ g(\theta_j, s_l) \big\}_{\substack{j=1,\dots,p \\ l=-q,\dots,q}}$$

where $s_l = h\, l = l/q$ and $\theta_j = \pi(j-1)/p$ (one more generally a set of p distinct angles). Then, the convolution $w_b * g$ is replaced by the discrete convolution

$$w_b \overset{h}{*} g(\theta_j, s) \overset{\text{def}}{=} h \sum_{l=-q}^{q} w_b(s - s_l)\, g(\theta, s_l)$$

and the back-projection part is performed by a quadrature rule of the form

$$\int_0^{2\pi} v(\theta)\, d\theta \cong \sum_{j=1}^{p} \alpha_{pj}\, v(\theta_j). \tag{3.26}$$

The crucial result is that [49] the computed result approximates well the given function f, under suitable assumptions. Furthermore if $p \simeq \pi\, q$ the algorithm can be implemented with $\mathcal{O}(q^3)$ operations.

We conclude by mentioning that many other methods have been developed to tackle the inverse problem for the Radon transform. One example being the Davison-Grünbaum method [13], whose theoretical aspects lead to interesting questions way beyond the applications of tomography [25].

4. Impedance Tomography

4.1 The Direct Problem

4.1.1 Physical Background

The importance of Laplace's equation in Mathematical Physics is due in part to its presence in a wide variety of distinct physical situations. It appears in electrostatics, as well as in more general electro-magnetic phenomena. It also arises in fluids, elasticity, as well as in probability. We shall now briefly review one of the contexts in electrostatics where one is lead to Laplace's equation.

Let's consider an experiment where we have a conducting body occupying a region $\Omega \subset \mathbb{R}^3$. The conductivity of such body is characterized at each point $x \in \mathbb{R}^3$ by the parameter $\sigma(x)$, which relates the electric field $E(x)$ (measured in V/m) to the density of flow of charges $J(x)$ (measured in Coulombs/sec $\times\, \mathrm{m}^2$) by means of

$$J = \sigma\, E. \tag{4.1}$$

The idea is shown in the Figure 4.1. The relation (4.1), which is obviously an

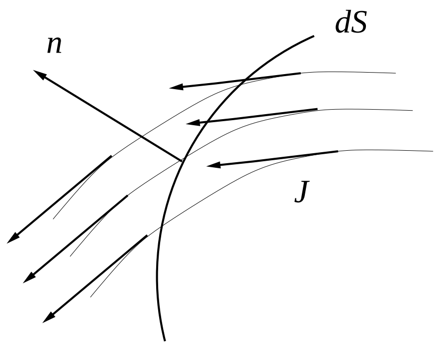

Fig. 4.1. Flow of charges through a surface element. The total flow in one unit of time is given by $J \cdot dS$. Usually $[J] =$ Coulombs/sec$\times \mathrm{m}^2$

assumption about the linearity of the relation between E and J, tells us that the rate of flows of charge through the surface dS is proportional to the intensity of the electric field.

In a more complex situation σ would be a matrix. We shall assume, however, that we are working with isotropic media.

Now, if the charges do not get destroyed or created inside a volume V it follows that

$$\mathrm{div} J = 0.$$

Furthermore, if the electric field E results from a potential u, i.e. we have that

$$E = -\nabla u,$$

then the potential u must satisfy

$$\nabla \cdot (\sigma \nabla u) = 0.$$

We stress that the underlying assumption in our present model is that one is working under a steady state situation with no sources or sinks. For more information on this we refer the reader to [53, 1, 17].

Now if the volume Ω is subjected to a forcing potential on its boundary $\partial\Omega$ given by a preassigned function f, we have a boundary value problem of the form

$$\begin{cases} \nabla \cdot (\sigma \nabla u) = 0 \\ u\big|_{\partial\Omega} = f \ . \end{cases} \tag{4.2}$$

The boundary condition $u\big|_{\partial\Omega} = f$ is called of Dirichlet type. We remark that to perform such an experiment one should be able to impose f by injecting energy in the system. This would typically be done by means of a source of energy outside the body, which should direct the current flow on the boundary. This flow is given in the outwards direction n by

$$I = -\sigma \frac{\partial u}{\partial n} = -\sigma \nabla u \cdot n. \tag{4.3}$$

The total energy per unit of time is given by

$$\int_{\partial\Omega} \sigma \frac{\partial u}{\partial n} u \, dS.$$

Another physically interesting situation is the prescription of the current $I(s)$, entering (or leaving) the region Ω through the boundary at s. Because of (4.3), prescribing $I(s)$ for $s \in \partial\Omega$ is equivalent to imposing a (Neumann type) boundary condition of the form

$$\frac{\partial u}{\partial n} = g,$$

where

$$g(s) = -\frac{1}{\sigma(s)}\, I(s)$$

with $n = n(s)$ the outwards unit normal at $s \in \partial\Omega$. Obviously the net amount of current entering Ω should be equal to net amount of current exiting it, in other words

$$\int_{\Omega} \sigma\, \frac{\partial u}{\partial n}\, dS = 0.$$

This gives a constraint that must be satisfied by $\partial u/\partial n$. We are thus lead to a Neumann type problem:

$$\begin{cases} \nabla \cdot (\sigma \nabla u) = 0 \\ \left.\dfrac{\partial u}{\partial n}\right|_{\partial\Omega} = g \end{cases} \tag{4.4}$$

Another important class of phenomena which leads to boundary value problems of the form (4.2) or (4.4) appears in potential theory.

4.2 Laplace's Equation in Riemannian Geometry

We shall now consider the equation

$$\nabla(\sigma \nabla u) = 0 \tag{4.5}$$

in a more general context of Riemannian geometry. This will not only shed some light on the geometrical meaning of (4.5) but also allow a simplification of many of the arguments.

We recall that given a Riemannian metric g_{ij} on a manifold M, one can naturally define the notion of gradient of $f\colon M \to \mathbb{R}$ as the vector field ∇f such that

$$(\nabla f, X) = X(f) = df(X)$$

for any vector field X on M.

In coordinates we have[1]

$$X = X^i \partial_i = X^i \frac{\partial}{\partial x^i}$$

and

$$X^i \partial_i f = df(X) = g_{ij}\, X^i (\nabla f)^j$$

Hence,

[1] The summation convention is enforced.

$$(\nabla f)^i = g^{ij}\,\frac{\partial f}{\partial x^j}$$

where g^{ij} is the (i,j)-th entry of the inverse matrix to the metric matrix $((g_{ij}))$.

The notion of divergence of a vector field can also be defined in an intrinsic way. For that it suffices to have a volume form defined on M. If ω denotes the volume form on M, then for any given vector field X on M we can associate the div$X$, which is a scalar field such that[2]

$$\mathcal{L}_X\,\omega = (\mathrm{div} X)\omega.$$

We recall that a volume form is a never vanishing element of the top exterior product $\Lambda^n M$, i.e. a never vanishing n-linear alternate differential form.

Since, $\omega \in \Lambda^n M$, it follows that $d\omega = 0$ and hence

$$(\mathrm{div} X)\omega = d(\omega \rfloor X).$$

Exercise 4.2.1. In $\mathbb{R}^n$, with the Euclidean structure, we have the usual volume form $dx^1 \wedge \cdots \wedge dx^n$. Show that in this case

$$\mathrm{div} X = \sum \frac{\partial X^i}{\partial x^i}\,\cdot$$

If (M,g) is a Riemannian manifold check that the natural volume form is given by

$$\omega = \sqrt{g}\,dx_1 \wedge \cdots \wedge dx_n$$

where $g = \det((g_{ij}))$.

Conclude that the divergence of the vector field X is given by

$$\begin{aligned} \mathrm{div} X &= g^{-1/2}\,\partial_j(g^{1/2}\,X^j) && (4.6)\\ &= \partial_j\,X^j + \frac{1}{2}\,g^{-1}\left(\frac{\partial g}{\partial x^j}\right)X^j. && (4.7) \end{aligned}$$

Show that once we have defined the geometric notions of gradient and divergence, it makes sense to talk about the Laplace-Beltrami operator div(grad u). In local coordinates it is given by

$$\mathrm{div}\,\mathrm{grad}\,u = g^{-1/2}\,\partial_j\,g^{1/2}\,g^{ij}\,\partial_i u.$$

For later use, we now consider Gauss theorem. Let M be a Riemannian manifold such that $\overline{M}$ is a compact manifold with smooth boundary ∂M and

[2] Recall Cartan's magic formula:

$$\mathcal{L}_X\,\alpha = d\alpha\rfloor X + d(\alpha\rfloor X)$$

which allows us to define Lie derivatives of k-forms.

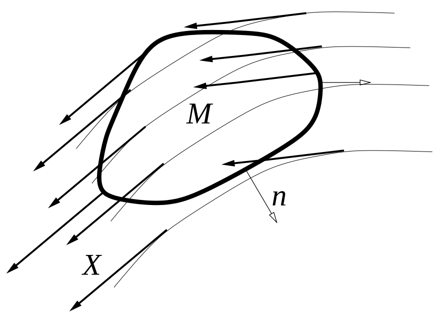

Fig. 4.2. Gauss divergence theorem

X is a smooth vector field on $\overline{M}$. Then, the classical Gauss divergence theorem takes the form

$$\int_M \mathrm{div} X \, dV = \int_{\partial M} (X, n) \, dS$$

where as usual n denotes the outgoing normal vector field to $\partial\Omega$. See figure 4.2. Let's recall that under the above hypothesis, it makes sense to talk about the L^2 inner product for vector fields on M

$$(X, Y)_{L^2(M)} = \int_M (X, Y)_x \, dV(x).$$

This extends naturally for complex vector fields and the natural Hermitean product. A fundamental fact, which we shall make use a number of times is that the grad and $-$div are formally adjoint operations of one another. More precisely, we have that for any smooth vector field X, and any test function u with compact support in M we have

$$(X, \mathrm{grad}\, u)_{L^2(M)} = -(\mathrm{div} X, u)_{L^2(M)} \, .$$

The same claim also holds if the support of X is contained in M.

Let's now discuss the geometric interpretation of the three-dimensional problem

$$\nabla \cdot (\sigma \, \nabla \, u) = 0. \tag{4.8}$$

If we consider in the region $\Omega \subset \mathbb{R}^3$ a metric conformally equivalent to the Euclidean one, namely

$$ds^2 = \sigma^2(x) \big((dx^1)^2 + (dx^2)^2 + (dx^3)^2 \big) \ , \tag{4.9}$$

then the Laplace-Beltrami operator associated this metric is given by

$$u \longmapsto \frac{1}{\sigma^3} \, \nabla(\sigma \, \nabla \, u).$$

So, equation (4.8) is exactly the Laplace-Beltrami equation associated to the metric (4.9).

4.3 The Dirichlet Problem for a Compact Manifold with Boundary

In this section we discuss the Dirichlet problem in a fairly general context of a Riemannian manifold with smooth boundary. Although our approach seems unduly abstract, this should not deter the reader unfamiliar with Riemannian geometry or Sobolev spaces. For excellent expositions on Riemannian geometry we refer the reader to [15, 68]. Also, a good review of such concepts can be found in M. Taylor's masterpiece [65] from where most of our exposition in this section is taken. We do this in order to make these notes as self contained as possible. In Appendix 2.1 the reader will find a review of Fourier transform and Sobolev spaces.

Let $\overline{M}$ be a smooth compact manifold with smooth boundary ∂M and interior M. We shall now discuss the fundamental result concerning the existence of solutions to the Dirichlet problem

$$\begin{cases} \Delta u = 0, \quad \text{in} \quad M \\ u\big|_{\partial M} = f \ . \end{cases} \tag{4.10}$$

Here $\Delta u = \operatorname{div}(\operatorname{grad} u)$ is the Laplace-Beltrami operator applied to u w.r.t. metric of M.

There are two main techniques in order to deal with such problems. The first one makes use of integral operator techniques. The second one, which we shall use now, takes advantage of Hilbert space techniques.

The purpose of this section is to give an idea of the proof of the following result.

Theorem 4.3.1. *The Dirichlet problem defines a map*

$$\begin{aligned} \mathrm{PI} : C^\infty(\partial\Omega) &\longrightarrow C^\infty(\overline{\Omega}) \\ f &\longmapsto u \end{aligned}$$

which has a unique continuous extension from $\mathcal{H}^s(\partial\Omega) \ \to \ \mathcal{H}^{s+1/2}(\Omega)$, $\forall\, s \geq 1/2$.

As a corollary it follows the existence of solutions to (4.10) with boundary data in Sobolev spaces.

The first ingredient in the proof of this result is an estimate which is of crucial relevance. It is called in some contexts Poincaré's inequality. See [38].

As above, if $\overline{\Omega}$ is a smooth, compact, Riemannian manifold with nonempty boundary $\partial\Omega$ then

$$||u||^2_{L^2(\Omega)} \leq C||du||^2_{L^2(\Omega)}\,, \quad \forall\, u \in C_0^\infty(\Omega).$$

The proof is a simple integration starting from a point on the boundary and joining x along a path $\gamma(x)$. The choice of the constant C will depend on an efficient way of choosing the path $\gamma(x)$. As a consequence, it follows that the

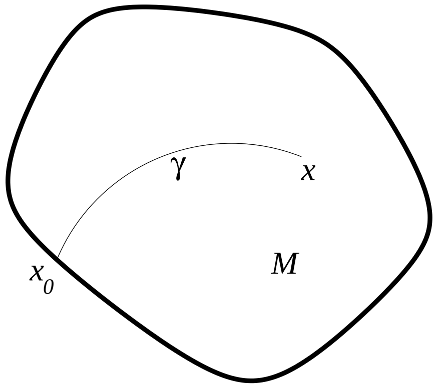

Fig. 4.3. The Poincaré's inequality construction

norm $||du||_{L^2(\Omega)}$ is *equivalent* to the norm $||u||_{\mathcal{H}^1(\Omega)}$ for $u \in \mathcal{H}_0^1(\Omega)$, which is the completion of the $C_0^\infty(\Omega)$ w.r.t. the $\mathcal{H}^1(\Omega)$ inner product.

We now study the operator Δ

$$\Delta\colon \mathcal{H}_0^1(\Omega) \to \mathcal{H}^{-1}(\Omega).$$

Notice that for $u \in C_0^\infty(\Omega)$

$$(-\Delta u, u) = ||du||^2_{L^2(\Omega)}\,.$$

Hence,

$$(-\Delta u, u) \geq C||u||^2_{\mathcal{H}^1(\Omega)}\,, \qquad \forall\, u \in \mathcal{H}_0^1(\Omega).$$

Therefore,

$$C||u||^2_{\mathcal{H}^1(\Omega)} \leq ||\Delta u||_{\mathcal{H}^{-1}(\Omega)}\, ||u||_{\mathcal{H}^1(\Omega)}$$

and thus

$$||\Delta u||_{\mathcal{H}^{-1}(\Omega)} \geq C||u||_{\mathcal{H}^1(\Omega)}\,, \qquad \forall\, u \in \mathcal{H}_0^1(\Omega). \tag{4.11}$$

As a corollary we get that

$$\Delta\colon \mathcal{H}_0^1(\Omega) \to \mathcal{H}^{-1}(\Omega) \tag{4.12}$$

is $1-1$ and onto. Indeed (4.11) gives immediately that Δ is $1-1$ with closed range. If the range $\operatorname{Ran}\Delta$ was not closed, we would have $\exists\, v \in (\mathcal{H}^{-1}(\Omega))^*$ such that $(\Delta u, v) = 0$, $\forall\, u \in \mathcal{H}_0^1(\Omega)$. But $\mathcal{H}^{-1}(\Omega)^* = \mathcal{H}_0^1(\Omega)$. Hence, upon taking $u = v$ we would get, $(\Delta v, v) = 0$, and thus $v = 0$.

Therefore, from now on we can talk about

$$T\colon \mathcal{H}^{-1}(\Omega) \to \mathcal{H}_0^1(\Omega)$$

which is the inverse to

$$\Delta\colon \mathcal{H}_0^1(\Omega) \to \mathcal{H}^{-1}(\Omega).$$

It is not difficult to check that T is symmetric w.r.t. the $L^2(\Omega)$ inner product, i.e.

$$(Tu, v) = (u, Tv).$$

Now, the space $\mathcal{H}_0^1(\Omega)$ is compactly imbedded in $L^2(\Omega)$, and hence $T\big|_{L^2(\Omega)}$ is a compact operator (Rellich's theorem).

Hence $T\big|_{L^2(\Omega)}$ (which we henceforth shall also denote by T) is a *compact self-adjoint* operator. By the Hilbert-Schmidt Spectral Theorem there exists an orthonormal basis $\{u_j\}_{j=0}^{\infty}$ of eigenvectors of T,

$$T\,u_j = -\mu_j u_j\,,$$

here $\mu_j > 0$ is s.t. $\mu_j \searrow 0$ as $j \to \infty$. The fact that the μ_i's are positive derives from the fact that Δ is negative, i.e. $(-\Delta u, u) = (du, du)$. We have established, in passing, the existence of an eigenvalue basis for the unbounded s.a. operator $-\Delta$ with $\mathrm{Dom}(-\Delta) = T(L^2(\Omega))$. This basis is crucial in understanding problems such as vibrating membranes on a bounded subdomain of $\mathbb{R}^n$ or the heat equation on M.

The next step is the following elliptic regularity estimate:

Theorem 4.3.2. *Let X be a vector field on $\overline{\Omega}$ and let $u \in \mathcal{H}_0^1(\Omega)$ s.t.*

$$Lu \stackrel{def}{=} \Delta u + Xu = f \in \mathcal{H}^{k-1}(\Omega),$$

for some $k = 0, 1, 2\cdots$. Then, $u \in \mathcal{H}^{k+1}(\Omega)$ and furthermore

$$||u||^2_{\mathcal{H}^{k+1}} \leq ||Lu||^2_{\mathcal{H}^{k-1}} + C||u||^2_{\mathcal{H}^k}\,, \quad \forall\, u \in \mathcal{H}^{k+1}(\Omega) \cap \mathcal{H}_0^1(\Omega).$$

The proof of this regularity estimate, will not be given here. We refer the interested reader to M. Taylor's book [65] page 304.

As an immediate corollary of the above theorem, it follows that the eigenfunctions $\{u_j\}_{j=0}^{\infty}$ of Δ belong to $C^\infty(\overline{\Omega})$.

Let's now complete the sequence of ideas in the proof of Theorem 4.3.1. For $f \in C^\infty(\overline{\Omega})$, we look at

$$\begin{cases} \Delta u = 0 \quad \text{on} \quad \Omega \\ u\big|_{\partial\Omega} = f\,, \end{cases} \tag{4.13}$$

and consider $F \in C^\infty(\overline{\Omega})$ such that $F\big|_{\partial\Omega} = f$. Now, the standard trick is to restate (4.13) as

$$\Delta v = g \stackrel{\text{def}}{=} -\Delta F, \qquad v\big|_{\partial\Omega} = 0. \tag{4.14}$$

Since $g \in C^\infty(\overline{\Omega})$, we get that

$$v = T\,g \in \mathcal{H}^1_0(\Omega)$$

and because of the elliptic regularity estimate Theorem 4.3.2, it follows that $v \in C^\infty(\overline{\Omega})$. We have therefore obtained way of going from $f \in C^\infty(\partial\Omega)$ to u a solution of (4.13).

It is usual to denote this map $f \mapsto u$ by PI, meaning the Poisson integral (this coincides with the Poisson integral in the case Ω is the disk in $\mathbb{R}^2$). Finally, the claim that PI admits a *unique* continuous extension

$$\mathrm{PI}\colon \mathcal{H}^s(\partial\Omega) \to \mathcal{H}^{s+1/2}(\Omega), \quad s \geq 1/2$$

is first established for discrete values of $s = k + 1/2,\ k = 0, 1, 2, \cdots$. To do this, we have $f \in \mathcal{H}^{k+1/2}(\partial\Omega)$ s.t. $F\big|_{\partial\Omega} = f$. This is done using the technique in Taylor's book of Prop. 1.7 page 274 [65]. See also Appendix 2.1. Then,

$$\mathrm{PI}\, f \stackrel{\text{def}}{=} F + v$$

where v by definition is the unique solution of

$$\Delta v = -\Delta F \in \mathcal{H}^{k-1}(\Omega), \quad v \in \mathcal{H}^1_0(\Omega).$$

This v, by the regularity result, belongs to $\mathcal{H}^{k+1}(\Omega)$. To get the estimate for other values of s one resorts to standard interpolation techniques.

One can patch together the above mentioned tricks to get:

Theorem 4.3.3. *Given $g \in \mathcal{H}^{k+1/2}(\partial\Omega)$ and $f \in \mathcal{H}^{k-1}(\Omega)$, the inhomogeneous Dirichlet problem*

$$\begin{cases} \Delta u = f \\ u\big|_{\partial\Omega} = g \end{cases}$$

has a unique solution $u \in \mathcal{H}^{k+1}(\Omega)$. Furthermore, for every $u \in \mathcal{H}^{k+1}(\Omega)$

$$||u||^2_{\mathcal{H}^{k+1}(\Omega)} \leq C\big(||\Delta u||^2_{\mathcal{H}^{k-1}(\Omega)} + ||u||^2_{\mathcal{H}^{k+1}(\partial\Omega)} + ||u||^2_{\mathcal{H}^{k}(\Omega)}\big).$$

The main result, Theorem 4.3.1 holds in much more general situations, which we however shall not delve in the present notes. The interested reader may consult [18]. We have throughout this section assumed that the boundary of our domain is smooth. Obviously, many of the results can be extended to the more practical situation where the boundary has "mild" nonsmooth curves. We shall not deal with such extensions here, and neither with the harder situation of "rough" boundaries. See [65] for that.

4.4 The Impedance Tomography Model

We shall now describe an example of an inverse problem that has attracted the attention of a large number of researchers. The reason for that are easy to explain:

1. It is easily stated.
2. It is very hard to solve, and in fact practical or useful solution are still actively sought nowadays.
3. It appears in a number of different physical contexts.

The problem can be stated as follows:

Let $\Omega \subset \mathbb{R}^n,\ \ n \geq 2$, be a bounded domain with smooth boundary.[3] Let $\sigma \in C^\infty(\overline{\Omega})$ be a positive function, and consider the Dirichlet problem

$$\begin{cases} \operatorname{div}(\sigma \operatorname{grad} u) = 0 \\ u\big|_{\partial\Omega} = f. \end{cases} \tag{4.15}$$

Under sufficient regularity on f, associated to the unique solution of (4.15) there exists a function

$$\sigma \frac{\partial u}{\partial n}$$

for n the normal at a given point in the boundary. As we saw in Section 4.1.1 its interpretation is the current through that point in the direction n.

The map

$$\Lambda_\sigma \colon f \mapsto \sigma \frac{\partial u}{\partial n}\bigg|_{\partial\Omega}$$

is called the Dirichlet to Neumann map, or the voltage to current map in the impedance tomography literature.

Since both the voltage and the current on the boundary are easily (at least theoretically) measured in a non invasive form the following questions arises naturally:

1. Does Λ_σ determine σ uniquely?
2. If so, how do we recover in a "practical" way σ from Λ_σ?

We shall see that although a lot of progress has been made in this last several years on both topics, they still have a lot of room for enthralling investigation. We start our discussion by remarking that we have been deliberately vague about the domain of definition for the map Λ_σ. In order to discuss that, we consider first the natural quadratic form:

[3] In its original formulation, Calderón [8] requested the boundary to be Lipschitzian. We shall not delve into such matters here, though many of our remarks still hold in that case.

$$Q_\sigma(\phi) = \inf_{\substack{w\big|_{\partial\Omega}=\phi \\ w\in\mathcal{H}^1(\Omega)}} \int_\Omega \sigma|\nabla w|^2\,dx\,.$$

In a short while, we shall show, using a simple variational argument that,

$$Q_\sigma(\phi) = \int_\Omega \sigma|\nabla u|^2\,dx \tag{4.16}$$

$$= \int_{\partial\Omega} \phi\sigma\,\frac{\partial u}{\partial n}\,dS \tag{4.17}$$

where u is the unique solution to the Dirichlet problem (4.15).

Because of (4.16) one can interpret $Q_\sigma(\phi)$ as the power (i.e. energy per unit of time) necessary to sustain the voltage ϕ on the boundary in a steady state situation. The quadratic form $Q_\sigma(\phi)$ is an unbounded quadratic form on $L^2(\partial\Omega)$.

It is not difficult to check that its domain is $\mathcal{H}^{1/2}(\partial\Omega)$. Because of a standard result in functional analysis (see Kato's treatise [39] page 331) there exists a unique self-adjoint operator associated to it whose domain is $\mathcal{H}^1(\partial\Omega)$. A simple computation shows that this corresponds to the operator Λ_σ defined above.

Calderón posed the following question: "decide whether σ is uniquely determined by Q_σ". As we mentioned above, the question still has many open facets despite all the efforts.

Consider the nonlinear functional

$$Q_\sigma(\phi) = \inf_{\substack{w\in\mathcal{H}^1(\Omega) \\ w\big|_{\partial\Omega}=\phi}} \int_\Omega \sigma|\nabla w|^2\,dx\,.$$

As we mentioned, this quadratic form has a couple of other characterizations involving the Dirichlet problem. Let

$$L_\sigma \stackrel{\text{def}}{=} \operatorname{div}(\sigma\,\operatorname{grad})$$

and consider the Dirichlet problem:

$$\begin{cases} L_\sigma\, u = 0 \\ u\big|_{\partial\Omega} = \phi\ . \end{cases} \tag{4.18}$$

Then,

$$Q_\sigma(\phi) = \int_\Omega \sigma|\nabla u|^2\,dx \tag{4.19}$$

and

$$Q_\sigma(\phi) = \int_{\partial\Omega} \phi\, \sigma \frac{\partial u}{\partial n}\, dS \ , \tag{4.20}$$

where u is the unique solution of (4.18).
The proof of the first claim (4.19):

Take $\psi \in \mathcal{H}_0^1(\Omega)$ and assume that there exists such a minimizing element

$$u \in \mathcal{H}^1(\Omega), \ \ u\big|_{\partial\Omega} = \phi \ .$$

Then,

$$\varepsilon \mapsto \int_\Omega \sigma |\nabla u + \varepsilon \nabla \psi|^2 \, dx$$

has a minimum at $\varepsilon = 0$, and hence

$$\int_\Omega \sigma\, \nabla u \cdot \nabla \psi \, dx = 0, \qquad \forall\, \psi \in \mathcal{H}_0^1(\Omega).$$

Now we integrate by parts (i.e. apply Green's formula) to get

$$\int_\Omega \nabla(\sigma\, \nabla\, u) \psi \, dx = 0$$

and hence, at least in the weak sense $L_\sigma\, u = 0$. So by elliptic regularity we get

$$\begin{cases} L_\sigma\, u = 0 \\ u\big|_{\partial\Omega} = \phi \ . \end{cases}$$

Now, to get (4.20) we use that

$$-\int_\Omega \operatorname{div}(\sigma \nabla u) u + \int_\Omega \operatorname{div}(\sigma u \nabla u) = \int_\Omega \sigma |\nabla u|^2.$$

Hence,

$$\begin{aligned} \int_\Omega \sigma |\nabla u|^2 \, dx &= \int_\Omega \operatorname{div}(\sigma u \nabla u)\, dx \\ &= \int_{\partial\Omega} \phi\, \sigma \frac{\partial u}{\partial n}\, dS. \end{aligned}$$

We shall now describe the first elements of the functional analytic tools introduced by Calderón to deal with the following basic question: In the neighborhood of a constant σ, say $\sigma = 1$ is the nonlinear map

$$\Phi\colon \sigma \mapsto Q_\sigma$$

locally injective, or at least: is $d\Phi$ injective?

In order to determine whether $d\Phi$ is injective, Calderón devised a very ingenious technique, which we shall describe shortly.

We shall denote by $\mathcal{Q}\left(\mathcal{H}^{1/2}(\partial\Omega)\right)$ the space of quadratic forms on $\mathcal{H}^{1/2}(\partial\Omega)$. It is endowed with the topology

$$||q||_{\mathcal{Q}} = \sup_{|||\phi|||\leq 1} |q(\phi)|$$

where

$$|||\phi|||^2 = \int_\Omega |\nabla u|^2\, dx, \quad u\big|_{\partial\Omega} = \phi, \quad \Delta u = 0 \text{ in } \Omega.$$

First we prove that the map Φ is analytic in a neighborhood of $\sigma = \sigma_0 > 0$. Without loss of generality, we shall assume henceforth that $\sigma_0 = 1$. This can be done because we can always change scales.

Proposition: (Calderón) *The map $\sigma \mapsto Q_\sigma$ is analytic in a neighborhood of $\sigma = 1$ as a map from $L^\infty(\overline{\Omega})$ to $\mathcal{Q}\left(\mathcal{H}^{1/2}(\partial\Omega)\right)$.*

Let's take $\sigma = 1 + \eta$, so that η is a small perturbation of the constant configuration $\sigma = 1$.

Before proving this result, we need the following:

Lemma: *Let $u = \mathrm{PI}\,\phi$ and set w the solution of*

$$\begin{cases} L_\sigma\, w = 0 \\ w\big|_{\partial\Omega} = \phi \end{cases} \tag{4.21}$$

take $w = u + v$, as the definition of v. Then, the map $\phi \mapsto v$ is given by

$$v = -\sum_{k=0}^{\infty} (-1)^k\, (T L_\eta)^k \cdot T L_\eta \cdot \mathrm{PI}\,\phi$$

where, $T\colon \mathcal{H}^{-1}(\Omega) \to \mathcal{H}^1_0(\Omega)$ is the inverse to $\Delta\colon \mathcal{H}^1_0(\Omega) \to \mathcal{H}^{-1}(\Omega)$, and $||\eta||_\infty\, ||T||_{\mathcal{H}^{-1}\to\mathcal{H}^1_0} < 1$.
Furthermore, $v \in \mathcal{H}^1_0(\Omega)$.

The proof of this lemma follows precisely the steps given in Calderón's paper: Rewrite the equation for v as

$$v + T\, L_\eta\, v = -T\, L_\eta\, u.$$

From the assumption $||\eta||_\infty\, ||T||_{\mathcal{H}^{-1},\mathcal{H}^1_0} < 1$ the Neumann series

$$-\left(\sum_{k=0}^{\infty} (-1)^k (T L_\eta)^k\right)$$

converges in the topology of $\mathcal{L}(\mathcal{H}^1(\Omega);\mathcal{H}^1(\Omega))$ and each term

$$L^\infty(\Omega) \ni \eta \mapsto (TL_\eta)^k \in \mathcal{L}(\mathcal{H}^1,\mathcal{H}^1)$$

is analytic in η.

Furthermore,

$$v = \Big(-\sum_{k=0}^{\infty} (-1)^k (TL_\eta)^k \Big) TL_\eta\, u$$

is such that

$$||v||_{\mathcal{H}^1(\Omega)} \leq \frac{||TL_\eta\, u||}{1-||T||\,||\eta||_\infty}$$
$$\leq \frac{||T||\,||\eta||_\infty ||\mathrm{PI}\phi||}{1-||T||\,||\eta||_\infty} \leq C||\phi||_{\mathcal{H}^{1/2}(\partial\Omega)} \ ,$$

where C is a constant. This concludes the proof of the Lemma.

Now let's look at the proof of the proposition. For w solution of (4.21) we have that

$$Q_\sigma(\phi) = \int_\Omega \sigma|\nabla w|^2\, dx \ =$$
$$= \int_\Omega \sigma|\nabla u|^2\, dx + 2\int_\Omega \sigma\, \nabla u\cdot\nabla v\, dx + \int_\Omega \sigma|\nabla v|^2\, dx.$$

Hence,

$$Q_\sigma(\phi) = Q_1(\phi) + \int_\Omega \eta|\nabla u|^2\, dx + \int_\Omega 2\nabla u\cdot\nabla v\, dx \ + \tag{4.22}$$
$$+ \int_\Omega 2\eta\, \nabla u\cdot\nabla v\, dx + \int_\Omega (1+\eta)|\nabla v|^2\, dx. \tag{4.23}$$

The claim now follows from the fact that the map

$$L^\infty(\Omega) \ni \eta \mapsto v \in \mathcal{H}^1(\Omega)$$

is analytic, and hence so is

$$\eta \mapsto \nabla v \in \mathcal{H}^0(\Omega).$$

Furthermore, if we set v_N as

$$v_N = \sum_0^N (-1)^k (TL_\eta)^k\, TL_\eta\, \mathrm{PI}(\phi),$$

using that $u = \mathrm{PI}(\phi)$ we have that

$$Q_1(\phi) + \int_\Omega \eta |\nabla u|^2 \, dx + \int_\Omega 2\eta \, \nabla u \cdot \nabla v_N \, dx + \int_\Omega (1+\eta) |\nabla v_N|^2 \, dx$$

converges in the norm of $\mathcal{Q}\left(\mathcal{H}^{1/2}(\partial\Omega)\right)$ to Q_σ. For that we have to use that the map $\phi \mapsto u = \mathrm{PI}\phi$ is bounded from $\mathcal{H}^{1/2}(\partial\Omega)$ to $\mathcal{H}^1(\Omega)$.

We close with some remarks about the physical significance of the different expression of the form Q_σ.

The minimization form $Q_\sigma(\phi) = \int_\Omega \sigma |\nabla u|^2 \, dx, \quad u\big|_{\partial\Omega} = \phi$ gives a variational characterization. It is telling us that $Q_\sigma[\phi]$ corresponds to the minimal dissipation of power[4] associated to the boundary value of the potential $u\big|_{\partial\Omega} = \phi$. In order for this infimum to be attained, the potential u must satisfy the stationary condition

$$\begin{cases} \operatorname{div}(\sigma \operatorname{grad} u) = 0 \\ u\big|_{\partial\Omega} = \phi \end{cases}$$

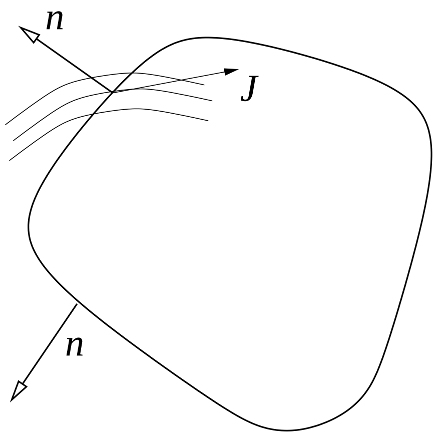

Fig. 4.4. The outwards normal and the flow of current

The form

$$\int_{\partial\Omega} \phi \, \sigma \, \frac{\partial u}{\partial n} \, dS$$

is telling us that in order to sustain the power dissipation

$$\int_\Omega \sigma |\nabla u|^2 \, dx$$

one has to provide a flow of power on the boundary which is given exactly by the density

[4] Recall $1/2 \; \sigma |E|^2 = 1/2 \; \sigma |\nabla u|^2$ is energy/time per unit of volume.

$$\sigma \frac{\partial u}{\partial n} \phi \, dS = \{\text{product of current crossing } dS\}$$
$$\times$$
$$\{\text{voltage on the point where } dS \text{ is located.}\}$$

4.4.1 The Injectivity of $d\Phi$

We shall now describe Calderón's argument to prove that $d\Phi$ is injective. We follow closely the exposition of [37, 36]. The first step is to compute the linearization of Φ at the point $\sigma = 1$. To do that we expand $Q_{1+\eta}$ as in the expression

$$Q_{1+\eta}(\phi) = \int_\Omega (1+\eta)|\nabla w|^2 \, dx,$$

where

$$\begin{cases} L_{1+\eta}\, w = 0 \\ w\big|_{\partial\Omega} = \phi \ . \end{cases}$$

Thus,

$$\begin{aligned} Q_{1+\eta}(\phi) &= \int_\Omega (1+\eta)|\nabla w|^2 \, dx \ = \\ &= \int_\Omega (1+\eta)(|\nabla u|^2 + 2\nabla u \cdot \nabla v + |\nabla v|^2) dx. \end{aligned}$$

But the expansion of v gives

$$v = -\left(\sum_{k=0}^{\infty} (-1)^k (TL_{1+\eta})^k \right) (TL_{1+\eta}\, u).$$

Note that u is independent of $\sigma = 1 + \eta$.
Hence, since $\Delta u = 0$ we get, after retaining[5] only the first order terms in η

$$Q_{1+\eta}(\phi) = Q_1(\phi) + \int_\Omega \eta |\nabla u|^2 + \mathcal{O}(\eta^2)$$

where $\mathcal{O}(\eta^2)$ is a term that goes to zero faster than $||\eta||_\infty^2$ uniformly for

$$|||\phi||| \le 1 \ .$$

In other words,

[5] Note that,

$$\int_\Omega 2\,\nabla u \cdot \nabla v \, dx = 2 \int_\Omega \operatorname{div}(v\,\nabla u) dx = 2 \int_{\partial\Omega} v\, \nabla u \cdot n \, dS = 0$$

$$d\Phi\big|_{\sigma=1} \cdot \eta = \int_\Omega \eta |\nabla u|^2 \, dx, \tag{4.24}$$

where u is the unique solution of

$$\begin{cases} \Delta u = 0 \\ u\big|_{\partial\Omega} = \phi \ . \end{cases}$$

Lemma: (Calderón). *The map* $d\Phi\big|_{\sigma=1}$ *is* $1-1$.

Proof. Let ϕ be an arbitrary element of $\mathcal{H}_0^{1/2}(\partial\Omega)$ and u the unique solution of

$$\begin{cases} \Delta u = 0 \\ u\big|_{\partial\Omega} = \phi \end{cases} \tag{4.25}$$

Because of (4.24) it is enough to show that if

$$\int_\Omega \eta |\nabla u|^2 \, dx = 0,$$

with $u = \mathrm{PI}\,\phi$ and arbitrary $\phi \in \mathcal{H}_0^{1/2}(\partial\Omega)$, then $\eta = 0$. By polarization, we may then assume that $d\Phi\big|_{\sigma=1} \eta = 0$ implies

$$\int_\Omega \eta \, \nabla u_1 \cdot \overline{\nabla u_2} \, dx = 0$$

for any arbitrary pair $u_i = \mathrm{PI}(\phi_i), \ \ i = 1, 2$. Calderón's ingenious idea, which was then extended in a number of other ensuing works is to construct a sufficiently large set of functions ϕ_i so as to guarantee that $\eta = 0$. We follow Calderón's original work, with minor modifications. Let

$$u_1(x) = \exp(i\xi \cdot x - \xi^\perp \cdot x) \tag{4.26}$$

and

$$u_2(x) = \exp(-i\xi \cdot x + \xi^\perp \cdot x) \tag{4.27}$$

where ξ and $\xi^\perp$ are vectors in $\mathbb{R}^n$ with the property that $\xi \cdot \xi^\perp = 0$ and of the same norm. Hence

$$\Delta u_j = 0, \quad j = 1, 2.$$

Indeed,

$$\sum \partial_k^2 \, u_{1,2} = \sum_k \Big(-\xi_k^2 + \xi_k^{\perp 2} \Big) e^{(\cdots)} = 0.$$

On the other hand,

$$\nabla u_1 \cdot \overline{\nabla u_2} = \sum_k (i\xi_k - \xi_k^\perp) \cdot (+i\xi_k + \xi_k^\perp) e^{2i\xi\cdot x} = -2|\xi|^2 e^{2i\xi\cdot x} \ .$$

and

$$\int_\Omega \eta \, \nabla u_1 \cdot \overline{\nabla u_2} \, dx = \int_\Omega -2\eta |\xi|^2 e^{2i\xi\cdot x} \, dx.$$

Hence,

$$\int_\Omega \eta(x) e^{2i\xi\cdot x} \, dx = 0, \quad \forall \, \xi \neq 0. \tag{4.28}$$

Equation (4.28) is telling us that the Fourier transform of η is 0 (except possibly at $\xi = 0$). But since $\eta \in L^\infty(\Omega)$ it follows that $\eta = 0$. *Q. E. D.*

4.4.2 Calderón's Method of Approximating the Conductivity

The first technique for approximating the conductivity σ from voltage-current measurements on the boundary was proposed by Calderón. We shall now present a small variant of the method, which was suggested in [37]. In fact, it deals with the Neumann-Dirichlet map, and goes as follows: Let's assume that the perturbation η is zero on a neighborhood of $\partial\Omega$, so that $\sigma \equiv 1$ in this neighborhood.

Let's now assume we pose the Neumann problem

$$\begin{cases} \nabla(\sigma \, \nabla \, u) = 0 \\ \sigma \, \dfrac{\partial u}{\partial n} = j \text{ on } \partial\Omega \ . \end{cases} \tag{4.29}$$

For this problem to admit a solution one must impose the compatibility

$$\int_{\partial\Omega} j(s) \, dS = 0.$$

Furthermore, in order to ensure that the solution is unique, one must normalize u by an additive constant. We choose it so that,

$$\int_{\partial\Omega} u(s) \, dS = 0.$$

Let's now assume that we have "full access to all possible boundary measurements."

Then, we consider a solution w_1 of (4.29) with $j_1 = n \cdot \nabla u_1$ with u_1 and u_2, as in (4.26) and (4.27), given by

$$u_1(x) = \exp(i\xi\cdot x - \xi^\perp\cdot x)$$
$$u_2(x) = \exp(-i\xi\cdot x + \xi^\perp\cdot x).$$

Then,

$$\begin{cases} \nabla(\sigma\nabla w_1) = 0 \\ \sigma\,\dfrac{\partial w_1}{\partial n} = j_1 = n\cdot\nabla u_1 = (i\xi - \xi^\perp)\cdot n\, e^{(i\xi\cdot x - \xi^\perp\cdot x)} \end{cases}$$

and

$$\begin{cases} \Delta u_2 = 0 \\ \dfrac{\partial u_2}{\partial n} = j_2 \end{cases}$$

We now use the divergence theorem[6] to get

$$\begin{aligned} 0 &= \int_\Omega (u_2\,\nabla(\sigma\,\nabla\bar w_1) - \bar w_1\,\nabla(\nabla u_2))dx \\ &= \int_\Omega \{\nabla(u_2\,\sigma\nabla\bar w_1 - \bar w_1\nabla u_2) - \eta\,\nabla u_2\cdot\nabla\bar w_1\}\,dx \\ &= \int_{\partial\Omega}\sigma\left(u_2\,\frac{\partial\bar w_1}{\partial n} - \bar w_1\,\frac{\partial u_2}{\partial n}\right)dS - \int_\Omega \eta\,\nabla u_2\cdot\nabla\bar w_1\,dx\,. \end{aligned}$$

Hence, using that $\sigma = 1$ on $\partial\Omega$ we get

$$C(\xi;\eta) \stackrel{\text{def}}{=} \int_{\partial\Omega}\left(u_2\,\frac{\partial\bar w_1}{\partial n} - \bar w_1\,\frac{\partial u_2}{\partial n}\right)dS = \int_\Omega \eta\,\nabla u_2\cdot\nabla\bar w_1\,dx\;. \tag{4.30}$$

Now, from one side the method of Section 2 gives that for $||\eta||_\infty \ll 1$

$$w_1 = u_1 + \mathcal{O}(\eta), \quad ||\eta||_\infty \to 0. \tag{4.31}$$

On the other hand, the quantity $C(\xi;\eta)$ of equation (4.30) is completely determined by external measurements. Plugging (4.31) into (4.30) we get

$$-2|\xi|^2\int_\Omega \eta(x)e^{-2i\xi\cdot x}\,dx \cong C(\xi;\eta)$$

i.e.

$$\widehat{\eta}(2\xi) \simeq -\frac{C(\xi;\eta)}{2|\xi|^2}\cdot$$

[6] Note that

$$u_2\,\nabla(\sigma\,\nabla\,w_1) = \nabla(u_2\sigma\,\nabla w_1) - \sigma\,\nabla u_2\cdot\nabla w_1\;,$$
$$w_1\,\nabla(\nabla\,u_2) = \nabla(w_1\,\nabla\,u_2) - \nabla w_1\cdot\nabla u_2\;.$$

As remarked by Calderón, this approximation works only for $||\eta||_\infty$ small. He in fact, determined conditions so that the error would be much smaller than $||\eta||_\infty$.

Isaacson and Isaacson [37] remarked that even in situations where $||\eta||$ is *not* small $C(\xi, \eta)$ could give relevant information. They did this by showing that a centered circular inhomogeneity could be imaged and precise knowledge about its radius obtained "even if its conductivity differed greatly from the background."

4.4.3 The Kohn-Vogelius Result

We have shown in the previous paragraph that the derivative of the Dirichlet-Neumann map is injective in a neighborhood of the constant conductivity configuration $\sigma = 1$.

We now turn our attention again to Calderón's question of whether the map $\Phi\colon \sigma \mapsto Q_\sigma$ is injective. One of the first results dealing with this question is due to Kohn and Vogelius, where the question is dealt with in a very elegant form [42, 41]. They have shown in [42] that if [7] $\sigma \in C^\infty(\overline{\Omega})$, and $\partial\Omega$ is C^∞, then Q_σ determines σ and all its derivatives at the boundary. This result, while providing an interesting answer to Calderón's question does *not* provide *the* answer. Even if we know "a priori" that a function is analytic, knowledge of its Taylor expansion at a given point is *not* a stable and effective way of reconstructing it. We remark that their technique could possibly be explored in devising more effective ways of obtaining information about σ. This is one interesting avenue, which to our knowledge has not been considered much.

The Kohn-Vogelius result can be stated as follows:

Theorem: (Kohn-Vogelius) *Let $0 < \varepsilon_0 \leq \sigma_i \in C^\infty(\overline{\Omega})$, $i = 1, 2$, and let $x^* \in \partial\Omega$, the boundary of Ω, which is also assumed C^∞. Suppose that there exists a neighborhood B of x^* in $\overline{\Omega}$ s.t. $\forall\, \phi \in \mathcal{H}^{1/2}(\partial\Omega)$ with* $\operatorname{supp}\phi \subset B$ *we have that*

$$Q_{\sigma_1}(\phi) = Q_{\sigma_2}(\phi)\ .$$

Then,

$$\partial^\alpha \sigma_1(x^*) = \partial^\alpha \sigma_2(x^*),$$

$\forall\, \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$, *$\alpha$ multi-index.*

[7] Actually, they only need to assume that σ is C^∞ in a neighborhood of the boundary. They further extended such results to piecewise C^∞ conductivities. See [41].

4.4.4 The Results of Nachman, Sylvester, and Uhlman

Further progress has been made in 1986 by Sylvester-Uhlman [64] who showed that

Theorem: (Sylvester-Uhlman). *Suppose that $n \geq 3$, and Ω is a domain with smooth boundary $\partial\Omega$. If $0 < \sigma_1$, $\sigma_2 \in C^\infty(\overline{\Omega})$, and*

$$Q_{\sigma_1}(\phi) = Q_{\sigma_2}(\phi), \qquad \forall\, \phi \in \mathcal{H}^{1/2}(\partial\Omega)$$

then $\sigma_1 = \sigma_2$.

Hence, not only the derivatives of σ at a given point on the boundary are determined by Q_σ but also σ itself. This result was further improved in the work of Nachman-Sylvester-Uhlmann by showing uniqueness for $\sigma \in C^{1,1}(\overline{\Omega})$. Here $C^{1,1}(\overline{\Omega})$ means C^1 with Lipschitz continuous derivatives.

Later on [47] Nachman showed the identifiability result for $\sigma \in C^{1,1}(\overline{\Omega})$ with $\partial\Omega$ a $C^{1,1}$ boundary. This was obtained as a by product of the techniques involving the reduced acoustic equation for time-harmonic waves with frequency ω:

$$\nabla \cdot \left(\frac{1}{\rho(x)}\, \nabla p(x) \right) + \omega^2\, k(x) p(x) = 0, \quad x \in \Omega,$$

where $0 < k_0 \leq k \in L^\infty(\Omega)$ and $\rho \geq \rho_0 > 0$, $\rho \in C^{1,1}(\overline{\Omega})$. Further details about this can be found in [9]. For the situation in dimension $n = 2$ see [48].

5. Diffuse Tomography

5.1 From X-Ray to Infra-Red Imaging

The use of infra-red (IR) radiation for non-invasive imaging is a potentially revolutionary development in several areas of medicine.

A few advantages of IR imaging are:

1. The contrast factor between tumor and tissue in the IR range is much larger than in the X-ray range.
2. In the IR range no ionizing radiation is used, thus eliminating a number of possible side effects.
3. The low energy of the radiation as compared with X-rays.

The ideas presented in this section had their origins with the effort of modeling in a simple and useful way the propagation of infra-red radiation in an unknown medium for the purpose of medical imaging [62]. In the past few years a number of groups have been pursuing this kind of approach [10, 3, 4, 63, 2, 55, 45] and a preliminary instrument was reported [54].

Recall from Chapter 3 that the propagation of radiation inside a medium is governed by Boltzmann's transport equation

$$\omega \cdot \nabla \Psi + \mu \Psi = q + \mu_s \int_{\mathbb{S}^2} \sigma(\omega, \omega') \Psi(\cdot, \omega') d\omega' \ . \tag{5.1}$$

In the case of traditional X-ray tomography, the radiation travels along straight lines, and one may neglect the scattering part. The imaging problem then reduces to the reconstruction of the function μ from its Radon transform. This was the subject of Chapter 3.

In retrospective we remark that in the case of X-ray the probing path of the particles is a straight line connecting the source to the detector. On the other hand, in order to deal with infra-red radiation in tissues one should consider a model that would take into consideration the contribution of all paths; or at least the most significant ones. This amounts to the need of studying the inverse problem for the Boltzmann transport equation including the previously neglected scattering part. This problem is substantially harder and although some important preliminary results were reported [50] there is still room for substantial practical improvement.

The next ingredient in our discussion is to introduce a discrete version of the transport equation, which is akin to the discretization performed in the ART algorithms described in Chapter 3.

5.2 The Diffuse Tomography Model

Our viewpoint, which will be detailed in Section 5.2.2 is based on first principles. It starts from the discretization of the medium into sufficiently small pixels or voxels [1]. Particles entering a voxel will either be absorbed by it or will move on to an adjacent one according to certain probabilities. Due to its simplicity, the models, as well as their associated imaging algorithms could be employed in a variety of physical situations besides photon propagation.

The long term strategy could be described in the following steps:

1. Start from a discrete version of the direct problem.
2. Study carefully the numerics of the associated small scale problems.
3. Understand the behavior of the inverse problem.
4. Devise strategies to handle more realistic problems.

We shall now describe the implementation of such strategy on a fairly simple model, which however displays very common features of many other inverses problems.

The plan of this section is the following:

In Section 5.2.1 we describe the direct model and mention a few of its variants, which would make the model more realistic. Section 5.2.2 is concerned with the description of the inverse problem. Section 5.3 deals with two classes of reconstruction algorithms. The first one is based on a minimization approach. The second one, which is suitable only for the the so-called isotropic case, yields the solution after a finite number of steps. Section 5.3.2 is concerned with relations between the discrete algorithms presented so far and the impedance tomography model. In Section 5.4 we present some numerical results, and discuss some implementation aspects of the algorithms. We close the chapter with some questions for further research.

5.2.1 The Discrete Model

We shall be concerned with two models for the movement of particles in an unknown medium. These models are special cases of more general ones, which

[1] Volume elements.

were discussed in [62, 29, 31, 30, 28]. The difference between these two models is essentially the number of directions that a particle may travel when moving inside the medium. As we shall explain below, both models will fall under the category of "isotropic" according to the terminology of [29]. The description given in the sequel follows closely that of [30]. For other aspects of the model and its generalizations see [23, 27, 26].

Before we go into the description of the models, let us remark that more realistic models should actually be in three dimensions instead of two dimensions. However, the process of going from two to three dimensions in these models is straightforward from the theoretical and conceptual point of view. From the practical viewpoint, the complexity of the computations increases substantially from two to three dimensions.

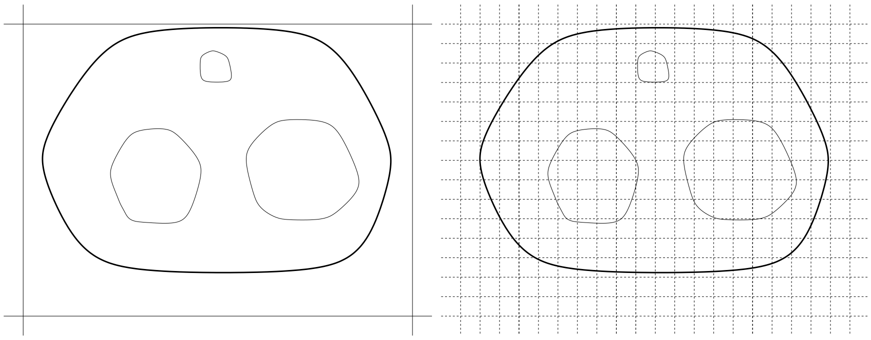

Fig. 5.1. The object and the superimposed grid

We start off by covering the object with a two-dimensional grid. This grid will be assumed to have a total of N^2 pixels, where N is the number of pixels on each side. By changing N one varies the resolution of the reconstruction, adjusting it to the desired one.

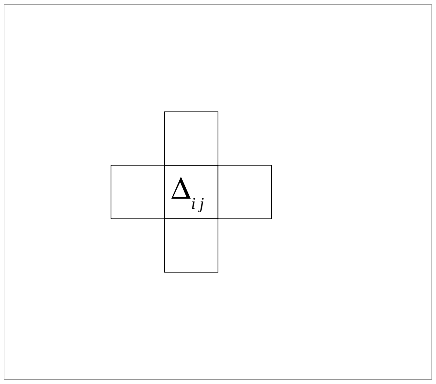

Fig. 5.2. The grid and its pixels

Around the grid we place an array of detectors, each capable of measuring the intensity of radiation leaving the grid from the pixel to which it is adjacent. Let us label the pixels by Δ_{ij} where $(i,j) \in D \stackrel{\text{def}}{=} \{(i,j)|1 \leq i,j \leq N\}$. By the boundary of the grid we understand the set ∂D defined by

$$\partial D = \{(i,j)|i = 1, \text{or } i = N, \text{or } j = 1, \text{or } j = N\} \ .$$

The pixels on ∂D are the so-called *external ones*, whereas the ones in $D - \partial D$ are called *internal*. Our measurements therefore will come from values obtained from external pixels. The photons are injected in the region to be imaged by means of sources that are also placed on the boundary ∂D. Once a photon is injected it undergoes a two-dimensional random motion with killing according to the following rules:

1. The photon is absorbed by pixel Δ_{ij} with probability $1 - w_{ij}$, where w_{ij} is the probability that the photon survives its visit to Δ_{ij}. We call w_{ij} the survival coefficient of Δ_{ij}.
2. Given that the photon survives its visit to Δ_{ij}, it is scattered to one of the adjacent sites, $\Delta_{i\pm1,j\pm1}$.

At this point we consider two propagation models:

Model 1: The particle moves with certain probabilities to one of the *four* sites: $\Delta_{i+1,j}$, $\Delta_{i-1,j}$, $\Delta_{i,j+1}$, or $\Delta_{i,j-1}$. These probabilities depend on the direction the particle entered $\Delta_{i,j}$. This is what we call the *four-direction* model.
Model 2: The particle moves with certain probabilities to one of the *eight* sites: $\Delta_{i+1,j}$, $\Delta_{i-1,j}$, $\Delta_{i,j+1}$, $\Delta_{i,j-1}$, $\Delta_{i+1,j+1}$, $\Delta_{i+1,j-1}$, $\Delta_{i-1,j+1}$, or $\Delta_{i-1,j-1}$. These probabilities depend on the direction the particle entered $\Delta_{i,j}$. This is the *eight-direction* model.

Obviously the transition probabilities mentioned above would lead to an immense number of parameters [28]. In the initial research developed on the subject these transition probabilities were assumed to depend only on the angle between the incoming direction and the outgoing direction [62]. The general case was studied in [24, 51, 52].

The model is called *isotropic* if all the transition probabilities from one direction into another are equal. Thus, in the two-dimensional four-direction model these probabilities are equal to 1/4. In the two dimensional eight-direction model they are 1/8.

We may recover the four-direction isotropic model from the general eight-direction one by making the probability of turning an angle of 0^o,90^o, 180^o, or 270^o equal to 1/4, and the other ones 0.

The four-direction and isotropic model was considered in [29, 31], whereas the eight-direction one was studied in [30].

We may summarize the main steps so far, in a more general situation as follows:

1. Divide the object into elements (pixels or voxels).
2. Assume that the radiation is composed of a large number of particles that enter and exit the object at different points. We control the entering point by placing a source at that point.
3. The movement is performed by going from one pixel to an adjacent one.
4. The particle may be annihilated by absorption within each pixel.
5. Once the particle enters a pixel and is not absorbed it has a probability

$$\left\{\begin{array}{c} f \\ b \\ s \end{array}\right\} \text{of going} \left\{\begin{array}{c} \text{forward} \\ \text{backwards} \\ \text{sideways} \end{array}\right\}$$

Note that, in the two-dimensional situation we have

$$b + 2s + f = 1, \tag{5.2}$$

whereas in the three-dimensional we have $b + 4s + f = 1$. So, each pixel is characterized by a triple (w, f, s), where w is the survival probability, f is the forward scattering probability, and s is the sideways scattering probability. In the two dimensional case, we shall say that the configuration (w, f, s, b) belongs to the physical region if (5.2) is satisfied with $f, s, b \geq 0$, and $w \in [0, 1]$. Mutatis mutandis for the three dimensional case.

5.2.2 The Direct Problem × Inverse Problem

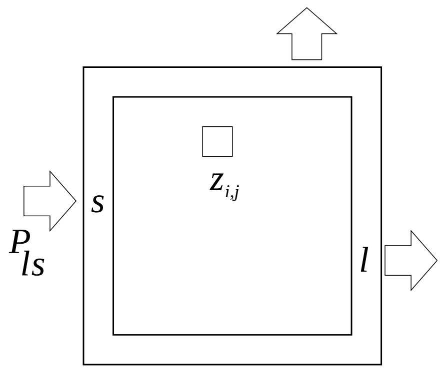

Fig. 5.3. The direct problem

The term "direct problem" refers to the following scenario:

Suppose we know the values of w_{ij}, f_{ij}, s_{ij} for $(i,j) \in D$. If we beam a known amount of radiation at source s find the relative intensity of radiation P_{ls} measured by detector l. See Figure 5.3.

We assume that $0 \leq w_{ij} < 1$, which is reasonable from physical considerations. We will now show that the direct problem can be solved in terms of a system of linear equations. The solution of the direct problem consists basically in the following steps:

1. Introducing internal (or "hidden") variables, defined by the probability that a particle will reach the detector l given it entered a certain internal pixel from a specified direction. See Figure 5.4.
2. Setting up a system of linear equations for these internal variables.
3. Solving this system of equations, which as we mentioned above has a unique solution.

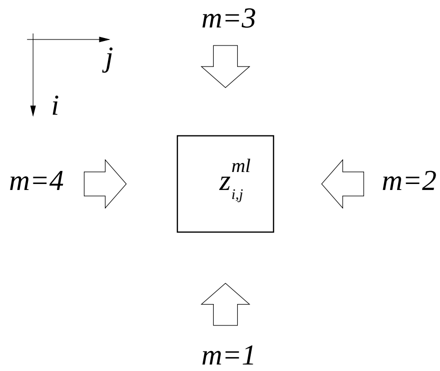

Fig. 5.4. The introduction of the auxiliary variables z_{ij}^{ml}

More precisely we proceed as follows: Given a distribution of parameters

$$(w_{ij}, f_{ij}, s_{ij})_{1 \leq i,j \leq N} \in (0,1)^{3N^2}$$

to find the probability of a particle injected at source s to be captured by detector l we construct a system of linear equations. See Figure 5.4. This system is constructed by writing the balance equation for each site $\Delta_{i,j}$. In the case of the four-direction model we have for an internal pixel

$$\begin{bmatrix} z_{ij}^{1,l} \\ z_{ij}^{2,l} \\ z_{ij}^{3,l} \\ z_{ij}^{4,l} \end{bmatrix} = w_{ij} \begin{bmatrix} f_{ij} & s_{ij} & b_{ij} & s_{ij} \\ s_{ij} & f_{ij} & s_{ij} & b_{ij} \\ b_{ij} & s_{ij} & f_{ij} & s_{ij} \\ s_{ij} & b_{ij} & s_{ij} & f_{ij} \end{bmatrix} \begin{bmatrix} z_{i-1,j}^{1,l} \\ z_{i,j-1}^{2,l} \\ z_{i+1,j}^{3,l} \\ z_{i,j+1}^{4,l} \end{bmatrix}, \tag{5.3}$$

where z_{ij}^{ml} denotes the probability of being captured by detector l given the particle visited site (i,j) entering from the direction m. For boundary pixels one has similar equations, but now one has to take into account the presence of detectors. In the boundary points, it is convenient to extend the definition of the variables z_{ij}^{ml} for $i, j \in \{0, N+1\}$ so as to take into account the presence of a detector. In the case of a detector, it would take the value 1.

At the end we get a huge but sparse system, whose coefficients depend on the distribution of parameters (w, f, s). It takes the form

$$A\mathbf{z}^l = \mathbf{v}^l \ , \tag{5.4}$$

where

$$\mathbf{z}^l = \text{ vector of } z_{ij}^{ml} \text{ with } 1 \leq i, j \leq N \text{ and } m = 1, \cdots, 4.$$

and l depends on the detector position. The RHS of this equation is constructed by setting certain values of the z_{ij}^{ml} to be 1 or 0 according to the presence of a detector close to a boundary pixel. Hence, it would depend on l.

It is now natural to ask whether the direct problem is well posed. The answer is provided by the following:

Proposition 5.2.1. *If $(w_{i,j}, f_{i,j}, s_{i,j}, b_{i,j})$ belongs to the physical region and $w_{i,j} < 1$ for all $1 \leq i, j \leq N$, then A is invertible and the solution of (5.4) depends analytically on each of the variables $(w_{i,j}, f_{i,j}, s_{i,j}, b_{i,j})$*

The proof is straightforward since under the assumptions, the matrix A is strictly diagonally-dominant, with ones along the main diagonal. The analytic dependence follows from the algebraic dependence of A on the variables $(w_{i,j}, f_{i,j}, s_{i,j}, b_{i,j})$ and its invertibility.

The statement and proof of a similar result for the eight-direction model of the previous section is totally analogous.

Furthermore, the system is very sparse and can be solved fairly fast, say by iterative methods [32].

Obviously, from the point of view of medical imaging the interesting problem is not the direct one but rather the inverse one. It goes as follows:

Given a set of intensity measurements P_{ls} obtained by using an array of detectors and varying the source position, determine the coefficients w_{ij} for $(i, j) \in D$. This corresponds to solving a fairly large number of *nonlinear* equations, many of which contain redundant information. For example, in the four-direction model we have

Proposition 5.2.2. *Given a distribution $(w_{i,j}, f_{i,j}, s_{i,j}, b_{i,j})_{1 \leq i,j \leq N}$ in the physical region, with $w_{i,j} < 1$, the measurement obtained by placing a detector at position l and a source at position s equals the one obtained by reversing their locations. In other words, we have*

$$P_{ls} = P_{sl} \ .$$

We refer the reader to [22] for a proof.

Because of the complexity of the equations, the hope for obtaining closed exact solutions is very small. One is therefore led to treating this problem numerically. Let's quickly discuss the complexity of the different problems we presented above.

In the 2D situation we have a total of N^2 pixels, and in principle $3N^2$ parameters to solve for. Once a detector is fixed, we have $4N^2$ internal and boundary probabilities z_{ij}^{ml}. The measurements would be from at most $16N^2$ pairs of source-detectors. However, from those only at most $8N^2 + 2N$ would be independent.

Remark 5.2.1. The model we are discussing here is a *highly* nonlinear version of traditional tomography. In the case of a discrete version of X-ray tomography we have that $\log(-P_{ls})$ would be a sum over the path joining l to s. In the model under consideration however, due to the contribution from all paths we have a highly nonlinear version of tomography.

Remark 5.2.2. For a three dimensional situation we have way more data. Thus, the complexity is substantially higher. Furthermore, in this case the amount of data for the solution of the inverse problem grows with the N^4, whereas the amount of parameters for which we are trying to solve grows with N^3.

5.3 Reconstruction Algorithms

We shall now describe two approaches that were used to attack the inverse problems discussed in the previous paragraph.

5.3.1 Minimization Approach

The first approach employed to deal with the inverse problem described in Section 5.2.2 was based on a minimization technique [62, 22, 30]. In general terms it goes as follows:

1. Start from an initial guess
$$(w_{ij}^0, f_{ij}^0, s_{ij}^0)_{1\le i,j\le N}$$
2. Using the direct problem solution we obtain a sequence of iterates for $n = 1, 2, \dots$:
$$(w_{ij}^n, f_{ij}^n, s_{ij}^n)_{1\le i,j\le N}$$
by means of a gradient type technique. This gradient can be computed by a difference scheme or by a direct method. The latter was described in [22] and turns out to be more efficient.

3. Stop when the error

$$E = \sum_{l=1}^{D} \sum_{s=1}^{S} (P_{ls} - C_{ls}^n)^2$$

falls below a certain threshold. Here C_{ls}^n denotes the computed solution for the configuration $(w_{ij}^n, f_{ij}^n, s_{ij}^n)$.

In practical reconstructions Levenberg-Marquardt minimization algorithm was used. See [46] and references therein for a description of the Levenberg-Marquardt method. The minimization technique worked pretty well only for small sizes of N.

The main drawback of the minimization technique is that as N increases the method becomes prohibitively slow and also local minima start to cause trouble. However, in many applications one might not want to solve for all parameters, but rather for a reduced set of parameters, and such methods could become competitive.

5.3.2 The Isotropic Case & the Dirichlet to Neumann map

As mentioned in the previous section, perhaps we are trying to obtain too much information from the interior. That is, we are trying to find the scattering probabilities as well as the absorption probability. Suppose we assume that some a priori knowledge on the medium is known. One idea would be to go for the *isotropic* situation, i.e,

$$f_{ij} = s_{ij} = b_{ij}$$

The first crucial remark is that in this case, one can drop the superscript m in equation (5.3). Let's denote by $z_{i,j}^d$ the probability that a particle entering the pixel Δ_{ij} will be captured by the detector d.

For interior pixels equation (5.3) becomes

$$z_{ij}^d - \frac{w_{ij}}{4}(z_{i-1,j}^d + z_{i+1,j}^d + z_{i,j-1}^d + z_{i,j+1}^d) = 0 \ . \tag{5.5}$$

The direct problem is therefore essentially that of solving a discrete version of the Laplacian plus potential. In fact, it can be shown that the inverse problem in this isotropic situation reduces to that of finding the potential from a discrete version of the Dirichlet-to-Neumann map [29]. In our opinion extending the connections between the methods in [29] and that of [11, 12] would shed light in both methods and lead to new algorithms.

We now turn to the inverse problem, and quote one of the main results of [29]. Namely, Grünbaum showed the following:

Theorem 5.3.1. *For a dense open set of values* $(w_{i,j})_{1\le i,j\le N} \in (0,1)^{N^2}$ *the inverse problem for the isotropic model in two dimensions has a unique solution that can be computed exactly in a constructive way after finitely many linear steps.*

The result above leads to a finite step algorithm, which was implemented and investigated in [31]. We shall now describe its main idea.

Let's assume that none of the survival probabilities w_{ij}, with $1 \le i,j \le N$, vanishes. The reason being that if we allow some of the values of w_{ij} to vanish we may be unable to reconstruct the interior of the object, as for example in the situation of a pixel enclosed by 4 other pixels with zero survival probability.

The strategy is to proceed sequentially by determining the values of w_{ij} along the p-th diagonal (i.e. the diagonal described by $i + j = p + 1$) and at the same time computing the values of $z^d_{i'j'}$ for (i', j') in the $(p+1)$-st diagonal (i.e. the diagonal given by $i' + j' = p + 2$). At each step we use only external measurements and values of w_{ij} and z^d_{ij} computed in the previous step. For convenience we take

$$v_{ij} \stackrel{\text{def}}{=} \frac{4}{w_{ij}}\ ,\ 1 \le i,j \le N. \tag{5.6}$$

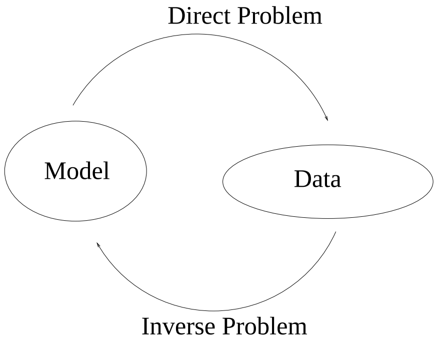

Fig. 5.5. The reconstruction method for the isotropic problem

In order to motivate the algorithms below let's start by reconstructing the values of v for the first two diagonals. Let $\mathcal{D}$ denote the set of detectors.

We start by assuming knowledge only of z^d_{ij} with $d \in \mathcal{D}$ and $(i,j) \in \partial D$. These values correspond to the measurements P_{ld}.

To find v_{11} we can use equation (5.5) with $(i,j) = (1,1)$. This gives

$$v_{11} z^d_{11} - (z^d_{01} + z^d_{21} + z^d_{10} + z^d_{12}) = 0\ . \tag{5.7}$$

But, $z^d_{21} = P_{d,(2,1)}$, $z^d_{12} = P_{d,(1,2)}$, $z^d_{01} = \delta_{(01),d}$, and $z^d_{10} = \delta_{(10),d}$. Here, as usual by $\delta_{l,s}$ we mean 1 if $l = s$ and 0 otherwise.

If we assume that all the w_{ij}'s are nonzero, it can be shown that the value of z^d_{11} is nonzero. To see this we can use the probabilistic interpretation of z^d_{ij} as the sum over all paths joining pixel (i,j) to the detector d of the products of the survival probabilities at each pixel along this path. Hence we can read off the value of v_{11} from (5.7). For practical computations in the presence of noise we can actually choose any set of measurements and perform a best fitting determination of v_{11}.

Now let's look at the values of v_{ij} on the second diagonal, assuming $N > 2$. We have

$$v_{12}z^d_{12} - (z^d_{02} + z^d_{22} + z^d_{11} + z^d_{13}) = 0 \ , \tag{5.8}$$

$$v_{21}z^d_{21} - (z^d_{20} + z^d_{22} + z^d_{11} + z^d_{31}) = 0 \ . \tag{5.9}$$

The crucial remark is that the only unknown values are v_{12}, v_{21}, and z^d_{22}.

Thus we can form a system of linear equations on these variables. If we take two distinct values, say d_1 and d_2, for the detectors, we get a system of 4 equations in the 4 unknowns v_{12}, v_{21}, $z^{d_1}_{22}$, and $z^{d_2}_{22}$. The condition for existence and uniqueness of the solutions of the system of equations thus formed is

$$\det \begin{bmatrix} -z^{d_1}_{12} & 0 & 1 & 0 \\ 0 & -z^{d_1}_{21} & 1 & 0 \\ -z^{d_2}_{12} & 0 & 0 & 1 \\ 0 & -z^{d_2}_{21} & 0 & 1 \end{bmatrix} \neq 0 \ . \tag{5.10}$$

In other words,

$$z^{d_1}_{12}z^{d_2}_{21} - z^{d_1}_{21}z^{d_2}_{12} \neq 0 \ .$$

In [29] it is shown that the set of vectors $(v_{ij})_{1\le i,j\le N}$ in $\mathbb{R}^{N^2}$ such that this condition is satisfied is a generic one. In fact, the "bad" complement is composed of lower dimensional algebraic sub-varieties.

The pattern for the arbitrary diagonal should be apparent by now. Suppose that we have found all the values of v_{ij} for $i+j < p+1$ and all the values of z^d_{ij} for $i+j \le p+1$ and $d \in \{d_1, \ldots, d_p\}$. We now set up a system of p^2 equations in p^2 unknowns. The unknowns being: $v_{i,p-i+1}$ for $1 \le i \le p$ and $z^d_{i+1,p-i+1}$ with $1 \le i \le p-1$ and $d \in \{d_1, \ldots, d_p\}$. Each of these equations is of the form

$$v_{ij}z^d_{ij} - (z^d_{i-1,j} + z^d_{i+1,j} + z^d_{i,j-1} + z^d_{i,j+1}) = 0 \ , \tag{5.11}$$

with $j = p - i + 1$. We remark that in the above equation the values of the quantities z^d_{ij}, $z^d_{i-1,j}$, and $z^d_{i,j-1}$ are already known. As in the $p = 2$ case, the solvability of this system is a generic property. See [29]. Once we solve this system of equations, we can go ahead and compute the values of z^d_{ij}, with $i + j \le p + 1$ and $d \notin \{d_1, \ldots, d_p\}$. One way of doing that is the following: **Procedure 1**

1. For each $i = 1, \ldots, p-1$, in this order:
   1.1. For each $j = 2, \ldots, p-i+1$, in this order:
   Take $z^d_{i+1,j} = v_{ij} z^d_{ij} - z^d_{i-1,j} - z^d_{i,j-1} - z^d_{i,j+1}$.

Note that this procedure works because we are using at each step values of z^d_{ij} and of v_{ij} that are known from the previous step.

Summarizing, at the p-th step, with $p \leq N$, we find the values of v_{ij} along the p-th diagonal and the values of z^d_{ij} along the $(p+1)$-st diagonal. The detectors are chosen in such a way that one has the determinant of the corresponding system nonzero. Once we have the values of v_{ij} on the p-th diagonal we can go ahead and find the values of z^d_{ij} for $i + j \leq p + 2$ for another detector. This provides us with values of z^d_{ij} for $p + 1$ detectors and for $i + j \leq p + 2$, and we can proceed inductively.

The following algorithm formalizes the discussion above:

Algorithm 1:

1. Find v_{11} from equation (5.7).
2. For each $1 < p \leq N$
   2.1. Solve the system of equations obtained by varying $1 \leq i \leq p$ and d in the set $\{(0,1), \ldots, (0,p)\}$ in equation (5.11).
   2.2. Find z^d_{ij} for $d = (0, p+1)$ and $i+j \leq p+2$ using the procedure described above.
3. Repeat steps 1 and 2 above starting from pixel (N, N) and working in the direction of the central diagonal. Alternatively, reflect all the measurements and values computed thus far on the central diagonal and repeat steps 1 and 2.

This algorithm was implemented using single precision (6-7 digits of accuracy) on a VAX 6420 and using the standard precision (14 digits of accuracy) of a CRAY XMP/14. The fact that we are using the minimal amount of information needed at each step leads to a method that is very sensitive to noise and to truncation errors.

Another algorithm, which takes into account the information from all detectors at each step, was proposed in [31].

Algorithm 2:

1. Find v_{11} from equation (5.7).
2. For each $1 < p \leq N$ set up an over-determined system of equations for the variables $(v_{i,p-i+1})_{1 \leq i \leq p}$ and
$$(z^d_{i+1,p-i+1})_{1 \leq i \leq p-1} \ , \ _{d \in \mathcal{D}} \ ,$$
   by taking $j = p - i + 1$ in equation (5.11), $1 \leq i \leq p - 1$, and $d \in \mathcal{D}$. Solve this over-determined system using a linear least-squares method, e.g. QR decomposition or SVD.

3. Repeat steps 1 and 2 after reflecting all the information on the central diagonal.

As in the case of Algorithm 1, this algorithm was also implemented on a VAX 6420 and on a CRAY XMP/14. In both practical implementations LINPACK's subroutines SQRDC and SQRSL [16] were used to perform the QR decomposition and to find the least-square solution of the linear systems.

5.4 Computational Results

The method of [29] was implemented and studied in [31] by means of two algorithms. The main conclusion being that it works very well for values of N up to 15 or so. However, as N increases the method requires the solution of systems of equations whose condition number could be very large. This makes the method sensitive to noise and called for some improvements of numerical nature. In [43], a variant of the methods considered in [29, 31] was implemented and discussed.

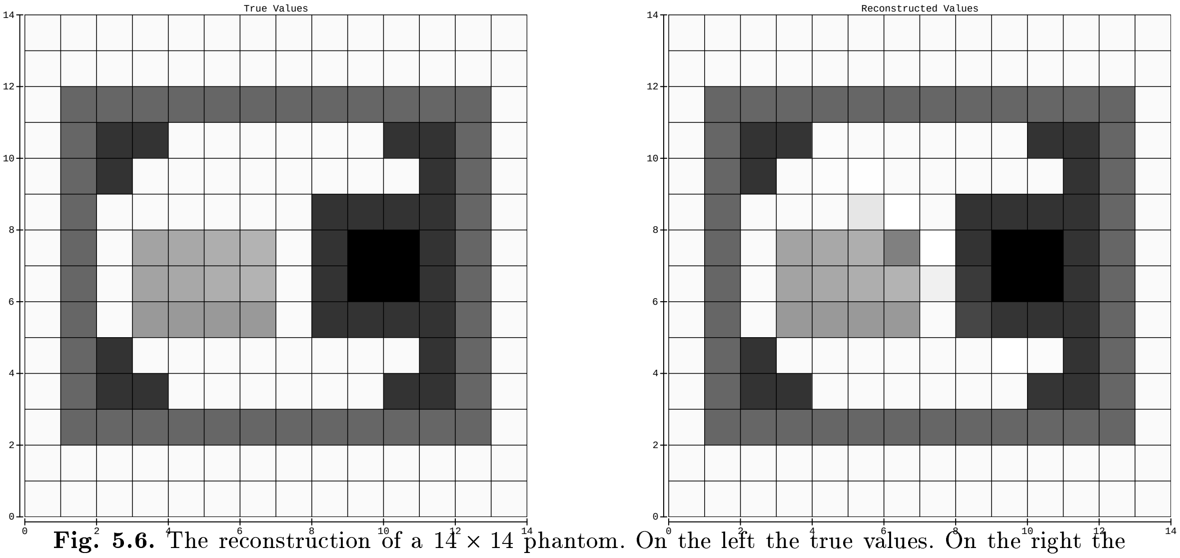

Fig. 5.6. The reconstruction of a 14×14 phantom. On the left the true values. On the right the reconstructed values

The main difficulty with the use of these algorithms for bigger, and more realistic problems, is due to the increase in the condition number at each step. This is probably due to two factors. First, a physically intrinsic one. The imaging gets harder and harder as we try to probe deep into the body from outside. Another, is related to the size of the matrices involved in the two methods proposed above.

On the other hand, there is no reason to go all the way to the middle diagonal. In fact, one can think about starting by "cutting" the four corners of the square, and then proceeding in a sort of “peeling” method. This idea was implemented in [43], and showed some improvement.

Let’s now turn back to the three dimensional case. It turns out that in this case, the same ideas above can be implemented. Here one proceeds along a plane perpendicular to the main diagonal of the cube. The amount of data allows us to actually choose which part of it to use in a judicious form. An initial version of this three dimensional analogue of the above peeling method was implemented in [59] and showed some promise for small size problems. Unfortunately, as before it was also plagued by large condition number difficulties.

We conclude this chapter by mentioning that in recent years, several groups have been developing research in discrete models for non-invasive reconstructions.

We mention briefly two of them that bear some relation to the discrete model mentioned here. The first one, whose references can be found in [45] deals with a discrete version of a model for photon migration in tissues. The second one, deals with the so-called high contrast case [6, 7].

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