A proof of Oseledets' theorem

1 Introduction

Let $f: M \longrightarrow M$ be a C^1 diffeomorphism on some compact finite-dimensional Riemannian manifold M. We say that $x \in M$ is a *regular point* for f if there are real numbers $\lambda_1 > \cdots > \lambda_l$ and a splitting $T_x M = E_1 \oplus \cdots \oplus E_l$ of the tangent space to M at x, such that

$$\lim_{n \to \pm\infty} \frac{1}{n} \log \|Df^n(x) u\| = \lambda_j \text{ for all } u \in E_j \setminus \{0\} \text{ and } 1 \le j \le l.$$

Observe that $\{\lambda_1, \ldots, \lambda_l\}$ must coincide with

$$\left\{ \lambda \in \mathbb{R} : \lambda = \lim_{n \to \pm\infty} \frac{1}{n} \log \|Df^n(x) u\| \text{ for some } u \in T_x M \setminus \{0\} \right\}$$

and we must also have

$$E_j = \{u \in T_x M : \frac{1}{n} \log \|Df^n(x) u\| \to \lambda_j \text{ both as } n \to \pm\infty\}$$

so that, when they exist, the reals $\lambda_1, \ldots, \lambda_l$ and the subspaces $E_1, \ldots, E_l$ are uniquely determined. We call $\lambda_1, \ldots, \lambda_l$ the *Lyapounov exponents* of f at x. It easy to see that the set $R(f)$ of regular points of f is invariant under iteration, i.e. $f(R(f)) = R(f)$, with $E_j(f(x)) = Df(x)\, E_j(x)$ and $\lambda_j(f(x)) = \lambda_j(x)$ for every j and all $x \in R(f)$.

While it is clear that the periodic points of f are always regular points, in general the set $R(f)$ is a topologically small subset of M: quite often it is meager (Baire first category) and it may even be finite. However, from a measure-theoretical point of view the situation is just opposite :

Theorem A ([O]) *The set* $R(f)$ total probability *in* M*, that is, it satisfies* $\mu(R(f)) = 1$ *for every* f*-invariant borelian probability measure in* M.

We prove this theorem, following [M], by deriving it from a more abstract result which we state as follows. Take M to be a compact metric space and $f: M \longrightarrow M$ to be a homeomorphism. Let $\pi: F \longrightarrow M$ be a finite-dimensional continuous vector bundle over M, endowed with a continuous Riemannian metric. Let $L: F \longrightarrow F$ be an isomorphism of continuous vector bundles covering f

(i.e. $\pi \circ L = f \circ \pi$), such that both L and L^{-1} have bounded norms. Denote by L_n the n-th iterate of L:

$$\begin{array}{ll} L_n(x) = L(f^{n-1}(x)) \circ \cdots \circ L(f(x)) \circ L(x) & \text{if } n > 0, \\ L_n(x) = L^{-1}(f^{-n+1}(x)) \circ \cdots \circ L^{-1}(f^{-1}(x)) \circ L^{-1}(x) & \text{if } n < 0. \end{array}$$

For $n_1, \ldots, n_l \geq 1$ define $\Lambda(n_1, \ldots, n_l)$ to be the set of points $x \in M$ such that the fibber F_x of F over x admits a splitting $F_x = E_1 \oplus \cdots \oplus E_l$ such that $\dim E_j = n_j$ for $1 \leq j \leq l$ and there are real numbers $\lambda_1 > \cdots > \lambda_l$ satisfying

$$\lim_{n \to \pm\infty} \frac{1}{n} \log \|L_n(x)\| = \lambda_j \text{ for every } u \in E_j \backslash \{0\} \text{ and } 1 \leq j \leq l.$$

Theorem 1.1 *a) For all $n_1, \ldots, n_l$, $\Lambda(n_1, \ldots, n_l)$ is a measurable subset of M. Moreover, for each $1 \leq j \leq l$, E_j is a measurable subbundle of the restriction of F to $\Lambda(n_1, \ldots, n_l)$ and $\Lambda(n_1, \ldots, n_l) \ni x \mapsto \lambda_j(x)$ is a measurable map.*

b) The set of regular points of L, given by $R(L) = \cup_{n_1, \ldots, n_l} \Lambda(n_1, \ldots, n_l)$, has total probability in M.

Clearly, this result contains Theorem A as a particular case: just take F to be the tangent bundle of M and L to be the morphism induced by Df.

We prove part a) of Theorem 1.1 in Section 2. Part b) is the crucial statement and its proof is given in Section 4. It uses preliminary results from Section 3 and is based on two main lemmas which we prove in Sections 5 and 6, respectively. In Section 7 we prove the following related result.

Proposition 1.2 *For $x \in R(L)$ denote $E^u(x) = \bigoplus_{\lambda_j > 0} E_j(x)$. Then*

$$\lim_{n \to +\infty} \frac{1}{n} \log |\det L^n(x) | E^u(x)| = \Sigma_{\lambda_j(x) > 0} (\lambda_j(x) \dim E_j(x))$$

and so, for every f-invariant probability measure μ,

$$\int \log |\det L(x) | E^u(x)| \, d\mu(x) = \int \Sigma_{\lambda_j(x) > 0} (\lambda_j(x) \dim E_j(x)) \, d\mu(x).$$

A result of Ruelle [R] asserts that (for $L = Df$) this last expression is always an upper bound for the entropy $h_\mu(f)$ of f with respect to μ.

2 Proof of Theorem 1.1: measurability

Let $n_1, \ldots, n_l$ be fixed. For $k \geq 1$ denote by $\mathcal{A}_k$ the set of $2l$-uples of rational numbers $\alpha_1 > \beta_1 > \cdots > \alpha_l > \beta_l$ with $(\alpha_j - \beta_j) < \frac{1}{k}$ for $1 \leq j \leq l$. For $m \geq 1$

and $(\alpha_1, \dots, \beta_l) \in \mathcal{A}_k$, let $\Lambda(m, \alpha_1, \dots, \beta_l)$ be the set of points $x \in M$ for which there is a splitting $F_x = F_1 \oplus \cdots \oplus F_l$ with $\dim F_j = n_j$ and

$$\exp(n\alpha_j) \|u\| \geq \|L_n(x)u\| \geq \exp(n\beta_j) \|u\| \tag{1}$$

$$\exp(-n\alpha_j) \|u\| \leq \|L_{-n}(x)u\| \leq \exp(-n\beta_j) \|u\| \tag{2}$$

for all $n \geq m$, $1 \leq j \leq l$, and $u \in F_j$. Observe that such a splitting is uniquely determined, for each $x \in \Lambda(m, \alpha_1, \dots, \beta_l)$, by

$$F_j = \{u \in F_x \colon \frac{\|L_n(x)u\|}{\|u\|} \leq \exp(n\alpha_j) \text{ and } \frac{\|L_{-n}(x)u\|}{\|u\|} \leq \exp(-n\beta_j) \text{ if } n \geq m\}.$$

Since (1) and (2) are closed conditions and the dimensions of the F_j are constant, we get that $\Lambda(m, \alpha_1, \dots, \beta_l)$ is closed and $\Lambda(m, \alpha_1, \dots, \beta_l) \ni x \mapsto F_j(x)$ is continuous for every $1 \leq j \leq l$. In particular, this proves that

$$\Lambda(n_1, \dots, n_l) = \bigcap_{k \geq 1} \bigcup_{(\alpha_1, \dots, \beta_l) \in \mathcal{A}_k} \bigcup_{m \geq 1} \Lambda(m, \alpha_1, \dots, \beta_l)$$

is a Borel set. To prove that the subbundles E_j are measurable on $\Lambda(n_1, \dots, n_l)$ it is now sufficient to show that for all $x \in \Lambda(n_1, \dots, n_l) \cap \Lambda(m, \alpha_1, \dots, \beta_l)$ we have $E_j(x) = F_j(x)$ for all $1 \leq j \leq l$. Note first that E_j must be contained in some F_k, $1 \leq k \leq l$, because all the vectors in E_j generate the same Lyapounov exponent λ_j, both for $n \to +\infty$ and for $n \to -\infty$. Using $\alpha_k \geq \lambda_j \geq \beta_k$, and recalling that $\lambda_1 > \cdots > \lambda_l$ and $\alpha_1 > \beta_1 > \cdots > \alpha_l > \beta_l$, one sees that $k = j$. Since $\dim E_j = n_j = \dim F_j$, this proves that $E_j = F_j$, as we claimed. Finally, the fact that $\lambda_j(x)$ is measurable in $\Lambda(n_1, \dots, n_l)$ follows immediately from

$$\lambda_j(x) = \lim_{n \to +\infty} \frac{1}{n} \log \|L_n(x) | E_j(x)\|.$$

The proof of Theorem 1.1a) is complete.

3 Sub-exponential growth

A measurable function $C \colon M \longrightarrow \mathbb{R}$ is said to have *subexponential growth* for a measure μ on M if

$$\lim_{n \to \pm\infty} \frac{1}{n} \log(C \circ f^n) = 0, \qquad \mu - \text{a.e.}$$

In this section we show that certain functions related to the iterates of L, and which play an important role in the proof of Theorem 1.1, have subexponential growth for every f-invariant probability measure.

Let μ be an f-invariant probability measure, E be an L-invariant measurable subbundle of F, and $\lambda \in \mathbb{R}$ be such that

$$\limsup_{n \to +\infty} \frac{1}{n} \log \|L_n(x)u\| \leq \lambda$$

for all $u \in E_x \backslash \{0\}$ and $\mu-$a.e. $x \in M$ (such a λ always exists, since we suppose $\|L\|$ to be bounded). Define, for $\varepsilon > 0$,

$$C_\varepsilon(x) = \sup \left\{ \frac{\|L_n(x)u\|}{\exp(n(\lambda+\varepsilon))\,\|u\|} : n \geq 0 \text{ and } u \in E_x \backslash \{0\} \right\}.$$

Proposition 3.1 *C_ε has subexponential growth for μ.*

In the proof of this proposition we use the following criterium for subexponential growth.

Lemma 3.2 *Let $f: M \longrightarrow M$ be a measurable mapping and μ be an f-invariant probability measure on M. Let $\varphi: M \longrightarrow \mathbb{R}$ be a measurable function such that $\varphi \circ f - \varphi$ is integrable. Then $\frac{1}{n}(\varphi \circ f^n) \to 0$ $\mu -$ a.e. as $n \to +\infty$.*

Proof: By Birkhoff's ergodic theorem applied to $(\varphi \circ f - \varphi)$, the sequence $\frac{1}{n}(\varphi \circ f^n)$ converges almost everywhere to some measurable function ψ. On the other hand, for each fixed $\delta > 0$,

$$\mu(\{x : \frac{1}{n}|\varphi \circ f^n(x)| \geq \delta\}) = \mu(f^{-n}\varphi^{-1}(-n\delta, n\delta)^c) = \mu(\varphi^{-1}(-n\delta, n\delta)^c) \to 0,$$

as $n \to +\infty$, which means that $\frac{1}{n}(\varphi \circ f^n)$ converges to 0 in measure. Therefore, $\frac{1}{n_k}(\varphi \circ f^{n_k}) \to 0$ almost everywhere, for some subsequence $n_k \to +\infty$. This proves that $\psi(x) = 0$ at $\mu -$ a.e. $x \in M$. □

Proof of Proposition 3.1: For $u \in E_x \backslash \{0\}$, let

$$C_\varepsilon(x,u) = \sup_{n \geq 0} \frac{\|L_n(x)u\|}{\exp(n(\lambda+\varepsilon))\,\|u\|}.$$

Then

$$C_\varepsilon(x,u) = \max \left\{ 1, \frac{\|L_n(x)u\|}{\exp(n(\lambda+\varepsilon))\,\|u\|} C_\varepsilon(f(x), L(x)u) \right\},$$

and so

$$a \leq \frac{C_\varepsilon(x,u)}{C_\varepsilon(f(x), L(x)u)} \leq \max\{1, b\},$$

where $a, b > 0$ are taken such that

$$a \leq \frac{\|L(x)u\|}{\exp(\lambda+\varepsilon)\,\|u\|} \leq b$$

for all x and $u \in E_x$. Then, immediately,

$$a \leq \frac{C_\varepsilon(x)}{C_\varepsilon(f(x))} \leq \max\{1, b\}$$

which ensures that $\log(C_\epsilon \circ f) - \log C_\epsilon$ and $\log(C_\epsilon \circ f^{-1}) - \log C_\epsilon$ are both bounded. In particular, they are integrable and so the proposition results from Lemma 3.2. □

The following consequence of Proposition 3.1 will be used several times in the proof of Theorem 1.1b).

Proposition 3.3 *Let E be an L-invariant measurable subbundle of F, $\lambda \in \mathbb{R}$, and μ be an f-invariant probability measure on M. Then*

$$\limsup_{n\to+\infty} \frac{1}{n} \log \|L_n(x)u\| \leq \lambda \qquad \textit{for } \mu - \textit{a.e. } x \in M \textit{ and all } u \in E_x \backslash \{0\}$$

if and only if

$$\limsup_{n\to-\infty} \frac{1}{n} \log \|L_n(x)u\| \leq \lambda \qquad \textit{for } \mu - \textit{a.e. } x \in M \textit{ and all } u \in E_x \backslash \{0\}.$$

Moreover, this statement remains true when $\limsup$ *and* $\leq$ *are replaced by* $\liminf$ *and* $\geq$ *(or by* $\lim$ *and* $=$*), respectively.*

Proof: We prove the 'only if' part, the other one being entirely analogous. Assume that

$$\limsup_{n\to+\infty} \frac{1}{n} \log \|L_n(x)u\| \leq \lambda \qquad \text{for } \mu - \text{a.e. } x \in M \text{ and all } u \in E_x \backslash \{0\}.$$

Let C_ε be as defined above. From

$$u = L_n(f^{-n}(x)) L_{-n}(x) u$$

we get

$$\|u\| \leq C_\varepsilon(f^{-n}(x)) \exp(n(\lambda + \varepsilon)) \|L_{-n}(x)u\|$$

and so

$$\frac{1}{n} \log \|u\| \leq \frac{1}{n} \log C_\varepsilon(f^{-n}(x)) + (\lambda + \varepsilon) + \frac{1}{n} \log \|L_{-n}(x)u\|$$

for all $n \geq 1$. By Proposition 3.1 it follows

$$0 \leq \liminf_{n\to+\infty} \frac{1}{n} \log \|L_{-n}(x)u\| + (\lambda + \varepsilon)$$

and so, since $\varepsilon > 0$ is arbitrary,

$$\limsup_{n\to-\infty} \frac{1}{n} \log \|L_n(x)u\| \leq \lambda,$$

as claimed. □

4 Proof of Theorem 1.1: total probability

To prove that $R(L)$ has total probability it suffices to show that $\mu(R(L)) = 1$ for every f-invariant *ergodic* probability measure on M. In what follows we always assume μ to be ergodic. Let

$$\lambda_1(L,x) = \limsup_{n\to+\infty} \frac{1}{n} \log \|L_n(x)\| \quad \text{and} \quad \lambda_1(L) = \int \lambda_1(L,x)\, d\mu(x).$$

Observe that $\lambda_1(L,x) \leq \sup_{x\in M} \|\mathrm{L}(x)\|$ and $\lambda_1(L,x) = \lambda_1(L)$ for $\mu -$ a.e. $x \in M$. Let G be the subbundle of F defined by

$$G_x = \{u \in F_x : \liminf_{n\to-\infty} \frac{1}{n} \log \|L_n(x)u\| \geq \lambda_1(L)\}.$$

The next lemma is a main step in the proof of the theorem, its proof will be given in the next section.

Lemma 4.1 *G is a measurable L-invariant subbundle with strictly positive dimension and*

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|L_n(x)u\| = \lambda_1(L)$$

for all $\mu -$ a.e. $x \in M$ and all $u \in G_x \backslash \{0\}$.

Somewhat more precisely, denoting by $\hat{\Lambda}(j)$ the set of points x for which $\dim G_x = j$, we shall show that these are measurable subsets of M and that the restriction of G to each $\hat{\Lambda}(j)$ is a measurable subbundle of the restriction of F to $\hat{\Lambda}(j)$. Since the dimension of G_x is an f-invariant function of x and we assume μ to be ergodic, we have $\mu(\Lambda_j) = 1$ for some (necessarily unique) j and we shall check that $j \geq 1$.

We write $G = F$ if $G_x = F_x$ for almost every $x \in M$. Note that in this case the proof of the theorem is complete. If $G \neq F$ we let $G^\perp$ be the orthogonal of G. Moreover, we let $p: F \longrightarrow G^\perp$ be the orthogonal projection and write $\hat{L} = p \circ L: G^\perp \longrightarrow G^\perp$.

Lemma 4.2 *If $F \neq G$ then $\lambda_1(\hat{L}) < \lambda_1(L)$.*

Proof: Observe first that

$$\|\hat{L}_n(x)u\| = \|p(L_n(x)u)\| \leq \|L_n(x)u\|,$$

as a consequence of the invariance of G. Let $\hat{G}$ be the subbundle of $G^\perp$ given by Lemma 4.1 for $\hat{L}$ and $\lambda_1(\hat{L})$ and let $u \in \hat{G}_x \backslash \{0\}$. Then

$$\begin{aligned}\lambda_1(\hat{L}) &= \lim_{n\to+\infty} \tfrac{1}{n} \log \|\hat{L}_n(x)u\| \leq \liminf_{n\to+\infty} \tfrac{1}{n} \log \|L_n(x)u\| \\ &\leq \limsup_{n\to+\infty} \tfrac{1}{n} \log \|L_n(x)u\| \leq \lambda_1(L).\end{aligned}$$

This proves that $\lambda_1(\hat{L}) \leq \lambda_1(L)$ and also that the equality would imply

$$\lim_{n\to+\infty} \frac{1}{n} \log \|L_n(x)u\| = \lambda_1(L)$$

for $u \in \hat{G}_x \backslash \{0\}$. Then, by Proposition 3.3, we would have

$$\lim_{n\to-\infty} \frac{1}{n} \log \|L_n(x)u\| = \lambda_1(L).$$

This would imply $u \in G_x$, contradicting the fact that $\hat{G}_x \cap G_x = \{0\}$. Therefore, it must be $\lambda_1(\hat{L}) < \lambda_1(L)$. $\square$

Now we invoke a second main lemma, whose proof is given in Section 6.

Lemma 4.3 *If $G \neq F$ then there is a measurable L-invariant subbundle H of F such that $G \oplus H = F$ and $\lambda_1(L|H) = \lambda_1(\hat{L}) < \lambda_1(L)$.*

Writing $E_1 = G$ and $H_1 = H$, we have shown that there exists an invariant splitting $F = E_1 \oplus H_1$ such that

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|L_n(x)u\| = \lambda_1 \text{ for all } u \in E_1 \backslash \{0\}$$

and $\lambda_1 > \lambda_1(L|H_1)$. Theorem 1.1b) now follows by repeated use of this procedure. Given any $j \geq 1$, suppose that we have already found an invariant splitting $F = E_1 \oplus \cdots E_i \oplus H_i$ with

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|L_n(x)u\| = \lambda_i \text{ for all } u \in E_i \backslash \{0\} \text{ and } \ 1 \leq i \leq j$$

and $\lambda_1 > \cdots > \lambda_j > \lambda_1(L|H_j)$. Then Lemma 4.1 allows us to find an invariant subbundle E_{j+1} of H_j such that

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|L_n(x)u\| = \lambda_{i+1} = \lambda_1(L|H_j) \text{ for all } u \in E_i \backslash \{0\}$$

and Lemma 4.3 yields an invariant splitting $H_j = E_{j+1} \oplus H_{j+1}$ such that $\lambda_1(L|H_{j+1}) < \lambda_{j+1}$. Since the subbundles E_j found in this way have positive dimension, $\dim H_j$ is strictly decreasing and so this process must come to an end after a finite number of steps.

5 Proof of Lemma 4.1

First we prove that G is a measurable subbundle. For the sake of clearness we divide this into three steps. Throughout we shall use the following elementary remark. Let $(A, \mathcal{A})$ and $(B, \mathcal{B})$ be measurable spaces and f be any map from A

to B. If there exists a countable covering $(A_n)_n$ of A by measurable sets, such that $f|A_n: A_n \longrightarrow B$ is measurable map for each $n \geq 1$, then $f: A \longrightarrow B$ is a measurable map.

Step 1: For $k \geq 1$ and $x \in M$ let

$$G_x(k) = \{u: F_x: \limsup_{n\to+\infty} \frac{1}{n} \log \|L_{-n}(x)u\| \leq -\lambda_1 + \frac{1}{k}\}.$$

Then define $M_k = \{x \in M: G_x(k) = G_x\}$. It is easy to check that $(M_k)_k$ covers M up to zero measure. Indeed, for any fixed $x \in M$, $(G_x(k))_k$ is a decreasing sequence of subspaces of F_x with $\cap_{k=1}^{\infty} G_x(k) = G_x$, thus there exists some $k_x \geq 1$ for which $G_x(k_x) = G_x$. We also claim that every M_k is a measurable set. In order to prove this we introduce the measurable functions

$$\begin{array}{ll} \lambda_1: M \longrightarrow \mathrm{I\!R}, & \lambda_1(x) = \limsup_{n\to+\infty} \frac{1}{n} \log \|L_n(x)\| \\ \phi: F \longrightarrow \mathrm{I\!R}, & \phi(x,u) = \limsup_{n\to+\infty} \frac{1}{n} \log \|L_{-n}(x)u\| \end{array}$$

and we also consider the bundle projection $\pi: F \longrightarrow M$. Clearly,

$$\begin{array}{lll} x \notin M_k & \Longleftrightarrow & \text{there is } u \in F_x \text{ such that } (\phi + \lambda_1\pi)(x,u) \in (0, 1/k] \\ & \Longleftrightarrow & x \in \pi((\phi + \lambda_1\pi)^{-1}((0,1/k])), \end{array}$$

that is, $M_k = M \backslash \pi\left((\phi + \lambda_1\pi)^{-1}((0,1/k])\right)$. As π maps measurable sets onto measurable sets, this proves our claim.

Step 2: Now we fix $k \geq 1$ and, for each $x \in M_k$, and $m \geq 1$, we define

$$G_x(k,m) = \{u: F_x: \|L_{-n}(x)u\| \leq m \exp(-n(\lambda_1 - \frac{1}{k}))\|u\|\}.$$

Note that $G_x(k) = \cup_m G_x(k,m)$ for all $x \in M_k$: the inclusion $\supset$ is obvious and $\subset$ follows from $G_x(k) = G_x(k+1)$. Now define $M_{k,m} = \{x \in M_k: G_x(k) = G_x(k,m)\} = \{x \in M_k: G_x(k,m) = G_x(k,l) \text{ for all } l \geq m\}$. Consider the measurable function

$$\psi_k: \pi^{-1}(M_k) \longrightarrow \mathrm{I\!R}, \qquad \psi_k(x,u) = \sup_{n>0} \frac{\|L_{-n}(x)u\|}{\|u\|} \exp(n(\lambda_1 - \frac{1}{k})).$$

Clearly

$$\begin{array}{lll} x \notin M_{k,m} & \Longleftrightarrow & \text{there are } u \in F_x \text{ and } l \geq m \text{ such that } m < \psi_k(x,u) \leq l \\ & \Longleftrightarrow & \text{there is } l \geq m \text{ such that } x \in \pi(\psi_k^{-1}((m,l])), \end{array}$$

that is, $M_{k,m} = M_k \backslash \cup_{l=m}^{\infty} \pi(\psi_k^{-1}((m,l]))$. This proves that every $M_{k,m}$ is a measurable set. In order to show that $(M_{k,m})_m$ covers M_k, let $x \in M_k$ and $\{u_1, \ldots, u_s\}$ be an orthogonal basis for $G_x(k)$. Take $m_1, \ldots, m_s \geq 1$ such that $u_i \in G_x(k, m_i)$, for all $1 \leq i \leq s$, and let $m = \max\{m_1, \ldots, m_s\}$. For every $u = \sum_{i=1}^{s} a_i u_i \in G_x(k)$ and every $n \geq 1$ we have

$$\begin{array}{ll} \|L_{-n}(x)u\| & \leq \sum_{i=1}^{s} m_i |a_i| \exp(-n(\lambda_1 - \frac{1}{k})) \\ & \leq m \exp(-n(\lambda_1 - \frac{1}{k})) \sum_{i=1}^{s} |a_i|. \end{array}$$

This gives $\|L_{-n}(x)u\| \leq m \exp(-n(\lambda_1 - \frac{1}{k}))C\,\|u\|$, where C depends only on the choice of the norm in F, and we conclude that $G_x(k) \subset G_x(k, [Cm+1])$, hence $x \in M_{k,[Cm+1]}$.

Step 3: Now we claim that G_x is lower semi-continuous on each $M_{k,m}$, that is, given any sequence $(x_i)_i$ in $M_{k,m}$ converging to some $x \in M_{k,m}$, we have $\lim G_{x_i} \subset G_x$. Indeed, let a sequence $u_i \in G_{x_i}$, $i \geq 1$, converge to some $u \in F_x$. Then

$$\|L_{-n}(x)u_i\| \leq m \exp(-n(\lambda_1 - \frac{1}{k}))\|u_i\|,$$

for all $i \geq 1$ and $n \geq 1$. Passing to the limit, $\|L_{-n}(x)u\| \leq m \exp(-n(\lambda_1 - \frac{1}{k}))\,\|u\|$, which implies $u \in G_x$ and thus proves the claim. It follows that each $M_{k,m,j} = \{x \in M_{k,m} \colon \dim G_x \geq j\}$ is a closed set and that G_x varies continuously with x on every $M_{k,m,j} \backslash M_{k,m,j+1} = \hat{\Lambda}(j) \cap M_{k,m}$. In view of the remark at the beginning of the section, this proves that each $G|\hat{\Lambda}(j)$ is a measurable subbundle of $F|\hat{\Lambda}(j)$.

Now we proceed to the second part of the proof, where we show that $\dim G_x > 0$ for $\mu -$ a.e. $x \in M$. In order to do this it suffices to check that for all $k \geq 1$ one has $G_x(k) \neq \{0\}$ at $\mu -$ a.e. $x \in M$. Let $k \geq 1$ be fixed and, for $m \geq 1$, define Y_m to be the set of points $x \in M$ such that there exists $u \in F_x \backslash \{0\}$ satisfying $\|L_{-n}(x)u\| \leq \exp(-n(\lambda_1 - \frac{1}{k}))\,\|u\|$ for $1 \leq n \leq m$. We claim that there is $\delta > 0$ such that

$$\mu(Y_m) \geq \delta, \qquad \text{for every } m \geq 1. \tag{3}$$

Note that this implies that the set $Y = \cap_{m=1}^{\infty} Y_m$ has positive measure and hence $\{x \in M \colon \dim G_x(k) > 0\}$ has total measure (because it is an f-invariant set containing Y), as we wanted to prove. Therefore, we are reduced to proving the claim above.

A main ingredient in the proof is the following result of Pliss.

Lemma 5.1 ([P]) *Given $\lambda \in \mathbb{R}$, $\varepsilon > 0$, and $A > 0$, there is $\delta = \delta(\lambda, \varepsilon, A) > 0$ such that given any finite sequence $a_0, \ldots, a_{N-1}$ in $\mathbb{R}$, with $\sum_{k=0}^{N-1} a_k \leq N\lambda$ and $a_k \leq A$ for all $0 \leq k \leq N-1$, there exist $l \geq N\delta$ and $0 \leq n_1 < \cdots < n_l \leq N-1$, such that*

$$\sum_{j=n}^{n_i-1} a_j \leq (n_i - n)(\lambda + \varepsilon), \qquad \textit{for all } 0 \leq n \leq n_i \textit{ and } 1 \leq i \leq l.$$

Proof: Denote $S(n) = \sum_{j=n}^{N-1}(a_j - (\lambda + \varepsilon))$ and take $n_1 < \cdots < n_l$ to be the elements of $\{0, \ldots, N-1\}$ which satisfy

$$S(n) \leq S(n_i) \qquad \text{for all } 0 \leq n < n_i. \tag{4}$$

Then, for $0 \leq n < n_i$,

$$\sum_{j=n}^{n_i-1} a_j = (S(n) - S(n_i)) + (n_i - n)(\lambda + \varepsilon) \leq (n_i - n)(\lambda + \varepsilon).$$

Therefore, we are left to estimate the value of l. To do this we observe first that $S(n_{i-1}) \geq S(n_i - 1)$ for all $i > 1$: otherwise there would be $n_{i-1} < m < n_i$ such that $S(n) \leq S(m)$ for all $1 \leq n < m$, contradicting the definition of $\{n_1, \ldots, n_l\}$. Then

$$S(n_{i-1}) \geq S(n_i) + (a_{n_i} - (\lambda + \varepsilon)) \geq S(n_i) - (|\lambda| + \varepsilon + A)$$

for all $i > 1$, and so

$$S(n_1) \geq S(n_l) - (l-1)(|\lambda| + \varepsilon + A).$$

Since

$$S(n_1) = S(0) = \sum_{k=0}^{N-1} a_k - N(\lambda+\varepsilon) \leq -N\varepsilon \text{ and } S(n_l) \geq S(N-1) = a_{N-1} - (\lambda+\varepsilon)$$

(because n_l is the largest element of $\{0, \ldots, N-1\}$ satisfying (4)), it follows that

$$-N\varepsilon \geq a_{N-1} - (\lambda + \varepsilon) - (l-1)(|\lambda| + \varepsilon + A) \geq -l(|\lambda| + \varepsilon + A).$$

Therefore, we may take $\delta = \varepsilon/(|\lambda| + \varepsilon + A)$. □

Starting the proof of (3), let $u \in F_x$ and $N \geq 1$ (large) be such that $\|L_N(x)u\| \geq \exp(N(\lambda_1 - \frac{1}{2k}))\,\|u\|$, and define

$$a_i = \log \left\| L^{-1}(f^{i+1}(x)) \left(\frac{L_{i+1}(x)u}{\|L_{i+1}(x)u\|} \right) \right\|, \qquad \text{for } 0 \leq i \leq N-1.$$

Note that $|a_i| \leq \log \left\| L^{-1} \right\|$ and

$$\sum_{j=0}^{N-1} a_j = \sum_{j=0}^{N-1} \log \frac{\|L_i(x)u\|}{\|L_{i+1}(x)u\|} = \log \frac{\|u\|}{\|L_N(x)u\|} \leq N(-\lambda_1 + \frac{1}{2k}).$$

Using Lemma 5.1, for $\lambda = -\lambda_1 + \frac{1}{2k}$, $\varepsilon = \frac{1}{2k}$, and $A = \log \left\| L^{-1} \right\|$, we get $0 \leq n_1 < \cdots < n_l \leq N-1$, with $l \geq N\delta$, such that

$$\log \frac{\|L_n(x)u\|}{\|L_{n_i}(x)u\|} = \sum_{j=n}^{n_i - 1} a_j \leq (n_i - n)(-\lambda_1 + \frac{1}{k})$$

for all $0 \leq n < n_i$. Denoting $v_i = L_{n_i}(x)u \in F_{f^{n_i}(x)}$, the last relation can be written

$$\|L_{n-n_i}(x)v_i\| \leq \exp((n - n_i)(\lambda_1 - \frac{1}{k})), \quad \text{for } 1 \leq n_i - n \leq n_i,$$

and this implies $f^{n_i}(x) \in Y_{n_i}$. Observing that $Y_{n_i} \subset Y_m$ when $n_i \geq m$, we conclude that

$$\frac{1}{N} \#\{0 \leq j < N \colon f^j(x) \in Y_m\} \geq \frac{l - m}{N} \geq \delta - \frac{m}{N}.$$

Taking N arbitrarily large, and using the ergodicity of μ, this gives that $\mu(Y_m) \geq \delta$, proving our claim.

Finally, we note that

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|L_n(x)u\| = \lambda_1, \qquad \text{for all } u \in G_x\backslash\{0\}$$

is a direct consequence of Proposition 3.3, $\limsup_{n\to+\infty} \frac{1}{n} \log \|L_n(x)u\| \leq \lambda_1$ for all $u \in F_x\backslash\{0\}$, and $\liminf_{n\to-\infty} \frac{1}{n} \log \|L_n(x)u\| \geq \lambda_1$, for all $u \in G_x\backslash\{0\}$. The proof of Lemma 4.1 is complete.

6 Proof of Lemma 4.3

Let Σ be the space of morphisms of measurable vector bundles $G^\perp \longrightarrow G$. We shall construct $A \in \Sigma$ such that $H = graph(A) = \{u + Au : u \in G^\perp\}$ is as in the statement of the lemma.

Let $\Phi: \Sigma \longrightarrow \Sigma$ be the graph transform, defined by $\Phi(A) = L^{-1} \circ A \circ \hat{L}$, and let $P = (L|G^\perp - \hat{L}): G^\perp \longrightarrow G$. Given $u \in G^\perp$,

$$L(u+Au) = Lu+LAu = \hat{L}u+Pu+LAu = \hat{L}u+A(\hat{L}u)+L(A-L^{-1}A\hat{L}+L^{-1}P)u.$$

Observe that the first term on the righthand side belongs in $G^\perp$, while the last two belong in G. Hence, H is L-invariant if and only if

$$A - \Phi(A) = -L^{-1}P. \tag{5}$$

Let $B = -L^{-1}P$. We shall show that there are $\lambda < 0$ and a measurable function $C: M \longrightarrow \mathbb{R}$ such that $\|\Phi^n(B)x\| \leq C(x)e^{\lambda n}$ for all $n \geq 0$ and $\mu -$a.e.. This ensures that $\sum_{n=0}^{\infty} \Phi^n(B)$ is convergent $\mu -$ a.e. and it is easy to see that $A = \sum_{n=0}^{\infty} \Phi^n(B)$ satisfies (5).

In order to find λ and C, let

$$K_\varepsilon(x) = \sup_{n\geq 0} \frac{\|\hat{L}_n(x)\|}{\exp\left(n(\lambda_1(\hat{L}) + \varepsilon)\right)} \quad \text{and} \quad C_\varepsilon(x) = \sup_{n\geq 0} \frac{\left\|\left(L^{-1}|G\right)_n(x)\right\|}{\exp\left(n(\lambda_1\left(L^{-1}|G\right) + \varepsilon)\right)}.$$

Then

$$\begin{aligned} \|\Phi^n(B)(x)\| &= \|(L^{-1}|G)_n(f^n x)B(f^n x)\hat{L}_n(x)\| \\ &\leq C_\varepsilon(f^n(x))\,\|B\|\,K_\varepsilon(x)\exp\left(n(\lambda_1(\hat{L}) + \lambda_1(L^{-1}|G) + 2\varepsilon)\right). \end{aligned}$$

The methods in Section 3 show that C_ε has subexponential growth, hence

$$D_\varepsilon(x) = \sup_{n\geq 0} \frac{C_\varepsilon(f^n x)}{\exp(n\varepsilon)}$$

is finite and

$$\|\Phi^n(B)(x)\| \leq \|B\|\, D_\varepsilon(x) K_\varepsilon(x) \exp\left(n(\lambda_1(\hat{L}) - \lambda_1(L) + 3\varepsilon)\right).$$

Hence, it suffices to take $\lambda = \lambda_1(\hat{L}) - \lambda_1(L) + 3\varepsilon$ and $C(x) = \|B\|\, D_\varepsilon(x) K_\varepsilon(x)$, where is $\varepsilon >$ is assumed to be small enough so that $\lambda < 0$

We are left to prove that $\lambda_1(L|H) = \lambda_1(\hat{L})$. In order to do that we take $\tilde{A}: G^\perp \longrightarrow H$ given by $\tilde{A}u = u + Au$. By construction $L \circ \tilde{A} = \tilde{A} \circ \hat{L}$, that is, $(L|H)(x) = \tilde{A}(fx)\hat{L}(x)(\tilde{A})^{-1}(x)$. Therefore,

$$\begin{aligned} \lambda_1(L|H,x) &= \limsup_{n\to+\infty} \tfrac{1}{n} \log \|(L|H)_n(x)\| \\ &= \limsup_{n\to+\infty} \tfrac{1}{n} \log \|\tilde{A}(f^n x)\hat{L}_n(x)(\tilde{A})^{-1}(x)\| \\ &\le \limsup_{n\to+\infty} \tfrac{1}{n} \log \|\tilde{A}(f^n x)\| + \limsup_{n\to+\infty} \tfrac{1}{n} \log \|\hat{L}_n(x)\| \\ &\quad + \limsup_{n\to+\infty} \tfrac{1}{n} \log \|(\tilde{A})^{-1}(x)\|. \end{aligned}$$

Clearly, the second term on the righthand side is equal to $\lambda_1(\hat{L})$, while the third one is null. Moreover,

$$\limsup_{n\to+\infty} \frac{1}{n} \log \|\tilde{A}(f^n x)\| = 0. \tag{6}$$

Indeed,

$$1 \le \|\mathrm{id}\| \le \|\tilde{A}(f^n x)\| = \|\mathrm{id} + A(f^n x)\| \le 1 + \|A(f^n x)\| \le 1 + \frac{C(f^n x)}{1 - e^\lambda}.$$

Using the same arguments as in Section 3 one sees that $D_\varepsilon(x)$ and $K_\varepsilon(x)$ have subexponential growth, hence the same holds for $C_\varepsilon(x)$. Combined with the last inequality, this proves (6) and it follows that $\lambda_1(L|H) \le \lambda_1(L)$. The reverse inequality is proved in the same way, using $\hat{L}(x) = (\tilde{A})^{-1}(fx)(L|H)(x)\tilde{A}(x)$ and also $\|(\tilde{A})^{-1}\| \le 1$ $\mu -$ a.e. (because $(\tilde{A})^{-1} = \pi|H: H \longrightarrow G^\perp$. The proof of the lemma is complete.

Remark: In general, $\|A\|$ is not a bounded function, which means that the angle $\angle(G, H)$ between the bundles G and H is not bounded away from zero. On the other hand, the previous estimates show that $\angle(G, H)$ always has subexponential growth (because $\|A\|$ does).

7 Growth of the Jacobian

We conclude by proving Proposition 1.2.

Given a finite-dimensional Hilbert space E, we define the *distortion* of a splitting $E = E_1 \oplus \cdots \oplus E_k$ to be $\delta = |\det \mathrm{id}: F \longrightarrow E|$, where $F = E_1 \perp \cdots \perp E_k$ is the orthogonal sum of the E_i. Note that if L is an isomorphism then the distortion δ_1 of the splitting $E = L(E_1) \oplus \cdots \oplus L(E_k)$ is given by

$$\delta_1 = \frac{|\det L|}{\prod_{i=1}^k |\det L|E_i|}\delta.$$

Denote by $\delta(x)$ the distortion of the decomposition $E^u(x) = E_1(x) \oplus \cdots E_k(x)$, where the $E_j(x)$ are ordered in such a way that $\lambda_1(x) > \cdots > \lambda_k(x) > 0$. Then

$$\frac{\delta(fx)}{\delta(x)} = \frac{|\det L|E^u(x)|}{\prod_{i=1}^k |\det L|E_i(x)|}.$$

Using the fact that L and L^{-1} are bounded (together with elementary relations between det and $\|\cdot\|$), we conclude that $\delta(fx)/\delta(x)$ is bounded away from zero and infinity. It follows, by the same arguments as in Section 3, that

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \delta(f^n x) = 0, \quad \mu-\text{a.e. } x \in M$$

that is, the distortion has subexponential growth for the measure μ. Then, from $|\det L_n|E^u(x)| = (\delta(f^n x)/\delta(x)) \prod_{i=1}^k |\det L_n|E_i(x)|$ we get

$$\lim_{n\to+\infty} \frac{1}{n} \log|\det L_n|E^u(x)| = \sum_{i=1}^k \lim_{n\to+\infty} \frac{1}{n} \log|\det L_n|E_i(x)|\,.$$

Moreover, $\lim_{n\to\pm\infty} \frac{1}{n} \log\|L_n(x)u\| = \lambda_i$ for all $u \in E_i(x)\setminus\{0\}$ is easily seen to imply $\lim_{n\to+\infty} \frac{1}{n} \log|\det L_n|E_i(x)| = \lambda_i \dim E_i(x)$ and this proves the first statement in the proposition. The second one is now a direct consequence of Birkhoff's ergodic theorem.

References

[M] R. Mañé, *Ergodic Theory and Differentiable Dynamics*, Springer Verlag, 1987.

[O] V. Oseledets, *A multiplicative ergodic theorem*, Trans. Moscow. Math. Soc. 19 (1968), 197-231.

[P] V. Pliss, *On a conjecture due to Smale*, Diff. Uravnenija, 8 (1972), 268-282.

[R] D. Ruelle, *An inequality for the entropy of differentiable maps*, Bull. Braz. Math. Soc. 9 (1978), 83-87.