On a class of submanifolds carrying an extrinsic totally umbilical foliation *

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Let $f\colon M^n \to \mathbb{R}^N$ be an isometric immersion of a riemannian manifold into euclidean space. A normal vector field η to f is called a *principal curvature normal* of f if $\eta(x)$ is a principal curvature normal at any $x \in M^n$, i.e., the *conformal nullity* subspace $\mathcal{E}_\eta(x) \subset T_xM$ associated to η given by

$$\mathcal{E}_\eta(x) = \{T \in T_xM\colon \alpha_f(T, X) = \langle T, X\rangle \eta, \text{ for all } X \in T_xM\},$$

is at least one-dimensional. Here $\alpha_f\colon TM \times TM \to T_f^\perp M$ stands for the second fundamental form of f with values in the normal bundle. If, in addition, $\mathcal{E}_\eta$ has constant dimension q everywhere, then η is said to be *proper of multiplicity* q. We call a proper principal curvature normal η *parallel* when it is parallel in the normal connection of f along $\mathcal{E}_\eta$. It is well known that the parallelism condition is automatic for multiplicity $q \geq 2$ (cf. [**Re**] or Proposition 8 below). Moreover, it is a standard fact that if η is a nonvanishing parallel principal curvature normal of multiplicity q, then $\mathcal{E}_\eta$ is an involutive distribution whose leaves are round q-dimensional spheres in $\mathbb{R}^N$. For hypersurfaces, admitting a parallel principal curvature normal reduces to having a principal curvature of constant multiplicity which is constant along the leaves of the corresponding eigenbundle.

Isometric immersions $f\colon M^n \to \mathbb{R}^N$ carrying principal curvature normals arise in several different geometric situations. For instance, it is a well-known fact (see [**Re**]) that f has flat normal bundle at $x \in M^n$ if and only if there exist principal curvature normals $\eta_1, \ldots, \eta_\ell$ at x such that the tangent space

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T_xM decomposes as an orthogonal direct sum of $\mathcal{E}_{\eta_1}, \ldots, \mathcal{E}_{\eta_\ell}$. Another important example occurs when M^n is conformally flat, $n \geq 4$ and $N \leq 2n-3$. In this case, it was shown in [**Mo**] that there is an open dense subset of M^n so that each connected component carries a proper principal curvature normal of multiplicity at least $2n - N$. For other geometric conditions implying the existence of principal curvature normals we refer to [**Ca**], [**CD**], [**AD**], [**DF**] and [**DT**].

In this paper we classify euclidean submanifolds carrying a parallel principal curvature normal η under the intrinsic additional assumption that the *(conformal) conullity* $\mathcal{E}_\eta^\perp$ associated to η is involutive and the leaves are extrinsic spheres in M^n in the sense of Nomizu ([**Nm**]). Our classification is conformal in nature, i.e., up to conformal transformations of the ambient space $\mathbb{R}^N$. A result due to Nolker ([**No**]) implies that the submanifold is rotational under the stronger hypothesis that the conullity is totally geodesic in the manifold.

Our result implies Cecil's local conformal classification of the Cyclides of Dupin, cf. [**Ce**$_1$] or [**Ce**$_2$]. In fact, it improves Cecil's result in that, for hypersurfaces with a principal curvature of multiplicity $n-1$ (which are precisely the conformally flat hypersurfaces when $n \geq 4$), we only require the curvature lines correspondent to the principal curvature of multiplicity one to be *circles* in M^n, that is, one-dimensional extrinsic spheres. Moreover, in contrast to Cecil's proof, which is based on Pinkall's local Lie geometric classification of the Cyclides of Dupin (cf. [**Pi**]) and thus uses the framework of Lie sphere geometry, ours is entirely done within euclidean geometry.

Among other applications, we show that a "generic" conformally flat euclidean submanifold in codimension two with nowhere flat normal bundle and integrable conullity $\mathcal{E}_\eta^\perp$ must be rotational. Here, η is the principal curvature normal given by the aforementioned result in [**Mo**]. We also give a complete description of the profiles of conformally flat rotational submanifolds of arbitrary codimension. Another application of our main result is a classification of the three–dimensional conformally flat hypersurfaces of euclidean space with three distinct principal curvatures whose curvature lines of one family are segments of circles or straight lines.

§1 The results.

A smooth distribution $\mathcal{U}$ on an n-dimensional riemannian manifold M^n is said to be *totally umbilical* if there exists a vector field δ in $\mathcal{U}^\perp$ such that

$$\nabla^h_T S = \langle T, S\rangle \delta, \quad \text{for all } \ T, S \in \mathcal{U},$$

where we write $Z = Z^v + Z^h$ according to the decomposition $TM = \mathcal{U} \oplus \mathcal{U}^\perp$. In this case, δ is called the *mean curvature* of $\mathcal{U}$. If δ vanishes identically, then $\mathcal{U}$ is said to be *totally geodesic*. The distribution is called *spherical* if it is totally umbilical and its mean curvature δ satisfies

$$\nabla^h_T \delta = 0, \quad \text{for all } \ T \in \mathcal{U}.$$

If $\mathcal{U}$ is totally geodesic, totally umbilical or spherical, then it is involutive and the leaves are, respectively, totally geodesic, totally umbilical or extrinsic spheres in M^n.

Theorem 1. *Let $f\colon M^n \to \mathbb{R}^N$ be an isometric immersion with a parallel principal curvature normal η of multiplicity q such that the conullity $\mathcal{E}_\eta^\perp$ is totally umbilical on M^n. If $q = n-1$, assume further that the integral curves of the conullity are circles in M^n. Then $f(M^n)$ is conformally congruent to an open subset of one of the following:*

i) a product $M^{n-q} \times \mathbb{R}^q$, where M^{n-q} is a submanifold of $\mathbb{R}^{N-q}$,

ii) a product $CM^{n-q} \times \mathbb{R}^{q-1}$, where $CM^{n-q} \subset \mathbb{R}^{N-q+1}$ is the cone over a submanifold M^{n-q} of the sphere $\mathbb{S}^{N-q} \subset \mathbb{R}^{N-q+1}$,

iii) a rotational submanifold over a submanifold M^{n-q} of $\mathbb{R}^{N-q}$.

Moreover, f has flat normal bundle if and only if the same holds for M^{n-q}.

Recall that the *rotational submanifold* N^n of $\mathbb{R}^N$ over M^{n-q} with *axis* $\mathbb{R}^{N-q-1}$ is the n-dimensional submanifold generated by the orbits of the points of M^{n-q} under the action of $SO(q+1)$. Here $M^{n-q} \subset \mathbb{R}^{N-q} \subset \mathbb{R}^N$ is a submanifold disjoint from the subspace $\mathbb{R}^{N-q-1} \subset \mathbb{R}^{N-q}$ and $SO(q+1)$ denotes the subgroup of $SO(N)$ which keeps $\mathbb{R}^{N-q-1}$ pointwise fixed. Notice that the hypothesis of the theorem for $q = 1$ is equivalent to assuming that the integral curves of $\mathcal{E}_\eta$ are segments of circles or straight lines in $\mathbb{R}^N$; cf. Proposition 8.

A hypersurface $f\colon M^n \to \mathbb{R}^{n+1}$ is called a *Cyclide of Dupin* of characteristic $(q, n{-}q)$ if it has everywhere only two principal curvatures of multiplicities q and $n-q$, respectively, which are constant along the correspondent eigenbundles; cf. [**Pi**]. It follows from Theorem 1 that any Cyclide of Dupin is conformally equivalent to a hypersurface of one of the three types in Theorem 1, where M^{n-q} is an $(n-q)$–dimensional sphere. This is Cecil's result referred to in the introduction.

In the case where $q = n-2$ in Theorem 1, we can replace the assumption on $\mathcal{E}_\eta^\perp$ by the weaker hypothesis that it is simply integrable if we impose on f the further restriction of having nowhere flat normal bundle, that is, not having flat normal bundle on any open subset.

Corollary 2. *Let $f\colon M^n \to \mathbb{R}^N$, $n \geq 4$, be an isometric immersion with nowhere flat normal bundle carrying a proper principal curvature normal η of multiplicity $n-2$ with integrable conullity. Then $f(M^n)$ is conformally congruent to an open subset of a submanifold of type i), ii) or iii) in Theorem 1 with nowhere flat normal bundle.*

Next, we consider isometric immersions of conformally flat manifolds. We start with a general result for immersions with flat normal bundle.

Proposition 3. *Let $f\colon M^n \to \mathbb{R}^N$ be an isometric immersion of a conformally flat manifold with flat normal bundle and a constant number of proper principal curvature normals $\eta_1, \ldots, \eta_\ell$. Then, the conullity $\mathcal{E}_{\eta_k}^\perp$ is integrable if the multiplicity of $\mathcal{E}_{\eta_k}$ is at least 2.*

We call an isometric immersion $f\colon M^n \to \mathbb{R}^N$, $n \geq 4$ and $N \leq 2n-3$, of a conformally flat manifold *generic* if $\mathcal{E}_\eta$ assumes everywhere its possible minimum dimension $2n-N$. Here η is the principal curvature normal of f given by Moore's result referred to in the introduction. By Proposition 3, the class of generic conformally flat submanifolds for which the conullity is integrable contains the class of generic conformally flat submanifolds with flat normal bundle satisfying the regularity assumption in the statement. The next result shows that for $N = n+2$ there is only one class of examples that belong to the former class but not to the latter.

Corollary 4. *Let $f\colon M^n \to \mathbb{R}^{n+2}$, $n \geq 4$, be a generic isometric immersion of a conformally flat manifold with nowhere flat normal bundle such that the conullity $\mathcal{E}_\eta^\perp$ is integrable. Then $f(M^n)$ is conformally congruent to a rotational submanifold over a surface $M^2 \in \mathbb{R}^4_+$ such that (M^2, g) has constant curvature -1, where g is the metric induced from the hyperbolic metric of constant sectional curvature -1 on $\mathbb{R}^4_+$ with the axis $\mathbb{R}^3$ as the hyperplane at infinity.*

Corollary 4 follows by putting together Corollary 2 and the following result of independent interest.

Proposition 5. *Let $f\colon M^n \to \mathbb{R}^N$ be a rotational submanifold over a submanifold $\varphi\colon M^{n-q} \to \mathbb{R}^{N-q}_+$. Then M^n is conformally flat if and only if one of the following possibilities holds:*

i) $n - q = 1$.

ii) $q = 1$ *and* (M^{n-q}, g) *has constant sectional curvature* K, *where* g *is the metric induced by* φ *from the hyperbolic metric of constant curvature* -1 *on* $\mathbb{R}^{N-q}_+$ *with the axis* $\mathbb{R}^{N-q-1}$ *as the hyperplane at infinity. Moreover,* $K \geq -1$ *if* $n \geq 4$ *and* $N \leq 2n-3$.

iii) $q \geq 2$, $n \geq 4$ *and* (M^{n-q}, g) *has constant sectional curvature* -1.

Theorem 1 and Proposition 5 yield the following result.

Corollary 6. *Let $f\colon M^3 \to \mathbb{R}^4$ be a connected conformally flat hypersurface with three distinct principal curvatures. Assume that the lines of curvature of one family are segments of circles or straight lines in $\mathbb{R}^4$. Then $f(M^3)$ is conformally congruent to an open subset of one of the following:*

i) a product $M^2 \times \mathbb{R}$, where M^2 is a surface in $\mathbb{R}^3$ of constant Gaussian curvature.

ii) a cone CM^2 over a surface M^2 in $\mathbb{S}^3$ of constant Gaussian curvature.

iii) a rotational hypersurface over a surface M^2 of constant Gaussian curvature in $\mathbb{R}^3_+$ endowed with the hyperbolic metric of constant negative sectional curvature.

Finally, we specialize to surfaces $f\colon M^2 \to \mathbb{R}^N$. For such an f notice that admitting a principal curvature normal is equivalent to having flat normal bundle. In this case, there are pointwise one or two principal curvature normals, the first case corresponding to an umbilical point. We have the following generalization of the classical classification of the Cyclides of Dupin in $\mathbb{R}^3$.

Corollary 7. *Let $f\colon M^2 \to \mathbb{R}^N$ be an isometric immersion with flat normal bundle and free of umbilical points. Assume that the lines of curvature of one family are segments of circles or straight lines in $\mathbb{R}^N$ and that those of the other family have constant geodesic curvature in M^2. Then $f(M^2)$ is conformally congruent to an open subset of one of the following:*

i) a product $\beta \times \mathbb{R}$, where β is a curve in $\mathbb{R}^{N-1}$,

ii) a cone over a curve β in the sphere $\mathbb{S}^{N-1}$,

iii) a rotational surface over a curve β in $\mathbb{R}^{N-1}$.

If in the above result we assume the lines of curvature of both families to be segments of circles or straight lines in $\mathbb{R}^N$, then f must be conformally congruent to a Cyclide of Dupin in an affine $\mathbb{R}^3 \subset \mathbb{R}^N$. For surfaces in $\mathbb{R}^3$, it was shown by Bonnet in 1867 ([**Bo**]) that the same conclusion of Corollary 7 holds under the weaker assumption that the lines of curvature of *both* families have constant geodesic curvature. A more geometric proof of this result is due to Ribaucour ([**Ri**], see also [**Da**], Vol III, pg. 121).

§2 The proofs.

We first discuss the main ingredients in the proof of Theorem 1, which are also of independent interest. Although they are essentially known in the literature, we provide complete proofs for the sake of completeness and simplicity. The first one is Reckziegel's basic result referred to in the introduction.

Proposition 8. *Let $f\colon M^n \to \mathbb{R}^N$ be an isometric immersion with a non-vanishing proper principal curvature normal η of multiplicity q. Then the following holds:*

i) $\mathcal{E}_\eta$ *is a spherical distribution on* M^n *whose leaves are* q*-dimensional round spheres in* $\mathbb{R}^N$ *if and only if* η *is parallel in the normal connection of* f *along* $\mathcal{E}_\eta$*.*

ii) If $q \geq 2$ *then* η *is parallel in the normal connection of* f *along* $\mathcal{E}_\eta$*.*

Proof: i) We write $\eta = \lambda\zeta$, where ζ has unit length. Assume first that η is parallel along $\mathcal{E}_\eta$ in the normal connection. Choose $S, T \in \mathcal{E}_\eta$, $X \in \mathcal{E}_\eta^\perp$ and $\xi \in T_f^\perp M$ such that $\xi \perp \eta$. We obtain that

$$(\lambda I - A_\zeta)\nabla_T S = \langle T, S\rangle \nabla\lambda \quad \text{and} \quad \langle A_\xi \nabla_T S, X\rangle = \lambda\langle T, S\rangle\langle \nabla_X^\perp \xi, \zeta\rangle \tag{1}$$

by taking the S-component of the Codazzi equations for (A_ζ, T, X) and (A_ξ, T, X). By this we mean taking the inner product with S of both sides of the Codazzi equations $\nabla A(\zeta, T, X) = \nabla A(\zeta, X, T)$ and $\nabla A(\xi, T, X) = \nabla A(\xi, X, T)$. This terminology is used throughout the paper.

It follows from (1) and $\mathcal{E}_\eta = \cap_{\gamma \in T_f^\perp M} \ker\, (A_\gamma - \langle \gamma, \eta\rangle I)$ that $\nabla_T S \in \mathcal{E}_\eta$ for any orthogonal pair S, T. This implies that $\mathcal{E}_\eta$ is totally umbilical with mean curvature vector δ satisfying

$$(\lambda I - A_\zeta)\delta = \nabla\lambda \tag{2}$$

and

$$\langle A_\xi \delta, X\rangle = \lambda\langle \nabla_X^\perp \xi, \zeta\rangle. \tag{3}$$

A straightforward computation using the Codazzi equation for (A_ζ, T, X) and (2) gives

$$\langle \nabla_T \delta, (\lambda I - A_\zeta)X\rangle = T\langle (\lambda I - A_\zeta)\delta, X\rangle - \langle \delta, \nabla_T(\lambda I - A_\zeta)X\rangle = 0. \tag{4}$$

A similar computation using (3) and the Codazzi equation for (A_ξ, T, X) yields

$$\langle \nabla_T \delta, A_\xi X\rangle = T\langle A_\xi \delta, X\rangle - \langle \delta, \nabla_T A_\xi X\rangle = \langle R^\perp(T, X)\xi, \zeta\rangle = 0. \tag{5}$$

We conclude from (4) and (5) that $\mathcal{E}_\eta$ is spherical. Now, denoting by $\widetilde{\nabla}$ the derivative in the ambient space, we have

$$\widetilde{\nabla}_T S = \nabla_T^v S + \langle T, S\rangle\sigma, \quad \text{where} \ \ \sigma := \delta + \eta. \tag{6}$$

Using that $\nabla^\perp_T\eta = 0$, we get that

$$\widetilde{\nabla}_T\sigma = \nabla_T\delta - A_\eta T = -\|\sigma\|^2 T. \tag{7}$$

It follows by a standard argument that the leaves of $\mathcal{E}_\eta$ are q-dimensional round spheres in $\mathbb{R}^N$. The converse is straightforward.

ii) The Codazzi equation for (A_ζ, T, S) gives

$$T(\lambda) = 0 \quad \text{and} \quad [S,T] \in \ker\,(\lambda I - A_\zeta),$$

whereas the Codazzi equation for (A_ξ, T, S) for ξ orthogonal to η yields

$$\nabla^\perp_T\zeta = 0 \quad \text{and} \quad [S,T] \in \ker\,A_\xi,$$

and the proof follows. ∎

For a given distribution $\mathcal{U}$ on M^n, in the following two results we agree that S, T (respectively, X, Y) are vector fields on $\mathcal{U}$ (respectively, on $\mathcal{U}^\perp$). Moreover, we denote by C the *splitting tensor* of $\mathcal{U}$ which assigns to each $T \in \mathcal{U}$ the endomorphism C_T of $\mathcal{U}^\perp$ given by

$$C_T X = -\nabla^h_X T.$$

Lemma 9. *Let $\mathcal{U}$ be a totally umbilical distribution on M^n with mean curvature vector δ. Then the following differential equations hold:*

$$(\nabla^h_T C_S)X = C_S C_T X + C_{\nabla^v_T S}X - R^h(T,X)S + \langle T, S\rangle\left(\langle X,\delta\rangle\delta - \nabla^h_X\delta\right), \tag{8}$$

$$(\nabla^h_X C_T)Y - (\nabla^h_Y C_T)X = C_{\nabla^v_X T}Y - C_{\nabla^v_Y T}X - R^h(X,Y)T - \langle [X,Y], T\rangle\delta. \tag{9}$$

Moreover, if $\mathcal{U} = \mathcal{E}_\eta$ is the distribution associated to a proper principal curvature normal η of an isometric immersion $f\colon M^n \to \mathbb{R}^N$, then (8) and (9) take, respectively, the form

$$(\nabla^h_T C_S)X = C_S C_T X + C_{\nabla^v_T S}X + \langle T, S\rangle\left(A_\eta X + \langle \delta, X\rangle\delta - \nabla^h_X\delta\right) \tag{10}$$

and

$$(\nabla^h_X C_T)Y - (\nabla^h_Y C_T)X = C_{\nabla^v_X T}Y - C_{\nabla^v_Y T}X - \langle [X,Y], T\rangle\delta. \tag{11}$$

Proof: The proof of (8) follows easily from

$$(\nabla^h_T C_S)X = -\nabla^h_T\nabla^h_X S - C_S\nabla^h_T X = -\nabla^h_T\nabla_X S + \langle T, \nabla_X S\rangle\delta - C_S\nabla^h_T X$$

and

$$\begin{aligned}
&\nabla^h_T\nabla_X S = R^h(T,X)S + \nabla^h_X\nabla_T S + \nabla^h_{[T,X]}S\\
&= R^h(T,X)S + \nabla^h_X\nabla^v_T S + \nabla^h_X\nabla^h_T S + \nabla^h_{[T,X]^v}S + \nabla^h_{\nabla^h_T X}S - \nabla^h_{\nabla^h_X T}S\\
&= R^h(T,X)S - C_{\nabla^v_T S}X + \nabla^h_X\langle T, S\rangle\delta + \langle [T,X], S\rangle\delta - C_S\nabla^h_T X - C_S C_T X.
\end{aligned}$$

To prove (9), we first compute

$$(\nabla^h_X C_T)Y = -\nabla^h_X\nabla^h_Y T - C_T\nabla^h_X Y = -\nabla^h_X\nabla_Y T - C_{\nabla^v_Y T}X + \nabla^h_{\nabla^h_X Y}T.$$

Therefore,

$$(\nabla^h_X C_T)Y - (\nabla^h_Y C_T)X = -R^h(X,Y)T - \nabla^h_{[X,Y]^v}T + C_{\nabla^v_X T}Y - C_{\nabla^v_Y T}X,$$

and the proof follows.

Assume now that $\mathcal{U} = \mathcal{E}_\eta$ as in the statement. Then (10) and (11) are consequences of (8), (9) and the Gauss equations $R(T,X)S = -\langle T, S\rangle A_\eta X$ and $R(X,Y)T = A_{\alpha_f(Y,T)}X - A_{\alpha_f(X,T)}Y = 0$. ∎

The next result also follows from Nolker's in [**No**].

Proposition 10. *Let $f\colon M^n \to \mathbb{R}^N$ be an isometric immersion with a non-vanishing proper principal curvature normal η which carries an ℓ-dimensional spherical distribution $\mathcal{U} \subset \mathcal{E}_\eta$ such that $\mathcal{U}^\perp$ is totally geodesic in M^n. Then f is a rotational submanifold over an $(n-\ell)$-dimensional submanifold of $\mathbb{R}^{N-\ell}$.*

Proof: Let δ denote the mean curvature of $\mathcal{U}$. Since (6) and (7) hold, then the leaves of $\mathcal{U}$ are ℓ-dimensional round spheres in $\mathbb{R}^N$. We claim that

$$\widetilde{\nabla}_X\sigma = \langle X, \delta\rangle\sigma. \tag{12}$$

To prove the claim, first observe that the splitting tensor of $\mathcal{U}$ vanishes identically since $\mathcal{U}^\perp$ is totally geodesic. Then (10) yields

$$\nabla_X\delta = A_\eta X + \langle X, \delta\rangle\delta. \tag{13}$$

The Codazzi equation for (A_ξ, T, X) with $\xi \perp \eta = \lambda\zeta$ implies that

$$\langle \alpha_f(X,\delta), \xi\rangle + \lambda\langle \nabla^\perp_X \zeta, \xi\rangle = 0, \tag{14}$$

whereas the first equation in (1) for $S = T$ gives

$$X(\lambda) = \lambda\langle X, \delta\rangle - \langle A_\zeta X, \delta\rangle. \tag{15}$$

We obtain the claim by replacing (13), (14) and (15) in

$$\widetilde{\nabla}_X \sigma = X(\lambda)\zeta - \lambda A_\zeta X + \lambda \nabla^\perp_X \zeta + \nabla_X \delta + \alpha_f(X, \delta).$$

Since $\mathcal{U}^\perp$ is totally geodesic, we have that $\widetilde{\nabla}_X T = \nabla_X T \in \mathcal{U}$. It follows using (6), (7) and (12) that the subspaces $L = \mathcal{U} \oplus \mathrm{span}\{\sigma\}$ containing the leaves of $\mathcal{U}$ are parallel in $\mathbb{R}^N$. Let $\Gamma = f + \|\sigma\|^{-2}\sigma$ be the submanifold generated by the centers of the leaves of $\mathcal{U}$. Using (12) we get

$$\Gamma_* X = X - \|\sigma\|^{-2}\langle X, \delta\rangle \sigma.$$

Since $\Gamma_*\mathcal{U}^\perp$ is orthogonal to L, we conclude that f is a rotational submanifold whose axis is an affine subspace $\mathbb{R}^{N-\ell-1}$ orthogonal to L. ∎

We are now in position to prove our main result.

Proof of Theorem 1: Throughout this proof we agree that $T \in \mathcal{E}_\eta$ and $X, Y \in \mathcal{E}_\eta^\perp$. We have from Proposition 8 that $\mathcal{E}_\eta$ is spherical with mean curvature δ. By assumption, there is a vector field $\beta \in \mathcal{E}_\eta$ such that

$$\nabla^v_X Y = \langle X, Y\rangle \beta. \tag{16}$$

Suppose that U is an open subset of M^n where one of the following holds everywhere: *a)* $\beta = 0 = \eta$, *b)* $\beta = 0 \neq \eta$, *c)* $\beta \neq 0 = \eta$, or *d)* $\beta \neq 0 \neq \eta$.

Assume first that either *a)* or *b)* holds. Then, it follows easily that each leaf of $\mathcal{E}_\eta^\perp$ is contained in an $(N-q)$-dimensional affine subspace orthogonal to $\mathcal{E}_\eta$ along the leaf. If η vanishes identically, then the leaves of $\mathcal{E}_\eta$ are q-dimensional parallel affine subspaces orthogonal to the affine subspaces containing the leaves of $\mathcal{E}_\eta^\perp$. Hence, f is as in part *i)* of the statement when *a)* holds. In case *b)* we have that f is as in part *iii)* by Proposition 10.

Assume now that β is nowhere zero in U and write $\beta = \mu\mathcal{T}$, where $\mathcal{T}$ has unit length. Suppose $q \geq 2$ and consider the orthogonal splitting $TM = \mathcal{U} \oplus \mathcal{U}^\perp$, where

$$\mathcal{U}^\perp = \mathrm{span}\{\mathcal{T}\} \oplus \mathcal{E}_\eta^\perp.$$

We claim that $\mathcal{U}^\perp$ is totally geodesic and that $\mathcal{U}$ is spherical when $\eta \neq 0$ and totally geodesic when $\eta = 0$. To prove the claim, first observe that (16) is equivalent to

$$C_T = \mu\langle \mathcal{T}, T\rangle I. \tag{17}$$

In the following, S and S' denote vector fields in $\mathcal{U}$. If $q \leq n-2$, replacing (17) in (11) easily gives

$$\langle \nabla_X \mathcal{T}, S\rangle = 0 \quad \text{and} \quad X(\mu) = 0, \tag{18}$$

whereas (18) holds by assumption for $q = n-1$. Moreover, we obtain from (10) that

$$\langle \nabla_{\mathcal{T}} \mathcal{T}, S\rangle = 0. \tag{19}$$

We conclude that $\mathcal{U}^\perp$ is totally geodesic from $\langle \nabla_{\mathcal{T}} X, S\rangle = -\langle S, \mathcal{T}\rangle\langle \delta, X\rangle = 0$, (16), (18) and (19).

Now, we have from (10) and (17) that

$$\mu\langle \nabla_S S', \mathcal{T}\rangle X = -\langle S, S'\rangle \left(A_\eta X + \langle \delta, X\rangle\delta - \nabla^h_X \delta\right). \tag{20}$$

If $\eta = 0$, then $\delta = 0$ and $\mathcal{E}_\eta$ is totally geodesic, hence also $\mathcal{U}$ is totally geodesic. If $\eta \neq 0$, it follows from (20) that we may write

$$\langle \nabla_S S', \mathcal{T}\rangle = \rho\langle S, S'\rangle \tag{21}$$

for some smooth function ρ, i.e., $\mathcal{U}$ is totally umbilical. Suppose first that $q \geq 3$. Then $\dim \mathcal{U} \geq 2$ and, the leaves of $\mathcal{E}_\eta$ being spheres in the ambient space, the same holds for the leaves of $\mathcal{U}$. Since $\mathcal{E}_\eta$ is spherical, it follows easily that also $\mathcal{U}$ is spherical. To conclude the proof of the claim it remains to show that the integral curves of $\mathcal{U}$ are circles when $q = 2$. Set $\delta = \alpha W$, where $\alpha = \|\delta\|$. Hence, $T(\alpha) = 0$ and $\nabla^h_T W = 0$. It follows using (18) that

$$\nabla_S W = -\alpha S \quad \text{and} \quad \nabla_W S = 0. \tag{22}$$

To show that $S(\rho) = 0$, we use that $\mu\rho = \langle A_\eta W, W\rangle - W(\alpha) - \alpha^2$, as follows from (20). On one hand, we get

$$S(\mu) = S\langle \nabla_W W, \mathcal{T}\rangle = \langle \nabla_S \nabla_W W, \mathcal{T}\rangle = \langle \nabla_W \nabla_S W, \mathcal{T}\rangle + \langle \nabla_{[S,W]} W, \mathcal{T}\rangle = 0 \tag{23}$$

using (22) and $\langle R(S,W)W, \mathcal{T}\rangle = 0$. On the other hand, we easily obtain from (22) and the Codazzi equation for (A_η, S, W) that

$$S\langle A_\eta W, W\rangle = \langle \nabla_S A_\eta W, W\rangle = 0. \tag{24}$$

Since $S(W(\alpha)) = [S, W](\alpha) = 0$ by (22), we conclude from (23) and (24) that $S(\rho) = 0$, as we wished.

By the claim, if $\eta = 0$ then $f(U)$ is a product $\widetilde{M}^{n-q+1} \times \mathbb{R}^{q-1}$, where $\widetilde{M}^{n-q+1} \subset \mathbb{R}^{N-q+1}$. If $\eta \neq 0$, the claim and Proposition 10 imply that f is a rotational submanifold over a submanifold $\widetilde{M}^{n-q+1} \subset \mathbb{R}^{N-q+1}$ with axis $\mathbb{R}^{N-q} \subset \mathbb{R}^{N-q+1}$. In both cases notice that $\widetilde{M}^{n-q+1}$ is a leaf of $\mathcal{U}^\perp$.

We now make a detailed study of $\widetilde{M}^{n-q+1}$. Observe that $\widetilde{M}^{n-q+1} = M^n$ when $q = 1$. We have that $\mathcal{E}_\eta^\perp$ is totally geodesic when $\eta = 0$. When $\eta \neq 0$ and $q \leq n-2$, it follows from (18) that $= \mathcal{E}_\eta^\perp$ is spherical. By assumption, this is also the case when $q = n-1$. On the other hand, we have from (19) that

$$\widetilde{\nabla}_{\mathcal{T}} \mathcal{T} = \delta + \eta := \gamma.$$

If $\eta = 0$, then also $\gamma = 0$, hence the integral curves of $\mathcal{T}$ are segments of straight lines in $\mathbb{R}^{N-q+1}$. When $\eta \neq 0$, it follows from $\widetilde{\nabla}_{\mathcal{T}} \gamma = -\|\gamma\|^2 \mathcal{T}$ that they are arcs of circles in $\mathbb{R}^{N-q+1}$. In both cases, we obtain using $\alpha_f(X, \mathcal{T}) = 0$ that $\mathcal{T}$ is parallel in the normal connection of a leaf of $\mathcal{E}_\eta^\perp$ in the ambient space. Moreover, it is an umbilical normal vector field with constant eigenvalue μ along the leaf. We conclude that each leaf of $\mathcal{E}_\eta^\perp$ lies in a $(N-q)$-dimensional sphere of radius μ^{-1} orthogonal to $\mathcal{T}$ in $\mathbb{R}^{N-q+1}$. Let $\mathcal{F}$ denote the family of such spheres. Their centers are parametrized by

$$F = f + \mu^{-1} \mathcal{T}. \tag{25}$$

Differentiating (25) we get

$$F_* \mathcal{T} = \mu^{-2} Z, \quad \text{where } \; Z = (\mu^2 - \mathcal{T}(\mu)) \mathcal{T} + \mu \gamma. \tag{26}$$

On the other hand, we obtain from (10) for $T = S = \mathcal{T}$ that

$$(\mu^2 - \mathcal{T}(\mu)) X = \nabla_X^h \delta - \langle \delta, X \rangle \delta - A_\eta X. \tag{27}$$

If $\eta = 0$ this implies that $Z = 0$, that is, $\mathcal{F}$ is a family of concentric spheres. Moreover, the straight lines containing the integral curves of $\mathcal{T}$ pass through their common center. We conclude that $\widetilde{M}^{n-q+1} \subset \mathbb{R}^{N-q+1}$ is the cone over a submanifold M^{n-q} of the sphere $\mathbb{S}^{N-q} \subset \mathbb{R}^{N-q+1}$, hence f is as in part *ii)* of the statement in case *c)*.

If $\eta \neq 0$, then (27) yields $\mu^2 - \mathcal{T}(\mu) = W(\alpha) - \alpha^2 - \lambda \langle A_\zeta W, W \rangle$, where $\lambda \zeta = \eta$. Thus,

$$Z = (W(\alpha) - \alpha^2 - \lambda \langle A_\zeta W, W \rangle) \mathcal{T} + \mu \gamma. \tag{28}$$

On one hand,

$$\mathcal{T}W(\alpha) = [\mathcal{T}, W](\alpha) = \nabla_{\mathcal{T}}W(\alpha) - \nabla_W\mathcal{T}(\alpha) = \mu W(\alpha). \tag{29}$$

On the other hand, taking the W component of the Codazzi equation for $(A_\zeta, W, \mathcal{T})$ yields

$$\mathcal{T}\langle A_\zeta W, W\rangle = \mu\langle A_\zeta W, W\rangle - \lambda\mu. \tag{30}$$

Using (28), (29) and (30) we easily obtain that $\widetilde{\nabla}_{\mathcal{T}}Z = \mu Z$. Therefore, F parametrizes a straight line r. Moreover, comparing (20) and (27) we get $\rho\mu = \mu^2 - \mathcal{T}(\mu)$, thus $Z = \mu\sigma$, where $\sigma = \rho\mathcal{T} + \delta + \eta$ is the mean curvature vector of the orbits of f, that is, the leaves of $\mathcal{U}$. Hence r is orthogonal to the axis $\mathbb{R}^{N-q} \subset \mathbb{R}^{N-q+1}$.

Observe that r is contained in the plane of each circle C containing an integral curve of $\mathcal{T}$, hence such planes intersect along r. In particular, each plane intersects the axis $\mathbb{R}^{N-q}$ orthogonally along a line s. We show now that s passes through the center O of C, that is, $O \in \mathbb{R}^{N-q}$. Let x be a point on C where $\mathcal{T}$ is parallel to $\mathbb{R}^{N-q}$. Since the normal vector γ to C at x is orthogonal to $\mathbb{R}^{N-q}$, all we need to show is that the distance from x to $\mathbb{R}^{N-q}$ equals the radius $\|\gamma\|^{-1}$ of C. Since the mean curvature vector $\sigma = \rho\mathcal{T} + \gamma$ of the orbits of f is everywhere orthogonal to $\mathbb{R}^{N-q}$, we must have $\rho(x) = 0$, hence, the distance from x to $\mathbb{R}^{N-q}$ is $\|\sigma(x)\|^{-1} = \|\gamma\|^{-1}$, as we wished.

For a fixed circle C containing an integral curve of $\mathcal{T}$, we consider separately the cases where r and C: i) Intersect at two points P_1, P_2; ii) Are tangent at some point P; iii) Are disjoint.

Suppose first that i) holds and consider an inversion I whose pole is, say, P_1. Then $I(r) = r$ and $I(C) = t$ is a straight line through $\widetilde{P}_2 = I(P_2) \in r$. Since each sphere $\mathcal{L}$ of $\mathcal{F}$ is orthogonal to C and r, we have that $I(\mathcal{L})$ is orthogonal to $I(C) = t$ and $I(r) = r$, hence $I(\mathcal{L})$ is a sphere with center at $\widetilde{P}_2 = t \cap r$. Therefore, $\widetilde{\mathcal{F}} = I(\mathcal{F})$ is a family of concentric spheres with center $\widetilde{P}_2$. Consider now another integral circle C' of $\mathcal{T}$. Since C' is orthogonal to each element of $\mathcal{F}$, we have that $I(C')$ is orthogonal to each sphere of $\widetilde{\mathcal{F}}$, hence $I(C')$ is a straight line through $\widetilde{P}_2$. In particular, this implies that C' intersects r at the same points P_1, P_2. We conclude that $I(\widetilde{M}^{n-q+1})$ is contained in a cone CM^{n-q}, where M^{n-q} is the image of a leaf of $\mathcal{E}_\eta^\perp$.

Assume now that ii) holds and consider an inversion I with pole P. Then $I(r) = r$ and $I(C) = t$ is a straight line parallel to r. Therefore, $\widetilde{\mathcal{F}} = I(\mathcal{F})$ is a family of parallel hyperplanes in $\mathbb{R}^{N-q+1}$ orthogonal to r. Given another integral circle C' of $\mathcal{T}$, we have that $I(C')$ must be a straight line parallel to r, since it is orthogonal to each hyperplane of $\widetilde{\mathcal{F}}$. In particular, it follows that C' must be tangent to r at the same point P. Hence, $I(\widetilde{M}^{n-q+1})$ is contained in a product $M^{n-q} \times \mathbb{R}$.

Finally, assume that $iii)$ holds. Let $\mathcal{H}$ be the hyperplane of $\mathbb{R}^{N-q+1}$ orthogonal to r through the center of C and let $\mathbb{S}^{N-q-1}$ be the sphere along which a fixed sphere $\mathbb{S}_0^{N-q}$ of $\mathcal{F}$ intersects $\mathcal{H}$. Consider an inversion whose pole is any point of $\mathbb{S}^{N-q-1}$. We have that $I(\mathcal{H}) = \mathcal{H}$ and $I(\mathbb{S}_0^{N-q}) = \mathbb{R}^{N-q}$ is a hyperplane of $\mathbb{R}^{N-q+1}$ which intersects $\mathcal{H}$ along $\mathbb{R}^{N-q-1} = I(\mathbb{S}^{N-q-1})$. Since $I(r)$ and $I(C)$ are circles orthogonal to $I(\mathcal{H}) = \mathcal{H}$ and to $I(\mathbb{S}_0^{N-q}) = \mathbb{R}^{N-q}$, both must be contained in planes orthogonal to $\mathbb{R}^{N-q-1}$ and must have their centers at $\mathbb{R}^{N-q-1}$. Given another sphere $\mathbb{S}^{N-q}$ of $\mathcal{F}$, we have that $I(\mathbb{S}^{N-q})$ is orthogonal to $I(r)$ and $I(C)$, hence $I(\mathbb{S}^{N-q})$ is a hyperplane of $\mathbb{R}^{N-q+1}$ containing $\mathbb{R}^{N-q-1}$. Therefore, $\widetilde{\mathcal{F}} = I(\mathcal{F})$ is a family of hyperplanes in $\mathbb{R}^{N-q+1}$ intersecting along $\mathbb{R}^{N-q-1}$. In particular, this implies that all spheres of $\mathcal{F}$ intersect along $\mathbb{S}^{N-q-1}$. Moreover, given another integral circle C' of $\mathcal{T}$, we have that $I(C')$ is orthogonal to any hyperplane of $\widetilde{\mathcal{F}}$, hence it is also a circle in a plane orthogonal to $\mathbb{R}^{N-q-1}$ with center at $\mathbb{R}^{N-q-1}$. In particular, C' does not intersect r. Therefore, $I(\widetilde{M}^{n-q+1})$ is a rotational submanifold with one-dimensional orbits and $\mathbb{R}^{N-q-1}$ as axis.

Now we determine $I(f(U))$ in each of the three cases above. In cases $i)$ or $ii)$, for each point $Q \in \widetilde{M}^{n-q+1}$ the image $I(\ell)$ of the leaf ℓ of $\mathcal{E}_\eta$ through Q is a q-dimensional subspace which intersects $\mathbb{R}^{N-q+1}$ orthogonally along the line $I(C)$. It follows that $I(f(U))$ is contained in a product $CM^{n-q} \times \mathbb{R}^{q-1}$ in case $i)$ or in a product $M^{n-q} \times \mathbb{R}^q$ in case $ii)$.

Consider now case $iii)$. We have seen that the center O of any circle C containing an integral curve of $\mathcal{T}$ belongs to the axis $\mathbb{R}^{N-q}$ of f. Hence, the euclidean subspace containing the sphere $\mathbb{S}^{N-q-1}$ along which all spheres of $\mathcal{F}$ intersect is precisely the axis $\mathbb{R}^{N-q}$. Our aim is to show that $g(U) = I(f(U))$ is a rotational submanifold whose axis is $\mathbb{R}^{N-q-1} = I(\mathbb{S}^{N-q-1})$. First we prove the following general fact.

Lemma 11. *Let h: $M^n \to \mathbb{R}^N$ be a rotational submanifold with axis $\mathbb{R}^{N-q}$ over a submanifold M^{n-q+1} of $\mathbb{R}^{N-q+1} \supset \mathbb{R}^{N-q}$ and let I be an inversion whose pole is any point in the axis. Then $I \circ h$ is also a rotational submanifold with the same axis over the image of M^{n-q+1} by I.*

Proof: The axis $\mathbb{R}^{N-q}$ is invariant under I. Moreover, the subspaces $\mathbb{R}^q$ containing the orbits of f are mapped onto spheres $\bar{\mathbb{S}}^q$ through the pole.

Since each subspace $\mathbb{R}^q$ is orthogonal to $\mathbb{R}^{N-q}$, the same holds for its image $\bar{\mathbb{S}}^q$. Hence, $\bar{\mathbb{S}}^q$ has its center on $\mathbb{R}^{N-q}$. Thus, the image $I(\mathbb{S}^{q-1})$ of each orbit $\mathbb{S}^{q-1} \subset \mathbb{R}^q$ of f lies on the intersection of $\bar{\mathbb{S}}^q = I(\mathbb{R}^q)$ with the cone over $\mathbb{S}^{q-1}$ with vertex at the pole. Therefore, $I(\mathbb{S}^{q-1})$ is also a sphere with center on $\mathbb{R}^{N-q}$ and contained in a subspace $\bar{\mathbb{R}}^q$ orthogonal to $\mathbb{R}^{N-q}$. ∎

By Lemma 11, we have that $g(U) = I(f(U))$ is a rotational submanifold over $I(\widetilde{M}^{n-q+1})$ with the same axis $\mathbb{R}^{N-q} \subset \mathbb{R}^{N-q+1}$ as f. Let $e_1, \ldots, e_N$ be an orthonormal basis of $\mathbb{R}^N$ such that $e_1, \ldots, e_{N-q+1}$ span $\mathbb{R}^{N-q+1}$ and e_{N-q+1} is orthogonal to the axis $\mathbb{R}^{N-q}$. Choose a coordinate system on $\mathbb{R}^N$ with respect to $e_1, \ldots, e_N$ with the origin in the axis $\mathbb{R}^{N-q}$. Then, g can be described parametrically as

$$g = (g_1, \ldots, g_{N-q}, g_{N-q+1}\phi), \tag{31}$$

where $g_i = g_i(x_1, \ldots, x_{n-q+1})$, $1 \leq i \leq N - q + 1$, parametrizes the profile $I(\widetilde{M}^{n-q+1})$ of g and $\phi(t_1, \ldots, t_{q-1})$ the unit $(q$–$1)$-dimensional sphere. On the other hand, we have seen that $I(\widetilde{M}^{n-q+1})$ is itself a rotational submanifold with one-dimensional orbits with axis $\mathbb{R}^{N-q-1}$. In terms of the parametrization of $I(\widetilde{M}^{n-q+1})$, this means that for some function ψ we have that

$$g_i = g_i(x_1, \ldots, x_{n-q}), \quad 1 \leq i \leq N - q - 1,$$

$$g_{N-q} = \psi(x_1, \ldots, x_{n-q}) \cos x_{n-q+1}, \quad g_{N-q+1} = \psi(x_1, \ldots, x_{n-q}) \sin x_{n-q+1}.$$

Therefore $g = (g_1, \ldots, g_{N-q}, \psi\bar{\phi})$, where

$$\bar{\phi}(t_1, \ldots, t_{q-1}, x_{n-q+1}) = (\cos x_{n-q+1}, \sin x_{n-q+1}\phi(t_1, \ldots, t_{q-1}))$$

is a parametrization of the unit q-dimensional sphere. We conclude that $g(U)$ is as in part *iii*).

We now argue that if there exists a non empty open subset $U \subset M^n$ such that U is an open subset of a product $M^{n-q} \times \mathbb{R}^q$, a product $CM^{n-q} \times \mathbb{R}^{q-1}$, a rotational submanifold over a submanifold M^{n-q} of $\mathbb{R}^{N-q}$ or the image by an inversion of an open subset of one of these submanifolds, then the same must hold for the entire submanifold M^n. In fact, in the first and third cases the leaves of $\mathcal{E}_\eta$ are open subsets of parallel affine subspaces or spheres contained in parallel affine subspaces, respectively. In the other cases, let β be given by (16). Then, the integral curves of β are arcs of straight lines through a common point in case *ii*) and arcs of circles contained in planes which intersect along a straight line r in the remaining ones. Moreover, the circles either are all tangent at a common point $P \in r$, intersect along two fixed points P and Q of r or do not intersect r. On the other hand, the leaves of $\mathcal{E}_\eta^\perp$ lie on $(N-q)$-dimensional spheres which are concentric in case $ii)$ and have their centers on a common line in the remaining ones. Furthermore, such spheres are either disjoint, are all tangent to a plane at a common point or intersect at a common $(N-q-1)$-dimensional sphere. Since $\mathcal{E}_\eta$ is a globally defined distribution on M^n by assumption, it follows easily that open subsets correspondent to any two of the above six possibilities can not be glued together.

Finally, the proof of the last statement follows from two elementary facts. One is that the normal curvature tensor of a submanifold is a conformal invariant; cf. [**Ch**]. The other one is that a rotational submanifold has flat normal bundle if and only if the same holds for its generating submanifold; cf. [**No**]. ∎

Proof of Corollary 2: The Codazzi equation yields

$$\begin{aligned}\langle C_T X, A_\xi Y\rangle = \langle \eta,\xi\rangle\langle\nabla_X Y, T\rangle - \langle A_\xi Y, \nabla_T X\rangle - \langle A_\xi X, \nabla_T Y\rangle \\ + T\langle A_\xi X, Y\rangle - \langle A_{\nabla_T^\perp \xi} X, Y\rangle,\end{aligned} \tag{32}$$

for any $T \in \mathcal{E}_\eta$, $X, Y \in \mathcal{E}_\eta^\perp$ and $\xi \in T_f^\perp M$. From the integrability of $\mathcal{E}_\eta^\perp$ it follows that C_T is symmetric for any $T \in \mathcal{E}_\eta$, and that the first term in the right hand side of (32) is symmetric in X and Y. Hence

$$\left[C_T, A_\xi|_{\mathcal{E}_\eta^\perp}\right] = 0. \tag{33}$$

Since $\mathcal{E}_\eta^\perp$ has dimension 2, at any point of M^n either there exists $T_0 \in \mathcal{E}_\eta$ such that $C_T = \langle T, T_0\rangle I$ for any $T \in \mathcal{E}_\eta$ or there is $T_1 \in \mathcal{E}_\eta$ such that C_{T_1} (is symmetric and) has two distinct real eigenvalues. If the latter possibility holds at some point, then it also holds in an open neighborhood U. It follows from (33) that f has flat normal bundle on U, contradicting our assumption. Hence, the first possibility holds everywhere, which is equivalent to $\mathcal{E}_\eta^\perp$ being a totally umbilical distribution on M^n. The conclusion now follows from Theorem 1. ∎

Proof of Proposition 3: The Codazzi equation gives

$$\langle\nabla_{X_i} X_j, X_k\rangle(\eta_j - \eta_k) = \langle\nabla_{X_j} X_i, X_k\rangle(\eta_i - \eta_k)$$

for all unit vectors $X_i \in \mathcal{E}_{\eta_i}$, $X_j \in \mathcal{E}_{\eta_j}$ and $X_k \in \mathcal{E}_{\eta_k}$ any $i \neq j \neq k \neq i$. In particular, $\mathcal{E}_{\eta_k}^{\perp}$ is integrable (regardless of the multiplicity of $\mathcal{E}_{\eta_k}$) if $\eta_i - \eta_k$ and $\eta_j - \eta_k$ are linearly independent for $i \neq j \neq k \neq i$. On the other hand, by Kulkarni's well-known criterion for conformal flatness,

$$K(X_1, X_2) + K(X_3, X_4) = K(X_1, X_3) + K(X_2, X_4)$$

for all orthonormal X_1, X_2, X_3, X_4, where $K(X_i, X_j)$ denotes the sectional curvature of the plane spanned by X_i and X_j. For $X_1 \in \mathcal{E}_{\eta_i}, X_2 \in \mathcal{E}_{\eta_j}$ and $X_3, X_4 \in \mathcal{E}_{\eta_k}$, we get that

$$\langle \eta_i - \eta_k, \eta_j - \eta_k \rangle = 0$$

for any pair $i \neq j$ with $i, j \neq k$, and this concludes the proof. ∎

Proof of Proposition 5: Consider the conformal diffeomorphism

$$\Theta \colon \mathbb{R}^N \to \mathbb{H}^{N-q} \times \mathbb{S}^q \subset \mathbb{L}^{N-q+1} \times \mathbb{R}^{q+1} = \mathbb{L}^{N+2}$$

given in terms of a pseudo–orthonormal basis $\{e_1, \ldots, e_{N+2}\}$ of standard flat Lorentzian space $\mathbb{L}^{N+2}$ with $\|e_1\| = 0 = \|e_{N-q+1}\|, \langle e_1, e_{N-q+1} \rangle = -1/2$ and $\langle e_i, e_j \rangle = \delta_{ij}$ if $i \neq 1, N-q+1$, by

$$\Theta(a_1, \ldots, a_N) = \left(\sum_{j=N-q}^{N} a_j^2 \right)^{-1/2} \left(1, a_1, \ldots, a_{N-q-1}, \sum_{j=1}^{N} a_j^2, a_{N-q}, \ldots, a_N \right).$$

Let f be parametrized by

$$\Psi(x, t) = (\varphi_1(x), \ldots, \varphi_{N-q-1}(x), \varphi_{N-q}(x)\phi(t)),$$

where $\varphi = (\varphi_1, \ldots, \varphi_{N-q})$ and ϕ parametrizes the unit sphere $\mathbb{S}^q \subset \mathbb{R}^{q+1}$. Then, $\Theta \circ \Psi \colon M^{n-q} \times \mathbb{S}^q \to \mathbb{H}^{N-q} \times \mathbb{S}^q$ satisfies

$$\Theta \circ \Psi = (\Phi \circ \varphi) \times id,$$

where $\Phi \colon \mathbb{R}_+^{N-q} \to \mathbb{H}^{N-q} \subset \mathbb{L}^{N-q+1}$, defined as

$$\Phi(x_1, \ldots, x_{N-q}) = x_{N-q}^{-1} \left(1, x_1, \ldots, x_{N-q-1}, \sum_{i=1}^{N-q} x_i^2 \right),$$

is an isometry between the half–space and hyperboloidal models of $\mathbb{H}^{N-q}$. Since Θ is conformal, the riemannian product $(M^{n-q}, g) \times \mathbb{S}^q$ must be conformally flat. The statement now follows from Proposition 2 of [**La**]. The restriction on K in part *ii*) is due to the nonimmersibility of a space form $M^m(K)$ into another space form $M^{m+p}(\tilde{K})$ when $m \geq 3$, $K < \tilde{K}$ and $p \leq m-2$. ∎

Proof of Corollary 6: Let e_1, e_2, e_3 denote the unit principal directions correspondent to the distinct principal curvatures $\lambda_1, \lambda_2, \lambda_3$, respectively. It was shown by E. Cartan (see [**La**], p. 84) that conformal flatness is equivalent to the relations

$$\langle \nabla_{e_i} e_j, e_k \rangle = 0 \tag{34}$$

and

$$(\lambda_j - \lambda_k) e_i(\lambda_i) + (\lambda_i - \lambda_k) e_i(\lambda_j) + (\lambda_j - \lambda_i) e_i(\lambda_k) = 0, \tag{35}$$

for all distinct indices i, j, k. It follows from Codazzi's equation and (34) that

$$\nabla_{e_i} e_i = \sum_{j \neq i} (\lambda_i - \lambda_j)^{-1} e_j(\lambda_i) e_j. \tag{36}$$

Assume for instance that $e_3(\lambda_3) = 0$. Then equation (35) yields

$$(\lambda_2 - \lambda_3) e_3(\lambda_1) = (\lambda_1 - \lambda_3) e_3(\lambda_2),$$

hence the distribution spanned by e_1 and e_2 is umbilic in M^3 from (36). By Theorem 1, we have that $f(M^3)$ is conformally congruent to an open subset of one of the following: *i*) a product $M^2 \times \mathbb{R}$, where M^2 is a surface in $\mathbb{R}^3$, *ii*) a cone CM^2 over a surface $M^2 \subset \mathbb{S}^3$, *iii*) a rotational hypersurface with axis $\mathbb{R}^2 \subset \mathbb{R}^3$ over a surface $M^2 \subset \mathbb{R}^3$. It follows from Propositions 1 and 2 of [**La**] and Proposition 5 that M^2 must be as stated. ∎

Remark 12. Corollary 6 can also be derived from the results in [**H-J**] (cf. pp. 328). The special case of conformally flat hypersurfaces with constant mean curvature and vanishing Gauss–Kroenecker curvature was also considered in [**Fu**], where they are shown to be cones over minimal Clifford tori.

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