A SPLITTING THEOREM FOR EUCLIDEAN SUBMANIFOLDS OF NONPOSITIVE SECTIONAL CURVATURE

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In **[F]** we showed that the index of relative nullity ν of an isometric immersion $f: M^n \to N^{n+p}$ between Riemannian manifolds with $K_M \leq K_N$ satisfies $\nu \geq n-2p$. Here, K_V stands for the sectional curvature of the manifold V and the *index of relative nullity* ν of f at $x \in M^n$ is the dimension of the relative nullity space of the second fundamental form α of f, i.e.,

$$\nu(x) = \dim \{X \in T_xM : \alpha(X,Y) = 0, \ \forall\, Y \in T_xM\}.$$

This result has several applications because of the strong restrictions that having $\nu > 0$ imposes on the manifold and the isometric immersion (see **[F]** and **[G]**, §3.2.2). A simple example shows that our estimate is sharp.

Example. Let $g_i : H_i^{n_i} \to \mathbf{R}^{n_i+1}$, $1 \leq i \leq p$, be nowhere flat Euclidean hypersurfaces of nonpositive sectional curvature. The Gauss equation implies that its index of relative nullity is $n_i - 2$. Therefore, the product manifold $M^n = H_1^{n_1} \times \cdots \times H_p^{n_p}$ has nonpositive sectional curvature and its product immersion $g = g_1 \times \cdots \times g_p$ into $\mathbf{R}^{n+p}$ verifies $\nu \equiv n-2p$. Notice that g has flat normal bundle.

The main purpose of this paper is to improve the above estimate to $\nu \geq n-p-1$ for irreducible Euclidean submanifolds with flat normal bundle. For an integer $r \geq 2$, we say that an isometric immersion $g : M^n \to \mathbf{R}^{n+p}$ *splits as an r–product* if $M^n = M_1^{n_1} \times \cdots \times M_r^{n_r}$ and there exist isometric immersions $g_j : M_j^{n_j} \to \mathbf{R}^{n_j+p_j}$, $1 \leq j \leq r$, $n_j \geq 1$, such that $g = g_1 \times \cdots \times g_r$.

Theorem 1. *Let $f : M^n \to \mathbf{R}^{n+p}$ be an isometric immersion with flat normal bundle into Euclidean space of a Riemannian manifold with nonpositive sectional curvature. Assume for some integer $2 \leq r \leq p$ that $\nu \leq n-p-r$ everywhere. Then there exists an open dense* *subset $\mathcal{U} \subset M^n$ such that $f|_{\mathcal{U}} : \mathcal{U} \to \mathbf{R}^{n+p}$ splits locally as an r–product of nowhere flat Euclidean submanifolds.*

Notice that the theorem implies that the above example is unique among those with flat normal bundle into Euclidean space and $\nu \equiv n - 2p$. Moreover, in this situation, from Theorem 1 of **[M]** we have that $f|_{\mathcal{U}}$ is locally isometrically rigid if and only if each factor is rigid.

In the following result, Ric_M denotes the Ricci curvature of M^n and Q_c^m a complete simply connected Riemannian manifold of constant sectional curvature c.

Corollary 2. *Let M^n be an immersed connected submanifold of Q_c^{n+p}, $p \leq \frac{n}{2}$, with $K_M \leq c$ and $Ric_M < c$. If the normal bundle is flat, then $c = 0$, $n = 2p$ and M^n splits locally as a p–product of surfaces in $\mathbf{R}^3$ of negative Gaussian curvature. Moreover, the splitting is global provided that M^n is a Hadamard manifold.*

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The proofs

We first introduce some definitions and notations. Let V^n and W^s be real vector spaces endowed with positive definite inner products, both denoted by $\langle\ ,\ \rangle$. We say that a vector valued symmetric bilinear map $\alpha : V^n \times V^n \to W^s$ is *nonpositive* if

$$\langle \alpha(X,X), \alpha(Y,Y) \rangle - \|\alpha(X,Y)\|^2 \leq 0, \quad \forall\ X, Y \in V^n.$$

The map α is *diagonalizable* if there exists an orthonormal basis $\{X_1, \ldots, X_n\}$ of V^n such that $\alpha(X_i, X_k) = 0$, for all $1 \leq i \neq k \leq n$. We denote by Δ the relative nullity space of α, i.e., $\Delta = \{X \in V^n : \alpha(X,Y) = 0,\ \forall\ Y \in V^n\}$, and set $\nu =$ dim Δ. By $\#A$ we mean de number of elements of the set A. From now on, j, j' stand for arbitrary integers

$$1 \leq j \leq r \quad \text{and} \quad 0 \leq j' \leq r.$$

Next, we extend Proposition 9 of **[F]** to the case of diagonalizable bilinear maps.

Proposition 3. *Let $\alpha : V^n \times V^n \to W^s$ be a nonpositive diagonalizable symmetric bilinear map with a diagonalizing orthonormal basis $\{X_1, \dots, X_n\}$. Assume that span $\{\eta_i = \alpha(X_i, X_i) : 1 \le i \le n\} = W^s$. Then $\nu \ge n - 2s$. Furthermore, if $\nu = n - s - r$, then there exists a partition $\{1, \dots, n\} = \bigcup_{j'=0}^{r} I_{j'}$ which for any $1 \le j \le r$ verifies:*

i) $\# I_j \ge 2$ and $L_j =$ span $\{\eta_i : i \in I_j\}$ is a subspace of dimension $\# I_j - 1$,

ii) there are $\# I_j$ negative numbers $\{a_j^i : i \in I_j\}$ such that $\sum_{i \in I_j} a_j^i \eta_i = 0$,

iii) $L_j \perp L_{j'}$, for all $j' \ne j$.

Proof. Since α is diagonalizable, its nonpositiveness is equivalent to

$$\langle \eta_i, \eta_k \rangle \le 0, \quad 1 \le i \ne k \le n. \tag{1}$$

Suppose that $\{\eta_1, \dots, \eta_s\}$ is a basis of W^s. Then,

$$\eta_k = \sum_{i=1}^{s} a_k^i \eta_i, \quad s + 1 \le k \le n.$$

We claim that $a_k^i \le 0$, for all $1 \le i \le s$ and $s + 1 \le k \le n$. To see this, fix k and set $P = P_k = \{i \le s : a_k^i \ge 0\}$. Hence, for $l \in P$,

$$0 \ge \langle \eta_k, \eta_l \rangle = \sum_{i=1}^{s} a_k^i \langle \eta_i, \eta_l \rangle \ge \sum_{i \in P} a_k^i \langle \eta_i, \eta_l \rangle.$$

Multipling by $a_k^l \ge 0$ and taking sum over $l \in P$, we obtain

$$\| \sum_{i \in P} a_k^i \eta_i \|^2 \le 0,$$

and the claim follows.

From the claim and (1), we get for all $1 \le l \ne k \le n - s$,

$$0 \le \sum_{i=1}^{s} a_{s+l}^i \langle \eta_i, \eta_{s+k} \rangle = \langle \eta_{s+l}, \eta_{s+k} \rangle \le 0. \tag{2}$$

Therefore, we have $(n-s)$ mutually *orthogonal* vectors $\{\eta_{s+1}, \ldots, \eta_n\}$ in an $s-$dimensional vector space. We may assume $\eta_{s+l} = 0$ for all $r+1 \leq l \leq n-s$ and some $r \leq s$. Thus $\nu = n-s-r \geq n-2s$.

Set $I_j = \{i \leq s : a^i_{s+j} < 0\} \cup \{s+j\}$ and $I_0 = \{i \leq n : i \notin I_j,\ 1 \leq j \leq r\}$. Take $L_{j'} =$ span $\{\eta_i : i \in I_{j'}\}$. From equations (1) and (2), we have that $u \notin I_j$ if and only if $\eta_u \in L_j^{\perp}$. Since for each $i \in I_j$ the proof holds replacing η_i by η_{s+j}, the orthogonality of $\{\eta_{s+1}, \ldots, \eta_{s+r}\}$ implies that $L_j \perp L_{j'}$, for all $j \neq j'$. This completes the proof. $\square$

From now on, $s(x)$ stands for the dimension of the *first normal space* of the isometric immersion f at $x \in M^n$, i.e.,

$$s(x) = \dim \mathcal{N}^1_f(x) = \dim \ \mathrm{span}\ \{\alpha(Y,Z) : Y, Z \in T_xM\}.$$

Corollary 4. *Let $f : M^n \to \mathbf{R}^{n+p}$ be an isometric immersion with flat normal bundle of a Riemannian manifold of nonpositive sectional curvature. Assume that $\nu \leq n-s-r$ everywhere for some integer $2 \leq r \leq p$. Then there exist locally in an open dense subset $\mathcal{V} \subset M^n$ an orthonormal smooth frame $\{X_1, \ldots, X_n\}$ of $\mathcal{V}$ and a partition $\{1, \ldots, n\} = \bigcup_{j'=0}^{r} I_{j'}$ which for any $1 \leq j \leq r$ verify:*

i) $\{X_1(x), \ldots, X_n(x)\}$ *diagonalizes the second fundamental form $\alpha(x)$ of f at $x \in \mathcal{V}$,*

ii) $L_j =$ *span* $\{\eta_i = \alpha(X_i, X_i) : i \in I_j\}$ *is a normal subbundle of dimension $\# I_j - 1 \geq 1$,*

iii) there are $\# I_j$ negative smooth functions $\{a^i_j : i \in I_j\}$ such that $\sum_{i \in I_j} a^i_j \eta_i = 0$,

iv) $L_j \perp L_{j'}$, *for all $j' \neq j$.*

Proof. Fix $x \in M^n$. Since the ambient space has constant sectional curvature, the flatness of the normal bundle is equivalent, by the Ricci equation, to the fact that $\alpha(x)$ is diagonalizable. Moreover, the curvature hypothesis on f is equivalent to $\alpha(x)$ being nonpositive by the Gauss equation. Hence, Proposition 3 applies to $\alpha(x)$. It is clear that $\nu(x)$, $s(x)$ and the dimensions of the normal subspaces $L_{j'}(x)$ are semicontinuous functions. Thus all of them are locally constant in an open dense subset $\mathcal{V}$. Now, we easily conclude the proof using Proposition 3 and the uniqueness, up to signs and permutations, of the (smooth) orthonormal basis which diagonalizes $\alpha|_{\Delta^{\perp} \times \Delta^{\perp}}$. $\square$

After studying the restrictions that the linear algebra of the Gauss equations imposes on the second fundamental form of the immersion, we now analize the differential implications of the Codazzi equations. With the notations of Corollary 4, let ∇ be the Levi–Civita connection of M^n and $\nabla^\perp$ the normal connection of f. Define locally on $\mathcal{V}$,

$$\mathcal{D}_j^0 = \mathcal{D}_j := \{X_i : i \in I_j\}, \quad \mathcal{D}_0^0 = \mathcal{D}_0 := \Delta^\perp \bigcap_{j=1}^{r} \mathcal{D}_j^\perp$$

and, for $m \geq 0$,

$$\mathcal{D}_{j'}^{m+1} := \text{ span } \{\nabla_X Y : X \in \mathcal{D}_{j'}, Y \in \mathcal{D}_{j'}^m\} \supseteq \mathcal{D}_{j'}^m.$$

Notice that $\dim \mathcal{D}_j = \# I_j \geq 2$ and that $\mathcal{D}_0$ can be trivial. Also, $u \in I_0$ and $X_u \in \Delta$ for all $s+r+1 \leq u \leq n$.

Lemma 5. *With the assumptions and notations of Corollary 4, we get for all $l, m \geq 0$:*

i) for all $X \in \mathcal{D}_0^\perp, \eta \in L_{j'}$, $\nabla_X^\perp \eta \in L_{j'}$,

ii) for all $V \in \mathcal{D}_j^\perp, W \in \mathcal{D}_j^m$, $\nabla_V W \in \mathcal{D}_j^m$,

iii) $\mathcal{D}_j^m \subset \mathcal{D}_j \oplus \mathcal{D}_0 \oplus \Delta$, and $\mathcal{D}_0^m \subset \mathcal{D}_0 \oplus \Delta$,

iv) for all $1 \leq j \neq k \leq r$, $\mathcal{D}_j^m \perp \mathcal{D}_k^l$.

Proof. i). Being the normal bundle flat, the Codazzi equations are

$$\nabla_{X_k}^\perp \eta_i = \langle \nabla_{X_i} X_i, X_k \rangle (\eta_i - \eta_k), \quad \forall\ 1 \leq i \neq k \leq n, \tag{3}$$

and

$$\langle \nabla_{X_i} X_k, X_u \rangle (\eta_u - \eta_k) = \langle \nabla_{X_k} X_i, X_u \rangle (\eta_u - \eta_i), \quad \forall\ 1 \leq i \neq u \neq k \leq n. \tag{4}$$

From equation (3), we obtain i) for $X \in \Delta$. Also from (3), we have that $\nabla_X^\perp \eta \in L_{j'} \oplus^\perp L_j$, for all $X \in \mathcal{D}_j$, $j \neq j'$. But from (3) and part iii) of Corollary 4, we conclude that $\nabla_X^\perp \xi \in L_j$, for all $X \in \mathcal{D}_j$ and $\xi \in L_j$. This proves i).

ii). From (1) and Corollary 4 iii) we easily have that the pair of vectors $\eta_u - \eta_k$ and $\eta_u - \eta_i$ are linearly independent for all $1 \leq i \neq u \neq k \leq s+r$. Thus, equation (4) yields

$$\nabla_{X_i} X_k \in \text{ span } \{X_i\} \oplus \Delta, \quad \forall\ 1 \leq i \neq k \leq s+r. \tag{5}$$

Also, equation (3) and part i) give

(6) $$\nabla_{X_i} X_i \in \mathcal{D}_0 \oplus \Delta, \ \ \forall\, i \in I_0, \quad \text{and} \quad \nabla_{X_k} X_k \in \mathcal{D}_j \oplus \mathcal{D}_0 \oplus \Delta, \ \ \forall\, k \in I_j.$$

Let us consider the case $m=0$. Taking $X_u \in \Delta$ in (4), we obtain using (5) that

$$\nabla_{X_i} X_k \in \ \text{span}\,\{X_i\} \quad \forall\, i \in I_{j'},\ k \in I_j, \ \ j \neq j'.$$

This, together with (6), proves ii) for all $V \in \mathcal{D}_j^{\perp} \cap \Delta^{\perp}$. Since the relative nullity distribution is integrable, the case $m=0$ follows from (4) for $V = X_i \in \Delta,\ \ k \in I_j,\ \ u \in I_{j'}$ and $j \neq j'$. We conclude the general case by induction on m using the Gauss equation, since, when $Z \in \mathcal{D}_j$,

$$\nabla_V \nabla_Z W = \nabla_Z \nabla_V W + \nabla_{[V,Z]} W \in \mathcal{D}_j^{m+1}.$$

iii) and iv). Both follow easily by induction using part ii). □

We are now in position to prove our main result.

Proof of Theorem 1. With the notations of Lemma 5, set

$$\Gamma_j := \mathcal{D}_j^{n-2p} \quad \text{and} \quad \Gamma_0 := \left(\bigoplus_{j=1}^{r} \Gamma_j\right)^{\perp} \subset \mathcal{D}_0 \oplus \Delta.$$

From Lemma 5 iv) and $\mathcal{D}_j^m \subseteq \mathcal{D}_j^{m+1}$, we obtain that $\mathcal{D}_j^m = \Gamma_j$, $m \geq n$. By semicontinuity, along an open dense subset $\mathcal{U} \subset \mathcal{V} \subset M^n$, all Γ_j's have locally constant dimension, say n_j. Hence, from Lemma 5 we have that all $\Gamma_{j'}$'s are smooth parallel orthogonal distributions on $\mathcal{U}$, that is, $\nabla_X Y \in \Gamma_{j'}$, for all $Y \in \Gamma_{j'}$, $X \in TM$. By the local de Rham's decomposition theorem we obtain locally that

$$\mathcal{U} = M_0^{n_0} \times M_1^{n_1} \times \cdots \times M_r^{n_r},$$

with $TM_{j'} = \Gamma_{j'}$. Observe that, since $\mathcal{D}_j \subset \Gamma_j$, by Corollary 4 iii) M_j is nowhere flat. However, M_0 can be flat, or even a point.

To conclude that f splits using the Main Lemma of **[M]**, we only need to show that $\alpha(X,Y) = 0$, for all $X \in \Gamma_{j'}$, $Y \in \Gamma_j$, $\ j \neq j'$. Observe first that from equation (3) and Lemma 5 iv) we have

(7) $$\nabla_X^{\perp} \xi \in L_j, \quad \forall\, X \in \Gamma_{j'}, \ \ \xi \in L_j, \ \ j \neq j'.$$

Part iii) of Lemma 5 gives $\alpha(X,Z)=0$, for all $X\in\Gamma_{j'},\ \ Z\in\mathcal{D}_j,\ j\neq j'$. Suppose by induction that $\alpha(X,W)=0$, for all $X\in\Gamma_{j'},\ \ W\in\mathcal{D}_j^m$. The Codazzi equation yields

$$\alpha(X,\nabla_Z W)=-\nabla_X^{\perp}\alpha(Z,W)+\alpha(\nabla_X Z,W)+\alpha(Z,\nabla_X W).$$

Lemma 5 ii) together with (7) imply that the right–hand side of the above belongs to L_j. The proof follows now from Lemma 5 iii) which gives $\alpha(X,\nabla_Z W)\in L_0$. $\square$

Proof of Corollary 2. The hypothesis on the Ricci curvature and the codimension imply that $\nu\equiv n-2p=0$. Hence, $\mathcal{D}_0=\Delta=\{0\}$. Observe that, since $s\equiv p$ and $\#I_j\equiv 2$, all the dimensions involved are now constant in M^n. We show next that $c=0$. Since the proofs of Corollary 4 and parts i) and ii) of Lemma 5 for $m=0$ do not depend on the (constant) sectional curvature of the ambient space, we have that all $\mathcal{D}_j$'s are parallel. Therefore, for unit $X\in\mathcal{D}_i,\ Y\in\mathcal{D}_j,\ 1\leq i\neq j\leq p$, we have $0=K_M(X,Y)=c$, where the last equality follows from the Gauss equation. The corollary is now a consequence of Theorem 1 and the global de Rham's decomposition theorem. $\square$

Final remarks

1) The relative nullity hypothesis of Theorem 1 can be weakened to $\nu(x)\leq n-s(x)-r$, since Corollary 4 and Lemma 5 hold with this assumption.

2) If $\mathcal{D}_0=\{0\}$, which is the case when, for example, $\nu=n-2s$, Lemma 5 i) implies that $\mathcal{N}_f^1$ is parallel. Therefore, the codimension of f reduces to s on $\mathcal{U}$.

3) Any Euclidean hypersurface $g:H^m\to\mathbf{R}^{m+1}$ of nonpositive curvature without flat points can be described locally by means of the Gauss parametrization in the following way (see **[D–G]** for details). Take a surface $\xi:V^2\to\mathbf{S}^m$ in the Euclidean unit sphere and a smooth function γ on V^2. The map $\Psi:T_\xi^{\perp}V\to\mathbf{R}^{m+1}$ given by

$$\Psi(v)=\gamma\xi+\mathrm{grad}\ \gamma+v \tag{8}$$

parametrizes g over the normal bundle of ξ, in the open set of normal vectors v which satisfies $\det(\gamma\mathrm{Id}+\mathrm{Hess}_\gamma-B_v)<0$. Here, B_v denotes the second fundamental operator of ξ in the direction v. In this parametrization, ξ is the Gauss map of g and $\gamma=\langle g,\xi\rangle$ is the support function. For a characterization of those hypersurfaces which are isometrically deformable see Sbrana **[S]** or Cartan **[C]**. Observe that any isometric immersion f in Theorem 1 with minimum index of relative nullity $\nu \equiv n-2p$ can be parametrized locally on $\mathcal{U}$ using the Gauss parametrization (8) for each factor.

References

[C] Cartan, É., *La déformation des hypersurfaces dans l'espace euclidien réel a n dimensions*, Bull. Soc. Math. France **44**, 65-99 (1916).

[D–G] Dajczer, M. and Gromoll, D., *Gauss parametrizations and rigidity aspects of submanifolds*, J. Diff. Geom. **22**, 1-12 (1985).

[F] Florit, L., *On submanifolds with nonpositive extrinsic curvature*, Math. Ann. **298**, 187-192 (1994).

[G] Gromov, M., "Partial Differential Relations", Berlin Heidelberg: Springer-Verlag 1986.

[M] Moore, J., *On isometric immersions of Riemannian products*, J. Diff. Geom. **5**, 159-168 (1971).

[S] Sbrana, V., *Sulla varietà ad $n-1$ dimensioni deformabili nello spazio euclideo ad n dimensioni*, Rend. Circ. Mat. Palermo **27**, 1-45 (1909).

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