On nonpositively curved euclidean submanifolds: splitting results II*

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Let $f : M^n \to \mathbb{Q}_c^{n+p}$ be an isometric immersion of an n–dimensional riemannian manifold into a simply connected space form of constant sectional curvature c. The *index of relative nullity* $\nu(x)$ of f at x is the dimension of the nullity space of the second fundamental form α of f, i.e.,

$$\nu(x) = \dim \operatorname{Ker} \alpha = \dim\{v \in T_xM : \alpha(v, w) = 0, \ \ \forall \ w \in T_xM\}.$$

The positiveness of this index is a very fundamental and useful data about the immersion since relative nullity distributions of submanifolds into space forms are integrable with totally geodesic leaves in both the submanifold and ambient space. Hence, for the theory of isometric immersions it is important to know any *a priori* estimates for this index, say, under some natural intrinsic condition. We choose for this condition to be bounds on sectional curvature K_M of the submanifold. Since there is no possible estimate for $K_M \geq c$ as the spheres in euclidean spaces show, we assume from now on that $K_M \leq c$.

In this context, for submanifolds with $K_M \equiv c$, the Chern-Kuiper inequality states that $\nu \geq n - p$. On the other hand, for $K_M \leq c$ it was shown in [**F1**] that $\nu \geq n - 2p$. This is a sharp inequality since a product of p nowhere flat nonpositively curved euclidean hypersurfaces satisfies the equality.

In [**F2**], the first author studied the flat normal bundle situation. It was shown under this assumption that, if $\nu = n-p-r$ for some integer $2 \leq r \leq p$, then any euclidean submanifold is (locally) a product of r submanifolds.

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Hence, for nonpositively curved *irreducible* euclidean submanifolds with flat normal bundle we have the improved inequality $\nu \geq n - p - 1$, which is also sharp because any local isometric immersion of the hyperbolic n–space into $\mathbb{R}^{2n-1}$ has flat normal bundle. The question now is if this strong condition on the normal bundle can be dropped.

In this direction, we proved in [**FZ**] that having minimal relative nullity index $\nu = n - 2p$ implies, in fact, that the normal bundle must be flat. Thus, we concluded that the only euclidean submanifolds with minimal relative index are (locally) products of nowhere flat hypersurfaces.

For the general case that $\nu = n - p - r$ with $2 \leq r \leq p - 1$, however, it does *not* hold that the submanifold is a product of r submanifolds, even for $\nu = n - 2p + 1$. The purpose of this paper is to completely understand the situation under this last assumption, giving rise to conjectures on the general case. We show that the submanifold is, in fact, locally a product of $p - 1$ submanifolds, but possibly contained in a flat hypersurface. More precisely, we prove the following

Theorem 1. *Let $f : M^n \to \mathbb{R}^{n+p}$ be an isometric immersion of a riemannian manifold with nonpositive sectional curvature. Assume $\nu = n - 2p + 1$ everywhere. Then, there is an open dense subset $\mathcal{V} \subset M^n$ such that each connected component $\mathcal{V}_\lambda$ of $\mathcal{V}$ satisfies $\mathcal{V}_\lambda = M_1^{n_1} \times \cdots \times M_{p-1}^{n_{p-1}}$ and either:*

1) *there is an isometric immersion $f_1 : M_1^{n_1} \to \mathbb{R}^{n_1+2}$ and nowhere flat hypersurfaces $f_i : M_i^{n_i} \to \mathbb{R}^{n_i+1}$, $2 \leq i \leq p-1$, such that*

$$f|_{\mathcal{V}_\lambda} = f_1 \times \cdots \times f_{p-1}, \qquad \textit{or}$$

2) *there are nowhere flat hypersurfaces $f_i : M_i^{n_i} \to \mathbb{R}^{n_i+1}$, $1 \leq i \leq p-1$, and a flat hypersurface $h : U \subset \mathbb{R}^{n+p-1} \to \mathbb{R}^{n+p}$ such that*

$$f|_{\mathcal{V}_\lambda} = h \circ (f_1 \times \cdots \times f_{p-1}).$$

It is interesting to observe that 1) in the above corresponds to the case that the relative nullity coincides with the nullity μ of the curvature tensor of the manifold and 2) to that of $\mu = \nu + 1$.

We should point out that the proof of the above result is much more difficult that the one in our previous work because it arises a double problem: to understand the "composition" case 2) and to deal with an undecomposable algebraic situation described latter. Both appear to be the key points to solve the general case.

For vanishig nullity we obtain global consequences:

Theorem 2. *Let* $f : M^n \to \mathbb{Q}_c^{n+p}$, $2p \leq n+1$, *be an isometric immersion of a riemannian manifold with* $K_M \leq c$ *and Ricci curvature* $Ric_M < c$. *Then,* $c = 0$ *and either:*

1) $n = 2p$, *and* f *splits locally as a product of* p *surfaces of* $\mathbb{R}^3$, *or*

2) $n = 2p-1$, *and* f *splits locally as a product of* $p-2$ *surfaces of* $\mathbb{R}^3$ *and* $M^3 \subset \mathbb{R}^5$,

that is, f *splits locally as a product of* $(n-p)$ *submanifolds. Moreover, the splitting is global provided that* M^n *is a Hadamard manifold.*

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§1 Some algebraic results.

Let V^n and W^p be real vector spaces of dimensions n and p, respectively. Suppose that V^n and W^p have positive definite inner products both denoted by $\langle\ ,\ \rangle$, and let $\alpha : V^n \times V^n \to W^p$ be a symmetric bilinear map. We say that α is *nonpositive* if its (sectional) *curvature* satisfies $K_\alpha \leq 0$, i.e.,

$$K_\alpha(X,Y) = \langle \alpha(X,X), \alpha(Y,Y)\rangle - \|\alpha(X,Y)\|^2 \leq 0, \qquad \forall\ X, Y \in V^n.$$

For $\xi \in W^p$, $X, Y \in V^n$, define $A_\xi \in \mathrm{End}(V^n)$ and $\alpha_X \in \mathcal{L}(V^n, W^p)$ by the formulas $\langle A_\xi X, Y\rangle = \langle \alpha_X(Y), \xi\rangle = \langle \alpha(X,Y), \xi\rangle$. We also interpret A_ξ as the symmetric bilinear form $A_\xi(X,Y) = \langle A_\xi X, Y\rangle$, giving sense to K_{A_ξ}. Observe that $K_{A_\xi} \leq 0$ is equivalent to A_ξ having at most two nonzero oppositely signed eigenvalues. Although the nonpositively of α has nothing to do with the inner product on V^n, we introduce it to see each A_ξ as an endomorphism in order to make the proofs more clear. Denote by $\mathcal{A}(\alpha)$ the set of asymptotic vectors of α: $\mathcal{A}(\alpha) = \{X \in V^n : \alpha(X,X) = 0\}$.

In the proof of the following lemma there are several arguments already contained in the proof of Proposition 4 in [**FZ**]. Thus, we refer to it for further details.

Lemma 3. *Let $\alpha : V^n \times V^n \to W^p$ be a nonpositive symmetric bilinear map such that $\nu = n - 2p + 1$. Then, there is a unit vector $\xi \in W^p$ so that* $\operatorname{rank} A_\xi \leq 2$ *and* $K_{A_\xi} \leq 0$.

Proof: Without loss of generality, we may assume $p \geq 3$ and $\nu = 0$. Hence $n = 2p - 1 \geq 5$. As in [**FZ**], we have for each $X \in \mathcal{A}(\alpha)$ the nonpositive symmetric bilinear map

$$\beta = \alpha|_{V' \times V'} \to \operatorname{Im}(\alpha_X)^\perp, \quad \text{with} \quad V' = \operatorname{Ker}(\alpha_X). \tag{1}$$

Fix such an X with $\operatorname{rank}(\alpha_X) = r = \min\{\operatorname{rank}(\alpha_X) : 0 \neq X \in \mathcal{A}(\alpha)\} > 0$. Noting that $X \in \operatorname{Ker}\beta$, set $r + s = \dim \operatorname{span}\{\operatorname{Im}(\alpha_Y) : Y \in \operatorname{Ker}\beta\}$ and denote by q the nullity of β, i.e., $q = \dim \operatorname{Ker}\beta$. Then, similar arguments as those of [**FZ**] yield

$$r + 2s - 1 \leq q = 1 \leq r + s.$$

Hence, $s = 0$ and r is either 1 or 2.

If $r = 1$, take $0 \neq \xi \in \operatorname{Im}(\alpha_X)$. Since $\langle \alpha(V' \times V'), \xi \rangle = 0$ and $V' \subseteq V^n$ has codimension 1, we conclude that $\operatorname{rank} A_\xi \leq 2$ with $K_{A_\xi} \leq 0$ as we wished.

Now assume $r = 2$. Since the nullity of β is $q = 1 = (n-2) - 2(p-2)$, by Proposition 4 of [**FZ**] β can be diagonalized. Thus, there exist a basis $\{e_1 = X, e_2, \ldots, e_{n-2}\}$ of V' and an orthonormal basis $\{\xi_1, \ldots, \xi_{p-2}\}$ of $(\operatorname{Im}\alpha_X)^\perp$, such that $\alpha(e_{2k}, e_{2k+1}) = \xi_i$, $k = 1, \ldots, p-2$, and $\alpha(e_i, e_j) = 0$ for any other $1 \leq i, j \leq n-2$. In particular,

$$K_{A_{\xi_1}}(e_2, e_3) < 0. \tag{2}$$

Let $\{\xi_{p-1}, \xi_p\}$ be an orthonormal basis of $W_1 = \operatorname{Im}(\alpha_X)$ and $e_{n-1}, e_n \in V^n$ so that $\alpha(X, e_{n-1}) = \xi_{p-1}$, $\alpha(X, e_n) = \xi_p$. For any $1 \leq i \neq j \leq p-2$, since $K_\alpha(e_{2i+\delta}, e_{2j+\delta'}) = 0$ (where $\delta, \delta' = 0, 1$), we have that

$$K_\alpha(e_{2j} + e_{2j+1}, e_{2i+\delta} + tZ) = 4t\langle \alpha(e_{2i+\delta}, Z), \xi_j \rangle - t^2 K_\alpha(e_{2j}, Z) \leq 0.$$

Thus, $\alpha(e_{2i+\delta}, V^n) \perp \xi_j$. Therefore, for $s = n-1, n$, replacing the vector e_s by $e_s - \sum_{i=1}^{p-2}(\langle \alpha(e_s, e_{2i}), \xi_i \rangle e_{2i+1} + \langle \alpha(e_s, e_{2i+1}), \xi_i \rangle e_{2i})$ we obtain for $V_1 = \operatorname{span}\{e_{n-1}, e_n\}$ that

$$\alpha(V' \times V_1) \subseteq W_1. \tag{3}$$

For each $Y \in V' \cap \mathcal{A}(\alpha)$, write $\alpha(Y, e_{n-2+i}) = B^Y_{i1}\xi_{p-1} + B^Y_{i2}\xi_p$, $i = 1, 2$. If we assume that B^Y has complex eigenvalues, we will arrive to a contradiction, as in the proof of $q = 1$ in Proposition 4 of [**FZ**]. We claim that B^Y is in fact symmetric. Replacing Y by $Y - \lambda X$, we may assume B^Y has an eigenvalue equal to zero. Suppose that $Z = z_1 e_{n-1} + z_2 e_n \neq 0$ satisfies $\alpha(Y, Z) = 0$, that is, $\sum_i z_i B^Y_{ij} = 0$. Since $\alpha(Y, X) = 0$, we have by (1) that $\alpha(X, Z) \perp \mathrm{Im}\,(\alpha_Y)$, so $\sum_j z_j B^Y_{ij} = 0$. These two linear systems yield the symmetry of B^Y.

Next, we want to change our frames in W_1 and V_1 so that B^{e_2} and B^{e_3} look specially simple. Replacing e_2 by $e_2 - \lambda X$, we may assume that B^{e_2} has a zero eigenvalue. So, pick a vector $e_{n-1} \in V_1$ so that $\alpha(e_2, e_{n-1}) = 0$ and $\|\alpha(X, e_{n-1})\| = 1$. Rotate the orthonormal basis $\{\xi_{p-1}, \xi_p\}$ of W_1 to obtain

$$\xi_{p-1} = \alpha(X, e_{n-1}).$$

Take the vector $e_n \in V_1$ such that $\alpha(X, e_n) = \xi_p$. Since $e_2 \in \mathcal{A}(\alpha)$, by (1) ξ_{p-1} is perpendicular to $\mathrm{Im}(\alpha_{e_2})$. So $\alpha(e_2, e_n) = c\xi_p$. Also replacing e_3 by $e_3 - \lambda X$ if necessary, we may assume that $\alpha(e_3, e_{n-1}) = a\xi_{p-1} + b\xi_p$, $\alpha(e_3, e_n) = b\xi_{p-1}$.

We are ready to show that A_{ξ_1} has rank 2. It suffices to show that $u = v = w = 0$, where u, v, and w are the ξ_1 components of $\alpha(e_{n-1}, e_{n-1})$, $\alpha(e_{n-1}, e_n)$ and $\alpha(e_n, e_n)$, respectively. For this purpose, let us consider

$$U = xe_1 + ye_2 + e_3, \quad S = ze_{n-1} + e_n.$$

Then we have

$$\langle \alpha(U, U), \alpha(S, S) \rangle = 2y(z^2 u + 2zv + w) \qquad \text{and}$$

$$\|\alpha(U, S)\|^2 = (xz + az + b)^2 + (x + cy + bz)^2.$$

So, by the nonpositivity of α we have

$$2y(z^2 u + 2zv + w) \leq (xz + az + b)^2 + (x + cy + bz)^2$$

for arbitrary real numbers x, y and z.

Taking $z = 0$ and $x = -cy$, we get $2yw \leq b^2$ for arbitrary y, so $w = 0$. Similarly, for $x = -a$ and $z \to \infty$ we get $2yu \leq b^2$, so $u = 0$ as well. The inequality now becomes $4yzv \leq (xz + az + b)^2 + (x + cy + bz)^2$. Choosing $z = \pm 1$, the inequality can be rewritten as

$$2x^2 + 2x(a + 2bz + cy) + (a^2 + 2b^2 + 2abz + c^2y^2 + 2bczy - 4zvy) \geq 0.$$

Since this holds for any x, the discriminant must be nonpositive:

$$c^2y^2 - 2y(ac + 4zv) + a^2 \geq 0, \qquad \forall\ y \in \mathbb{R}.$$

So, we obtain $2v^2 + aczv \leq 0$. Choosing $z = 1$ or $z = -1$ so that $aczv \geq 0$, we conclude that $v^2 \leq 0$, hence $v = 0$. This and (2) conclude the proof. ∎

The following is our main algebraic result.

Lemma 4. *Let $\alpha : V^n \times V^n \to W^p$ be a nonpositive symmetric bilinear map such that $\nu = n - 2p + 1$. Assume that rank $A_\xi \neq 1$, for all $\xi \in W^p$. Then, there is an orthonormal basis $\{\xi_1, \ldots, \xi_p\}$ of W^p so that, for all $1 \leq i \neq j \leq p-1$, $1 \leq k \leq p-2$, we have* $\operatorname{rank} A_{\xi_i} = 2$ *with* $K_{A_{\xi_i}} \leq 0$, $\operatorname{Im} A_{\xi_i} \cap \operatorname{Im} A_{\xi_j} = \operatorname{Im} A_{\xi_k} \cap \operatorname{Im} A_{\xi_p} = 0$*, and either*

1) $\dim(\operatorname{Im} A_{\xi_{p-1}} + \operatorname{Im} A_{\xi_p}) = 3$*, or*

2) $\operatorname{rank} A_{\xi_p} = 2$ *with* $K_{A_{\xi_p}} \leq 0$ *and* $\operatorname{Im} A_{\xi_{p-1}} \cap \operatorname{Im} A_{\xi_p} = 0$.

Moreover, the sets $\{\xi_1, \ldots \xi_{p-2}\}$ in case 1) and $\{\xi_1, \ldots, \xi_p\}$ in case 2) are unique (up to signs and permutations).

Proof: Of course we need to prove the lemma only for $\nu = 0$. Let us proceed by induction on p. Since there is nothing to prove for $p = 1, 2$, assume that the lemma holds for some $p \geq 2$ and let see for $p+1$. By Lemma 3 there is a unit vector $\xi \in W^{p+1}$ such that $\operatorname{rank} A_\xi = 2$ and $K_{A_\xi} \leq 0$. Set $T^{n-2} = \operatorname{Ker} A_\xi$ and

$$\widetilde{\alpha} = \alpha - \langle A_\xi \cdot, \cdot \rangle \xi : V^n \times V^n \to \widetilde{W}^p = (\operatorname{span}\{\xi\})^\perp.$$

Since $\operatorname{Ker} \alpha = T^{n-2} \cap \operatorname{Ker} \widetilde{\alpha}$, we have that $\widetilde{\nu} := \dim \operatorname{Ker} \widetilde{\alpha} \leq 2$. In case of $\widetilde{\nu} = n - 2p + 1 = 2$, define $\xi_1 = \xi$ and apply the inductive hypothesis to $\widetilde{\alpha}|_{T \times T}$ to easily conclude that the Lemma holds for $p+1$. Thus, assume from now on that $\widetilde{\nu} \leq 1$.

For each $e \in V^n \backslash T^{n-2}$ take the hyperplane $V_e^{n-1} := T^{n-2} \oplus \operatorname{span}\{e\} \subset V^n$ and the symmetric bilinear map

$$\alpha' = \widetilde{\alpha}|_{V_e \times V_e} : V_e^{n-1} \times V_e^{n-1} \to \widetilde{W}^p$$

with $\nu' = \dim \operatorname{Ker} \alpha'$. It is clear that $K_{\alpha'} \leq 0$ since $\operatorname{rank} A_\xi|_{V_e} \leq 1$.
Claim. There is $e \in V^n \setminus T^{n-2}$ such that $\nu' = (n-1) - 2p = 0$.

For $\widetilde{\nu} = 1$, any $e \notin \operatorname{Ker} \widetilde{\alpha} + T^{n-2}$ satisfies the claim. So, assume that $\widetilde{\nu} = 0$. Set $\gamma = \widetilde{\alpha}|_{T \times T}$ and $\Delta_\gamma = \operatorname{Ker} \gamma$. First we show that

$$\dim \left(+_{X \in \Delta_\gamma} \operatorname{Im} \alpha'_X \right) \geq \dim \Delta_\gamma. \tag{4}$$

To see this, take a plane $L^2 \subset V^n$ so that $L^2 \cap T^{n-2} = 0$ and set $\beta = \alpha|_{L\times\Delta_\gamma}$ and $U = L^2 \oplus \Delta_\gamma$. We now apply the techniques of the proof of Proposition 8 of [**F1**] to $\alpha|_{U\times U}$. It is clear that $\Delta_\gamma \subset \mathcal{A}(\alpha|_{U\times U})$. If $Y_0 \in RE(\beta) \subset L^2$, we obtain as in the proof of Proposition 8 of [**F1**] that

$$\alpha(Y, X) = 0, \qquad \forall\, Y \in L^2,\ X \in \operatorname{Ker}\beta(Y_0) \subset \Delta_\gamma.$$

We conclude that $\operatorname{Ker}\beta(Y_0) \subset \operatorname{Ker}\alpha = 0$, i.e., $\{\alpha(X_1, Y_0), \ldots, \alpha(X_s, Y_0)\}$ are linearly independent for a basis $\{X_1, \ldots, X_s\}$ of Δ_γ. Thus, equation (4) follows.

Now, take $X \in \Delta_\gamma, Y \in T^{n-2}, Z \in V^n$. Hence, $K_\alpha(X, Y) = 0$ and then $K_\alpha(X + tZ, Y) = 2t\langle\alpha(X, Z), \alpha(Y, Y)\rangle + t^2K(Z, Y) \leq 0, t \in \mathbb{R}$. Therefore, $\langle\alpha(X, Z), \alpha(Y, Y)\rangle = 0$, or equivalently, the nonpositive symmetric bilinear map γ satisfies

$$\gamma : T \times T \to \big(\operatorname{span}\{\xi\} \oplus (+_{X\in\Delta_\gamma}\operatorname{Im}\alpha'_X)\big)^\perp.$$

From Proposition 9 of [**F1**] and (4) we obtain

$$\dim\Delta_\gamma \geq (n-2) - 2\left(p - \dim(+_{X\in\Delta_\gamma}\operatorname{Im}\alpha'_X)\right) \geq 2\dim\Delta_\gamma - 1,$$

that is, $\dim\Delta_\gamma \leq 1$.

For $\dim\Delta_\gamma = 1$, using that $\widetilde{\nu} = 0$ choose $e \in V^n \setminus T^{n-2}$ such that $\widetilde{\alpha}(e, \Delta_\gamma) \neq 0$. It is easy to verify that $\nu' = 0$ in this case.

For $\dim\Delta_\gamma = 0$, take $e_1, e_2 \in V^n \setminus T^{n-2}$ linearly independent. If $\nu' > 0$ for both e_i's, we have from $\dim\Delta_\gamma = 0$ that $\nu' = 1$ in both cases and that there are $t_i \in T^{n-2}$ so that $\alpha(e_i + t_i, V_{e_i}) = 0,\ i = 1, 2$. But $\alpha(e_1, e_2) \neq 0$ since $\nu = 0$. We easily verify that $e = e_1 + e_2$ satisfies the claim, which is now completely proved.

We conclude from the above claim and from Proposition 4 of [**FZ**] applied to α' that there is an orthonormal frame $\{\xi_1, \ldots, \xi_p\}$ of $\widetilde{W^p}$ such that each shape operator A'_{ξ_i} of α', $1 \leq i \leq p$, satisfies $\operatorname{rank} A'_{\xi_i} = 2$ with $K_{A'_{\xi_i}} \leq 0$ and

$$V_e^{n-1} = \operatorname{Im} A'_{\xi_1} \oplus \cdots \oplus \operatorname{Im} A'_{\xi_p}. \tag{5}$$

If $\widetilde{\nu} = n - 2p = 1$ take $\xi_{p+1} = \xi$. We have from $\nu = 0$ that $E = \operatorname{Im} A_\xi \cap V_e^{n-1}$ is a line. If $E \not\subset \operatorname{Im} A_{\xi_k}$ for all $1 \leq k \leq p$, we conclude that α is as in case 2). Otherwise, it is as in case 1) and the lemma for $p+1$ holds in this situation. From now on assume that $\widetilde{\nu} = 0$, the only remaining case.

Pick $1 \le i \ne j \le p$ and $x_i \in I_i^2 := \bigcap_{k \ne i} \operatorname{Ker} A'_{\xi_k} \subset V_e^{n-1}$. Choose $e_0 \in V^n \setminus V_e^{n-1}$. If A stands for the matrix of A_ξ under $\{e_0, x_i, x_j\}$, from $\operatorname{rank} A \le 2$ and $\operatorname{rank} A_\xi|_{V_e \times V_e} \le 1$ we easily obtain that $A_{02}A_{11} - A_{12}A_{01} = 0$. Hence,

$$g(t) = K_\alpha(x_i, te_0 + x_j) = t^2 K_\alpha(x_i, e_0) + 2t\langle \alpha(e_0, x_j), \xi_i\rangle \langle A'_{\xi_i} x_i, x_i\rangle \le 0.$$

We conclude that $\langle \alpha(e_0, x_j), \xi_i\rangle = 0$. In view of (5), we can replace e_0 by $e_0 + \sum_{i=1}^p a_i x_i$ for suitable $a_i \in \mathbb{R}$ and $x_i \in I_i^2$ as we did to obtain (3) to conclude

$$\widetilde{\alpha}(e_0, V_e^{n-1}) = 0. \tag{6}$$

Now, from $\widetilde{\nu} = 0$ and the above we get $\eta := \widetilde{\alpha}(e_0, e_0) \ne 0$. Moreover, if $l = \#\{i \le p : \langle \eta, \xi_i\rangle \ne 0\}$, we have from (5) that A'_η has exactly l positive eigenvalues. On the other hand, from (6) we have that

$$\langle A'_\eta X, X\rangle = K_{\widetilde{\alpha}}(e_0, X) = K_\alpha(e_0, X) \le 0, \qquad \forall\ X \in T^{n-2} \subset V_e^{n-1}. \tag{7}$$

Thus, $l = 1$, say,

$$\widetilde{\alpha}(e_0, e_0) = \lambda \xi_p \ne 0.$$

Then, from $\operatorname{rank} A'_{\xi_p} = 2$ and (6) we get $e_0 \in \bigcap_{j \le p-1} \operatorname{Ker} A_{\xi_j}$, $e_0 \notin \operatorname{Ker} A_{\xi_p}$ and $\operatorname{rank} A_{\xi_p} = 3$. Moreover, (7) yields that $\operatorname{Ker} A_{\xi_p} \subset T^{n-2} = \operatorname{Ker} A_\xi$. Setting $\xi_{p+1} := \xi_p$ and $\xi_p := \xi$, we conclude from (5) that

$$V^n = \operatorname{Im} A_{\xi_1} \oplus \cdots \oplus \operatorname{Im} A_{\xi_{p-1}} \oplus \operatorname{Im} A_{\xi_{p+1}},$$

with $\operatorname{Im} A_{\xi_p} \subset \operatorname{Im} A_{\xi_{p+1}}$. The lemma is proved. ∎

Remark. The lemma just says that algebraic bilinear maps as above fall into two types. On one hand, the 'decomposable' type 1), that is, V^n decomposes as $V_1 \oplus \cdots \oplus V_{p-1}$ so that $\alpha(V_i, V_i) \perp \alpha(V_j, V_j)$ and $\alpha(V_i, V_j) = 0$ for $i \ne j$. On the other hand, the 'undecomposable' type 2), that needs some additional argument to be treated in order to obtain a splitting (cf. Lemma 6).

§2 The splitting.

We now show how the algebraic results of the last section give rise to the (local) splitting of the tangent bundle of submanifolds, as the one of Theorem 1, into distributions. We deal first with the case of nullity $\mu = \dim \operatorname{Ker} R = \nu + 1$ (R is here the curvature tensor of M^n), showing that they are locally compositions.

Proposition 5. *Let $f : M^n \to \mathbb{R}^{n+p}$ be an isometric immersion of a simply connected nonpositively curved riemannian manifold with $\nu = n - 2p + 1$. Suppose there is a unit normal vector field ξ such that* $\operatorname{rank} A_\xi \leq 1$. *Then, there exists an isometric immersion $\tilde{f} : M^n \to V \subset \mathbb{R}^{n+p-1}$ which splits locally along an open dense subset as a product of $(p-1)$ nowhere flat hypersurfaces, and a flat hypersurface $h : V \subset \mathbb{R}^{n+p-1} \to \mathbb{R}^{n+p}$, so that $f = h \circ \widetilde{f}$.*

Proof: Denote by ∇ and $\nabla^\perp$ the tangent and normal connections, respectively. Set $\widetilde{\alpha} = \alpha - \langle A_\xi \ \cdot, \cdot \rangle \xi$. First, notice that $\Delta_\alpha = \Delta_{\widetilde{\alpha}} \cap \operatorname{Ker} A_\xi$ and, from the Gauss equation and the rank hypothesis on A_ξ, that $K_{\widetilde{\alpha}} = K_\alpha \leq 0$. Thus $\widetilde{\alpha}$ is nonpositive and satisfies Gauss equation. Observe that

$$\operatorname{Im} A_\xi \not\subset \Delta_{\widetilde{\alpha}}^\perp. \tag{8}$$

On the contrary, from $\Delta_\alpha = \Delta_{\widetilde{\alpha}}$ and Proposition 9 of [**F1**] we conclude that $n - 2p - 1 = \nu_\alpha = \nu_{\widetilde{\alpha}} \geq n - 2(p-1)$, which is a contradiction. Thus, we have that

$$\nu_{\widetilde{\alpha}} = \nu_\alpha + 1 = n - 2(p-1). \tag{9}$$

Hence, by Proposition 4 of [**FZ**] we get an orthonormal frame $\{\xi_1, \ldots, \xi_{p-1}\}$ of $L = \operatorname{span}\{\xi\}^\perp \subset TM^\perp$ such that

$$\Delta_{\widetilde{\alpha}}^\perp = \operatorname{Im} A_{\xi_1} \oplus \cdots \oplus \operatorname{Im} A_{\xi_{p-1}}, \quad \text{with} \quad \operatorname{rank} A_{\xi_i} = 2,\ 1 \leq i \leq p-1. \tag{10}$$

In particular, since ξ and $\{\xi_1, \ldots, \xi_{p-1}\}$ are unique up to signs and permutations, we can choose them (locally) smooth. Therefore, $\widetilde{\alpha}$ is also smooth.

The Codazzi equation for A_ξ gives

$$A_\xi[X, Y] = A_{\nabla^\perp_X \xi} Y - A_{\nabla^\perp_Y \xi} X \in \Delta_{\widetilde{\alpha}}^\perp, \quad \forall\ X, Y \in \operatorname{Ker} A_\xi.$$

This, (8) and (10) imply that

$$\langle \nabla^\perp_X \xi, \xi_j \rangle A_{\xi_j} Y = \langle \nabla^\perp_Y \xi, \xi_j \rangle A_{\xi_j} X, \quad \forall\ X, Y \in \operatorname{Ker} A_\xi.$$

Again by (8) and (10) we easily conclude that

$$\nabla^\perp_X\xi = 0, \quad \forall\ X \in \operatorname{Ker} A_\xi, \tag{11}$$

which yields

$$\langle A_\xi X, Z\rangle \nabla^\perp_Y\xi = \langle A_\xi Y, Z\rangle \nabla^\perp_X\xi, \quad \forall\ X, Y, Z \in TM. \tag{12}$$

Let ∇' be the connection on L induced by $\nabla^\perp$. Then, for all $X, Y, Z \in TM$

$$\begin{aligned}
&\nabla'_Y\widetilde{\alpha}\,(X, Z) - \widetilde{\alpha}(\nabla_Y X, Z) - \widetilde{\alpha}(X, \nabla_Y Z)\\
&\quad = \big(\nabla^\perp_Y(\alpha(X,Z) - \langle A_\xi X, Z\rangle\xi) - \alpha(\nabla_Y X, Z) - \alpha(X, \nabla_Y Z)\big)_L\\
&\quad = \big(\nabla^\perp_Y\alpha(X,Z) - \alpha(\nabla_Y X, Z) - \alpha(X, \nabla_Y Z)\big)_L - \langle A_\xi X, Z\rangle\nabla^\perp_Y\xi\\
&\quad = \nabla'_X\widetilde{\alpha}(Y,Z) - \widetilde{\alpha}(\nabla_X Y, Z) - \widetilde{\alpha}(Y, \nabla_X Z),
\end{aligned}$$

where the last equality followed from the Codazzi equation for α and (12). Thus, $\widetilde{\alpha}$ satisfies Codazzi equation.

Now, denote by $R^\perp$ and R' the curvature tensors of $\nabla^\perp$ and ∇', respectively, take $\eta \in L$ and set $\psi(X) = \langle\nabla^\perp_X\xi, \eta\rangle$. Noting that (11) implies $\operatorname{Ker} A_\xi \subset \operatorname{Ker}\,\psi$, we thus have for all $X, Y \in TM$ that

$$\begin{aligned}
(R^\perp(X,Y)\eta)_L &= \big(\nabla^\perp_X(\nabla'_Y\eta - \psi(Y)\xi) - \nabla^\perp_Y(\nabla'_X\eta - \psi(Y)\xi)\big)_L - \nabla'_{[X,Y]}\eta\\
&= R'(X,Y)\eta - \psi(Y)\nabla^\perp_X\xi + \psi(X)\nabla^\perp_Y\xi \;=\; R'(X,Y)\eta.
\end{aligned}$$

We conclude that $\{\widetilde{\alpha}, \nabla'\}$ satisfies Gauss, Codazzi and Ricci equations for constant sectional curvature zero. Therefore, by the fundamental theorem of submanifolds, there exists an isometric immersion $\tilde{f} : M^n \to \mathbb{R}^{n+p-1}$ with second fundamental form $\widetilde{\alpha}$. From (9) and Theorem 1 of [**FZ**] we have that along an open dense subset $\widetilde{f}$ is locally a product of $(p-1)$ nowhere flat hypersurfaces with nonpositive sectional curvature.

Take a unit vector field Z_0 which spans $\operatorname{Im} A_\xi$. Then, $\widetilde{\nabla}_{Z_0}\xi = \lambda Z_0 + \gamma_0 \neq 0$ by (8), for some $\gamma_0 \in TM^\perp$. Let $\Gamma \subset TM \oplus L$ be the rank $(p-1)$ subbundle transversal to TM defined as

$$\Gamma = (L \cap \operatorname{span}\{\gamma_0\}^\perp) + \operatorname{span}\{\|\gamma_0\|^2 Z_0 - \lambda\gamma_0\}.$$

By definition, Γ satisfies that

$$\langle\widetilde{\nabla}_{Z_0}\mu, \xi\rangle = -\langle\widetilde{\nabla}_{Z_0}\xi, \mu\rangle = 0, \quad \forall\ \mu \in \Gamma.$$

Thus, from (11) we conclude that $\widetilde{\nabla}_X \mu \in TM \oplus L$, for all $X \in TM, \mu \in \Gamma$. The proof is now a consequence of Theorem 5 of [**DT**] applied to Γ. ∎

In the following result, we apply Lemma 4 to the second fundamental form α of an isometric immersion as the one of Theorem 1 which is nowhere a composition. The purpose is to show that the 'undecomposable' type 2) of Lemma 4 cannot occur for such an α.

Lemma 6. *Let $f : M^n \to \mathbb{R}^{n+p}$ be an isometric immersion with nonpositive sectional curvature and $\nu = n-2p+1$ which is nowhere a composition in the sense of Proposition 5. Then, there is an open dense subset $\mathcal{W} \subset M^n$ along which there is (locally) a smooth orthonormal normal frame $\{\xi_1, \ldots, \xi_p\}$ such that* $\operatorname{rank} A_{\xi_i} = 2$ *with $K_{A_{\xi_i}} \le 0,\ \ 1 \le i \le p-1$, and*

$$T\mathcal{W} = \Delta_\alpha \oplus \operatorname{Im} A_{\xi_1} \oplus \cdots \oplus \operatorname{Im} A_{\xi_{p-2}} \oplus (\operatorname{Im} A_{\xi_{p-1}} + \operatorname{Im} A_{\xi_p}). \tag{13}$$

Proof: First observe that there is no open subset $V \subset M^n$ so that, for each $x \in V$, there is a normal unit vector $\xi(x) \neq 0$ with $\operatorname{rank} A_{\xi(x)} \le 1$. On the contrary, from (10) we have that the direction determined by ξ is unique, thus smooth. Hence, from Proposition 5, we conclude that f is locally a composition on V, which is a contradiction. This implies that the open subset $\mathcal{W} \subset M^n$ along which there is no direction of rank one is also dense. This set is, in fact, the set along which the nullity μ of the curvature tensor of M^n is minimal: $\mathcal{W} = \mu^{-1}(n-2p+1)$.

For each $x \in \mathcal{W}$, we have the decomposition of Lemma 4 for the second fundamental form $\alpha(x)$ of f at x. Thus, we only need to show that the set $\mathcal{S} \subset \mathcal{W}$ along which the decomposition is of type 2) is empty.

First, notice that $\mathcal{S}$ is open. Assume that $\mathcal{S}$ is not empty, take $x \in \mathcal{S}$ and the (smooth by uniqueness) orthonormal frame $\{\xi_1, \ldots, \xi_p\}$ of $TM^\perp$ given by Lemma 4 in a neighborhood $U \subset \mathcal{S}$ of x. Observe now that

$$L = \big(\operatorname{Im} A_{\xi_1} \oplus \cdots \oplus \operatorname{Im} A_{\xi_{p-1}}\big) \cap \operatorname{Im} A_{\xi_p}$$

is a line bundle since $\nu = n-2p+1$. Changing order in the frame if necessary, we have that there is an integer $1 \le r \le p$ and smooth $0 \neq Y_j \in \operatorname{Im} A_{\xi_j}$ for $r \le j \le p$ such that

$$\sum_{j=1}^{p} A_{\xi_j} Y_j = 0, \tag{14}$$

where $Y_1 = \cdots = Y_{r-1} = 0$. Notice that this equation is unique, up to scalar multiplication. Observe that the assumption on $\mathcal{S}$ is equivalent to $r \leq p-2$.

From now on, we take arbitrary $1 \leq i \neq j \leq p$ and denote by ϕ_{ij} the skew-symmetric normal connection 1-forms $\phi_{ij}(Z) = \langle \nabla^{\perp}_Z \xi_i, \xi_j \rangle$. We only need to show that $\phi_{ip} = 0$ for all $i < p$. If this is the case, from the Ricci equations we get $[A_{\xi_i}, A_{\xi_p}] = 0$. Hence, $\operatorname{Im} A_{\xi_i} \subset \operatorname{Ker} A_{\xi_p}$ since $\operatorname{Im} A_{\xi_i} \cap \operatorname{Im} A_{\xi_p} = 0$. Therefore $L = 0$, a contradiction.

From the Codazzi equation for A_{ξ_i} we have

$$A_{\xi_i}[X,Y] + \sum_{j \neq i}^{p} A_{\xi_j} \widehat{\phi_{ij}}(X,Y) = 0, \qquad \forall\ X, Y \in \operatorname{Ker} A_{\xi_i}, \tag{15}$$

where $\widehat{\phi_{ij}}(X,Y) := \phi_{ij}(X)Y - \phi_{ij}(Y)X$. Observe that

$$\widehat{\phi_{ij}}(\operatorname{Ker} A_{\xi_i} \times \operatorname{Ker} A_{\xi_i}) = \begin{cases} \operatorname{Ker} A_{\xi_i} \cap \operatorname{Ker} \phi_{ij} & \text{if } \operatorname{Ker} A_{\xi_i} \not\subset \operatorname{Ker} \phi_{ij} \\ 0 & \text{otherwise,} \end{cases} \tag{16}$$

which is either an $(n-3)$–dimensional subspace or 0. We claim that it is always 0.

Assume the contrary for some $i' \neq j'$. Denoting by π_j the orthogonal projection to $\operatorname{Im} A_{\xi_j}$, (14) and (15) easily yield that, for each i, there is a 2-form $\lambda_i \in \Lambda^2(\operatorname{Ker} A_{\xi_i})$ such that $\pi_j[\ ,\ ] = \lambda_i Y_i$ and

$$\pi_j \widehat{\phi_{ij}} = \lambda_i Y_j \quad \text{ on } \quad \operatorname{Ker} A_{\xi_i} \times \operatorname{Ker} A_{\xi_i}. \tag{17}$$

Since along $\mathcal{S}$ it holds that $\operatorname{Ker} A_{\xi_i} + \operatorname{Ker} A_{\xi_j} = (\operatorname{Im} A_{\xi_i} \cap \operatorname{Im} A_{\xi_j})^{\perp} = T_xM$, we get $\pi_j(\operatorname{Ker} A_{\xi_i}) = \operatorname{Im} A_{\xi_j}$. By dimension reasons, this, (16) and (17) give

$$0 \neq \pi_{j'}(\operatorname{Ker} A_{\xi_{i'}} \cap \operatorname{Ker} \phi_{i'j'}) = \lambda_{i'}(\operatorname{Ker} A_{\xi_{i'}} \times \operatorname{Ker} A_{\xi_{i'}}) Y_{j'}.$$

Thus, $\lambda_{i'} \neq 0$ and $j' \geq r$. But (17) for i' and $j \geq r$ gives

$$\pi_j \widehat{\phi_{i'j}}(\operatorname{Ker} A_{\xi_{i'}} \times \operatorname{Ker} A_{\xi_{i'}}) = \operatorname{span}\{Y_j\} \neq 0.$$

Therefore, (16) yields $\pi_j(\operatorname{Ker} A_{\xi_{i'}} \cap \operatorname{Ker} \phi_{i'j}) = \operatorname{span}\{Y_j\}$. By dimension reasons we conclude that

$$\operatorname{Ker} A_{\xi_{i'}} \cap \operatorname{Ker} \phi_{i'j} = \operatorname{Ker} A_{\xi_{i'}} \cap \left(\operatorname{Ker} A_{\xi_j} \oplus^{\perp} \operatorname{span}\{Y_j\} \right), \qquad \forall\ j \geq r. \tag{18}$$

On the other hand, we have from (17) that

$$\operatorname{Ker} A_{\xi_{i'}} \cap \operatorname{Ker} \phi_{i'j} \cap (\operatorname{span}\{Y_j\})^{\perp} \subset \operatorname{Ker} \lambda_{i'}.$$

Since $\lambda_{i'} \neq 0$ is skew-symmetric, we obtain equality above and, by (18), $\operatorname{Ker} \lambda_{i'} = \operatorname{Ker} A_{\xi_{i'}} \cap \operatorname{Ker} A_{\xi_j}$, for all $j \geq r$. Hence,

$$\begin{aligned} n-4 &= \dim \operatorname{Ker} \lambda_{i'} = \dim(\operatorname{Ker} A_{\xi_{i'}} \bigcap_{j\geq r} \operatorname{Ker} A_{\xi_j}) \\ &\leq n - \dim\left(+_{j\geq r} \operatorname{Im} A_{\xi_j}\right) = 2(r-1) \leq 2p-6, \end{aligned}$$

a contradiction because $n-2p+1 = \nu \geq 0$. Thus, $\operatorname{Ker} A_{\xi_i} \subset \operatorname{Ker} \phi_{ij}$ for all i, j.

To conclude the proof, observe that the skew-symmetry of ϕ_{ip} gives $T_xM = \operatorname{Ker} A_{\xi_i} + \operatorname{Ker} A_{\xi_p} \subset \operatorname{Ker} \phi_{ip}$. Therefore, $\phi_{ip} = 0$. ∎

The following general splitting result has its own interest (compare with Lemma 5 of [**F2**]).

Proposition 7. *Let $f : M^n \to \mathbb{Q}_c^{n+p}$ be an isometric immersion of a riemannian manifold which has k distributions $\mathcal{D}_1, \ldots, \mathcal{D}_k$ such that*

$$TM = (\mathcal{D}_1 \oplus \cdots \oplus \mathcal{D}_k) \oplus^{\perp} \Delta \quad \text{and} \quad \alpha(\mathcal{D}_i, \mathcal{D}_j) = 0, \qquad \forall\ 1 \leq i \neq j \leq k.$$

Assume that $W_i^{p_i} = \operatorname{span} \alpha(\mathcal{D}_i, \mathcal{D}_i)$, $1 \leq i \leq k$, are mutually orthogonal parallel normal subbundles. Then:

1) *If $c = 0$, there exists an open dense subset $\mathcal{U}$ so that $\mathcal{U}$ is locally a riemannian product $\mathcal{U}_\lambda = M_1^{n_1} \times \cdots \times M_k^{n_k}$ with $\mathcal{D}_i \subset TM_i$. In addition, there are isometric immersions $f_i : M_i^{n_i} \to \mathbb{R}^{n_i+p_i}$ such that*

$$f|_{\mathcal{U}_\lambda} = f_1 \times \cdots \times f_k.$$

2) *For $\Delta = 0$, we must have $c = 0$. In addition, the above local splitting holds for the whole $\mathcal{U} = M^n$.*

Proof: Of course we can suppose that $k = 2$. Along this proof, denote by i, j arbitrary indexes $1 \leq i \neq j \leq 2$.

Claim 1): $\mathcal{D}_1 \perp \mathcal{D}_2$. Set

$$K_i = \Delta^\perp \bigcap_{w \in W_i} \operatorname{Ker} A_w, \quad I_i = \operatorname{span}\{\operatorname{Im} A_w : w \in W_i\} = K_i^\perp \cap \Delta^\perp.$$

From $\alpha(\mathcal{D}_1, \mathcal{D}_2) = 0$ and $W_1 \perp W_2$ we get $\mathcal{D}_i \subseteq K_j$. But $\Delta^\perp = \mathcal{D}_1 \oplus \mathcal{D}_2 \subseteq K_1 \oplus K_2 \subseteq \Delta^\perp$. Thus, $\mathcal{D}_i = K_j$.

For $\eta_i \in W_i$, Ricci equation and the assumption on W_1, W_2 imply that $[A_{\eta_1}, A_{\eta_2}] = 0$. Thus, $A_{\eta_i} A_{\eta_j} = 0$ since $K_1 \cap K_2 = 0$. In other words, $I_i \subseteq K_j = \mathcal{D}_i$. Hence $\Delta^\perp = I_1 \oplus I_2 \subseteq K_1 \oplus K_2 = \Delta^\perp$. This implies $I_i = \mathcal{D}_i$ and proves Claim 1) since $\mathcal{D}_1 = I_1 \perp K_1 = \mathcal{D}_2$.

Set $\mathcal{D}_i^0 = \mathcal{D}_i$ and

$$\mathcal{D}_i^{r+1} := \operatorname{span}\{\nabla_X Y : X \in \mathcal{D}_i,\ Y \in \mathcal{D}_i^r\} \supseteq \mathcal{D}_i^r.$$

Claim 2): $\mathcal{D}_i^r \perp \mathcal{D}_j$, for all $r \geq 0$. By induction suppose that the Claim holds for r and take $X \in \mathcal{D}_j$, $Y \in \mathcal{D}_i$ and $Z \in \mathcal{D}_i^r$. The Codazzi equation says

$$\alpha(X, \nabla_Y Z) = \nabla_Y^\perp \alpha(X, Z) - \nabla_X^\perp \alpha(Y, Z) + \alpha([X, Y], Z) + \alpha(Y, \nabla_X Z).$$

The parallelism of the W_k's and the inductive hypothesis imply that the right hand side of the above equation belongs to W_i. This proves the claim, because the left hand side belongs to W_j.

Claim 3): $\nabla_V Z \in \mathcal{D}_i^r$, for all $V \in \mathcal{D}_i^\perp$, $Z \in \mathcal{D}_i^r$, $r \geq 0$.

By induction, take $X, Y \in \mathcal{D}_j$ and $Z \in \mathcal{D}_i$. From Claim 2) we obtain that

$$\nabla_X Z \in \mathcal{D}_i \oplus \Delta \tag{19}$$

since $\langle \nabla_X Z, Y \rangle = -\langle Z, \nabla_X Y \rangle = 0$. The claim is now clear for $\Delta = 0$. Observe also that $c = 0$ in this case since $0 = K_M(X, Z) = c\,\|X\|^2 \|Z\|^2$ by the Gauss equation.

For $U \in \Delta$, Codazzi equation implies that $\alpha(X, \nabla_Z U) = \alpha(Z, \nabla_X U) = 0$, since both belong to different W_k's. Hence $\nabla_X U \in \mathcal{D}_j \oplus \Delta$, which in view of (19) implies that

$$\nabla_X Z \in \mathcal{D}_i. \tag{20}$$

Again from the Codazzi equation we get $\alpha([X, U], Z) = \alpha(\nabla_U Z, X) = 0$, or equivalently, $\nabla_U Z \in \mathcal{D}_i$ since Δ is a totally geodesic distribution. This proves the claim for $r = 0$. The claim is now a consequence of Gauss equation and the inductive process since, if $Z \in \mathcal{D}_i^r$, $S \in \mathcal{D}_i$ and $V \in \mathcal{D}_i^\perp$, we have

$$\nabla_V \nabla_S Z = \nabla_S \nabla_V Z + \nabla_{[V,S]} Z \quad \in \; \mathcal{D}_i^{r+1}.$$

Claim 4): $\mathcal{D}_i^r \perp \mathcal{D}_j^m$, for all $r, m \geq 0$.

The case $m = 0$ is just Claim 2). Take $V \in \mathcal{D}_j$, $Y \in \mathcal{D}_j^m$ and $Z \in \mathcal{D}_i^r$. The claim now follows from Claim 3) by induction since

$$\langle Z, \nabla_V Y \rangle = -\langle \nabla_V Z, Y \rangle = 0.$$

Set

$$\Gamma_i := \mathcal{D}_i^n, \qquad \Gamma_0 := (\Gamma_1 \oplus \Gamma_2)^\perp \subset \Delta,$$

and define $\mathcal{U}$ as the open dense subset where all the Γ_i's have locally constant dimension. Observe that $\mathcal{U} = M^n$ if $\Delta = 0$. Along the connected components $\mathcal{U}_\lambda$ of $\mathcal{U}$, we conclude from Claim 4) that all the Γ_k's are smooth parallel mutually orthogonal distributions. By the local de Rham's decomposition theorem we obtain locally that

$$\mathcal{U}_\lambda = M_0 \times M_1 \times M_2,$$

with $TM_i = \Gamma_i \supseteq \mathcal{D}_i$. The splitting of f is a consequence of the Main Lemma in [**M**] and the proposition is proved. ∎

Finally, we have the necessary tools to prove our main results.

Proof ot Theorem 1: Take $\mathcal{U}'$ the interior of the set where $\mu = \nu + 1 = n - 2p + 2$. By Proposition 5, we have that this is also the maximal open subset along which f is locally a composition, so it is as in case 2) of the theorem. On the other hand, from Lemma 6 we get locally decomposition (13) along the open set $\mathcal{W}$ of minimal nullity $\mu = \nu$. It also says that $\mathcal{U}' \cup \mathcal{W}$ is dense in M^n. Along $\mathcal{W}$, using Lemma 6, set

$$\begin{aligned} &\mathcal{D}_i = \Delta^\perp \textstyle\bigcap_{j \neq i} \operatorname{Ker} A_{\xi_j}, \quad W_i = \operatorname{span}\{\xi_i\}, \quad 1 \leq i \leq p-2, \quad \text{and} \\ &\mathcal{D}_{p-1} = \Delta^\perp \textstyle\bigcap_{j \leq p-2} \operatorname{Ker} A_{\xi_j}, \quad W_{p-1} = \operatorname{span}\{\xi_{p-1}, \xi_p\}. \end{aligned} \tag{21}$$

It is clear from (13) that the $\mathcal{D}_i$'s satisfy the hypothesis of Proposition 7. Suppose now that the W_i's are parallel along $\mathcal{W}$. We conclude from Proposition 7 that, locally along an open dense subset $\mathcal{U} \subset \mathcal{W}$, f is as in part 1) of Theorem 1. Since $\mathcal{V}=\mathcal{U}'\cup\mathcal{U}$ is also dense in M^n, the theorem will be proved. We now use the notations and definitions of the proof of Lemma 6 to show that W_j is parallel for $1\leq j\leq p-2$. Hence, W_{p-1} is parallel as well.

Fix such a $1\leq j\leq p-2$ and $x\in\mathcal{W}$. From (13) and (15) we get $\mathrm{Im}\,\widehat{\phi_{ij}}\subset\mathrm{Ker}\,A_{\xi_j}$ at x. For dimension reasons, this and (16) yield that

$$\mathrm{Ker}\,A_{\xi_i}\subset\mathrm{Ker}\,\phi_{ij},\quad \forall\ 1\leq i\leq p.$$

Hence, for $j\neq i\leq p-2$ we have $T_x\mathcal{W}=\mathrm{Ker}\,A_{\xi_i}+\mathrm{Ker}\,A_{\xi_j}\subset\mathrm{Ker}\,\phi_{ij}$. Thus,

$$\phi_{ij}=0,\quad \forall\ 1\leq i\neq j\leq p-2. \tag{22}$$

Let $\tau=\tau_j:\mathrm{Ker}\,A_{\xi_j}\to W^2=\mathrm{span}\,\{\xi_{p-1},\xi_p\}$ be the map $\tau(Z)=\nabla^\perp_Z\xi_j$. In view of (13), the Codazzi equation for A_{ξ_j} implies that

$$A_{\tau Y}X=A_{\tau X}Y,\quad \forall\ X,Y\in\mathrm{Ker}\,A_{\xi_j}.$$

If $0<l:=\mathrm{rank}\,\tau_j$, we get from the above that $\mathrm{Ker}\,\tau\subset\mathrm{Ker}\,A_\xi$, for all $\xi\in\mathrm{Im}\,\tau$. Therefore,

$$n-2-l=\dim\mathrm{Ker}\,\tau\leq\dim\left(\bigcap\nolimits_{\xi\in\mathrm{Im}\,\tau}\mathrm{Ker}\,A_\xi\right)\leq n-1-l,$$

that is, $\mathrm{Ker}\,\tau\subset\mathrm{Ker}\,A_{\xi_j}$ has codimension ≤ 1 into $\bigcap_{\xi\in\mathrm{Im}\,\tau}\mathrm{Ker}\,A_\xi$. We conclude that $\mathrm{Im}\,A_{\xi_j}\cap(+_{\xi\in\mathrm{Im}\,\tau}\mathrm{Im}\,A_\xi)\neq 0$, a contradiction with (13). Thus, $l=0$ and, by (22), W_j is parallel. ∎

Proof of Theorem 2: The hypothesis on the Ricci curvature implies that $\mu=\nu=0$. The codimension assumption and Proposition 9 of [**F1**] yield $n=2p$ or $n=2p-1$. The first case is Theorem 2 of [**FZ**]. For the second case, following the proof of Theorem 1 we get that $\mathcal{W}=M^n$ by Lemma 6. Also, $\mathcal{U}'=M^n$. The theorem is now a consequence of part 2 of Proposition 7 and the global de Rham's decomposition theorem. ∎

Final remarks. 1) Although f in 2) of Theorem 1 is not a product, it can be described by means of the Gauss parametrization (see [**DG**] for details). To do this, first proceed as in [**FZ**] for each hypersurface f_i in the decomposition. To take care of the flat hypersurface h, use that any flat nowhere totally geodesic hypersurface is (locally) just the "twisted" normal bundle of an arc length parametrized spherical curve $c : I \to \mathbb{S}^N$: take $r \in C^\infty(I)$ and $h : Tc^\perp \subset T_c\mathbb{S}^{N\perp} \to \mathbb{R}^{N+1}$ as

$$h(\xi(s)) = r(s)c(s) + r'(s)c'(s) + \xi(s).$$

The image $h(T_sc^\perp \setminus S)$ is a flat hypersurface with gauss map c, where S is the singular set of h, $S = \{\xi \in T_sc^\perp : \langle \xi, c'' \rangle = r + r''\}$.

2) In 1) of Theorem 1, f can fail to have flat normal bundle only because f_1, generically, has nonflat normal bundle since it has relative nullity $n_1 - 3$ and codimension two. In case 2), the immersion h is what 'destroys' the flatness of the normal bundle of the product immersion.

3) In view of the our results and those mentioned on the introduction, we believe that the following are also true for euclidean submanifolds f with $K_M \leq 0$ and $\nu = n - p - r$, for some $2 \leq r \leq p$:

$\circ$ The equality between nullity indexes $\nu = \mu$ implies the splitting of f as a product of r submanifolds. In fact, the next is stronger.

$\circ$ Locally, there are r euclidean submanifolds $f_i : M_i^{n_i} \to \mathbb{R}^{n_i+p_i}$ with relative nullity $\nu_i = n_i - p_i - 1$, and a flat submanifold $h : U \subset \mathbb{R}^{n+q} \to \mathbb{R}^{n+p}$, $q = \sum_{i=1}^r p_i$, such that $f = h \circ (f_1 \times \cdots \times f_r)$ splits.

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