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On submanifolds with nonpositive extrinsic curvature

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1 Introduction

For an isometric immersion between Riemannian manifolds $f: M^n \to N^{n+p}$, the Gauss equation says that the (sectional) *extrinsic curvature* of M^n in N^{n+p} at $x \in M^n$ for a plane $\sigma \subset T_xM$, $K_f(\sigma) := K_M(\sigma) - K_N(\sigma)$, is given by

$$K_f(\sigma) = \langle \alpha(X, X), \alpha(Y, Y) \rangle - \|\alpha(X, Y)\|^2 ,$$

where α is the second fundamental form of the immersion and $\{X, Y\}$ any orthonormal basis of σ. Superscripts always means dimension.

Chern and Kuiper [3] have shown that $\nu \geqq n - p$ at the points where the extrinsic curvature vanishes. Here $\nu(x)$ is the dimension of the subspace

$$\Delta(x) = \operatorname{Ker} \alpha(x) = \{X \in T_xM : \alpha(X, Y) = 0, \forall Y \in T_xM\}$$

and is called the *index of relative nullity* of f at x. It is a well known fact that the positiveness of the index of relative nullity imposes strong conditions on the metric of the submanifold and on the structure of the immersion. Therefore, it is a natural question to ask what happens if the extrinsic curvature is merely nonpositive.

In that direction, Borisenko had shown in [2] that at points where $K_f \leqq 0$, the index of relative nullity verifies $\nu \geqq n - p^2 - p$. The main purpose of this paper is to prove the following improvement of Borisenko's result

Theorem 1. *Let $f: M^n \to N^{n+p}$ be an isometric immersion between Riemannian manifolds. Suppose that at $x_0 \in M^n$ we have $K_f(x_0) \leqq 0$. Then $\nu(x_0) \geqq n - 2p$.*

The following example shows that our estimate in Theorem 1 is sharp.

Example. Let $U^2 \subset R^3$ be a surface in the euclidean space with negative Gaussian curvature at $x_0 \in U^2$. Then the product immersion of p factors $U^2 \times \ldots \times U^2 \to R^{3p}$ satisfies $\nu(x_0, \ldots, x_0) = n - 2p = 0$.

The strong restrictions that $\nu > 0$ imposes on an isometric immersion allow us to find several applications of Theorem 1. The following corollary is an improvement of Theorem 3 in [2], where a much stronger quadratic hypothesis for the codimension is needed.

Corollary 2. *Let $f: M^n \to S_c^{n+p}$ be an isometric immersion of a complete Riemannian manifold into the euclidean sphere of constant sectional curvature c. If $K_M \leqq c$ and $2p < n - \nu_n$, then f is totally geodesic.*

In the above statement ν_n is defined as $\nu_n = \max\{k : \rho(n-k) \geqq k+1\}$, where $\rho(n)$ is given by $\rho((\text{odd})2^{4d+b}) = 8d + 2^b$, with d being any nonnegative integer and $b = 0, 1, 2, 3$. Some values of ν_n are: $\nu_n = n -$ (highest power of $2 \leqq n$) for $n \leqq 24$, $\nu_n \leqq 8d - 1$ for $n < 16^d$ and $\nu_{2^a} = 0$.

At least for some dimensions, the hypothesis in the codimension in the above corollary cannot be improved to $2p < n$. For example, the simplest of Cartan's isoparametric hypersurfaces, i.e., the unit normal bundle of the Veronesse surface in S_1^4, is a compact non totally geodesic submanifold of S_1^4 with curvature less or equal than one.

We have the following for isometric immersions of Riemannian products.

Corollary 3. *Let $M^n = N_1^{n_1} \times N_2^{n_2}$ be the product of two Riemannian manifolds. Suppose that there exists $(x, x') \in M^n$ such that $K_{N_1}(x)$, $K_{N_2}(x') \leqq c$, a positive constant. Then, there is no isometric immersion of M^n into S_c^{n+p} for $2p < n$.*

Corollaries 2 and 3 also hold if we replace the ambient space by any manifold of constant sectional curvature c.

By Q_c^n (resp. CQ_c^n) we denote the standard real (resp. complex) simply connected space form of constant sectional (resp. holomorphic) curvature c and real (resp. complex) dimension n. Dajczer and Rodríguez [5] have shown that any isometric immersion of a Kähler manifold with $\nu > 0$ everywhere into $CQ_c^N, c \neq 0$, must be holomorphic. From the proof of that theorem and our main result we conclude the following statement.

Corollary 4. *Let M^{2n} be a Kähler manifold and $x_0 \in M^{2n}$ such that $K_M(x_0) \leqq c \neq 0$. If $p < n$, then there exists no isometric immersion of M^{2n} into Q_c^{2n+p}.*

Further applications of our main result for isometric immersions of Kähler manifolds will be given in Sect. 3.

2 The proof of Theorem 1

Let V^n and W^p be real vector spaces of dimensions n and p respectively. Suppose that W^p has a positive definite inner product $\langle , \rangle$, and let $\alpha: V^n \times V^n \to W^p$ be a symmetric bilinear map with nonpositive curvature, i.e.,

$$K_\alpha(X, Y) = \langle \alpha(X, X), \alpha(Y, Y) \rangle - \|\alpha(X, Y)\|^2 \leqq 0 ,$$

for all $X, Y \in V^n$. We recall the following version of a well-known result due to Otsuki [9].

Lemma 5. *With the above assumptions, for any subspace $S \subset V^n$ with $\dim S > p$ there exists $0 \neq X \in S$ such that $\alpha(X, X) = 0$.*

We say that $T \subseteq V^n$ is an *asymptotic subspace* of α if $\alpha(X, Y) = 0$ for all $X, Y \in T$. We denote the set of asymptotic vectors of α by $A(\alpha)$. One of the main tools in the proof of Theorem 1 is the following generalization of Lemma 5.

Proposition 6. *Let* $\alpha: V^n \times V^n \to W^p$ *be a symmetric bilinear map with* $K_\alpha \leqq 0$. *Then, there exists an asymptotic subspace* $T \subset V^n$ *of* α *such that* $\dim T \geqq n - p$.

Proof. For each $X_0 \in V^n$, we define a linear transformation $\alpha(X_0): V^n \to W^p$ by

$$\alpha(X_0)(Y) = \alpha(X_0, Y).$$

Given $X_0 \in A(\alpha)$, we denote

$$V_1 = V_1(X_0) = \operatorname{Ker} \alpha(X_0), \quad W_1 = W_1(X_0) = \{\operatorname{Im} \alpha(X_0)\}^\perp$$

and define $\alpha_1 = \alpha|_{V_1 \times V_1}$.

With the above notations, we claim that $\operatorname{Im} \alpha_1 \subseteq W_1$. To prove the claim, take $Z \in V_1$, $Y \in V^n$. Then, since $X_0 \in A(\alpha)$,

$$\begin{aligned} K_\alpha(X_0 + tY, Z) &= \langle 2t\alpha(X_0, Y) + t^2\alpha(Y, Y), \alpha(Z, Z)\rangle - t^2 \|\alpha(Y, Z)\|^2 \\ &= 2t\langle \alpha(X_0, Y), \alpha(Z, Z)\rangle + t^2 K_\alpha(Y, Z), \end{aligned}$$

for all $t \in \mathbf{R}$. Then, we have that $\langle \alpha(X_0, Y), \alpha(Z, Z)\rangle = 0$, for all $Z \in V_1$, $Y \in V^n$, because $K_\alpha \leqq 0$. The claim follows easily using the symmetry of α.

The above claim allows us to make an inductive process as follows. Set $V_0 = V$ and $W_0 = W$. Given $k \geqq 0$, for the symmetric bilinear map of nonpositive curvature $\alpha_k = \alpha|_{V_k \times V_k}: V_k \times V_k \to W_k$, define

$$r_k = \max \{\dim \operatorname{Im} \alpha_k(X): X \in A(\alpha_k)\},$$

and suppose that if $k \geqq 1$, $n_k = \dim V_k = n - \sum_{i=0}^{k-1} r_i$ and $p_k = \dim W_k = p - \sum_{i=0}^{k-1} r_i$. Picking $X_k \in A(\alpha_k)$ such that $\dim \operatorname{Im} \alpha_k(X_k) = r_k$, set

$$V_{k+1} = V_{k+1}(X_0, \ldots, X_k) = \operatorname{Ker} \alpha_k(X_k),$$

and then

$$n_{k+1} = \dim V_{k+1} = n - \sum_{i=0}^{k} r_i.$$

The above claim implies that $\operatorname{Im} \alpha_{k+1} \subseteq W_{k+1}$, where

$$W_{k+1} = W_{k+1}(X_0, \ldots, X_k) = \{\operatorname{Im} \alpha_k(X_k)\}^\perp \subseteq W_k,$$

and $\alpha_{k+1} = \alpha_k|_{V_{k+1} \times V_{k+1}}$.

Since $0 \leqq p_{k+1} = \dim W_{k+1} = p - \sum_{i=0}^{k} r_i$, there exists a positive integer m such that $r_m = 0$. This tells us that $A(\alpha_m) = \operatorname{Ker} \alpha_m$. Set $T = \operatorname{Ker} \alpha_m$. By Lemma 5, for all subspace $S \subset V_m$ such that $\dim S > p_m$, we have that $S \cap T = S \cap A(\alpha_m) \neq \{0\}$. Hence, $\dim T \geqq n_m - p_m = n - p$. Moreover, since $\alpha_m = \alpha|_{V_m \times V_m}$, then T is an asymptotic subspace of α and this concludes the proof. □

Let $\beta: V' \times V \to V''$ be a bilinear map. We say that $X \in V'$ is a *regular element* of β if $\dim \operatorname{Im} \beta(X) = \max \{\dim \operatorname{Im} \beta(Z): Z \in V'\}$. The set of regular elements of β is denoted by $\operatorname{RE}(\beta)$. For the proof of Proposition 8 we need the following result which is essentially due to Moore [8].

Lemma 7. *Let* $\beta: V' \times V \to V''$ *be a bilinear map and* $Y_0 \in \operatorname{RE}(\beta)$. *Then,*

$$\beta(Y, \operatorname{Ker} \beta(Y_0)) \subseteq \operatorname{Im} \beta(Y_0)$$

for all $Y \in V'$.

Proof. Let $Z_1, \ldots, Z_r$ be vectors in V with $r = \dim \operatorname{Im} \beta(Y_0)$ and

$$\operatorname{Im} \beta(Y_0) = \operatorname{span}\{\beta(Y_0, Z_j),\ 1 \leqq j \leqq r\}\ .$$

It is easy to see that the vectors $\beta(Y_0 + tY, Z_j)$, $1 \leqq j \leqq r$, are linearly independent except for a finite number of values of t. Hence, they generate a family of r-dimensional subspaces that varies continuously with t if $|t| < \varepsilon$, for some $\varepsilon > 0$. But if $Z \in \operatorname{Ker} \beta(Y_0)$, then $\beta(Y_0 + tY, Z) = t\beta(Y, Z)$. Therefore, by continuity, $\beta(Y, Z) \in \operatorname{Im} \beta(Y_0)$. □

Proposition 8. *Let $\alpha: V^n \times V^n \to W^p$ be a symmetric bilinear map with $K_\alpha \leqq 0$. Let T be an asymptotic subspace of α. Then $v \geqq \dim T - p$.*

Proof. Let $T' \subseteq V^n$ be a subspace such that $T' \oplus T = V^n$, and define $\beta: T' \times T \to W^p$ by $\beta = \alpha|_{T' \times T}$. Take $Y_0 \in \operatorname{RE}(\beta)$, $Z \in T$, $Z' \in \operatorname{Ker} \beta(Y_0) \subseteq T$ and $Y \in T'$. Using only the assumption on T, we have for all $s, t \in \mathbf{R}$ that

$$K_\alpha(Y_0 + tZ, Y + sZ') = \langle \alpha(Y_0, Y_0) + 2t\alpha(Y_0, Z), \alpha(Y, Y) + 2s\alpha(Y, Z') \rangle$$
$$- \|\alpha(Y_0 + tZ, Y + sZ')\|^2\ .$$

Since $\alpha(Y_0, Z') = 0$, we get that

$$K_\alpha(Y_0 + tZ, Y + sZ') = K_\alpha(Y_0, Y) - t^2 \|\alpha(Z, Y)\|^2$$
$$+ 2t(\langle \alpha(Y_0, Z), \alpha(Y, Y) \rangle - \langle \alpha(Y_0, Y), \alpha(Z, Y) \rangle)$$
$$+ 2s(\langle \alpha(Y_0, Y_0), \alpha(Y, Z') \rangle + 2t \langle \alpha(Y_0, Z), \alpha(Y, Z') \rangle)\ ,$$

which is linear in s. This implies, in view of the hypothesis on K_α, that

$$\langle \alpha(Y_0, Y_0), \alpha(Y, Z') \rangle + 2t \langle \alpha(Y_0, Z), \alpha(Y, Z') \rangle = 0\ ,$$

for all $t \in \mathbf{R}$, which says that $\langle \alpha(Y_0, Z), \alpha(Y, Z') \rangle = 0$. From the arbitrariness of Z and Z', it follows that

$$\beta(Y, \operatorname{Ker} \beta(Y_0)) \subseteq \{\operatorname{Im} \beta(Y_0)\}^\perp\ ,$$

for all $Y \in T'$. This, together with Lemma 7, tells us that $\alpha(Y, X) = 0$, for all $Y \in T'$, $X \in \operatorname{Ker} \beta(Y_0)$. But since $\operatorname{Ker} \beta(Y_0) \subseteq T$, we obtain

$$\operatorname{Ker} \beta(Y_0) \subseteq \operatorname{Ker} \alpha\ .$$

Then,

$$v \geqq \dim \operatorname{Ker} \beta(Y_0) = \dim T - \dim \operatorname{Im} \beta(Y_0) \geqq \dim T - p\ ,$$

which concludes the proof. □

The proof of Theorem 1 follows immediately from the Gauss equation and the next result.

Proposition 9. *Let $\alpha: V^n \times V^n \to W^p$ be a symmetric bilinear map such that $K_\alpha \leqq 0$. Then $v \geqq n - 2p$.*

Proof. It is clear from Propositions 6 and 8. □

3 Some applications

First of all, we give the proofs of the three corollaries stated in the Introduction.

Proof of Corollary 2. It is a well-known fact (see e.g. [7]) that when the ambient space has constant sectional curvature, i.e., $N^{n+p} = N_c^{n+p}$, the minimum relative nullity distribution of an isometric immersion $f: M^n \to N_c^{n+p}$ is smooth and integrable with totally geodesic leaves in both M^n and N_c^{n+p}. If in addition M^n is complete, then the leaves are also complete. Now, the proof is a direct consequence of Theorem 1 in [6] and our Theorem 1. □

Proof of Corollary 3. Suppose that such an immersion exists. Since the product manifold satisfies $K_M \leqq c$ at (x, x'), by Theorem 1, there exists a unit vector $X \in T_{(x,x')}M$ such that $K_M(X, Y) \equiv c > 0$, for all unit vector $Y \in T_{(x,x')}M$ normal to X. But this is a contradiction because $K_M(Z, Z') = 0$ if $Z \in T_x N_1$ and $Z' \in T_{x'} N_2$. □

The following is the same argument as the one in the proof of Theorem 3 in [5].

Proof of Corollary 4. Suppose that such an immersion exists and call it f. Composing f with the totally geodesic and totally real inclusion $i: Q_c^{2n+p} \to CQ_c^{2n+p}$, we conclude from Theorem 1 that $\nu_{(i \circ f)}(x_0) > 0$. But the proof of the main result in [5] tells us that $T_{x_0}M$ must be J invariant, where J is the almost complex structure of M. This is a contradiction, because $i \circ f$ is totally real. □

In [1], Abe had shown that any holomorphic isometric immersion of a complete Kähler manifold into the complex projective space with $\nu > 0$ everywhere must be totally geodesic. But by the main result of [5], the holomorphic hypothesis is superfluous. Therefore, from Theorem 1, we conclude:

Corollary 10. *Let $f: M^{2n} \to CP_c^{n+p}$, $2p < n$, be an isometric immersion of a complete Kähler manifold into the complex projective space. Suppose that $K_f \leqq 0$. Then $M^{2n} = CP_c^n$ and f is a totally geodesic inclusion.*

Given an isometric immersion of a Riemannian manifold into the euclidean space with $\nu > 0$ everywhere, it is an interesting question to ask whether the relative nullity distribution gives rise to an euclidean factor of the submanifold. The next result follows directly from Theorems 3 and 4 of [4]. We say that the scalar curvature s of a Riemannian manifold M has *subquadratic grow along geodesics* if it satisfies $\lim_{t \to \infty} \frac{s(t)}{t^2} = 0$, where t is the parameter of any geodesic γ and $s(t)$ is the scalar curvature of M at $\gamma(t)$.

Corollary 11. *Let $f: M^{2n} \to R^{2n+p}$ be a minimal isometric immersion of a complete Kähler manifold of nonpositive sectional curvature. Suppose that $p < n$ and one of the following holds:*

a) *s has subquadratic grow along geodesics, or*

b) *there exists $x_0 \in M^{2n}$ where all the holomorphic curvatures of planes in $\Delta(x_0)^\perp$ are negative.*

Then $M^{2n} = N^{2p} \times R^{2(n-p)}$ and $f = f_1 \times \mathrm{id}$ splits.

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