Complete real Kähler submanifolds in codimension two*

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Abstract

Minimal isometric immersions $f : M^{2n} \to \mathbb{R}^{2n+2}$ in codimension two from a complete Kähler manifold into Euclidean space had been classified in [**DG2**] for $n \geq 3$. In this note we describe the non–minimal situation showing that, if f is real analytic but not everywhere minimal, then f is a cylinder over a real Kähler surface $g : N^4 \to \mathbb{R}^6$, that is, $M^{2n} = N^4 \times \mathbb{C}^{n-2}$ and $f = g \times id$ split, where $id\colon \mathbb{C}^{n-2} \cong \mathbb{R}^{2n-4}$ is the identity map. Moreover, g can be further described.

§1. Introduction

By a *real Kähler Euclidean submanifold* we mean a smooth (C^∞) isometric immersion $f : M^{2n} \to \mathbb{R}^{2n+p}$ from a Kähler manifold M^{2n} of real dimension $2n$ into Euclidean space. As expected, the Kähler structure imposes strong restrictions on the immersion. In fact, the hypersurface situation ($p = 1$) is well understood, both locally and globally. Locally, by means of an explicit parametrization ([**DG1**]), while if M^{2n} is assumed to be complete, we showed in [**FZ4**] that f must be a *cylinder* over a complete orientable surface $g\colon N^2 \to \mathbb{R}^3$, that is, $M^{2n} = N^2 \times \mathbb{C}^{n-1}$ and $f = g \times id$ split, where $id\colon \mathbb{C}^{n-1} \cong \mathbb{R}^{2n-2}$ is the identity map.

In codimension $p = 2$ the problem becomes far more interesting. Although few results were known until now in the local case unless the immersion has rank at most two ([**DF2**]), the complete case for dimension $n \geq 3$ is well understood for *minimal* immersions. Here, f is either a holomorphic complex hypersurface under an identification $\mathbb{R}^{2n+2} \cong \mathbb{C}^{n+1}$, or a cylinder over a complete minimal real Kähler surface $g\colon N^4 \to \mathbb{R}^6$, or it is essentially completely holomorphically ruled, i.e., M^{2n} is the total space of a holomorphic vector bundle over a Riemann surface, and f maps each fiber onto a linear

*Mathematics Subject Classification (2000): Primary 53C40; Secondary 53B25.

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Research partially supported by CNPq.

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subvariety in $\mathbb{R}^{2n+2}$ ([**DR**]). The paper [**DG2**] was devoted to give a precise description of the latter case in terms of a Weierstrass-type representation. Interesting explicit examples were then given.

Our main purpose here is to understand the general (analytic) situation, that is, to drop the minimality assumption on f and hence to complete the global classification by showing that, if non–minimal, f must be a cylinder over a real Kähler surface in $\mathbb{R}^6$:

Theorem 1. *Let $f\colon M^{2n} \to \mathbb{R}^{2n+2}$, $n \geq 2$, be a complete analytic real Kähler Euclidean submanifold. If f is not everywhere minimal, then $M^{2n} = N^4 \times \mathbb{C}^{n-2}$ and $f = g \times id$ split, for some complete real Kähler Euclidean surface $g\colon N^4 \to \mathbb{R}^6$.*

Although we believe it is superfluous, the analyticity assumption appears since we do not know at this point how to deal with the possible gluing phenomenon. In general, we are able to show that there is an open subset $\mathcal{U} \subset M^{2n}$ such that $\mathcal{U} = V^4 \times \mathbb{C}^{n-2}$ and the restriction $f|_{\mathcal{U}} = g \times id$ is a cylinder over some $g\colon V^4 \to \mathbb{R}^6$.

By Hartman's Theorem ([**H**]), a complete flat Euclidean submanifold in codimension two is a cylinder over a flat surface $g\colon N^2 \to \mathbb{R}^4$, although the decomposition $f = g \times id\colon M^{2n} = N^2 \times \mathbb{R}^{2n-2} \to \mathbb{R}^{2n+2}$ does not need to be a Kähler one. The local case has been classified in [**DF1**].

The surface case $n = 2$ in codimension two is not well understood if the immersion is minimal. It was shown in [**DG3**] that there is a family of irreducible (i.e., neither a product of two surfaces in $\mathbb{R}^3$ nor a cylinder over a surface in $\mathbb{R}^4$) complete minimal real Kähler surfaces in $\mathbb{R}^6$ which are neither holomorphic nor complex ruled. However, we can describe the non–minimal situation:

Theorem 2. *Let $f\colon M^4 \to \mathbb{R}^6$ be a complete irreducible analytic real Kähler Euclidean surface that is not everywhere minimal. Then, $M^4 = N^2 \times \mathbb{C}$, and there is an open dense subset $W \subset M^4$ such that, along each connected component V of W, the restriction $f|_V$ is a composition of analytic isometric immersions. That is, V is a real Kähler Euclidean hypersurface, $g\colon V \to \mathbb{R}^5$, and $f|_V = h \circ g$, where $h\colon \mathcal{U} \subseteq \mathbb{R}^5 \to \mathbb{R}^6$ is a flat hypersurface, for some open subset $\mathcal{U} \subseteq \mathbb{R}^5$ with $g(V) \subset \mathcal{U}$.*

In some situations, we can assure that the whole f is globally a composition; see Remark 11. Moreover, since the hypersurface situation is parametrically understood, $f|_V$ above can now also be parametrically described. More importantly, the proof of Theorem 2, being local in nature, contains the ingredients that allow to give a *local* parametric classification of the (not necessarily analytic) everywhere non–minimal real Kähler Euclidean submanifolds in codimension two; cf. Remark 12.

Theorem 1 is based on the main result in this paper that holds for any (not necessarily analytic) complete real Kähler Euclidean submanifold, in any codimension:

The complex relative nullity foliation is always holomorphic.

The complex relative nullity distribution D is just the maximal complex spaces contained in the relative nullity Δ of the immersion; that is, $D = \Delta \cap J\Delta$, where J stands for the Kähler structure of M^{2n} and Δ for the kernel of the second fundamental form. It is easy to check that, on the open subset where D attains its minimal dimension, D is an integrable distribution with complete totally geodesic leaves in both M^{2n} and the Euclidean ambient space. The holomorphicity of D then imposes strong restrictions on the immersion that, in codimension two, allow us to easily deduce Theorem 1.

§2. The complex relative nullity foliation

In this section, we will discuss some general properties about the complex relative nullity foliation of any complete real Kähler Euclidean submanifold. The main result is that the foliation is always a holomorphic one. We then apply this property to the codimension two case to easily get Theorem 1. Hopefully the main result might be useful in other contexts as well.

Let $f : M^{2n} \to \mathbb{R}^{2n+p}$ be a complete real Kähler Euclidean submanifold. We will denote by Δ_x the *relative nullity* of f at $x \in M$, that is, the nullity space of the second fundamental form $\alpha(x)$ of f at x, $\Delta_x = \{Z \in T_xM : \alpha(Z, T_xM) = 0\}$, and by $\nu(x)$ the *index of relative nullity* of f at x, i.e., $\nu(x) = \dim_{\mathbb{R}} \Delta_x$. Let $U \subset M$ be the open set where ν attains its minimum value ν_0,

$$U = \{x \in M \mid \nu(x) = \nu_0\}.$$

It is a well-known fact (see e.g. [**F**]) that Δ is smooth and integrable in U with totally geodesic leaves in both M^{2n} and $\mathbb{R}^{2n+p}$. If, in addition, M is complete, then the leaves are also complete and thus each one is mapped by f onto a linear subvariety (that is, a translation of linear subspace) of $\mathbb{R}^{2n+p}$.

Let us define the complex subspaces $D_x \subset T_xM$ by

$$D_x = \Delta_x \cap J\Delta_x,$$

where J is the almost complex structure of M, and by $\nu'(x) = \dim_{\mathbb{C}} D_x$ its complex dimension. Observe that $D = \Delta \cap \Delta_J$ as well, where Δ_J is the *pluriharmonic nullity* of f defined in [**FZ3**] by $\Delta_J := \{Z \in TM : \alpha(JZ, Y) = \alpha(Z, JY), \forall\, Y \in TM\}$.

Let ν'_0 be the minimum value of $\nu'(x)$ for all $x \in U$, and $U_0 \subseteq U$ be the open subset where $\nu' = \nu'_0$. Clearly, since J is parallel, D is also smooth, integrable and totally geodesic in U_0 and its leaves are again complete. Therefore, each leaf is isometric to $\mathbb{C}^{\nu'_0}$ and is mapped by f onto a linear subvariety in $\mathbb{R}^{2n+p}$. We will call the leaves of D in U_0 the *complex relative nullity* foliation of f from now on.

Let us write $r = n - \nu'_0$ and fix any $x \in U_0$. Let V be the space of type $(1, 0)$ tangent vectors at x, that is, V is the complex subspace of $(T_xM) \otimes \mathbb{C}$ defined as $V = \{v - iJv : v \in T_xM\}$. Denote by $W = W_x \cong \mathbb{C}^r$ the complex linear subspace of V perpendicular to D_x, that is,

$$W \oplus \overline{W} = D_x^\perp \otimes \mathbb{C}\,.$$

Let $C\colon D \times D^\perp \to D^\perp$ be the *twisting tensor* (also called the *splitting tensor*) of the totally geodesic foliation D defined by $C_T X = C(T,X) = -(\nabla_X T)_{D^\perp}$, where ∇ stands for the Levi-Civita connection of M and $(\ )_{D^\perp}$ for the orthogonal projection onto $D^\perp$. Fix $T \in D$ and write for the complexified operator C_T,

$$C_T(e_i) = \sum_{j=1}^{r} (A_{ij} e_j + \overline{B_{\bar{i}j}}\, \overline{e_j}),$$

for a basis $\mathcal{B} = \{e_1, \dots, e_r\}$ of W. We need a basic property of the twisting tensor:

Lemma 3. *Let $f : M^{2n} \to \mathbb{R}^{2n+p}$ be a complete real Kähler submanifold and $x \in U_0$. Then, for any $T \in D_x$ and any $\lambda \in \mathbb{C}$, the matrix*

$$\tilde{C} = \begin{bmatrix} \lambda A & \overline{\lambda B} \\ \lambda B & \overline{\lambda A} \end{bmatrix}$$

has no non-zero real eigenvalues.

Proof: Write $\lambda = a + ib$ and take a geodesic γ with $\gamma'(0) = aT + bJT$. From $JD = D$ we have $C_{JT} = JC_T$. So, under the frame $\{e_i, \overline{e_i}\}$ for $D^\perp \otimes \mathbb{C}$, the twisting tensor $C_{\gamma'(0)}$ is represented by the above $2r \times 2r$ matrix. Since D is totally geodesic and is contained in the nullity space of the curvature tensor, it is easy to check that $C = C_{\gamma'(t)}$ satisfies the Riccati equation $C' = C^2$. Hence, since the leaves of D are complete, $C_{\gamma'(0)}$ cannot have any non-zero real eigenvalue, just like the case for conullity operators as observed by Abe (cf. [**A**]): the solution of the above Riccati equation is $C(t) = C(0)(I - tC(0))^{-1}$. ∎

Let us now recall the following decomposition of the second fundamental form α of f at $x \in M$ (see [**FHZ**]). Extend α bilinearly over $\mathbb{C}$, and still denote it by α,

$$\alpha : (T_x M) \otimes \mathbb{C} \times (T_x M) \otimes \mathbb{C} \to T_x^\perp M \otimes \mathbb{C}\,.$$

Using that $(T_x M) \otimes \mathbb{C} = V \oplus \overline{V}$, we can write

$$H = \alpha|_{V \times \overline{V}} \qquad \text{and} \qquad S = \alpha|_{V \times V}$$

for the $(1,1)$ and $(2,0)$ parts of α, respectively. Let $W' \subset V$ be the complex linear subspace given by

$$W' = \ker H \ \cap \ \ker S.$$

Hence, $D \otimes \mathbb{C} = W' \oplus \overline{W'}$ and $V = W \oplus W'$. With respect to the basis $\mathcal{B}$, we have that A and B in Lemma 3 are $r \times r$ complex matrices, while H and S are Hermitian and complex symmetric matrices, respectively, with values in the (complexification of the) normal space of f at x. These operators satisfy the following compatibility conditions:

Lemma 4. *At any $x \in U_0$ and under any basis $\mathcal{B}$ of W, the matrices AS and BH are always symmetric. Moreover, it holds that $AH = SB^t$.*

Proof: Note that under a base change, say $e_i' = \sum_{j=1}^r P_{ij} e_j$, for $P \in GL(r, \mathbb{C})$, the matrices A, B, H, S change to

$$PAP^{-1}, \quad \overline{P}BP^{-1}, \quad PHP^*, \quad PSP^t, \tag{1}$$

respectively. So the symmetry of AS and BH, as well as the identity $AH = SB^t$, are independent of the choice of the frame.

Let $\{e_i, e_\alpha\}$ be a local unitary frame of type $(1,0)$ tangent vector fields near x, such that $\{e_i\}$ gives a basis of W in a neighborhood of x. Here and below we will use the convention that Latin indices i, j,... will run from 1 to r, while Greek indices α, β,... will run between $r+1$ and n. Also, let a, b,... run through the full range, from 1 to n.

Let g and θ be the matrices of metric and connection of M under the frame $\mathcal{B}$. Also, write

$$\xi_{a\mu} = \langle \tilde{\nabla} e_a, w_\mu \rangle,$$

where $1 \leq \mu \leq p$, $\{w_1, \ldots, w_p\}$ is an orthonormal frame of the normal bundle of f near x, and $\tilde{\nabla}$ is the covariant differentiation in $\mathbb{R}^{2n+p}$. we get

$$\xi_{\alpha\mu} = 0, \quad \xi_{i\mu} = \sum_{k=1}^r (S^\mu_{ik} \varphi_k + H^\mu_{i\overline{k}} \overline{\varphi_k}),$$

where $\{\varphi_a\}$ is the dual coframe of $\{e_a\}$. Fix any α, write $e_\alpha = \frac{1}{2}(T_\alpha - \sqrt{-1} J T_\alpha)$ and $C_{T_\alpha}(e_i) = \sum_{j=1}^r (A^\alpha_{ij} e_j + \overline{B^\alpha_{\overline{i}j}}\, \overline{e_j})$. From now on, for the sake of simplicity, we omit the superscript μ for S and H. Then we obtain

$$\theta_{\alpha i} = \langle \nabla e_\alpha, e_i \rangle = - \sum_{j=1}^r (A^\alpha_{ji} \varphi_j - B^\alpha_{\overline{j}i} \overline{\varphi_j}).$$

By the Codazzi equation, we get

$$\begin{aligned} 0 \;=\; -d\xi_{\alpha\mu} \;&=\; - \sum_{i=1}^r (\theta_{\alpha i}\, \xi_{i\mu}) \\ &= \sum_{i,j,k=1}^r (A^\alpha_{ji}\, \varphi_j + B^\alpha_{\overline{j}i}\, \overline{\varphi_j}) \wedge (S_{ik}\, \varphi_k + H_{i\overline{k}}\, \overline{\varphi_k}) \\ &= \sum_{j,k=1}^r \left((A^\alpha S)_{jk}\, \varphi_j \wedge \varphi_k + (B^\alpha H)_{\overline{j}\overline{k}}\, \overline{\varphi_j} \wedge \overline{\varphi_k} + (A^\alpha H - S (B^\alpha)^t)_{j\overline{k}}\, \varphi_j \wedge \overline{\varphi_k} \right). \end{aligned}$$

We conclude that $A^\alpha S$ and $B^\alpha H$ are both symmetric, and $A^\alpha H = S(B^\alpha)^t$ for all α. ∎

Next, we observe that the operators A, B also satisfy a compatibility condition with the curvature tensor R of M.

Lemma 5. *For any $1 \leq i,j,k,l \leq r$, the components A and B of C_T satisfy that*

$$\sum_{p=1}^{r} A_{ip} R_{p\bar{j}k\bar{l}} = \sum_{p=1}^{r} A_{kp} R_{p\bar{j}i\bar{l}}, \qquad \text{and} \qquad \sum_{p=1}^{r} B_{\bar{i}p} R_{p\bar{j}k\bar{l}} = \sum_{p=1}^{r} B_{\bar{j}p} R_{p\bar{i}k\bar{l}}.$$

Proof: Recall that, since M is Kähler, $R(V,V) = R(\overline{V},\overline{V}) = 0$. Moreover, the relative nullity, and hence D, is always contained in the nullity of R by the Gauss equation. So, the second Bianchi identity gives us that $0 = (\nabla_{e_i} R)(T, \overline{e}_j, e_k, \overline{e}_l) - (\nabla_{e_k} R)(T, \overline{e}_j, e_i, \overline{e}_l) = R(C_T e_i, \overline{e}_j, e_k, \overline{e}_l) - R(C_T e_k, \overline{e}_j, e_i, \overline{e}_l)$, which is the first relation we wanted to prove. The proof of the second one is similar. ∎

Note that Lemma 5 holds true under any basis $\mathcal{B}$ of W, not necessarily an orthogonal one. The symmetry in Lemma 5 was observed in [**WZ**], where the situation is intrinsic. In our case here, the manifold M may not have nonpositive or nonnegative bisectional curvature, and the complex relative nullity D, although is always contained in the nullity of M, may not coincide with the nullity. So, in order for us to exploit techniques of the proof of Theorem A in [**WZ**], we need more symmetry conditions on the components A and B of the twisting tensor.

By the Gauss equations, the curvature tensor R of M is given by

$$\langle R(X,\overline{Y})Z,\overline{U}\rangle = \langle H(X,\overline{U}), H(Z,\overline{Y})\rangle - \langle S(X,Z), \overline{S(Y,U)}\rangle,$$

for all $X,Y,Z,U \in V$. Let us introduce the tensor $\hat{R}$ by

$$\langle \hat{R}(X,\overline{Y})Z,\overline{U}\rangle = \langle H(X,\overline{U}), H(Z,\overline{Y})\rangle + \langle S(X,Z), \overline{S(Y,U)}\rangle,$$

also with $\hat{R}(V,V) = \hat{R}(\overline{V},\overline{V}) = 0$. It has all the symmetries of R, i.e., it is a curvature-like tensor. Taking a unitary basis $\{w_1, , \ldots, w_r\}$ of W, the Ricci tensor of M is given by

$$Q(X,Y) = \mathrm{Ric}(X,\overline{Y}) = \sum_{j=1}^{r} \langle R(X,\overline{w}_i)w_i,\overline{Y}\rangle = \sum_{j=1}^{r} \langle R(X,\overline{Y})w_i,\overline{w}_i\rangle = (Q^H - Q^S)(X,Y),$$

with $Q^H(X,Y) = \sum_{i=1}^{r} \langle H(X,\overline{w}_i), \overline{H(Y,\overline{w}_i)}\rangle$ and $Q^S(X,Y) = \sum_{i=1}^{r} \langle S(X,w_i), \overline{S(Y,w_i)}\rangle$. Notice that the corresponding $\hat{Q} = Q^H + Q^S$ for $\hat{R}$ is positive definite on W, since both Q^H and Q^S are positive semidefinite, and W is the orthogonal complement of the common nullity of H and S.

Since AS and BH are symmetric by Lemma 4, Lemma 5 still holds true if we replace R by $\hat{R}$. That is, we have:

Lemma 6. *For any $1 \leq i,j,k,l \leq r$, the components A, B of the twisting tensor satisfy*

$$\sum_{p=1}^{r} A_{ip} \hat{R}_{p\bar{j}k\bar{l}} = \sum_{p=1}^{r} A_{kp} \hat{R}_{p\bar{j}i\bar{l}}\,, \qquad \text{and} \qquad \sum_{p=1}^{r} B_{\bar{i}p} \hat{R}_{p\bar{j}k\bar{l}} = \sum_{p=1}^{r} B_{\bar{j}p} \hat{R}_{p\bar{i}k\bar{l}}.$$

In particular, the matrix $B\hat{Q}$ is always symmetric.

The last sentence is the result of contracting the second identity by the inverse of the metric, $g^{\bar{l}k}$. We have all the ingredients to deduce our main result:

Theorem 7. *For any complete real Kähler Euclidean submanifold $f : M^{2n} \to \mathbb{R}^{2n+p}$, the complex relative nullity $D = \Delta \cap J\Delta$ in U_0 is a holomorphic foliation, that is, $B = 0$.*

Proof: Consider a basis $\mathcal{B}$ such that $\hat{Q}(e_i, e_j) = \delta_{ij}$. Under $\mathcal{B}$, by the last assertion in Lemma 6, B becomes a complex symmetric matrix. By (1), it is well known that there is a basis $\mathcal{B}'$ under which B is diagonal with nonnegative diagonal entries. The proof now ends just as the one for Theorem A in [**WZ**], in view of Lemma 3 and Lemma 4. ∎

Let us put together what we know in general about the complex relative nullity distribution. For this, denote by $\operatorname{Im} H = \operatorname{span} H = \operatorname{span}\{H(X, \overline{Y}) : X, Y \in V\}$.

Corollary 8. *In the situation of Theorem 7, for any $T \in D$, A is nilpotent, AS is symmetric, and $AH = 0$; that is, the complexified twisting tensor C_T satisfies that*

$$\alpha(C_T X, Y) = \alpha(X, C_T Y) \in (\operatorname{Im} H)^{\perp}, \quad \forall\, X, Y \in W, \tag{2}$$

$$C_T \colon W \to W, \qquad C_T^r = 0, \qquad \operatorname{Im} C_T \subset \ker H. \tag{3}$$

In other words, for the real *twisting tensor $C_T \colon D^{\perp} \to D^{\perp}$ we have that*

$$\alpha(C_T X, Y) = \alpha(X, C_T Y), \quad \forall\, X, Y \in TM, \tag{4}$$

$$C_T \circ J = J \circ C_T, \qquad C_T^r = 0, \qquad \operatorname{Im} C_T \subset \Delta_J. \tag{5}$$

Proof: It follows from Theorem 7, Lemma 3, Lemma 4, and the relation

$$\langle S_{ij}, H_{k\bar{s}} \rangle = \langle S_{kj}, H_{i\bar{s}} \rangle \quad \forall\, i, j, k, s, \tag{6}$$

that is an easy consequence of the curvature symmetries; see (2) in [**FHZ**]. ∎

Corollary 9. *With the hypothesis of Theorem 7, assume that one of the following holds:*

$$(a)\ \ker H \subseteq \ker S, \qquad or \qquad (b)\ \dim \operatorname{Im} H \geq p - 1.$$

Then, $C = 0$. That is, each connected component U_i of U_0 is isometric to a product

$$U_i = N^{2r} \times \mathbb{C}^{n-r}, \qquad r \leq p,$$

and $f|_{U_i} = f' \times id$ split, where $f' \colon N^{2r} \to \mathbb{R}^{2r+p}$ is a real Kähler Euclidean submanifold, and $id \colon \mathbb{C}^{n-r} \cong \mathbb{R}^{2(n-r)}$ is the identity map.

Proof: If (a) holds, then ker $H = W'$ and by (3) C must vanish. Assume that (b) holds but (a) does not. By (6) we have that

$$S(\ker H, V) \subset (\operatorname{Im} H)^{\perp}. \tag{7}$$

Hence, dim $(\operatorname{Im} H)^{\perp} = 1$, and there is $Z \in \ker H$ such that $S(Z, V) = (\operatorname{Im} H)^{\perp}$. Consider the hyperplane $L = \ker S(Z, \cdot\,) \subset V$. Since from the curvature symmetries the relation $\langle S_{ij}, S_{ks}\rangle = \langle S_{kj}, S_{is}\rangle$ always holds (see (3) in [**FHZ**]), we get $S(L, V) \subset \operatorname{Im} H$ and

$$W' = L \cap \ker H. \tag{8}$$

Hence, by (2) we obtain that $\alpha(C_T L, V) \in \operatorname{Im} H \cap (\operatorname{Im} H)^{\perp} = 0$ since H is Hermitian. This yields $L \subset \ker C$ by the last relation in (3). Since C_T is nilpotent, we conclude again from the last relation in (3) that $\operatorname{Im} C_T \subset L \cap \ker H$. Therefore, by (8), $C = 0$.

The estimate on r follows from (8) and Lemma 7 of [**FHZ**], where it was proved that dim ker $H \geq n -$ dim Im H. ∎

Remark 10. We point out that, where $\ker H$ attains its minimal possible complex dimension $n - p$, it holds that $\ker H \subseteq \ker S$ (see Lemma 7 in [**FHZ**]).

Proof of Theorem 1: It follows from Corollary 9 (b) and the fact that f is minimal if and only if $H = 0$ (cf. Remark 8 in [**FHZ**]). ∎

§3. Real Kähler surfaces and the local case in codimension two

Here we basically argue locally to understand real Kähler Euclidean surfaces in $\mathbb{R}^6$, giving the proof of Theorem 2. As a side effect, we obtain a local classification of all real Kähler Euclidean submanifolds in codimension two.

Proof of Theorem 2: By Corollary 9, $r = 2 - \nu_0' = 2$ and then $D = 0$. Since f is not minimal, we get that $\nu_J = \dim_{\mathbb{C}} \Delta_J = 0$ or 1 almost everywhere. In the first case, the composition structure of f follows from Theorem 1 in [**FZ3**], with nullity index $\mu = 2$ of the curvature tensor of M^4 and relative nullity $\nu = 1$ almost everywhere. Hence, assume also that $\nu_J = 1$ almost everywhere. The next arguments hold along (connected components of) an open dense subset by the analyticity of f.

Consider ξ a unit (analytic) vector field spanning the line bundle $\operatorname{Im} H$. Observe that we can choose ξ to be real since $H(X, \overline{X})$ is real, $X \in V$. Take $\{\xi, \eta\}$ an orthonormal basis of the normal space of f, with corresponding shape operators A_ξ and A_η. By (6) we have that

$$1 \leq \operatorname{rank} A_\xi \leq 2, \tag{9}$$

and that (7) holds. Hence, trace $A_\eta = 0$ and $A_\eta \neq 0$ since $D = 0$.

The curvature tensor symmetries also imply that $\langle S_{ij}, S_{ks}\rangle = \langle S_{kj}, S_{is}\rangle$, for all i, j, k, s (cf. (3) in [**FHZ**]). In terms of the complex bilinear form $s(X, Y) = \langle S(X, Y), \eta\rangle$, this is equivalent to $s(X,Y)s(Z,W) = s(X,W)s(Z,Y)$, for all $X,Y,Z,W \in V$. Taking $Y = X$ with $s(X,X) \neq 0$ and $s(Z,X) = 0$, we conclude from $\operatorname{Im} H \perp \eta$ that

$$\operatorname{rank} A_\eta = 2. \tag{10}$$

Since $D = 0$, we also get from (9) and (10) that $\operatorname{Im} A_\xi \not\subset \operatorname{Im} A_\eta$, and, since μ is even, by the Gauss equation we have that

$$\operatorname{Im} A_\eta \cap \operatorname{Im} A_\xi = 0. \tag{11}$$

The fact that $\mu = 0$ is then equivalent, by Gauss equation, to $\operatorname{rank} A_\xi = 2$. In this case, we claim that f would split as a product of two surfaces in $\mathbb{R}^3$, which is a contradiction.

To prove the claim, consider $\xi_1 = \xi, \xi_2 = \eta$, and take $X, Y \in \operatorname{Ker} A_{\xi_i}$. The Codazzi equation for A_{ξ_i} says that

$$A_{\xi_i}[X,Y] = (-1)^i A_{\xi_j}(\psi(X)Y - \psi(Y)X), \qquad 1 \leq i \neq j \leq 2, \tag{12}$$

where $\psi(X) = \langle \nabla^\perp_X \xi, \eta \rangle$. From this, (10), (11) and $\operatorname{rank} A_\xi = 2$ we easily get that $\operatorname{Ker} A_{\xi_i}$ is integrable and that ξ_i is parallel, $i = 1, 2$. Then, by Ricci equation, we obtain that $\operatorname{Ker} A_{\xi_i} = \operatorname{Im} A_{\xi_j}$. On the other hand, Codazzi equation for $X \in \operatorname{Ker} A_{\xi_i}$ and $Y \in \operatorname{Im} A_{\xi_i}$ gives $\nabla_X A_{\xi_i} Y = A_{\xi_i}[X,Y] \in \operatorname{Im} A_{\xi_i}$. Therefore, we have the decomposition $TM = \operatorname{Im} A_\xi \oplus \operatorname{Im} A_\eta$ into orthogonal parallel distributions. The claim now follows from the local de Rham decomposition Theorem and the Main Lemma in [**M**].

So, we must have, again, $\mu = 2$ and $\nu = 1$, with $\operatorname{rank} A_\xi = 1$. In this situation, we obtain the (local) composition structure of f as follows. By (11) and (12) for $\xi_i = \xi$, $\xi_j = \eta$, we obtain that

$$\operatorname{Ker} A_\xi \subset \operatorname{Ker} \psi. \tag{13}$$

Thus, it is easy to check that A_η is a Codazzi tensor, and then, since $\operatorname{rank} A_\xi = 1$, there is a Euclidean hypersurface g whose second fundamental form is A_η. Now, by (13), we have $(\tilde{\nabla}_{TM}\xi) \cap \operatorname{span}\{\eta\} = 0$. Therefore, we conclude that f is a composition from Proposition 8 in [**DF3**]. Observe that the fact that the line bundles spanned by ξ and η are analytic implies that the immersions h and g also are.

It remains only to argue that $M^4 = N^2 \times \mathbb{C}$ if $\mu = 2$. It is well known that the nullity Γ of the curvature tensor of any Riemannian manifold is an integrable totally geodesic distribution, in any open subset where $\mu = \dim \Gamma$ is constant. Moreover, along the (open) set where μ is minimal, in our case $\mu^{-1}(2)$, the leaves are complete. The twisting tensor $\hat{C}$ of Γ also satisfies the Riccati equation $\hat{C}'_T = \hat{C}^2_T$ for any $T \in \Gamma$, since $R(X,T)T = 0$ for all $X \in TM$. Again by the completeness of the leaves of Γ, this equation has no real eigenvalues. But since Γ is J-invariant, the same argument that of Lemma 3 gives $\hat{C} = 0$. The global splitting follows from the analyticity of f. ∎

Remark 11. We conclude that $f|_W$ in Theorem 2 is itself a composition if f is an embedding. But we do not know if such an f itself is always a composition on the whole M^4, $f = h \circ g$. However, if f is nowhere flat, then $\operatorname{rank} A_\eta = 2$ everywhere. So, the normal subbundles spanned by ξ and η are globally well defined. Hence, if M^4 is simply connected, g is also globally well defined and then it is a cylinder over a surface in $\mathbb{R}^3$ by [**FZ4**]. The same holds if f is nowhere minimal because ξ would never vanish, but in this case, since α decomposes regularly, h is also globally well defined, and thus $f = h \circ g$ is globally a composition; see Proposition 8 in [**DF3**].

Remark 12. The proof of Theorem 2 contains all the ingredients to obtain the *local* classification of nowhere minimal real Kähler Euclidean submanifolds in codimension two $f: M^{2n} \to \mathbb{R}^{2n+2}$, not necessarily analytic. In terms of the index of nullity μ of M^{2n} and index of relative nullity $\nu \leq \mu$ of f, the restriction of f to each connected component V of an open dense subset $W \subset M^{2n}$ must then be:

- $\mu = 2n$: this is the flat case, classified parametrically in Theorem 13 in [**DF1**];
- $\mu = \nu = 2n-2$: either f reduces codimension and is then a hypersurface (classified parametrically in [**DG1**]), or is a cylinder over a surface in $\mathbb{R}^4$. Otherwise, by Theorems 25 and 27 in [**DF2**], it would be minimal, admitting a Weierstrass-type representation;
- $\mu = 2n-2$, $\nu = 2n-3$: similarly as in Theorem 2, f is a composition, so it reduces to the hypersurface situation;
- $\mu = 2n-4$: here f is a product of two Euclidean hypersurfaces (see Theorem 1 in [**FZ3**]).

Observe that $\mu < 2n-4$ cannot occur, since the submanifold would then be a holomorphic hypersurface in $\mathbb{C}^{n+1} \cong \mathbb{R}^{2n+2}$ by [**D**], and hence minimal. Therefore, the local classification problem of real Kähler Euclidean submanifolds f in codimension two reduces, aside from the gluing phenomena, to the minimal case with $\nu = \mu = 2n-4$. However, we recall that any minimal real Kähler submanifold $f: M^{2n} \to \mathbb{R}^N$ is the real part of a holomorphic complex submanifold $g: M^{2n} \to \mathbb{C}^N = \mathbb{R}^N \oplus \mathbb{R}^N$, $f = \mathrm{Re}\, g$; cf. Theorem 1.11 in [**DG1**] and Remark 8 in [**FHZ**].

References

[**A**] Abe, K.: *On a class of hypersurfaces of* $\mathbb{R}^{2n+1}$. Duke Math. J. **41** (1974), 865–874.

[**D**] Dajczer, M.: *A characterization of complex hypersurfaces in* $\mathbb{C}^m$. Proc. Amer. Math. Soc. **105** (1989), no. 2, 425–428.

[**DF1**] Dajczer, M., Florit, L.: *On conformally flat submanifolds.* Comm. Anal. Geom. **4** (1996), 261–284.

[**DF2**] Dajczer, M., Florit, L.: *A class of austere submanifolds.* Illinois J. Math. **45** (2001), 735–755.

[**DF3**] Dajczer, M., Florit, L.: *Compositions of isometric immersions in higher codimension.* Manuscripta Math. **105** (2001), 507–517.

[**DG1**] Dajczer, M., Gromoll, D.: *Real Kaehler submanifolds and uniqueness of the Gauss map.* J. Differential Geom. **22**, 13–28 (1985).

[**DG2**] Dajczer, M., Gromoll, D.: *The Weierstrass representation for complete minimal real Kaehler submanifolds of codimension two.* Invent. Math. **119** (1995), 235–242.

[**DG3**] Dajczer, M., Gromoll, D.: *Complete minimal Kähler surfaces in* $\mathbb{R}^6$. Ann. Global Anal. Geom. **15** (1997), no. 6, 539–541.

[**DR**] Dajczer, M., Rodríguez, L.: *Complete real Kähler minimal submanifolds.* J. Reine Angew. Math. **419** (1991), 1–8.

[**F**] Ferus, D.: *On the completeness of nullity foliations.* Mich. Math. J. **18** (1971), 61-64.

[**FHZ**] Florit, L., Hui, W., Zheng, F.: *On real Kähler Euclidean submanifolds with non-negative Ricci or holomorphic curvature.* J. Eur. Math. Soc. (JEMS) **7** (2005), 1-11.

[**FZ1**] Florit, L., Zheng, F.: *On nonpositively curved euclidean submanifolds: splitting results.* Comm. Math. Helv. **74** (1999), 53–62.

[**FZ2**] Florit, L., Zheng, F.: *On nonpositively curved euclidean submanifolds: splitting results, II.* J. Reine Angew. Math. **508** (1999), 1–15.

[**FZ3**] Florit, L., Zheng, F.: *A local and global splitting result for real Kähler Euclidean submanifolds.* Arch. Math. (Basel) **84** (2005), 88–95.

[**FZ4**] Florit, L., Zheng, F.: *Complete real Kähler Euclidean hypersurfaces are cylinders.* To appear in Ann. Inst. Fourier (Grenoble). Preprint at http://www.preprint.impa.br/Shadows/SERIE_A/2004/274.html

[**H**] Hartman, P.: *On isometric immersions in Euclidean space of manifolds with non-negative sectional curvature.* Trans. Amer. Math. Soc. **147** (1970), 529–540.

[**M**] Moore, J.: *On isometric immersions of riemannian products.* J. Differential Geom. **5** (1971), 159–168.

[**WZ**] H. Wu and F. Zheng.: *Compact Kähler manifolds with nonpositive bisectional curvature.* J. Differential Geom. **61** (2002), 263–287.