ON EXTENSIONS OF INFINITESIMAL DEFORMATIONS[1]

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The classical Allendoerfer's local rigidity result assures that any isometric immersion $f : M^n \to Q_c^{n+p}$ with type number $\rho_f \geq 3$ everywhere is isometrically rigid. Here and throughout the paper, M^n stands for a connected $n-$dimensional Riemannian manifold and Q_c^N denotes a complete simply connected Riemannian manifold of constant sectional curvature c. On the other hand, Dajczer and Rodríguez ([**1**]) have shown that the same type number condition also guarantees infinitesimal rigidity. Thus, "generically speaking", isometric rigidity and infinitesimal rigidity are the same property in low codimension.

A stronger concept of isometric rigidity was considered by Dajczer and Tojeiro in [**2**]. Given an isometric embedding $f : M^n \to Q_c^{n+1}$ and an isometric immersion $g : M^n \to Q_c^{n+p}$, they proved, under some weak regularity conditions on g, that g must be a *composition* $g = h \circ f$, with $\rho_f \geq p + 2$ everywhere. Here $h : U \subset Q_c^{n+1} \to Q_c^{n+p}$ is an isometric immersion and U an open subset containing $f(M)$. In view of [**1**], it is thus natural to look for an infinitesimal version of this stronger isometric rigidity result. The main purpose of this paper is to show that, in this context, any infinitesimal deformation of the composition g must be the restriction to $f(M)$ of an infinitesimal deformation of the extension h, when we restrict ourselves to the case of codimension $p = 2$.

To state our main result, we first need some definitions. We say that an infinitesimal deformation Z of a composition $g = h \circ f$ *admits an extension along* h, if $Z = \overline{Z} \circ f$, where $\overline{Z}$ is an infinitesimal deformation of $h|_{U'}$, for some open subset $f(M) \subset U' \subset U$. The isometric immersion g is said to be *1-regular* if the normal space spanned by its second fundamental form has constant dimension.

Theorem 1. *Let $f : M^n \to Q_c^{n+1}$ be an isometric embedding and let $h : U \subset Q_c^{n+1} \to Q_c^{n+2}$ be an isometric immersion such that $f(M) \subset U$. Assume that the composition*

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$g = h \circ f$ is 1-regular and that either $\rho_f \geq 5$ or $\rho_f \geq 4$ and the sectional curvature of M^n verifies $K_M \geq 0$ everywhere. Let Z be an infinitesimal deformation of g. Then, one of the following holds:

1) *$g = i \circ f$, where $i : Q_c^{n+1} \to Q_c^{n+2}$ is a totally geodesic inclusion, and Z admits an extension $\overline{Z}$ along i, or*
2) *Z admits a unique extension $\overline{Z}$ along h.*

The theorem is no longer true if the composition g is not 1-regular. Consider an hypersurface M^n of Euclidean space such that T_pM disconnects M^n in more than two components for some $p \in M^n$. Now, define a C^∞ infinitesimal deformation Z of M^n to be zero except in one component where it is nonzero arbitrary close to p. If h is the circular cylinder with leaves of relative nullity being the hyperplanes parallel to T_pM, we can choose the connected component of $M^n \setminus T_pM$ where Z is nonzero, in such a way that h_*Z cannot be extended along h.

Without any regularity assumption, we conclude from Theorem 1 the following.

Corollary 2. *Let $f : M^n \to Q_c^{n+1}$ and $h : U \subset Q_c^{n+1} \to Q_c^{n+2}$ be isometric immersions such that $f(M) \subset U$ and that either $\rho_f \geq 5$ or $\rho_f \geq 4$ and $K_M \geq 0$ everywhere. Then, there exists an open and dense subset $V \subset M^n$ such that any infinitesimal deformation Z of $h \circ f|_V$ can be extended along h.*

If the isometric immersion h has arbitrary codimension, the situation is more complicated. Nevertheless, for this case we provide an equivalent condition to the existence of an extension $\overline{Z}$ along h of an infinitesimal deformation Z of the composition g. We also give a characterization of all possible extensions.

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1. Preliminaries

For what follows, we will need some definitions. Given an isometric immersion $f : M \to N$ between Riemannian manifolds, a *deformation of* f is a smooth map $F : I \times M \to N$, where $I \subset \mathbf{R}$ is an open interval around the origin, and the maps $F_t = F(t, \cdot)$ are isometric immersions from M to N with $F_0 = f$. It is easy to verify (cf.[**4**], Chapter 12) that the variational vector field Z along f, $Z = \frac{\partial F}{\partial t}|_{t=0}$, satisfies

$$(1) \qquad \langle \overline{\nabla}_V Z, V \rangle = 0, \quad \forall\, V \in TM,$$

where $\overline{\nabla}$ is the Levi-Civita connection of the metric $\langle\, , \,\rangle$ of N. We call any vector field Z along f which satisfies (1) an *infinitesimal deformation of* f. This terminology arise from the fact that the map $\overline{F}(t,x) = exp_{f(x)}(tZ(x))$ is a deformation of first order of f, being *exp* the exponential map of the ambient space N. For example, if $N = R^m$,

$$\|\overline{F}_{t*} X\|^2 = \|X\|^2 + t^2 \|\overline{\nabla}_X Z\|^2, \quad \forall X \in TM.$$

We say that the infinitesimal deformation Z is *trivial* if it is the restriction to M of a Killing field of the ambient space N. The existence of nontrivial infinitesimal deformations means that M is *infinitesimally deformable* in N.

Let $g : M^n \to Q_c^{n+p}$ be an isometric immersion with second fundamental form $\alpha_g : TM \times TM \to T_gM^{\perp}$. Denote by $N_1^g(x)$ the *first normal space of* g *at* x, i.e. the normal subspace defined as

$$N_1^g(x) = \text{span } \{\alpha_g(X,Y) : X, Y \in T_xM\}.$$

Set $s_g(x) := \dim\, N_1^g(x)$. Notice that, in the context of Theorem 1, for each $x \in M^n$ we have $s_g(x) = 1$ or 2. We say that g is *1-regular* if the semicontinuous function s_g remains constant on M^n. For each normal vector $\xi \in T_xM^{\perp}$, we denote by A_ξ^g the endomorphism of T_xM defined by

$$\langle A_\xi^g X, Y \rangle = \langle \alpha_g(X,Y), \xi \rangle.$$

We denote by $\Delta_g(x)$ the *relative nullity* of g at x, that is, the subspace of T_xM given by

$$\Delta_g(x) = \{X \in T_xM : \alpha_g(X,Y) = 0, \text{ for all } Y \in T_xM\}.$$

It is well known that in an open subset $W \subset M^n$ where the dimension of Δ_g is constant, Δ_g is a smooth and integrable distribution whose leaves are totally geodesic in both W and Q_c^{n+p}.

If the Riemannian manifold M^n has constant sectional curvature c, the Gauss equation says that the symmetric bilinear map α_g is *flat*, i.e. for all $X, Y \in TM$,

$$\langle \alpha_g(X,X), \alpha_g(Y,Y) \rangle = \|\alpha_g(X,Y)\|^2.$$

In this case, the classical Chern-Kuiper inequality implies that dim $\Delta_g \geq n-p$ everywhere (cf.[**4**], Chapter 11).

2. Infinitesimal rigidity

Sbrana has shown in [**3**] that there is a large family of hypersurfaces of R^{n+1} with type number $\rho = 2$, which are locally isometrically rigid but are infinitesimally deformable. On the other hand, the classical algebraic condition which assures local isometric rigidity for any codimension also assures local infinitesimal rigidity. In fact, Dajczer and Rodríguez ([**1**]) have shown that any isometric immersion $f : M^n \to R^N$ into Euclidean space with type number $\rho_f \geq 3$ must be infinitesimally rigid, that is, admits only trivial infinitesimal deformations. For later use, we give an extension of the above result to the context of arbitrary simply connected space forms.

Theorem 3. *Let $f : M^n \to Q_c^N$ be an isometric immersion such that $\rho_f \geq 3$ everywhere. Then f is infinitesimally rigid.*

This result follows, like Theorem 2 in [**1**], from the next statement where we consider the ambient space totally umbilically included in the $(N+1)-$dimensional Euclidean space (respectively, the $(N+1)-$dimensional Lorentzian space) for $c > 0$ (respectively, for $c < 0$).

Proposition 4. *Let Z be an infinitesimal deformation of the isometric immersion f : $M^n \to Q_c^N$. Consider the maps $G_t : M^n \to Q_c^N$, $t \in I$, defined as*

$$G_t(x) = \frac{1}{\sqrt{1 + ct^2\|Z(x)\|^2}}(f(x) + tZ(x)).$$

Then,

a) G_t *is an immersion and* G_t *and* G_{-t} *induce the same metric, for all* $t \in I$.

b) *If* f *is substantial and, for some* $t_0 \neq 0$, G_{t_0} *and* G_{-t_0} *are congruent in* Q_c^N, *then* Z *is trivial.*

Proof: The map G_t differs from the one of Theorem 1 in **[1]** only by the scalar factor $\frac{1}{\sqrt{1+ct^2\|Z(x)\|^2}}$, and the proof is very similar. We get *a*) from

$$\|G_{t*}(X)\|^2 = \frac{1}{1+ct^2\|Z(x)\|^2}\left(\|X\|^2 + t^2\|\overline{\nabla}_X Z\|^2 - t^4\frac{\langle\overline{\nabla}_X Z, Z\rangle^2}{1+ct^2\|Z(x)\|^2}\right).$$

Then, *b*) follows from the fact that all operators involved in the proof of the theorem in **[1]** are linear. Details are left to the reader. □

3. The condition of extendibility

In this section, we provide a condition equivalent to the existence of an extension $\overline{Z}$ along h of an infinitesimal deformation Z of the composition $g = h \circ f$. Here, g may have arbitrary codimension. From now on, ∇ and $\overline{\nabla}$ denote the Levi-Civita connections of M^n and Q_c^{n+p} respectively. All proofs in this paper will be done for the case $c = 0$, being the other cases similar.

Proposition 5. *Let* $f : M^n \to Q_c^{n+1}$ *be an isometric embedding and let* $h : U \subset Q_c^{n+1} \to Q_c^{n+p}$ *be an isometric immersion with* $f(M) \subset U$. *Assume that* $g = h \circ f$ *is 1-regular and that the relative nullity of* h *has constant dimension on* U *with leaves transversal to* M^n. *Then, an infinitesimal deformation* Z *of* g *admits an extension along* h *if and only if there is* $\sigma \in L^\perp \subset T_g M^\perp$ *such that*

$$\langle A^g_\sigma X, Y\rangle = \langle \overline{\nabla}_X\overline{\nabla}_Y Z - \overline{\nabla}_{\nabla_X Y} Z, \overline{N}\,\rangle + c\langle X, Y\rangle\langle Z, \overline{N}\rangle,$$

for all $X, Y \in TM$. *Here,* L *is the line bundle generated by* $\overline{N} = h_* N$, *being* N *an unit vector field normal to* f.

Proof: Suppose now that M^n is orientable. Set $\gamma = \alpha_h|_{TM\times TM}$. First, we see that $\mathrm{Ker}\ \gamma = \Delta_h \cap TM$. In fact, from the flatness of α_h, we have for all $X \in \mathrm{Ker}\ \gamma$, $Y \in TR^{n+1}$, that

$$\|\alpha_h(X, Y)\|^2 = \langle\alpha_h(X, X), \alpha_h(Y, Y)\rangle = \langle\gamma(X, X), \alpha_h(Y, Y)\rangle = 0.$$

Consequently, being TM a subbundle of $T_f R^{n+1}$ of codimension one, our assumption of transversality is equivalent to the existence of a smooth vector field $X_0 \in (\text{Ker } \gamma)^{\perp} \subset TM$ such that $\eta = N + X_0 \in \Delta_h$. Clearly, there exists a positive continuous function $\lambda : M^n \to \mathbf{R}$ such that the map $y : \mathbf{R} \times M^n \to U'$ given by

$$y(t,x) = f(x) + t\eta(x),$$

is a parametrization of a tubular neighborhood $U' \subset U$ of $f(M)$, when restricted to the subset $\{(t,x) \in \mathbf{R} \times M^n : |t| < \lambda(x)\}$. This parametrization plays an essential rôle in the proof. In fact, since Δ_h gives rise to a totally geodesic foliation of $h(U')$, we have that

$$h(t,x) = g(x) + t\overline{\eta}(x),$$

being $\overline{\eta} = h_* \eta$. Therefore, any smooth extension $\overline{Z}$ of Z must have the form

$$\overline{Z}(t,x) = Z(x) + W(t,x),$$

where $W(0,x) = 0$ for all $x \in M^n$.

Set $A^f = A^f_N$ and denote by $'$ the covariant derivative of a vector field with respect to the parameter t. We claim that the fact that $\overline{Z}$ is an infinitesimal deformation of h is equivalent to the following three equations:

$$\langle W, \overline{\eta} \rangle = 0, \tag{2}$$

$$\langle W', I_t(X) \rangle + \langle \overline{\nabla}_X Z + \overline{\nabla}_X W, \overline{\eta} \rangle = 0, \tag{3}$$

$$\langle \overline{\nabla}_X W, I_t(X) + t\langle A^f X, X_0 \rangle \overline{\eta} \rangle + t\langle \overline{\nabla}_X Z, \overline{\nabla}_X \overline{\eta} \rangle = 0, \tag{4}$$

for all $X \in TM$ where, for small values of t, I_t is the isomorphism of TM given by $I_t = Id + tl$, being

$$l(X) = \overline{\nabla}_X \overline{\eta} - \langle A^f X, X_0 \rangle \, \overline{\eta} = \nabla_X X_0 - A^f X - \langle A^f X, X_0 \rangle X_0. \tag{5}$$

To prove the claim, first observe that equation (2) is a consequence of $W(0,x)=0$ and

$$0=\langle\overline{\nabla}_{\partial t}\overline{Z},h_*(\partial t)\rangle=\langle\overline{\nabla}_{\partial t}W,h_*(\partial t)\rangle=\langle W',\overline{\eta}\rangle.$$

Equation (3) now follows using (2),

$$\begin{aligned}0&=\langle\overline{\nabla}_{\partial t}\overline{Z},h_*(X)\rangle+\langle\overline{\nabla}_X\overline{Z},h_*(\partial t)\rangle\\&=\langle W',X+t\overline{\nabla}_X\overline{\eta}\rangle+\langle\overline{\nabla}_XZ+\overline{\nabla}_XW,\,\overline{\eta}\,\rangle\\&=\langle W',I_t(X)\rangle+\langle\overline{\nabla}_XZ+\overline{\nabla}_XW,\,\overline{\eta}\rangle.\end{aligned}$$

Finally, we obtain (4) from

$$\begin{aligned}0&=\langle\overline{\nabla}_X\overline{Z},h_*(X)\rangle\\&=\langle\overline{\nabla}_XW,h_*X\rangle+\langle\overline{\nabla}_XZ,h_*X\rangle\\&=\langle\overline{\nabla}_XW,I_t(X)+t\langle A^fX,X_0\rangle\overline{\eta}\rangle+t\langle\overline{\nabla}_XZ,\overline{\nabla}_X\overline{\eta}\rangle,\end{aligned}$$

and the claim follows easily.

Differentiating equation (3) with respect to t and using (2) yields

$$\begin{aligned}0&=\langle W'',I_t(X)\rangle+\langle W',I_t'(X)\rangle+\langle\overline{\nabla}_XW',\overline{\eta}\rangle\\&=\langle W'',I_t(X)\rangle+\langle W',l(X)\rangle-\langle W',\overline{\nabla}_X\overline{\eta}\rangle\\&=\langle W'',I_t(X)\rangle.\end{aligned}$$

By (2), this is equivalent to

$$W(t,x)=tW_0(x)+W_1(t,x),$$

where $W_0\in TM\oplus L$ and $W_1\in L^\perp$. Observe that the parallel decomposition $T_{h(t,x)}R^{n+p}=\{T_{g(x)}M\oplus L(g(x))\}\oplus L^\perp(g(x))$ allows us to differentiate with respect to t in each factor without leaving it. Using the above, (2) and (5), we easily see that equation (3) is equivalent to

$$\langle W_0,Y\rangle+\langle\overline{\nabla}_YZ,\overline{\eta}\,\rangle=0,\quad\forall\;Y\in TM. \tag{6}$$

Observe that W_0 is univocally determined by (2) and (6).

It remains to compute W_1 from (4). By a long but straightforward computation, using (2) and (6), we get

$$\begin{aligned}
0 &= t\langle\overline{\nabla}_X W_0, I_t(X)\rangle - \langle W_1, \gamma(X, I_t(X))\rangle - t^2\langle A^f X, X_0\rangle\langle W_0, l(X)\rangle \\
&\quad + t\langle\overline{\nabla}_X Z, l(X) + \langle A^f X, X_0\rangle\overline{\eta}\,\rangle \\
&= -\langle W_1, \gamma(X, I_t(X))\rangle - t\{\langle W_0, \nabla_X I_t(X) + \langle A^f X, I_t(X)\rangle\overline{N}\rangle \\
&\quad + \langle\overline{\nabla}_X\overline{\nabla}_{I_t(X)} Z, \overline{\eta}\,\rangle + \langle\overline{\nabla}_{I_t(X)} Z, l(X) + \langle A^f X, X_0\rangle\overline{\eta}\,\rangle\} \\
&\quad + t^2\langle A^f X, X_0\rangle\langle\overline{\nabla}_{l(X)} Z, \overline{\eta}\,\rangle + t\langle\overline{\nabla}_X Z, l(X) + \langle A^f X, X_0\rangle\overline{\eta}\,\rangle \\
&= -\langle W_1, \gamma(X, I_t(X))\rangle - tH(X, I_t X) - t^2\langle\overline{\nabla}_{l(X)} Z, l(X)\rangle,
\end{aligned}$$

being H the symmetric (0,2) tensor on TM defined by

$$H(X,Y) = \langle\overline{\nabla}_X\overline{\nabla}_Y Z - \overline{\nabla}_{\nabla_X Y} Z + \langle A^f X, Y\rangle\overline{\nabla}_{X_0} Z, \overline{\eta}\,\rangle.$$

Since Z is an infinitesimal deformation the last term vanishes. Setting $-W_1 = t\mu + W_2$, with $\mu \in N_1^g \cap L^\perp$ and $W_2 \in N_1^{g\perp} \subset T_g M^\perp$, it follows that

$$\langle A^g_\mu X, Y\rangle = \langle\mu, \gamma(X,Y)\rangle = H(X,Y), \quad \forall\ X, Y \in TM. \tag{7}$$

Observe that μ is independent from t and determined by (7).

In summary, the last equation together with (2) and (6) tell us that the extendibility of Z is equivalent to the existence of a vector field $\mu \in L^\perp$ satisfying (7).

Consider the tensor field S defined by

$$S_{ijk} = \langle\overline{\nabla}_{X_i}\overline{\nabla}_{X_j} Z - \overline{\nabla}_{\nabla_{X_i} X_j} Z, X_k\rangle + \langle A^f X_i, X_j\rangle\langle\overline{\nabla}_{X_k} Z, \overline{N}\rangle + \langle\overline{\nabla}_{X_k} Z, \gamma(X_i, X_j)\rangle.$$

Then S is skew-symmetric in the second two components and symmetric in the first two. The first statement follows from equation (1), namely,

$$\begin{aligned}
S(X,Y,Y) &= \langle\overline{\nabla}_X\overline{\nabla}_Y Z, Y\rangle - \langle\overline{\nabla}_{\nabla_X Y} Z, Y\rangle + \langle\overline{\nabla}_Y Z, \alpha_g(X,Y)\rangle \\
&= \langle\overline{\nabla}_X\overline{\nabla}_Y Z, Y\rangle + \langle\overline{\nabla}_Y Z, \nabla_X Y\rangle + \langle\overline{\nabla}_Y Z, \alpha_g(X,Y)\rangle \\
&= \langle\overline{\nabla}_X\overline{\nabla}_Y Z, Y\rangle + \langle\overline{\nabla}_Y Z, \overline{\nabla}_X Y\rangle \\
&= X\langle\overline{\nabla}_Y Z, Y\rangle = 0.
\end{aligned}$$

The second statement is due to the vanishing of the curvature tensor of the ambient space. We conclude that $S=0$ from

$$S_{ijk}=-S_{ikj}=-S_{kij}=S_{kji}=S_{jki}=-S_{jik}=-S_{ijk}.$$

Now, setting $\sigma=\mu+(\overline{\nabla}_{X_0}Z)^{L^\perp}$, where $(\ )^{L^\perp}$ denotes the orthogonal projection on $L^\perp$, the proof of the proposition when M^n is orientable follows easily from (7) and $S(X,Y,X_0)=0$.

In the general case, the condition of the proposition is equivalent to the local extendibility of Z. But, from (2), (6) and (7), we have uniqueness of $\overline{Z}(t,x)$ in $T_xM\oplus N_1^g(x)$. Thus, we obtain the global extendibility of Z from the embedding hypothesis on f and the proof is complete. $\square$

According to (2), (6) and (7), any extension $\overline{Z}$ of Z in the parametrization y must have the form

$$\overline{Z}(t,x)=Z(x)+t\Psi(x)+\Phi_t(x),$$

with $\Psi\in TM\oplus N_1^g$ univocally determined by Z, and $\Phi_t\in(TM\oplus N_1^g)^\perp$ being any smooth vector field such that $\Phi_0=0$. From this, we obtain the following consequence.

Corollary 6. *With the same assumtions of Proposition 5, let $\overline{Z}_1$ and $\overline{Z}_2$ be infinitesimal deformations of h such that $\overline{Z}_1\circ f=\overline{Z}_2\circ f$. If $N_1^g=T_gM^\perp$, then $\overline{Z}_1=\overline{Z}_2$ in a neighborhood of $f(M)$.*

Remark. If the transversality condition fails in an open subset $V\subset M^n$, it follows from the flatness of α_h that Ker $\gamma=\Delta_h$ on V. Thus, V must be $n-p+1$ ruled, since the Chern-Kuiper inequality for the flat bilinear map γ says that dim Ker $\gamma\geq n-(p-1)$. In particular, $\rho_f\leq 2(p-1)$ on V.

4. The proof of Theorem 1

Proof: Let N and $\overline{N}$ be as in Proposition 5, and consider $\overline{N}^\perp$ so that $\{\overline{N},\overline{N}^\perp\}$ is an orthonormal basis of $T_gM^\perp$. Set

$$B=A^g_{\overline{N}^\perp},\quad \overline{B}=A^h_{\overline{N}^\perp}.$$

Observe that $B = \pi \circ \overline{B}|_{TM}$, being π the the orthogonal projection onto TM. Moreover, from the beginning of the proof of Proposition 5, we have that Ker $B =$ Ker $\overline{B} \cap TM$. Let us denote by ∇' the connection of R^{n+1}. We have to consider two cases:

Case 1: $s_g \equiv 1$. Here, A^f and B are linearly independent. Since $\rho_f \geq 4$ and rank $B \leq 1$, we have $N_1^g = L$, that is, $B = 0$. Thus, h must be totally geodesic in $f(M)$. Otherwise, if there exists $p \in M^n$ such that $T_pM =$ Ker $\overline{B}$, the same holds in a neighborhood V of p. But this would imply that V must be totally geodesic in R^{n+1}, which is not the case in view of our assumptions. Then, $\overline{N}^{\perp}$ is parallel along g. We conclude from the isometric rigidity of f that $g = i \circ f$, where $i : R^{n+1} \to R^{n+2}$ is a totally geodesic inclusion.

Setting $i_*(Z_0) = Z - \langle Z, \overline{N}^{\perp} \rangle \overline{N}^{\perp}$, we have that

$$\langle \nabla'_X Z_0, f_* X \rangle = \langle \overline{\nabla}_X Z, g_* X \rangle = 0, \quad \forall\ X \in TM,$$

that is, Z_0 is an infinitesimal deformation of f. Hence, by Theorem 3, we have that $Z_0 = Cf + v$ is trivial. Then, it follows from Proposition 5 that Z can be extended along i, since

$$\begin{aligned}
\langle \overline{\nabla}_X \overline{\nabla}_Y Z - \overline{\nabla}_{\nabla_X Y} Z, \overline{N} \rangle &= \langle \nabla'_X \nabla'_Y Z_0 - \nabla'_{\nabla_X Y} Z_0, N \rangle \\
&= \langle \nabla'_X CY - C \nabla_X Y, N \rangle \\
&= \langle A^f X, Y \rangle \langle CN, N \rangle \\
&= 0,
\end{aligned}$$

for all $X, Y \in TM$. In particular, $Z = (V + \varphi \overline{N}^{\perp}) \circ f$, being V a Killing field of R^{n+1} and φ a smooth function. Notice that, in this case, we only made use of $\rho_f \geq 3$.

Case 2: $s_g \equiv 2$. Uniqueness follows from Corollary 6 and $N_1^g = T_gM^{\perp}$. Thus, we only need to prove the result locally, since f is an embedding. In particular, we can suppose that the parametrization y and the vector fields η and $\overline{\eta}$ of the proof of Proposition 5 are globally defined on M^n. We have that

$$\eta = X_0 + N = \phi e_0 + N \in \text{ Ker } \overline{B},$$

where $e_0 \in$ Im B is unitary and ϕ a smooth function. Next, we use the fundamental equations of the isometric immersions to obtain more information. We claim that Ker B

is integrable, and that

$$\nabla_X e_0 + \phi A^f X - \langle A^f X, X_0\rangle e_0 = 0, \tag{8}$$

$$X(\phi^2+1) = 2\langle A^f X, X_0\rangle(\phi^2+1), \tag{9}$$

where $X \in \text{Ker } B$ is arbitrary.

To prove the claim, first observe that

$$\nabla^\perp_Y \overline{N}^\perp = \langle \overline{\nabla}_Y \overline{N}^\perp, \overline{N}\rangle \overline{N} = \langle BY, X_0\rangle \overline{N}, \quad \forall\, Y \in TM,$$

since $\eta \in \text{Ker } \overline{B}$. Here, $\nabla^\perp$ denotes the normal connection on $T_g M^\perp$ induced by $\overline{\nabla}$. Hence, the Codazzi equation for $X \in \text{Ker } B$, $Y \in TM$, is

$$\nabla_X(BY) - B(\nabla_X Y) = -B(\nabla_Y X) - \langle BY, X_0\rangle A^f X.$$

In particular, taking $Y \in \text{Ker } B$, we conclude that Ker B is integrable. For $Y = e_0$, we get

$$a\nabla_X e_0 + X(a)e_0 = -a\langle \nabla_{e_0} X, e_0\rangle e_0 - a\phi A^f X,$$

or equivalently,

$$\{X(a) + a\langle \nabla_{e_0} X + \phi A^f X, e_0\rangle\} e_0 = -a\{\nabla_X e_0 + \phi A^f X - \phi\langle A^f X, e_0\rangle e_0\},$$

where $a \neq 0$ is given by $Be_0 = ae_0$. Since the right hand side belongs to Ker B, both sides must vanish. This proves (8) and yields

$$X(a) = -a\langle \nabla_{e_0} X + \phi A^f X, e_0\rangle. \tag{10}$$

For $X \in \text{Ker } B \subset \text{Ker } \overline{B}$, we have by the Ricci equation,

$$\begin{aligned} 0 &= \langle R^\perp(X, e_0)\overline{N}^\perp, \overline{N}\rangle + \langle [A^f, B]X, e_0\rangle \\ &= \langle \nabla^\perp_X \nabla^\perp_{e_0} \overline{N}^\perp + \nabla^\perp_{\nabla_{e_0} X} \overline{N}^\perp, \overline{N}\rangle - \langle BA^f X, e_0\rangle \\ &= \langle \nabla^\perp_X (a\phi \overline{N}) + \langle \nabla_{e_0} X, e_0\rangle a\phi \overline{N}, \overline{N}\rangle - a\langle A^f X, e_0\rangle. \end{aligned} \tag{11}$$

From (10) and (11) it follows that

$$
\begin{aligned}
0 &= X(a\phi) + a\phi\langle \nabla_{e_0} X, e_0\rangle - a\langle A^f X, e_0\rangle \\
&= aX(\phi) - a(\phi^2+1)\langle A^f X, e_0\rangle,
\end{aligned}
$$

and we have (9). The claim has been proved.

From (5), (8) and (9) we conclude that

$$
l(X) = -(\phi^2+1)(A^f X - \langle A^f X, e_0\rangle e_0) \in \text{Ker } B, \tag{12}
$$

for all $X \in \text{Ker } B$. Let $j : \mathbf{R}\times R^{n-1} \to W \subset M^n$ be a local parametrization of M^n such that, for each $t \in \mathbf{R}$, $H_t = j(t, R^{n-1})$ is contained in a leaf of the foliation defined by Ker B. Observe that each leaf $f(H_t)$ is also given by the intersection of $f(W)$ with the hyperplane of relative nullity Ker $\overline{B}_t$ = Ker $\overline{B}(f(t,x))$, which only depends on t. If $\xi = \frac{1}{\sqrt{\phi^2+1}}(e_0 - \phi N)$, we also have that $\xi(t,x) = \xi(t)$, since ξ spans the orthogonal complement Λ of Ker $\overline{B}\circ f$ in R^{n+1}. This allows the orthogonal and smooth decomposition along g,

$$
T_{g(t,x)}R^{n+2} = h_*(f(t,x))\left(\text{Ker } \overline{B}_t \oplus \Lambda_t\right) \oplus L_t^{\perp},
$$

being $L_t^{\perp}$ the line bundle spanned by $\overline{N}^{\perp}(t) = \overline{N}^{\perp}(f(t,x))$. According to this decomposition, set

$$
Z = h_*(U_0 + b\xi) + \psi\overline{N}^{\perp},
$$

where $U_0 \in \text{Ker } \overline{B}_t$. Define $U = U_0 + b\xi$. For each $t \in \mathbf{R}$, let $f_t : H_t \to \text{Ker } \overline{B}_t$ be the restriction of f to the leaf H_t. Equation (1) implies that

$$
\langle \nabla'_X U_0, f_{t*}X\rangle = \langle \nabla'_X U, f_*X\rangle = \langle \overline{\nabla}_X Z, g_*X\rangle = 0, \quad \forall\ X \in \text{Ker } B, \tag{13}
$$

that is, U_0 is an infinitesimal deformation of f_t. By the definition (5) of l, the second fundamental form of the isometric immersion f_t in the hyperplane Ker $\overline{B}_t$ is

$$
A^{f_t}_{\eta} = -l|_{\text{Ker } B}\ . \tag{14}
$$

In view of (12), $A^{f_t}_{\eta}$ is just the orthogonal projection on Ker B of $A^f|_{\text{Ker } B}$. Since both operators are symmetric, it is easy to see that rank $A^{f_t}_{\eta} \geq$ rank $A^f - 2$. Moreover, if A^f is semidefinite, we can improve the inequality to rank $A_\eta^{f_t} \geq$ rank $A^f - 1$. Hence, from our assumption on ρ_f, we have that $\rho_{f_t} =$ rank $A_\eta^{f_t} \geq 3$ everywhere in H_t. Thus, from (13) and Theorem 3, we conclude that U_0 is trivial along each leaf. Then, in the parametrization j, we have

$$U_0(t,x) = C_t f_t(x) + v_t, \tag{15}$$

being C_t a one parameter family of skew-symmetric endomorphisms of R^{n+1} so that $C_t\xi(t) = 0$, and $v_t \in$ Ker $\overline{B}_t$.

Now, as we saw in the proof of Proposition 5, Z admits an extension if and only if there exists $\mu \in L^\perp$ such that equation (7) holds for all $X, Y \in TM$. Since, in our case, g has codimension 2, we have that $A_\mu^g = \langle \mu, \overline{N}^\perp \rangle B$, where rank $B = 1$. Thus, we conclude that Z admits an extension if

$$\langle \overline{\nabla}_X \overline{\nabla}_Y Z - \overline{\nabla}_{\nabla_X Y} Z + \langle A^f X, Y\rangle \overline{\nabla}_{X_0} Z, \overline{\eta}\rangle = 0,$$

for all $Y \in TM$, $X \in$ Ker B. But, since $\eta \in$ Ker B, we get

$$\begin{aligned}
&\langle \overline{\nabla}_X \overline{\nabla}_Y Z - \overline{\nabla}_{\nabla_X Y} Z + \langle A^f X, Y\rangle \overline{\nabla}_{X_0} Z, \overline{\eta}\rangle \\
&\qquad = \langle \overline{\nabla}_X \{h_*(\nabla'_Y U) + (Y(\psi) + \langle \overline{B}U, Y\rangle)\overline{N}^\perp - \psi h_* \overline{B} Y\}, \overline{\eta}\rangle \\
&\qquad\quad - \langle h_* \nabla'_{\nabla_X Y} U + \langle A^f X, Y\rangle h_* \nabla'_{X_0} U, \overline{\eta}\rangle \\
&\qquad = \langle \nabla'_X \nabla'_Y U - \nabla'_{\nabla_X Y} U + \langle A^f X, Y\rangle \nabla'_{X_0} U, \eta\rangle + \langle \psi \nabla'_X(\overline{B}Y), \eta\rangle \\
&\qquad = \langle \nabla'_X \nabla'_Y (U_0 + b\xi) - \nabla'_{\nabla_X Y}(U_0 + b\xi) + \langle A^f X, Y\rangle \nabla'_{X_0}(U_0 + b\xi), \eta\rangle,
\end{aligned}$$

where the last equality follows from the Codazzi equation for h. From the vanishing of the curvature tensor of R^{n+1}, we have that Z admits an extension if, for all $X \in$ Ker B, $Y \in TM$,

$$\begin{aligned}
0 &= \langle \nabla'_X \nabla'_Y U_0 - \nabla'_{\nabla_X Y} U_0 + \langle A^f X, Y\rangle \nabla'_{X_0} U_0, \eta\rangle \\
&\quad + \langle X(b)\nabla'_Y \xi - b\{\langle \nabla_Y X, e_0\rangle - \phi\langle A^f X, Y\rangle\}\nabla'_{e_0}\xi, \eta\rangle \\
&= \langle \nabla'_X \nabla'_Y U_0 - \nabla'_{\nabla_X Y} U_0 + \langle A^f X, Y\rangle \nabla'_{X_0} U_0, \eta\rangle \qquad (16) \\
&\quad + \langle Y, e_0\rangle \{X(b) + b\langle X, \nabla_{e_0} e_0 + A^f X_0\rangle\}\langle \nabla'_{e_0}\xi, \eta\rangle,
\end{aligned}$$

where the last equality follows from equation (8) for the Ker B component of Y.

First, we verify from equation (15) that (16) holds for all $X, Y \in \text{Ker } B$. In fact,

$$
\begin{aligned}
(17) \qquad & \langle \nabla'_X \nabla'_Y U_0 - \nabla'_{\nabla_X Y} U_0 + \langle A^f X, Y \rangle \nabla'_{X_0} U_0, \eta \rangle \\
&= \langle \nabla'_X C_t Y - C_t(\nabla_X Y - \langle A^f X, Y \rangle X_0), \eta \rangle \\
&= \langle C_t(\nabla'_X Y - \nabla_X Y - \langle A^f X, Y \rangle X_0), \eta \rangle \\
&= -\langle A^f X, Y \rangle \langle C_t \eta, \eta \rangle \\
&= 0.
\end{aligned}
$$

Hence, to conclude that Z can be extended we only need to show that (16) also holds for $Y = e_0$.

Set $T(j(t,x)) = j_*(\frac{\partial}{\partial t}, 0)(t,x)$. Equations (1) and (15) yield

$$
\begin{aligned}
(18) \qquad 0 &= \langle \overline{\nabla}_X Z, T \rangle + \langle \overline{\nabla}_T Z, X \rangle \\
&= \langle \nabla'_X U, T \rangle + \langle \nabla'_T U, X \rangle \\
&= \langle C_t X + X(b)\xi, T \rangle + \langle C'_t f + v'_t + C_t T + b'\xi + b\xi', X \rangle \\
&= X(b) \langle \xi, T \rangle + \langle C'_t f + v'_t + b\xi', X \rangle,
\end{aligned}
$$

for all $X \in \text{Ker } B$. Thus, for all $X, Y \in \text{Ker } B$,

$$
\begin{aligned}
0 &= Y\{X(b)\langle \xi, T \rangle + \langle C'_t f + v'_t + b\xi', X \rangle\} \\
&\quad - X\{Y(b)\langle \xi, T \rangle + \langle C'_t f + v'_t + b\xi', Y \rangle\} \\
&= [Y, X](b)\langle \xi, T \rangle + \langle C'_t f + v'_t + b\xi', [Y, X] \rangle \\
&\quad + X(b)\langle \xi, \nabla'_Y T \rangle - Y(b)\langle \xi, \nabla'_X T \rangle \\
&\quad + \langle C'_t Y + Y(b)\xi', X \rangle - \langle C'_t X + X(b)\xi', Y \rangle \\
&= 2Y(b)\langle \xi', X \rangle - 2X(b)\langle \xi', Y \rangle + 2\langle C'_t Y, X \rangle,
\end{aligned}
$$

where the last equality follows from the fact that Ker B is integrable and $\langle \xi', X \rangle = T\langle \xi, X \rangle - \langle \xi, \nabla'_T X \rangle = -\langle \xi, \nabla'_X T \rangle$, for all $X \in \text{Ker } B$. Again by (18), we obtain

$$
(19) \qquad \langle \xi, T \rangle \langle C'_t Y, X \rangle = \langle \xi', X \rangle \langle C'_t f + v'_t, Y \rangle - \langle \xi', Y \rangle \langle C'_t f + v'_t, X \rangle,
$$

or equivalently,

$$\langle \xi, T\rangle C_t'Y = \langle C_t'f + v_t', Y\rangle \xi' - \langle \xi', Y\rangle (C_t'f + v_t') + \omega(Y)\eta + \theta(Y)\xi, \tag{20}$$

for all $Y \in \text{Ker } B$, and some $\omega, \theta \in (\text{Ker } B)^*$.

We claim that equation (16) for $Y = e_0$ is equivalent to $\omega = 0$. First, observe that by (9) we get, for all $X \in \text{Ker } B$,

$$\begin{aligned} \langle \xi', X\rangle &= -\langle \xi, \nabla_X' T\rangle = -X\left(\frac{\langle T, e_0\rangle}{\sqrt{\phi^2+1}}\right) \\ &= \frac{1}{\sqrt{\phi^2+1}}(\langle T, e_0\rangle\langle A^f X, X_0\rangle - X\langle T, e_0\rangle). \end{aligned} \tag{21}$$

Since $\langle \xi', X\rangle = \langle T, e_0\rangle\langle \nabla_{e_0}'\xi, X\rangle = \langle \xi, T\rangle\langle \nabla_{e_0}e_0 + A^f X_0, X\rangle$, equation (16) for $Y = \langle T, e_0\rangle e_0 \neq 0$ becomes

$$\begin{aligned} 0 &= \langle T, e_0\rangle\langle \nabla_X'\nabla_{e_0}'U_0 - \nabla_{\nabla_X e_0}'U_0 + \langle A^f X, X_0\rangle\nabla_{e_0}'U_0, \eta\rangle \\ &\quad + \{X(b) + b\langle X, \nabla_{e_0}e_0 + A^f X_0\rangle\}\langle \xi', \eta\rangle \\ &= \langle \nabla_X'\left(C_t'f + v_t' + \langle T, e_0\rangle C_t e_0\right) - \frac{X\langle T, e_0\rangle}{\langle T, e_0\rangle}(C_t'f + v_t'), \eta\rangle \\ &\quad - \langle C_t\nabla_X(\langle T, e_0\rangle e_0), \eta\rangle + \langle A^f X, X_0\rangle\langle (C_t'f + v_t' + \langle T, e_0\rangle C_t e_0), \eta\rangle \\ &\quad + \langle \xi, T\rangle^{-1}(X(b)\langle \xi, T\rangle + b\langle \xi', X\rangle)\langle \xi', \eta\rangle. \end{aligned} \tag{22}$$

Hence, by (18) and (21), we get

$$\begin{aligned} 0 &= \langle C_t'X + C_t\{\nabla_X'(\langle T, e_0\rangle e_0) - \nabla_X(\langle T, e_0\rangle e_0) + \langle A^f X, X_0\rangle\langle T, e_0\rangle e_0\}, \eta\rangle \\ &\quad + \langle \xi, T\rangle^{-1}\langle \langle \xi', X\rangle(C_t'f + v_t') - \langle C_t'f + v_t', X\rangle\xi', \eta\rangle \\ &= \langle C_t'X + \langle A^f X, e_0\rangle\langle T, e_0\rangle C_t\eta, \eta\rangle \\ &\quad + \langle \xi, T\rangle^{-1}\langle\, \langle \xi', X\rangle(C_t'f + v_t') - \langle C_t'f + v_t', X\rangle\xi', \eta\rangle, \end{aligned}$$

and the claim follows from the skew-symmetricity of C_t.

Denoting by ∇^t the connection of the leaf H_t and differentiating (20) with respect to

$X \in \text{Ker } B$, we get

$$
\begin{aligned}
0 &= \langle \xi, T \rangle \{ X \langle \xi, T \rangle C'_t Y + \langle \xi, T \rangle C'_t \nabla'_X Y - \langle C'_t X, Y \rangle \xi' - \langle C'_t f + v'_t, \nabla'_X Y \rangle \xi' \\
&\quad + \langle \xi', \nabla'_X Y \rangle (C'_t f + v'_t) + \langle \xi', Y \rangle C'_t X - X(\omega(Y))\eta - \omega(Y) \nabla'_X \eta - X(\theta(Y))\xi \} \\
&= -\langle \xi', X \rangle \langle \xi, T \rangle C'_t Y + \langle \xi, T \rangle^2 C'_t \left(\nabla^t_X Y - \frac{\langle Y, l(X) \rangle}{\phi^2 + 1} \eta \right) \\
&\quad + \langle \xi', X \rangle \langle C'_t f + v'_t, Y \rangle \xi' - \langle \xi, T \rangle \langle C'_t f + v'_t, \nabla^t_X Y - \frac{\langle Y, l(X) \rangle}{\phi^2 + 1} \eta \rangle \xi' \\
&\quad + \langle \xi, T \rangle \langle \xi', \nabla^t_X Y - \frac{\langle Y, l(X) \rangle}{\phi^2 + 1} \eta \rangle (C'_t f + v'_t) - \langle \xi', Y \rangle \langle \xi', X \rangle (C'_t f + v'_t) \\
&\quad - \{ X(\omega(Y)) + \omega(X) + \omega(Y) \langle A^f X, X_0 \rangle \} \eta - \langle \xi, T \rangle \omega(Y) l(X) - X(\theta(Y))\xi.
\end{aligned}
$$

Using (19) and (20) several times, we obtain

$$
(\phi^2 + 1)\omega(Y) l(X) + \langle Y, l(X) \rangle V + \theta_1(X)\xi = 0, \quad \forall\ X \in \text{Ker } B,
$$

where $V = \langle \xi, T \rangle C'_t \eta - \langle C'_t f + v'_t, \eta \rangle \xi' + \langle \xi', \eta \rangle (C'_t f + v'_t)$. Therefore, if $\omega \neq 0$,

$$
l(\text{Ker } B) \subset \ \text{span}\{V, \xi\} \cap \ \text{Ker } B,
$$

which implies that rank $l|_{\text{Ker } B} \leq 1$. But this is not the case, since, by (14), rank $l|_{\text{Ker } B} = \rho(f_t) \geq 3$. Thus, $\omega = 0$ and Z can be extended. □

5. Final remarks.

1. When in case 1 of Theorem 1, Z may not admit an extension along h. To see this, take h such that $f(M)$ contains just one point x_0 of the boundary of the open set of nontotally geodesic points of h. Any smooth function $\varphi : M^n \to \mathbf{R}$ defines a normal infinitesimal deformation $\varphi \overline{N}^{\perp}$ of g, but it is easy to see that the existence of an extension of such infinitesimal deformation along h imposes conditions on φ at x_0.

2. It is natural to conjecture that the conclusion in Theorem 1 remains true just with the type number assumption $\rho_f \geq 4$.

3. The main difficulty to extend Theorem 1 for any codimension $p-1$ of h, $2 \leq p \leq n-2$, with the type number assumption $\rho_f \geq p+2$, is due to the fact that it is not clear how to find the section σ of Proposition 5. Nevertheless, we expect Theorem 1 to extend to any codimension.

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