A counterexample to a conjecture on flat bilinear forms

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Abstract. We provide a counterexample to a conjecture on the dimension of the nullity of a flat symmetric bilinear form.

The algebraic theory of flat bilinear forms was developed by J. D. Moore after the seminal work of E. Cartan on exteriorly orthogonal quadratic forms as a tool to treat the "rigidity problem" for submanifolds; see [**5**] and references therein. An $\mathbb{R}$-bilinear form $\beta\colon \mathbb{V}^n \times \mathbb{V}^n \to \mathbb{W}^{p,q}$ into a vector space endowed with an indefinite inner product of type (p,q) is said to be *flat* if

$$\langle \beta(X,Y), \beta(Z,W)\rangle - \langle \beta(X,W), \beta(Z,Y)\rangle = 0 \quad \text{for all } \ X,Y,Z,W \in \mathbb{V}^n.$$

One main goal of the theory is to estimate the dimension of the nullity space

$$N(\beta) = \{X \in \mathbb{V}^n : \beta(X,Y) = 0 : Y \in \mathbb{V}^n\}$$

of a given β that is assumed to be onto, that is, $\mathbb{W}^{p,q} = \text{span}\{\beta(X,Y) : X,Y \in \mathbb{V}^n\}$.

In [**1**] the following result for a symmetric bilinear form was proved.

Theorem 1. *Let $\beta\colon \mathbb{V}^n \times \mathbb{V}^n \to \mathbb{W}^{q,q}$ be a flat symmetric bilinear form. If $q \le 5$ and β is onto then $\dim N(\beta) \ge n - 2q$.*

A proof of the preceding result for $q = 2$ is contained in the argument by Cartan in [**2**]. It was conjectured around 1984 by the first author of this paper that the same estimate holds for arbitrary dimension q. A positive answer to the conjecture would have important consequences. For instance, the isometric and conformal rigidity results in [**1**] would hold after dropping the restriction on the codimension. Moreover, an extension of the results in [**3**] and [**4**] for arbitrary codimension with the same bounds would be possible. However, we give next a counterexample that shows that the conjecture is already false for $q = 6$ and that there is no linear estimate.

Theorem 2. *For a given $\tau \in \mathbb{N}$ with $\tau \ge 3$ set $2p = \tau(\tau+1)$. Then, there is an onto flat symmetric bilinear form $\beta\colon \mathbb{V}^n \times \mathbb{V}^n \to \mathbb{W}^{p,p}$ such that $\dim N(\beta) = n - 2p - \binom{\tau}{3}$.*

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Proof: Denote $L = \{1, 2, \ldots, \tau\}$, $I = (L \times L)/S(2)$ and $J = (L \times L \times L)/S(3)$, where $S(n)$ is the group of permutations of n elements. Then, $\#I = p$ and $\#J = m := \binom{\tau+2}{3}$. For $a \in L$, $k = [(i,j)] \in I$ and $s = [(u,v,w)] \in J$, we say that $a \in s$ if $a \in \{u,v,w\}$, and define $* : L \times I \to J$ by $a * k = [(a,i,j)]$. Then either $a \notin s$ or there is a unique $k \in I$ such that $a * k = s$.

Let $\mathbb{V}^n = \mathbb{R}^\tau \oplus \mathbb{R}^m$, and take bases $\{y_1, \ldots, y_\tau\}$ and $\{n_s : s \in J\}$ of $\mathbb{R}^\tau$ and $\mathbb{R}^m$, respectively. Let $B_1 = \{e_r : r \in I\}$ and $B_2 = \{\hat{e}_r : r \in I\}$ be two basis of $\mathbb{R}^p$, and consider on $\mathbb{W}^{p,p} = \mathbb{R}^{2p}$ the metric of type (p,p) given by $\langle e_r, e_s \rangle = \langle \hat{e}_r, \hat{e}_s \rangle = 0$, $\langle e_r, \hat{e}_s \rangle = \delta_{r,s}$ for all $r, s \in I$. The (ordered) union of the bases B_1 and B_2 is called a pseudo-orthonormal basis of $\mathbb{W}^{p,p}$. Define a symmetric bilinear map β as follows:

$$
\begin{aligned}
\beta(n_s, n_r) &= 0, && r, s \in J,\\
\beta(y_i, y_j) &= \hat{e}_{[(i,j)]}, && i, j \in L,\\
\beta(y_i, n_s) &= 0, && \text{if } \ i \notin s,\\
\beta(y_i, n_s) &= e_k, && \text{if } \ i \in s \ \text{ and } \ i * k = s.
\end{aligned}
$$

To prove that $N(\beta) = 0$, take $x = y + n \in N(\beta)$ with $y = \sum_{j=1}^{\tau} a_j y_j$ and $n = \sum_{s \in J} b_s n_s$. Then $a_i = \langle \beta(x, y_i), e_{[(i,i)]} \rangle = 0$, and $b_s = \langle \beta(x, y_u), \hat{e}_{[(v,w)]} \rangle = 0$ for $s = [(u,v,w)]$. To see that β is flat just observe that $\langle \beta(y_i, y_j), \beta(y_t, n_s) \rangle = \delta_{s,[(i,j,t)]}$ is symmetric in $i, j, t \in L$. ∎

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