Communications in
Analysis and Geometry
Volume 25, Number 4, 751–797, 2017

Classification of codimension two deformations of rank two Riemannian manifolds

Luis A. Florit and Guilherme M. de Freitas

The purpose of this work is to close the local deformation problem of rank two Euclidean submanifolds in codimension two by describing their moduli space of deformations. In the process, we provide an explicit simple representation of these submanifolds, a result of independent interest by its applications. We also determine which deformations are genuine and honest, allowing us to find the first known examples of honestly locally deformable rank two submanifolds in codimension two. In addition, we study which of these submanifolds admit isometric immersions as Euclidean hypersurfaces, a property that gives rise to several applications to the Sbrana-Cartan theory of deformable Euclidean hypersurfaces.

Among the most fundamental properties of a structure defined on a certain class of objects is its rigidity, in a broad sense: whether the structure exists on the given class at all, when it does if it is unique, or, when not unique, to somehow understand its moduli space, i.e., the space of deformations of the structure. In submanifold theory, the corresponding concept is that of isometric rigidity, and starts by asking if a fixed Riemannian manifold admits, either locally or globally, an isometric immersion in a given ambient space, usually the Euclidean space. And, when it does exist, to try to classify all its isometric immersions in that ambient space.

Well-known results in the field include the Nash-Gromov-Rocklin embedding theorems, which state that any smooth Riemannian manifold M^n admits an isometric immersion in the Euclidean space $\mathbb{R}^{n+p}$ with codimension $p \sim n^2/2$. This implies that the rigidity question for submanifolds only makes sense for relatively small codimensions, at least bounded in terms of the dimension of the manifold. One of the main characteristics of rigidity problems in submanifold theory is that, except for the lowest dimensions, the difficulties do not depend on the dimension of the manifold, but rather they grow very fast with the codimension.

In this work we are interested in the local deformation problem of the theory. A well-known result and the starting point of the subject is the Beez-Killing theorem, that states that a Euclidean hypersurface is locally rigid provided that the number of nonzero principal curvatures, called the *rank* of the hypersurface, is at least three. This result has had several generalizations, like the ones in [1], [2], [3] and [19]. The general idea behind these works is that, in order for a submanifold in low codimension to possess noncongruent deformations, its second fundamental form and curvature tensor must be highly degenerate. Several of this kind of results were unified and generalized in [7], where the notion of *genuine rigidity* was introduced (see Section 1 for definitions). This concept relies on the idea that, as we discard congruent submanifolds when analyzing rigidity, we should also discard deformations that are induced by deformations of a bigger dimensional submanifold containing the original one.

On the other hand, flat hypersurfaces have at most rank one, and the moduli space of their local isometric deformations is well understood: they are parametrized in a very simple and geometric way, through their Gauss map, by smooth arbitrary regular curves in their ambient space $\mathbb{R}^{n+1}$. In this work we will see that this is also the case for flat submanifolds in codimension two. Therefore, we argue here that we should also discard compositions with flat submanifolds since these are well understood in low codimension, giving rise to the concept of *honest rigidity.*

From the discussion above we conclude that the interesting local deformation phenomena for hypersurfaces arise only for those that have rank two. A century ago, V. Sbrana and E. Cartan described the rank two locally deformable hypersurfaces, the so-called *Sbrana-Cartan hypersurfaces*, by showing that they split into four classes, according to their space of deformations. However, despite their classification, a deep question remained open for almost a century, namely, the very existence of examples in the least deformable *discrete* class, whose members admit, precisely, only one noncongruent deformation. This question was answered for any dimension in [9] by means of a very geometric construction carried out in codimension two that provided a large family of examples in this discrete class. More precisely, it was shown that the transversal intersection of two flat hypersurfaces in general position gives rise to a Sbrana-Cartan hypersurface of the discrete class, together with its two unique isometric immersions as a Euclidean hypersurface. We call these hypersurfaces *Sbrana-Cartan hypersurfaces of intersection type.*

Since the problem beyond hypersurfaces is quite involved, until recently nothing similar to the Sbrana-Cartan theory for codimension higher than one had been attempted, not even in codimension two. For the compact case, the main result in [13] says that, if $n \geq 5$, they are nowhere genuinely deformable once certain mild singularities for the extensions are allowed. However, the necessity of considering singular extensions was not established in [13], and we will address this question here. A genuine rigidity theory considering also singular extensions was carried out recently in [15] that is used to extend the main result in [13].

Now, in the search for interesting local deformation phenomena it is natural to begin such study with the Euclidean submanifolds in codimension two whose second fundamental forms or curvature tensors are very degenerate, that is, those with rank two. We point out that the rank condition, in this setting, is essentially an intrinsic property, and agrees with the rank of the curvature tensor of the manifold.

When substantial and irreducible, rank two submanifolds in codimension two naturally divide into three classes, *elliptic*, *parabolic* and *hyperbolic*, according to the number 0, 1 or 2 of independent normal directions whose shape operators have rank one. In particular, in the hyperbolic case we have a fundamental function, the *main angle* between these two normal directions, that plays a key role in this work. To our surprise, elliptic and parabolic submanifolds were shown to be honestly rigid in [8] without the need of a thorough understanding of their space of deformations. However, the study of the deformations of hyperbolic submanifolds remained elusive, among other reasons, by the lack of a good representation of them.

And this is precisely the main goal of this work: to describe locally in a convenient way all these hyperbolic submanifolds (Theorem 16) and to characterize their moduli spaces of deformations (Theorem 21), hence closing the classification of the local deformation problem of rank two Euclidean submanifolds in codimension two. We will see that, in contrast to the elliptic and parabolic cases, there are hyperbolic submanifolds that are honestly deformable.

One of the main objectives of this work is to call the attention to the other side of the coin of genuine and honest rigidity, that remained somehow hidden: although it is a good idea to discard deformations that arise from submanifolds in lower codimension when dealing with rigidity, one can instead study non genuine deformations but in higher codimension to obtain information about the deformations in the codimension we are interested in. The last two sections of this paper show that this twist is indeed fruitful. Therefore, we should not simply ignore all non honest and non genuine deformations, but instead it is important to understand them.

To accomplish our task, we proceed as follows. First, in Section 2 we show that all the data relevant to the study of the deformations of a hyperbolic submanifold in codimension two project to the leaf space of its totally geodesic foliation of relative nullity, that is a smooth surface. We then use this to show in Section 3 that these hyperbolic submanifolds admit a very simple and geometric representation in terms of their *polar surfaces*. This result is interesting in its own right, beyond rigidity problems, since it provides an explicit tool to construct interesting classes of submanifolds. For example, as an immediate consequence of this parametrization, we obtain a local description of all flat submanifolds in codimension two in terms of Euclidean surfaces with flat normal bundle, which were classified in [14]. This construction is not only simpler to use and more explicit, but also more elegant than the one found in [5].

In Section 4 we provide a full set of 6 functions of a given hyperbolic submanifold that are invariant under deformations, which will substantially simplify our work. This is a somehow different approach than the usual ones that deal with deformability of submanifolds. We then make use of this set to obtain our main result in Section 5: the description of the moduli space of deformations. It turns out that this space is given as pairs of functions of one variable satisfying a single equation that, surprisingly, depends only on the metric of the polar surface. This is not to say that this space is always easy to compute. Indeed, the richness of the deformation phenomena arises from the way that these two functions are entangled in the equation.

We determine in Section 6 which deformations are honest, and how the ones that are not extend. As a simple consequence, we will conclude the necessity of considering singular extensions for the global rigidity problem for compact manifolds, as established in [13]. In Section 7 we characterize which hyperbolic submanifolds in codimension two admit an isometric immersion as a Euclidean hypersurface, a result that should be seen as the converse to the main result in [11], and that also has applications to the Sbrana-Cartan theory.

Finally, we use this machinery to give three applications. First, in Section 8 we characterize all deformations that preserve the main angle. It turns out that all of them are not honest nor genuine, yet they provide applications to the Sbrana-Cartan theory, in particular, giving new examples of the interesting classes. Then, in Section 9 we fully recover the main result in [9] cited above about the construction of the Sbrana-Cartan hypersurfaces of intersection type. More importantly, we compute their moduli space of deformations as submanifolds in codimension two, where they naturally live. We will see that generically they are also honestly rigid, except in one situation where the moduli space is precompact and diffeomorphic to a line. These are the first known examples of honestly locally deformable rank two Euclidean submanifolds in codimension two.

We end this introduction by pointing out that the techniques used in this work can be easily extended to elliptic and parabolic submanifolds in codimension two by using complex conjugate coordinates instead of real ones, in the same spirit as in [11]. Therefore, a unified approach of the rigidity and deformation phenomena of rank two submanifolds in codimension two, and in fact in space forms, can be easily carried out with the techniques presented here. However, since as we pointed out elliptic and parabolic submanifolds are honestly rigid, by simplicity of the presentation and to avoid conceptual duplications we choose to restrict ourselves to the hyperbolic case in Euclidean space.

1. Preliminaries

In this section we recall basic facts about rank two submanifolds and their splitting tensors, and we introduce the concept of honest rigidity, which is a slight extension of that of genuine rigidity and better suits the study of deformation phenomena when the submanifold admits an isometric immersion in lower codimension.

Along this paper, M^n will denote an n-dimensional Riemannian manifold, and $f\colon M^n \to \mathbb{R}^{n+p}$ an isometric immersion, always in codimension $p = 2$ except in this section. By a *deformation* of f we simply mean another non-congruent isometric immersion of M^n into the same ambient space $\mathbb{R}^{n+p}$.

1.1. Honest rigidity

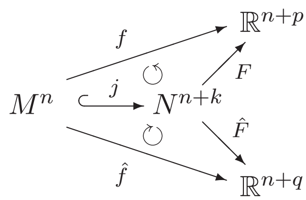
To study the deformation problem for such an f we need the following.

Definition 1. We say that $f : M^n \to \mathbb{R}^{n+p}$ is a *composition* (of g) if there is an open subset $U \subset \mathbb{R}^{n+r}$ with $r < p$, and isometric immersions $h : U \subset \mathbb{R}^{n+r} \to \mathbb{R}^{n+p}$ and $g : M^n \to \mathbb{R}^{n+r}$ with $g(M^n) \subset U$ such that $f = h \circ g$.

It is important to observe that all flat Euclidean hypersurfaces, as h above for $r = p - 1$, can be easily parametrized and classified using just an arbitrary curve in the unit sphere $\mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$ (the Gauss image of h), together with a function of one variable, by means of the Gauss parametrization; see e.g. (15) in [9]. In fact, as we will see in Corollary 18 below, in this paper we also describe easily flat Euclidean submanifolds in codimension two. Therefore, compositions are not interesting when studying rigidity in codimension two and three, since all these deformations arise from the ones for the submanifold g, hence reducing the codimension of the problem. This is one of the reasons why we want to discard compositions, in the same way we discard congruences.

Key concepts in this work are those of genuine rigidity and genuine deformations introduced in [7] and extended to the conformal realm in [16]. As we pointed out, the complexity of rigidity problems in submanifold theory grows very fast with the codimension. Moreover, if we have inclusions $M^n \subset N^{n+k} \subset \mathbb{R}^{n+p}$, deformations of the lower codimensional submanifold N^{n+k} in $\mathbb{R}^{n+p}$ induce obvious deformations of M^n in $\mathbb{R}^{n+p}$. In order to discard these simpler deformations we proceed as follows.

Given another isometric immersion $\hat{f} : M^n \to \mathbb{R}^{n+q}$, $q \in \mathbb{N}$, we say that the pair $\{f.\hat{f}\}$ *extends isometrically* if there are an isometric embedding $j : M^n \hookrightarrow N^{n+k}$ into a Riemannian manifold N^{n+k}, $k \geq 1$, and a pair of isometric immersions $F : N^{n+k} \to \mathbb{R}^{n+p}$ and $\hat{F} : N^{n+k} \to \mathbb{R}^{n+q}$, such that $f = F \circ j$ and $\hat{f} = \hat{F} \circ j$, that is, when the following diagram commutes:



Accordingly, we say that $\hat{f}$ as above is a *genuine deformation* of f (in codimension q), or simply that $\{f, \hat{f}\}$ is a *genuine pair*, if $\{f|_U, \hat{f}|_U\}$ does not extend isometrically on any open subset $U \subset M^n$.

Observe that this concept (locally) extends that of compositions when $k = p < q$, and that of congruence when $k = p = q$. This allowed to unify and generalize in [7] and [16] several known rigidity results that seemed different in nature. The main results in those two papers are that, in sufficiently low codimensions, the members of a genuine pair have to be mutually ruled, with a special kind of rulings of large dimension.

The associated rigidity concept is the following.

Definition 2. An isometric immersion $f : M^n \to \mathbb{R}^{n+p}$ is *genuinely rigid* if, for any given isometric immersion $\hat{f} : M^n \to \mathbb{R}^{n+p}$, there is an open dense subset $U \subset M^n$ such that the pair $\{f|_U, \hat{f}|_U\}$ extends isometrically.

Although this concept is appropriate when M^n admits no lower codimensional isometric immersion g as in Definition 1, when it does, we automatically have all the compositions $h \circ g$ for each flat submanifold h, and therefore we cannot expect to have genuine rigidity. On the other hand, as we saw, these compositions in low codimension are also well understood, and thus we want to discard them as well. Therefore, this justifies us to say that a genuine deformation $\hat{f} : M^n \to \mathbb{R}^{n+p}$ of f is *honest* if it is nowhere a composition. Accordingly to this, we introduce our next concept.

Definition 3. We say that f is *honestly rigid* if its only genuine deformations are compositions along an open dense subset.

We do not require in this work for M^n to admit no isometric immersion as an Euclidean hypersurface, as it was done in Theorem 1 in [8]. This has two main reasons. First, we can and we will use the machinery developed in this paper to obtain information about the theory of deformable hypersurfaces; see Sections 8 and 9 for examples of this situation. Secondly, the concept of honest rigidity better fits the deformation problem in codimensions bigger than one and avoids this hypothesis. For example, the cited hypothesis in Theorem 1 in [8] becomes now unnecessary, since its proof actually shows the stronger result that, indeed, any elliptic rank two Euclidean submanifold in codimension two is honestly rigid, even if it is also a Euclidean hypersurface. Analogously, Theorem 4 in [8] implies that any parabolic rank two Euclidean submanifold in codimension two is honestly rigid. We will show in Section 9 that this is not the case for hyperbolic submanifolds in codimension two.

1.2. Rank two and hyperbolic submanifolds

Let R be the curvature tensor of M^n. We denote by $\Gamma(x)$ the *nullity* of M^n at $x \in M^n$,

$$\Gamma(x) := \{X \in T_xM : R(X,Y) = 0 \text{ for all } Y \in T_xM\}.$$

The *rank* of M^n at x is the integer $n - \dim \Gamma(x)$, which is constant on connected components of an open dense subset of M^n. Now, given an isometric immersion $f : M^n \to \mathbb{R}^{n+2}$ of M^n, we denote by $\Delta(x)$ the *relative nullity* of f at x, that is, the nullity space of the second fundamental form $\alpha = \alpha_f$ of f at x,

$$\Delta(x) = \{X \in T_xM : \alpha(X,Y) = 0 \text{ for all } Y \in T_xM\}.$$

We call the *rank* of f at x the integer $n - \dim \Delta(x)$. By the Gauss equation for f it is immediate that $\Delta(x) \subset \Gamma(x)$, and so the rank of f is pointwise greater than or equal to the rank of M^n. It is well-known that both Γ and Δ are smooth, integrable totally geodesic distributions on M^n (along the connected components of an open dense subset of M^n where they have constant dimension). In addition, Δ is totally geodesic also in the ambient Euclidean space.

Since our work is local in nature, we will assume whenever necessary and without further mention that all distributions that appear as images or kernels of tensors have constant dimension. This will not bring any problem since our rigidity concepts are required to hold locally almost everywhere by definition. In fact, we will see in the last section that submanifolds that deform in very different forms can be glued smoothly, although non analytically, in quite complicated ways, and then we cannot expect any local classification to hold everywhere.

In particular, the following proposition shown in [8] tells us that we can restrict ourselves to rank two immersions when studying deformations of rank two Riemannian manifolds.

Proposition 4. *Let $f : M^n \to \mathbb{R}^{n+2}$ be an isometric immersion of a Riemannian manifold of rank two. If at a certain point $x \in M^n$ it holds that* $\dim \Delta(x) \neq n-2$, *then $f|_W$ is a composition in some open neighborhood W of x.*

We denote by $N_f^1(x)$ the *first normal space* of f at x,

$$N_f^1(x) = \operatorname{span}\{\alpha(X,Y) : X, Y \in T_xM\} \subset T_x^\perp M.$$

If M^n is nowhere flat, then by the Gauss equation $\dim N_f^1 \geq 1$. If, on the other hand, $\dim N_f^1 = 1$, it is easy to see using Codazzi equation that N_f^1 is parallel, and hence $f(M^n)$ is contained in an affine hyperplane of $\mathbb{R}^{n+2}$, being, in particular, a composition. But even if M^n is flat, $\dim N_f^1 < 2$ also implies that f is a composition of a totally geodesic inclusion $M^n \subset \mathbb{R}^{n+1}$ by Proposition 9 in [7]. Therefore, we also have the following, where f to be *full* means that $N_f^1 = T_f^\perp M$ everywhere.

Proposition 5. *If $f : M^n \to \mathbb{R}^{n+2}$ is any isometric immersion that is nowhere a composition, then* $\dim N_f^1 = 2$ *almost everywhere.*

As it is the case with any totally geodesic distribution, Δ possesses its *splitting tensor* $C : \Delta \times \Delta^\perp \to \Delta^\perp$ given by

$$C(T,X) = C_T X = -(\nabla_X T)_{\Delta^\perp}.$$

By the Codazzi equation, for any vector ξ in the normal bundle of f, the corresponding shape operator A_ξ of f, that we always consider restricted to $\Delta^\perp$, satisfies that

$$A_\xi \circ C_S = C_S^t \circ A_\xi, \tag{1}$$

for any $S \in \Delta$. In particular, when M^n has rank two and f is nowhere a composition, by Proposition 4 we have $\Delta = \Gamma$, the splitting tensor of Δ

agrees with the one for the nullity Γ of M^n, and therefore it is intrinsic; see [9] for details.

Remark 6. Given a rank two Riemannian manifold M^n with splitting tensor D of Γ, if $d := \dim(\operatorname{Im} D \subset \operatorname{End}(\Gamma^\perp)) = 4$, condition (1) clearly implies that M^n admits no rank 2 isometric immersion in Euclidean space. In particular, by Proposition 4, M^n cannot be a Euclidean submanifold in codimensions 1 or 2. On the other hand, if $d = 3$, also from (1) we see that any rank 2 isometric immersion f of M^n must satisfy $\dim N_f^1 = 1$, and thus by Proposition 5 it must be a composition almost everywhere.

Recall that f is *surface-like* if, along each connected component U_λ of an open dense subset U of M^n, there is a surface V^2 such that U_λ splits as a Riemannian product $U_\lambda \subset V^2 \times \mathbb{R}^{n-2}$ if the splitting tensor C vanishes (respectively, $U_\lambda \subset CV^2 \times \mathbb{R}^{n-3}$ if $C \neq 0$), and $f|_{U_\lambda} = (g \times Id_{\mathbb{R}^{n-2}})|_{U_\lambda}$ splits (respectively, $f|_{U_\lambda} = (\mathcal{C}g \times Id_{\mathbb{R}^{n-3}})|_{U_\lambda}$), for some isometric immersion $g : V^2 \to \mathbb{R}^4$ (respectively, $g : V^2 \to \mathbb{S}^4$, and $\mathcal{C}g$ stands for the cone over g, $\mathcal{C}g(x,t) = tg(x)$). It is easy to see that f is surface-like if and only if it holds everywhere that $C_T = \mu(T)I$ for all $T \in \Delta$; cf. Lemma 6 in [9].

Now, for a rank two immersion f as above, since its codimension is two, for a given basis $X, Y \in \Delta^\perp(x)$ there are $a, b, c \in \mathbb{R}$ such that

$$a\alpha_f(X,X) + 2c\alpha_f(X,Y) + b\alpha_f(Y,Y) = 0. \tag{2}$$

Following [6], a nowhere surface-like rank two Euclidean submanifold $f : M^n \to \mathbb{R}^{n+2}$ is called *hyperbolic* (respectively, *parabolic* or *elliptic*) if it holds everywhere that $\dim N_f^1 = 2$, and $ab - c^2 < 0$ (respectively, $ab - c^2 = 0$ or $ab - c^2 > 0$), a condition that is independent of the given basis. Of course, these concepts make perfect sense in any codimension, but in this paper we always reserve the term 'hyperbolic (parabolic, elliptic) submanifold' to those in codimension two, except for surfaces. We can choose the basis $\{X_1, X_2\}$ such that (2) takes the form

$$\alpha_f(X_1,X_1) - \epsilon\alpha_f(X_2,X_2) = 0, \tag{3}$$

where $\epsilon = 1$ (respectively, $\epsilon = 0$, $\epsilon = -1$). Moreover, the pairs $a_1X_1 + a_2X_2$, $a_1X_2 + \epsilon a_2X_1$ also satisfy (3) and, up to signs, there are no others. Then let $J : \Delta^\perp \to \Delta^\perp$ be the (unique up to sign) linear map defined by $JX_1 = X_2$ and $JX_2 = \epsilon X_1$. In particular, $J^2 = \epsilon I$. We conclude that f is hyperbolic (respectively, parabolic or elliptic) if and only if there is a linear map $J :$

$\Delta^\perp \to \Delta^\perp$ such that $J^2 = \epsilon I$ with $\epsilon = 1$ (respectively, $\epsilon = 0$ or $\epsilon = -1$), and $\alpha_f(X, JY) = \alpha_f(JX, Y)$, for all $X, Y \in \Delta^\perp$, or, equivalently,

$$A_\xi \circ J = J^t \circ A_\xi, \quad \forall\, \xi \in T_f^\perp M. \tag{4}$$

In particular, it is easy to check that $\epsilon + 1$ is the number of linearly independent normal directions whose corresponding shape operators have rank one. Moreover, by (1),

$$\{C_S : S \in \Delta\} \subset \operatorname{span}\{I, J\}. \tag{5}$$

Hence, since f is always assumed to be nowhere surface-like, by (5) the endomorphism J above is also intrinsic, and so is the property of being hyperbolic, parabolic or elliptic. In other words, by Remark 6, if a nowhere surface-like rank two Riemannian manifold M^n admits a Euclidean isometric immersion in codimension two that is nowhere a composition, then $d \leq 2$, $\Delta = \Gamma$, the immersion has to be either hyperbolic, parabolic or elliptic, (5) holds, and accordingly we call M^n itself hyperbolic, parabolic or elliptic. This justifies the intrinsic flavour of the title of this work. In particular, if a deformation of a hyperbolic submanifold is not hyperbolic, then it is somewhere a composition, and hence it is not honest.

Remark 7. As we pointed out, it is very easy to classify all the compositions in codimension two of a given hypersurface $M^n \subset \mathbb{R}^{n+1}$. On the other hand, the genuine isometric immersions of such an M^n in codimension two were classified in Proposition 3 and Theorem 4 of [7] for the parabolic case, and in Theorem 1 of [11] for the remaining elliptic and hyperbolic cases.

Remark 8. The local understanding of rank two Riemannian manifolds is also important from an intrinsic point of view. Indeed, by Theorem A in [17], such a complete Riemannian manifold with finite volume is always surface-like. This is true even allowing rank less or equal than two. In particular, this shows that Nomizu's conjecture, which states that a complete locally irreducible semi-symmetric space of dimension at least three must be locally symmetric, that is well-known to be false, is actually true for complete manifolds with finite volume.

1.3. The Sbrana-Cartan theory

As we pointed out, a Euclidean hypersurface is locally rigid when its rank is greater than or equal to three, and highly deformable but well understood when flat, i.e., when its rank is at most one. At the beginning of the last century, V. Sbrana [18] and a few years later E. Cartan [4], independently and with different techniques, worked out the remaining interesting case, the hypersurfaces with rank two. That is, they classified nowhere flat locally deformable Euclidean hypersurfaces, $f : M^n \to \mathbb{R}^{n+1}$, for $n \geq 3$, extending earlier works by Schur and Bianchi; see [9] and references therein. According to this classification, these hypersurfaces, now called *Sbrana-Cartan hypersurfaces*, are (locally) divided into four classes.

The first two classes of Sbrana-Cartan hypersurfaces are obvious and highly deformable: the surface-like and the ruled ones. The first ones deform as their surfaces do in $\mathbb{R}^3$ or $\mathbb{S}^3$, while the space of deformations of a ruled hypersurface can be naturally parametrized by the set of smooth real functions in one variable.

Therefore the actually interesting Sbrana-Cartan hypersurfaces belong to the two remaining *continuous* and *discrete* classes, and thus are the ones that demand the hard work in the theory. The ones in the continuous class admit precisely a one parameter family of deformations, while the ones in the discrete class have just only one noncongruent deformation.

Since the beginning of the theory several families of examples of the continuous class were known. In fact, the bulk of Sbrana and Cartan works is concentrated on the study of this class. For example, those whose Gauss map is a minimal surface in the sphere belong to this class. Another large set of examples is given by minimal hypersurfaces of rank two that have a one parameter associated family of deformations like minimal surfaces do ([12]).

However, until very recently not a single example of the discrete class was known, nor even if this class was actually empty. A large set of explicit examples of the discrete class was then explicitly constructed and characterized in [9] in a very geometric way: as the transversal intersection of two generic flat hypersurfaces. Although the construction is quite natural, the actual computations are long and involved. It turns out that these submanifolds in codimension two are hyperbolic, and in Section 9 we will recover this result, in a much simpler way, by using the machinery developed in this work. More interestingly, we will compute all the moduli spaces of deformations of these submanifolds in codimension two where they naturally live, finding the first known examples of honestly deformable Euclidean submanifolds of rank and codimension two.

Nothing similar to the local results for hypersurfaces due to Sbrana and Cartan had been known for codimension higher than one until recently. Locally, rank two elliptic and parabolic submanifolds were shown to be honestly rigid in [8], and it was not clear what would happen for hyperbolic submanifolds. We will show here that, in contrast, hyperbolic submanifolds are not genuinely rigid.

Now, for global rigidity, [13] is devoted to show that any pair of isometric immersions in codimension two of a compact Riemannian manifold M^n is nowhere genuine, provided certain mild singular extensions are allowed. Indeed, the authors had to consider hyperbolic Sbrana-Cartan hypersurfaces $N^{n+1} \subset \mathbb{R}^{n+2}$ together with their singular set, $\Sigma^n \subset N^{n+1}$, which is itself, in fact, a regular deformable hyperbolic submanifold in $\mathbb{R}^{n+2}$. They called them *generalized Sbrana-Cartan hypersurfaces*. Actually, when deforming N^{n+1} its singular set Σ^n also deforms, and M^n could very well share an open subset with Σ^n. More in the spirit of this work, we can think of Σ^n as a regular submanifold in $\mathbb{R}^{n+2}$ that extends as a Sbrana-Cartan hypersurface $N^{n+1} \subset \mathbb{R}^{n+2}$, but allowing singularities along Σ^n. Yet, the actual necessity of considering singular extensions was not established in [13], because the pair of immersions could also extend regularly. Precisely this kind of singularities will appear again in our Theorem 26 below, and as a consequence we will conclude that it is indeed necessary to consider singular extensions to obtain this strong genuine rigidity in codimension two for compact manifolds; cf. Corollary 29.

Remark 9. We point out that the main result in [13] is not actually correct as it is stated in that paper. Indeed, by the same reason discussed above, singular flat extensions may appear when the submanifold has an open subset of flat points with first normal spaces of dimension less than two. The omission is due to some minor gaps in some lemmas. However, all gaps are fixed once singular flat extensions are allowed, hence the result remains valid if we consider the flat extensions together with their singularities; see Corollary 4 in [15]. In any case, these gaps do not affect our work since we will use these lemmas only when applied to nonflat submanifolds with two dimensional first normal spaces.

Recently, a genuine rigidity theory taking into account singularities was developed in [15], where the main result in [13] (with the gaps fixed) is easily obtained and even extended in a unified proof together with the classical Sacksteader's rigidity theorem of compact hypersurfaces, and a proof that compact hypersurfaces are singularly genuinely rigid up to codimension 3.

1.4. Shared dimension of a pair of curves

For later use, we introduce an elementary property about curves in Euclidean space. Given two curves α_1 and α_2 in $\mathbb{R}^N$ we define the *shared dimension between* α_1 *and* α_2, denoted by $\overline{I}(\alpha_1, \alpha_2)$, as the smallest integer k for which there is an orthogonal decomposition in affine subspaces, $\mathbb{R}^N = \mathbb{V}_1 \oplus \mathbb{V}^k \oplus \mathbb{V}_2$, satisfying $\operatorname{span}(\alpha_i) \subset \mathbb{V}_i \oplus \mathbb{V}^k$, $i = 1, 2$, where $\operatorname{span}(\alpha)$ stands for the smallest affine linear subspace which contains the image of α. We call $\mathbb{V}^k$ the *shared subspace between* α_1 *and* α_2. Of course, generically, $\overline{I}(\alpha_1, \alpha_2) = N$, and $\overline{I}(\alpha_1, \alpha_2) = 0$ if and only if the surface $g(u, v) = \alpha_1(u) + \alpha_2(v)$ has flat normal bundle, when $\alpha_1', \alpha_1'', \alpha_2', \alpha_2''$ are independent. We need the following elementary characterization of $\overline{I}(\alpha_1, \alpha_2)$.

Lemma 10. *Given two curves $\alpha_1(u)$ and $\alpha_2(v)$ in $\mathbb{R}^N$, the integer $\overline{I}(\alpha_1, \alpha_2)$ agrees with the minimum integer k such that $\langle \alpha_1'(u), \alpha_2'(v)\rangle$ can be written as a sum of the form $\sum_{i=1}^k a_i(u) b_i(v)$ for certain smooth functions of one variable a_i, b_i, $1 \le i \le k$.*

Proof. It is clear by definition that $k \le \overline{I}(\alpha_1, \alpha_2)$. To prove the opposite inequality, define $\tilde\alpha_1 = (\alpha_1, \int_0^u a_1(s)ds, \dots, \int_0^u a_k(s)ds)$ and $\tilde\alpha_2 = (\alpha_2, -\int_0^v b_1(s)ds, \dots, -\int_0^v b_k(s)ds)$ as orthogonal curves in $\mathbb{R}^{N+k}$. So, $\mathbb{R}^{N+k} = \tilde{\mathbb{V}}_1^{n_1} \oplus^\perp \tilde{\mathbb{V}}_2^{n_2}$ with $\operatorname{span}(\tilde\alpha_i) \subset \tilde{\mathbb{V}}_i^{n_i}$, $i = 1, 2$. Consider $\mathbb{V}_i = \tilde{\mathbb{V}}_i^{n_i} \cap (\mathbb{R}^N \times \{0\}) \subset \mathbb{R}^N$, and complete to an orthogonal decomposition, $\mathbb{R}^N = \mathbb{V}_1 \oplus \mathbb{V} \oplus \mathbb{V}_2$. By construction, $\operatorname{span}(\alpha_i)$ is orthogonal to $\mathbb{V}_j$, $1 \le j \ne i \le 2$, and $\dim \mathbb{V} = N - \dim \mathbb{V}_1 - \dim \mathbb{V}_2 \le N - (n_1 - k) - (n_2 - k) = k$. □

We will actually need the local version of this concept. Since $\overline{I}$ does not increase when we restrict the domains of the curves, we define the *local shared dimension between* α_1 *and* α_2 as the integer-valued function

$$I(\alpha_1, \alpha_2)(u, v) := \lim_{\epsilon \to 0} \overline{I}\left(\alpha_1|_{(u-\epsilon, u+\epsilon)}, \alpha_2|_{(v-\epsilon, v+\epsilon)}\right).$$

Since $I(\alpha_1, \alpha_2)$ is clearly semicontinuous, it is constant along connected components of an open dense subset of the parameters (u, v).

2. Projecting the data

In this section we show that all the data relevant to the study of the rigidity of a hyperbolic submanifold actually project to the leaf space of the relative nullity foliation, $L^2 = M^n/\Delta$. Notice that L^2 is a smooth surface when working locally on M^n, while globally it may only fail to be Hausdorff. The objects and notations introduced in this section will be used throughout the whole work.

Let $f : M^n \to \mathbb{R}^{n+2}$ be a hyperbolic submanifold. We denote by $\{\xi_1, \xi_2\} \subset T_f^\perp M$ and $\{Y_1, Y_2\} \subset \Delta^\perp$ local smooth unit frames such that

$$\ker A_{\xi_i} = \Delta \oplus \operatorname{span}\{Y_i\}, \quad i = 1, 2, \tag{6}$$

where we also require that the nonzero eigenvalue λ_i of A_{ξ_i} is positive. Let $0 < \theta < \pi$ and $0 < \omega < \pi$ be the angles between ξ_1 and ξ_2, and between Y_1 and Y_2, respectively. The function θ, which we call the *main angle* of f, plays a crucial role in this work. Fix the orientation on $T_f^\perp M$ and $\Delta^\perp$ determined by these bases, and complete to local smooth oriented orthonormal frames $\{\xi_i, \eta_i\} \subset T_f^\perp M$ and $\{X_i, Y_i\} \subset \Delta^\perp$, i.e.,

$$\sin(w) X_i = (-1)^{i+1}(\cos(w) Y_i - Y_j), \quad \sin(\theta)\eta_i = (-1)^i(\cos(\theta)\xi_i - \xi_j), \tag{7}$$

$1 \le i \ne j \le 2$. Observe that, since we are in codimension two and $\operatorname{rk} A_{\xi_1} = 1$, Gauss equation reduces to $\operatorname{scal} = \det(A_{\eta_i}|_{\Delta^\perp})$, where scal denotes the (non-normalized) scalar curvature of M^n. Then from (6) and (7) we obtain

$$\operatorname{scal} = -\sin^2(\omega)\,\frac{\cos(\theta)}{\sin^2(\theta)}\lambda_1\lambda_2. \tag{8}$$

Finally, denote by ψ^i the normal connection 1-form associated to the frame $\{\xi_i, \eta_i\}$,

$$\psi^i(X) = \langle \nabla_X^\perp \xi_i, \eta_i \rangle.$$

Although the normal connection 1-form depends on the chosen orthonormal frame, its differential is independent since by (7) we have

$$\psi^2 = \psi^1 + d\theta. \tag{9}$$

We proceed to obtain information about the evolution of the geometric data described above along the relative nullity foliation Δ. As a consequence, we will be able to project all the data relevant to the understanding of the isometric deformation problem to its leaf space

$$\pi : M^n \to L^2 := M^n/\Delta.$$

Recall that L^2 is intrinsic when M^n is nowhere flat, since it coincides with M^n/Γ.

Proposition 11. *With the above notations, the following holds for* $1 \leq i \neq j \leq 2$*:*

i) ξ_i *is constant in* $\mathbb{R}^{n+2}$ *along the leaves of* Δ*;*

ii) Y_i *is constant in* $\mathbb{R}^{n+2}$ *along the leaves of* Δ*;*

iii) $\{Y_1, Y_2\}$ *is an eigenbasis of the hyperbolic structure on* $\Delta^\perp$*, and the splitting tensor of* Δ *satisfies that* $C_T Y_i = T(\ln \lambda_j) Y_i$*, for all* $T \in \Delta$*;*

iv) There exist functions $\overline{\theta}, \overline{\omega}$ *and a 1-form* $\overline{\psi^i}$ *on* L^2 *such that* $\theta = \overline{\theta} \circ \pi$*,* $\omega = \overline{\omega} \circ \pi$ *and* $\psi^i = \pi^* \overline{\psi^i}$*.*

Proof. The Codazzi equation for $(A_{\xi_i}, T \in \Delta, Z \in \Delta^\perp)$ yields

$$\nabla_T A_{\xi_i} Z - A_{\xi_i}[T, Z] - \psi^i(T) A_{\eta_i} Z = 0. \tag{10}$$

This for $Z = Y_i \in \ker A_{\xi_i}$ gives

$$A_{\xi_i}[T, Y_i] = -\psi^i(T) A_{\eta_i} Y_i. \tag{11}$$

Since $Y_i \in \ker A_{\xi_i}$ and f has rank two, we have that $0 \neq A_{\eta_i} Y_i \in \operatorname{Im} A_{\xi_j}$. However, since $\operatorname{Im} A_{\xi_1} \cap \operatorname{Im} A_{\xi_2} = 0$, both sides of (11) vanish, and then

$$\psi^i(T) = 0, \quad \forall T \in \Delta, \tag{12}$$

$$[T, Y_i] \in \ker A_{\xi_i} \quad \forall T \in \Delta. \tag{13}$$

Equation (12) says that ξ_i is parallel along Δ with respect to the normal connection, which by the Weingarten formula implies (i).

By the above, (10) reduces to

$$\nabla_T A_{\xi_i} Z = A_{\xi_i}[T, Z], \tag{14}$$

which shows that the line bundle $\operatorname{Im} A_{\xi_i}$ is parallel along Δ with respect to the Levi-Civita connection ∇ of M^n. In particular, the unitary vector field X_i, and then also Y_i, are parallel along Δ with respect to ∇, which proves (ii) by the Gauss formula.

Now, (ii) says that $[T, Y_i] = -\nabla_{Y_i} T$. Taking the orthogonal projection onto $\Delta^\perp$ of this relation we conclude from (13) that $\{Y_1, Y_2\}$ is a common eigenbasis for all splitting tensors and thus, by (5), for the hyperbolic structure as well since f is nowhere surface-like. In other words, there are 1-forms

b_j on Δ such that

$$C_T Y_j = b_j(T) Y_j, \quad \forall T \in \Delta. \tag{15}$$

A straightforward computation using (7) gives

$$b_j(T) = \langle C_T X_i, X_i \rangle.$$

On the other hand, setting $Z = X_i$ in (14) and using $A_{\xi_i} X_i = \lambda_i X_i$ we obtain that the right-hand side of the above equation equals $T(\ln \lambda_i)$, from which (iii) follows.

By (i) and (ii) and the definition of θ and ω both angles are constant along Δ, so they project to L^2. Finally, the Ricci equation for $(\xi_i, \eta_i, T \in \Delta, Z \in \Delta^\perp)$ yields

$$d\psi^i(T, Z) = 0, \ \forall T \in \Delta, \ Z \in \Delta^\perp,$$

which alongside (12) implies that ψ^i is projectable by Corollary 12 in [11]. □

We often identify projectable functions and 1-forms with their respective projections, when there is no risk of confusion. This will be further clarified in the next section after Theorem 16.

We show next that (ii) and (iii) above allow us to rescale Y_1 and Y_2 so that the resulting frame projects to a coordinate frame on L^2. These coordinates will be used throughout this work, and in particular will be useful to prove the existence of polar surfaces.

Proposition 12. *There exist smooth positive functions μ_1 and μ_2 on M^n and a coordinate system (u_1, u_2) on L^2 such that the frame $\{Z_1, Z_2\}$ defined by $Z_i = \mu_i Y_i$ satisfies*

$$\partial_{u_i} \circ \pi = \pi_* \circ Z_i, \quad i = 1, 2.$$

Proof. According to Proposition 10 in [11], the necessary and sufficient condition for the vector fields Z_i to be projectable is that

$$[Z_i, T] \in \Delta, \quad \forall T \in \Delta, \tag{16}$$

whereas the projections $\pi_* \circ Z_i$ come from a local coordinate system if, additionally,

$$[Z_1, Z_2] \in \Delta. \tag{17}$$

From Proposition 11-(ii) and (iii), we have that (16) reduces to $T(\mu_i) = -b_i(T)\mu_i$, with $b_i = d(\ln\lambda_j)|_\Delta$, $1 \le i \ne j \le 2$, while condition (17) can be written as $Y_j(\mu_i) = -r_i\mu_i$, for r_1 and r_2 defined by $[Y_1, Y_2] + r_1 Y_1 - r_2 Y_2 \in \Delta$. In other words, we must show that the first order system of PDEs

$$d(\ln\mu_i)|_\Delta = -b_i\,, \quad Y_j(\ln\mu_i) = -r_i \tag{18}$$

is integrable.

Consider the distribution $\Omega_i = \Delta \oplus \operatorname{span}\{Y_j\}$ on M^n, $1 \le i \ne j \le 2$. Since Y_j is parallel along Δ and an eigenvector of all C_T, $T \in \Delta$, we have from the integrability of Δ that Ω_i is also integrable. Define on Ω_i the 1-form σ_i by

$$\sigma_i|_\Delta = -b_i, \quad \sigma_i(Y_j) = -r_i.$$

Since all our considerations are local, the integrability condition of (18) translates into the exactness of σ_i. Since the 1-form b_i on Δ is exact, $d\sigma_i|_{\Delta\times\Delta} = 0$. Thus, it suffices to show that $d\sigma_i(T, Y_j) = 0$, or equivalently,

$$Y_j(b_i(T)) - T(r_i) = b_i(\nabla^v_{Y_j}T) - r_i b_j(T), \quad \forall\, T \in \Delta, \tag{19}$$

since, by Proposition 11-(ii),

$$\sigma_i([T, Y_j]) = -\sigma_i(\nabla_{Y_j}T) = \sigma_i(C_T Y_j) - \sigma_i(\nabla^v_{Y_j}T) = b_i(\nabla^v_{Y_j}T) - r_i b_j(T),$$

where ∇^h and ∇^v stand for the connections induced by ∇ on $\Delta^\perp$ and Δ, respectively.

It is well-known and easy to see that the Codazzi equation for f implies that the splitting tensor C itself is a Codazzi tensor, that is, $(\nabla^h_{Y_i}C_T)Y_j - C_{\nabla^v_{Y_i}T}Y_j = (\nabla^h_{Y_j}C_T)Y_i - C_{\nabla^v_{Y_j}T}Y_i$. This can be easily rewritten using (15) and (7) as

$$Y_j(b_i(T)) - b_i(\nabla^v_{Y_j}T) = (-1)^i \sin(\omega)^{-1}(b_i(T) - b_j(T))G_i, \tag{20}$$

where $G_i = \langle \nabla_{Y_i}Y_j, X_j\rangle$. By (7) we express r_i in terms of G_1 and G_2 as

$$r_i = (-1)^i \sin(\omega)^{-1}(G_i - \cos(\omega)G_j). \tag{21}$$

Now, as $T \in \Delta = \Gamma$, we have from Proposition 11-(ii) that

$$0 = \langle R(T, Y_i)Y_j, X_j\rangle = \langle \nabla_T\nabla_{Y_i}Y_j, X_j\rangle - b_i(T)G_i = T(G_i) - b_i(T)G_i. \tag{22}$$

Since ω and X_i are constant along Δ, differentiate (21) along Δ and use (22) to conclude that $T(r_i) - r_i b_j(T)$ agrees with the right-hand side of (20). This proves (19), as wished.

Therefore, the system (18) is integrable, which means that μ_i can be arbitrarily prescribed along a fixed integral curve γ of Y_i and then extended along each leaf of Ω_i through γ as a solution of (18). $\square$

3. Polar surfaces and the parametrization

In this section we show that the normal space of a hyperbolic submanifold is always integrable via a so-called polar surface. We use this to recover any hyperbolic submanifold from its polar surface through a very simple and explicit parametrization. As an important application, we will classify all flat Euclidean submanifolds in codimension two in a simple and explicit way.

Given a hyperbolic submanifold $f : M^n \to \mathbb{R}^{n+2}$, we will show first the (local) existence of a surface $g : L^2 = M^n/\Delta \to \mathbb{R}^{n+2}$ that integrates its normal space in the sense that $T_f^\perp M = \pi^*(T_g L)$. We follow all the notations and definitions of the previous section and, for convenience and from now on, $u = u_1$ and $v = u_2$ as subindexes will denote the corresponding partial derivatives for the local coordinates (u, v) constructed in Proposition 12.

Assume that L^2 is simply-connected, and suppose that there is such a surface g. Then, $g_u = c\,\overline{\xi}_1 + a\,\overline{\xi}_2$ and $g_v = b\,\overline{\xi}_1 + d\,\overline{\xi}_2$, for certain functions a, b, c, d over L^2. Differentiating the first equation with respect to v and the second with respect to u, and projecting orthogonally onto $T^\perp_{g(\pi(x))}L \supseteq \Delta_f^\perp(x)$ we get $cA_{\xi_1}Z_2 = dA_{\xi_2}Z_1$. Then $c = d = 0$ since $A_{\xi_1}Z_2$ and $A_{\xi_2}Z_1$ are linearly independent. So consider the following first order system of PDEs over L^2:

$$g_u = a\,\overline{\xi}_2\,, \qquad g_v = b\,\overline{\xi}_1. \tag{23}$$

Differentiating the first equation in (23) with respect to v gives

$$g_{uv} \circ \pi = a_v \circ \pi\ \xi_2 + a \circ \pi\ \tilde{\nabla}_{\pi_* \circ Z_2}\overline{\xi}_2 = a_v \circ \pi\ \xi_2 + a \circ \pi\ \tilde{\nabla}_{Z_2}\xi_2.$$

Since $Z_2 \in \ker A_{\xi_2}$ and the normal connection form $\psi^2 = \langle \nabla^\perp_\bullet \xi_2, \eta_2 \rangle$ projects to $\overline{\psi^2}$, we get

$$g_{uv} = a_v\,\overline{\xi}_2 + a\,\overline{\psi^2}(\partial_v)\overline{\eta}_2, \tag{24}$$

and analogously for the second equation in (23),

$$g_{vu} = b_u \,\overline{\xi}_1 + b\,\overline{\psi^1}(\partial_u)\overline{\eta}_1. \tag{25}$$

Since both bases are equally oriented, we get that the integrability conditions for (23) are

$$\begin{aligned}\sin(\overline{\theta})\, a_v &= \overline{\psi^1}(\partial_u)\, b - \cos(\overline{\theta})\overline{\psi^2}(\partial_v)\, a,\\ \sin(\overline{\theta})\, b_u &= \cos(\overline{\theta})\overline{\psi^1}(\partial_u)\, b - \overline{\psi^2}(\partial_v)\, a.\end{aligned}$$

From $\sin(\overline{\theta}) > 0$ we conclude that this system always has solutions $a \neq 0, b \neq 0$ and so by (23) there exists a regular surface $g : L^2 \to \mathbb{R}^{n+2}$, with

$$\pi^*(T_g L) = T_f^\perp M, \quad \text{and} \quad \pi^*(N_g^1) = \Delta_f^\perp. \tag{26}$$

In addition,

$$E = a^2, \quad G = b^2, \quad F = ab\,\cos(\overline{\theta}) \tag{27}$$

are the coefficients of the first fundamental form of g. Moreover, g is also hyperbolic in the sense that $\dim N_g^1 = 2$ and $\alpha_g(\partial_u, \partial_v) = 0$ for the second fundamental form α_g of g, since $g_{uv} \in T_g L$. Recall that a coordinate system (u, v) on L^2 such that $\alpha_g(\partial_u, \partial_v) = 0$ everywhere is called *conjugate*. Following [6] we call such a surface g a *polar surface of f*, and from now on we consider on L^2 the metric induced by a (fixed) polar surface g of f, as in (27).

Remark 13. The 1-forms $\overline{\psi^i}$ are tangent connection forms for g, so $d\overline{\psi^i} = K_g\; dA$, where K_g denotes the Gaussian curvature of g and dA its area element. Since $\psi^i = \pi^*\overline{\psi^i}$, we have for the normal curvature 2-form $R_f^\perp = d\psi^i$ of f that

$$R_f^\perp = (K_g \circ \pi)\; \pi^* dA.$$

In particular, g is flat if and only if f has flat normal bundle. Furthermore, since θ is the angle between the conjugate directions of g, we also have that f is flat if and only if g has flat normal bundle, that is, $\theta \equiv \pi/2$, or, equivalently, $F \equiv 0$.

Remark 14. Hyperbolic surfaces in $\mathbb{R}^k$ are trivial to construct and classify locally: they are simply given by k generic solutions of a fixed wave equation. Indeed, let $U \subset \mathbb{R}^2$ be an open set with coordinates (u, v), and let Γ^u, Γ^v :

$U \to \mathbb{R}$ be two arbitrary smooth functions. Consider the second order linear wave differential operator

$$Q = \partial_u \circ \partial_v - \Gamma^u \partial_u - \Gamma^v \partial_v. \tag{28}$$

Take k smooth functions $g = (g_1, \dots, g_k)$ that are solutions of the wave equation $Q = 0$, which are *independent* in the sense that g_u, g_v, g_{uu}, g_{vv} are pointwise linearly independent. Then, $g : U \to \mathbb{R}^k$ is a regular surface, $\dim N_g^1 = 2$, and

$$g_{uv} - \Gamma^u g_u - \Gamma^v g_v = 0, \tag{29}$$

thus g is hyperbolic. In this sense, there is no geometry involved in the construction of hyperbolic surfaces. Notice also that Γ^u, Γ^v are automatically Christoffel symbols of g since $\nabla^g_{\partial_u} \partial_v = \Gamma^u \partial_u + \Gamma^v \partial_v$. In particular, Q is related with the Hessian of the metric induced by g by the relation $Q(\rho) = \mathrm{Hess}_\rho(\partial_u, \partial_v)$.

Remark 15. By (7), (24), (25), (27) and (29) it holds that

$$\psi^1(\partial_u) = \sin(\theta)\sqrt{E/G}\,\Gamma^u, \quad \text{and} \quad \psi^2(\partial_v) = -\sin(\theta)\sqrt{G/E}\,\Gamma^v. \tag{30}$$

Using (9) we see that these two equations recover the normal connection of f in terms of the metric of g. In particular, although a polar surface of a given hyperbolic submanifold is not unique, the functions $F/\sqrt{EG} = \cos(\theta)$ as well as $\sqrt{E/G}\,\Gamma^u$ and $\sqrt{G/E}\,\Gamma^v$ coincide for all its polar surfaces.

Next we proceed to show how to recover any hyperbolic submanifold f from a polar surface g of it by providing a simple parametrization of f that depends only on g and a smooth function over L^2 satisfying the same wave equation as g.

Consider $h : L^2 = M^n/\Delta \to M^n$ any local *cross-section* to the relative nullity foliation, that is, $h^*(\Delta) \oplus T_h L = h^*(TM)$. Since the leaves of relative nullity are mapped by f to (open subsets of) affine $(n-2)$-dimensional subspaces, (26) implies that

$$\Psi : (N_g^1)^\perp \subset T_g^\perp L \to \mathbb{R}^{n+2}, \qquad \Psi(\mu) = (f \circ h) + \mu$$

for any smooth section μ of $(N_g^1)^\perp$ parametrizes, at regular points, our submanifold f. Notice that, with the notations of Remark 14,

$$(N_g^1)^\perp = \mathrm{span}\,\{g_u, g_v, g_{uu}, g_{vv}\}^\perp.$$

We will conclude the parametrization by describing $h' := f \circ h$ in terms of g and a smooth function over L^2.

By (26), the map h' is characterized by the property

$$f_*(h^*(\Delta)) \oplus T_{h'}L = h^*(T_f M) = T_g^\perp L. \tag{31}$$

Decompose the position vector $h' \in \mathbb{R}^{n+2} = T_g L \oplus T_g^\perp L$. The component in $(N_g^1)^\perp$ of h' plays no role in (31) (or, equivalently, it can be absorbed in the parametrization Ψ), so we write

$$h' = X + \eta, \quad X \in T_g L, \quad \eta \in N_g^1.$$

But with this decomposition (31) is equivalent to $\nabla^g_\bullet X = A^g_\eta$. In particular, X is the gradient of a certain function ρ, $X = \nabla\rho$, whose Hessian satisfies

$$\mathrm{Hess}_\rho = A^g_\eta. \tag{32}$$

Since $\dim N_g^1 = 2$, by dimension reasons, given a function ρ on L there exists (a unique) $\eta \in N_g^1$ for which (32) holds if and only if $\mathrm{Hess}_\rho(\partial_u, \partial_v) = 0$. In other words, by Remark 14, ρ satisfies the same linear PDE as the coordinate functions of g, i.e., $Q(\rho) = 0$ for Q as in (28).

This was the final ingredient for the main result in this section, that is interesting in its own right:

Theorem 16. *Let $g = (g_1, \ldots, g_{n+2}) : W \subset \mathbb{R}^2 \to \mathbb{R}^{n+2}$ be $(n+2)$ independent solutions of a wave equation* (28)*, $Q(g) = 0$, and let ρ be another solution, $Q(\rho) = 0$. Then, g is a hyperbolic surface once we endow W with the metric induced by g, and there is a unique $\eta_\rho \in N_g^1$ such that $\mathrm{Hess}_\rho = A^g_{\eta_\rho}$. Moreover, the map*

$$\Psi : (N_g^1)^\perp \subset T_g^\perp L \to \mathbb{R}^{n+2}, \quad \Psi(\mu_x) = (\nabla\rho + \eta_\rho)(x) + \mu_x, \tag{33}$$

parametrizes, at regular points, a hyperbolic n-dimensional submanifold of $\mathbb{R}^{n+2}$ for which g is a polar surface.

Conversely, any hyperbolic Euclidean submanifold in codimension two can be locally parametrized in this way.

Proof. We have already proved the converse claim. For the direct statement, first notice that, by construction, the submanifold Ψ has g as a polar surface at its regular points, i.e., $\pi^*(T_g L) = T_\Psi^\perp M$ for $\pi : M^n = (N_g^1)^\perp \to L^2$

and M^n endowed with the metric induced by Ψ. In fact, since g is constructed from independent solutions, $\dim N^1_\Psi = 2$. In addition, g_u and g_v are independent normal vector fields to Ψ whose shape operators have rank one since $(g_v)_u \circ \pi = (g_u)_v \circ \pi = g_{uv} \circ \pi \in \pi^*(T_g L) = T^\perp_\Psi M$. Therefore, Ψ is hyperbolic. $\square$

In a local trivialization $(u, v, t_1, \ldots, t_{n-2})$ of $(N^1_g)^\perp$ determined by a moving frame $\{\eta_1, \ldots, \eta_{n-2}\}$ of $(N^1_g)^\perp$ along a conjugate local coordinate system (u, v) of L^2, Ψ can be written as

$$\Psi(u, v, t_1, \ldots, t_{n-2}) = (\nabla\rho + \eta_\rho)(u, v) + \sum_{i=1}^{n-2} t_i \eta_i(u, v).$$

We will consider from now on this as the *standard coordinate system* of our hyperbolic submanifolds. In particular, the coordinate vector fields ∂_u, ∂_v as well as differentiation with respect to u and v makes sense now also on M^n, and the fact that u and v are considered as coordinate functions in both M^n and L^2 will not cause any confusion. For example, since (u, v) are conjugate coordinates for g, the coordinate vector fields ∂_u and ∂_v are also conjugate for Ψ, i.e., $\alpha_\Psi(\partial_u, \partial_v) = 0$. Moreover, that a certain function $h = h(u, v, t_1, \ldots, t_{n-2})$ on M^n projects to its leaf space L^2 means simply that it does not depend on the coordinates $t_1, \ldots, t_{n-2}$. So we will always denote with the same symbol $h = h(u, v)$ the projection of h to L^2.

Remark 17. From Theorem 16 it follows that f splits a Euclidean factor if and only if g is not substantial, that is, if g reduces codimension. Moreover, it is easy to check that f is surface-like if and only if g has substantial conformal codimension 2, i.e., the image of g is contained in a 4-dimensional umbilical submanifold of the ambient space.

As an immediate consequence of Theorem 16 and Remark 13 we have a parametric description of all generic rank two Euclidean submanifolds f in codimension two with flat normal bundle in terms of Euclidean hyperbolic flat surfaces. Generic here means simply that $\dim N^1_f = 2$ everywhere, a condition that, by Proposition 5, is automatic if N^1_f has constant dimension and f is not a composition. More interestingly, in a dual way and also by Remark 13, we can characterize all generic flat Euclidean submanifolds in codimension two:

Corollary 18. *Let $g : L^2 \to \mathbb{R}^{n+2}$ be any surface with flat normal bundle and principal coordinates (u, v), and $\rho : L^2 \to \mathbb{R}$ any smooth function satisfying* $\mathrm{Hess}_\rho(\partial_u, \partial_v) = 0$*. Then, the map* (33) *parametrizes, at regular points, a flat n-dimensional submanifold of $\mathbb{R}^{n+2}$ for which g is a polar surface. Conversely, any generic flat submanifold in codimension two can be locally parametrized this way.*

The construction in Corollary 18 is much simpler, direct and explicit than the one given in Theorem 13 in [5]. While the latter depends on an elusive kind of surfaces in the sphere, called of 'type C', any surface in $\mathbb{R}^k$ with flat normal bundle parametrized by lines of curvature can be described explicitly from k arbitrary solutions of a certain simple linear integrable system of PDEs thanks to the beautiful construction due to E. Ferapontov in [14] (see also [10] for the generalization of this construction to arbitrary dimensions). In particular, Corollary 18 tells us what all those surfaces ξ of type C really are: they are just the principal directions of any surface g with flat normal bundle, i.e., $\xi = g_u/\|g_u\| : L^2 \to \mathbb{S}^{n+1}$.

Remark 19. As already pointed out, one of the main reasons to discard compositions when studying Euclidean rigidity in codimension two is the existence of a very simple local classification of flat Euclidean hypersurfaces (via the Gauss parametrization). Now Corollary 18 also justifies discarding compositions when working in codimension 3, as well as the introduction of our concept of honest rigidity.

4. The 6 invariants of a hyperbolic submanifold

In this section we extract the main data of a Euclidean hyperbolic submanifold in codimension two and show how it completely determines its deformations by means of a set of six invariants.

Let $f : M^n \to \mathbb{R}^{n+2}$ be a nowhere flat hyperbolic submanifold, and $L^2 = M^n/\Delta$ its nullity leaf space. Following the notations of Section 2, we set $s := \sin^2(\theta)$ for the main angle θ of f, and, in a standard coordinate system, we define the *main symbols* of f by

$$\Lambda^u := \frac{\psi^2(\partial_u)}{\tan(\theta)}, \quad \Lambda^v := -\frac{\psi^1(\partial_v)}{\tan(\theta)}. \tag{34}$$

We can easily express Λ^u, Λ^v and s, which by Proposition 11-(iv) can and will be seen as functions over L^2, in terms of the first fundamental form of the polar surface g. Indeed, from (9) and (30) we get

$$\Lambda^u = \frac{F}{G}\Gamma^u + \frac{s_u}{2s}, \quad \Lambda^v = \frac{F}{E}\Gamma^v + \frac{s_v}{2s}, \quad s = 1 - \frac{F^2}{EG}, \tag{35}$$

where E,F and G are the coefficients of the first fundamental form of g in the coordinate system (u, v). Notice that, by Remark 15, the expressions in (35) do not depend on the particular choice of a polar surface of f. We set

$$\mathcal{S}(f) := \{\theta, \Lambda^u, \Lambda^v, \kappa^u, \kappa^v\},$$

where

$$\kappa^u := \lambda_1^2/s, \quad \kappa^v := \lambda_2^2/s.$$

Observe that by (9) and $\lambda_1, \lambda_2 > 0$, we can recover λ_1, λ_2, ψ^1 and thus the second fundamental form and normal connection of f from $\mathcal{S}(f)$. The Fundamental Theorem of Submanifolds hence says that $\mathcal{S}(f)$ determines f itself when M^n is simply-connected.

Our following result provides a set of six functions that both determine and remain invariant under deformations $\hat{f}$ of f that are nowhere compositions. Observe that this is a slightly different approach to the usual ones that deal with the study of deformations. From now on, we add a hat to indicate the objects of $\hat{f}$ corresponding to those of f. In particular, $\hat{\theta}$ refers to the main angle of $\hat{f}$, and $\hat{\Lambda}^u, \hat{\Lambda}^v$ to its main symbols.

Proposition 20. *Let $f : M^n \to \mathbb{R}^{n+2}$ be a nowhere flat hyperbolic submanifold. Then, the functions $\mathcal{G}$, $\mathcal{C}_1^u$, $\mathcal{C}_1^v$, $\mathcal{C}_2^u$, $\mathcal{C}_2^v$ and $\mathcal{R}$ given by*

$$\begin{gathered} \mathcal{G} = \cos(\theta)\sqrt{\kappa^u\kappa^v}, \\ \mathcal{C}_1^u = \kappa^u\Lambda^u, \quad \mathcal{C}_1^v = \kappa^v\Lambda^v, \\ \mathcal{C}_2^u = \frac{\kappa_u^u}{\kappa^u} + 2\Lambda^u, \quad \mathcal{C}_2^v = \frac{\kappa_v^v}{\kappa^v} + 2\Lambda^v, \\ \mathcal{R} = \rho_{uv} + \cos^2(\theta)\,\rho_u\rho_v - \Lambda^v\rho_u - \Lambda^u\rho_v, \end{gathered} \tag{36}$$

where $\rho = \ln(|\tan(\theta)|)$, are invariant, i.e., such functions are preserved by any hyperbolic deformation of f. Moreover, the ratios $\tau^u := \hat{\kappa}^u/\kappa^u$ and $\tau^v := \hat{\kappa}^v/\kappa^v$ project to L^2.

Conversely, if M^n is simply-connected, given functions $\tau^u > 0$, $\tau^v > 0$, $0 < \hat{\theta} < \pi$, $\hat{\Lambda}^u$ and $\hat{\Lambda}^v$ on L^2 such that $\hat{\mathcal{S}} = \{\hat{\theta}, \hat{\Lambda}^u, \hat{\Lambda}^v, \hat{\kappa}^u := \tau^u\kappa^u, \hat{\kappa}^v := \tau^v\kappa^v\}$ satisfies system (36), there exists an isometric immersion $\hat{f} : M^n \to \mathbb{R}^{n+2}$ with $\mathcal{S}(\hat{f}) = \hat{\mathcal{S}}$.

Proof. Since M^n is nowhere flat and $\hat{f}$ has rank two, $\hat{\Delta} = \Gamma = \Delta$ and $\hat{C} = C$ everywhere. Moreover, since $\hat{f}$ is nowhere surface-like, by (5) $\hat{f}$ is hyperbolic with $\hat{J} = J$. In particular, by Proposition 11-(*iii*), $\hat{Z}_i = Z_i$, $i = 1, 2$, and hence $\hat{\omega} = \omega$, so that the invariance of $\mathcal{G}$ is simply a restatement of the Gauss equation (8) for $\Delta^\perp$ and $\hat{f}$. It also follows from Proposition 11-(*iii*) that τ^u and τ^v project to L^2.

From (6) and (7), the Codazzi equation for $(A_{\xi_1}, \partial_u, \partial_v)$ is

$$\sin(\theta)\nabla_{\partial_u} A_{\xi_1}\partial_v = -\cos(\theta)\psi^1(\partial_u)A_{\xi_1}\partial_v - \psi^1(\partial_v)A_{\xi_2}\partial_u. \tag{37}$$

Setting $\delta_i = \hat{\lambda}_i/\lambda_i$, we have that $\hat{A}_{\hat{\xi}_i} = \delta_i A_{\xi_i}$ and (37) for $\hat{f}$ gives

$$\begin{aligned} &\sin(\hat{\theta})((\delta_1)_u A_{\xi_1}\partial_v + \delta_1 \nabla_{\partial_u} A_{\xi_1}\partial_v) \\ &\quad = -\cos(\hat{\theta})\hat{\psi}^1(\partial_u)\delta_1 A_{\xi_1}\partial_v - \hat{\psi}^1(\partial_v)\delta_2 A_{\xi_2}\partial_u. \end{aligned} \tag{38}$$

Using (37) in (38) and $\sin(\theta) \neq 0$ we get

$$\begin{aligned} &\Big(\delta_1 \sin(\hat{\theta})\psi^1(\partial_v) - \delta_2 \sin(\theta)\hat{\psi}^1(\partial_v)\Big) A_{\xi_2}\partial_u \\ &= \Big((\delta_1)_u \sin(\theta)\sin(\hat{\theta}) + \delta_1 \sin(\theta)\cos(\hat{\theta})\hat{\psi}^1(\partial_u) \\ &\quad - \delta_1 \sin(\hat{\theta})\cos(\theta)\psi^1(\partial_u)\Big) A_{\xi_1}\partial_v. \end{aligned}$$

Since $A_{\xi_1}\partial_v$ and $A_{\xi_2}\partial_u$ are linearly independent everywhere, we get from the invariance of $\mathcal{G}$ the invariance of $\mathcal{C}_1^v$ and $(\ln \delta_1)_u = \cot(\theta)\psi^1(\partial_u) - \cot(\hat{\theta})\hat{\psi}^1(\partial_u)$, which is clearly equivalent to the invariance of $\mathcal{C}_2^u$ by (9). Similarly, the invariance of $\mathcal{C}_1^u$ and $\mathcal{C}_2^v$ is equivalent to the Codazzi equation for $(\hat{A}_{\hat{\xi}_2}, \partial_u, \partial_v)$. Notice that, as shown in the proof of Proposition 11, Codazzi and Ricci equations for vectors in Δ and $\Delta^\perp$ are just the projectability onto L^2 of the functions involved and the determination of the splitting tensor, so they provide no additional information.

For the last invariant, observe first that $\tilde{\mathcal{R}} = \cot(\theta)d\psi^1(\partial_u, \partial_v)$ is intrinsic. Indeed, by (7) and the Ricci equation,

$$\begin{aligned} \tilde{\mathcal{R}} &= \cot(\theta)\langle[A_{\xi_1}, A_{\eta_1}]\partial_u, \partial_v\rangle \\ &= \frac{\cos(\theta)}{\sin^2(\theta)}\langle A_{\xi_1}\partial_v, A_{\xi_2}\partial_u\rangle = \|\partial_u\|\|\partial_v\|\cos(\omega)\,\mathrm{scal}, \end{aligned}$$

where for the last equality we used (8). We conclude the invariance of $\mathcal{R}$ from this and the previous invariants, since it is straightforward to check that $2\mathcal{R} = 2\tilde{\mathcal{R}} + (\mathcal{C}_2^u)_v + (\mathcal{C}_2^v)_u - (\ln \mathcal{G}^2)_{uv}$. This completes the proof of the direct statement.

Conversely, given 5 smooth functions $\tau^u > 0, \tau^v > 0$, $\hat{\theta} \in (0, \pi)$, $\hat{\Lambda}^u$ and $\hat{\Lambda}^v$ on L^2 satisfying (36) for $\hat{\kappa}^u = \tau^u \kappa^u$ and $\hat{\kappa}^v = \tau^v \kappa^v$, we construct the second fundamental form and the normal connection 1-form for a new isometric immersion as follows. We endow $T_f^\perp M$ with the new metric that keeps $\hat{\xi}_1 = \xi_1$ and $\hat{\xi}_2 = \xi_2$ unitary but making an angle $\hat{\theta}$ instead of θ. Then we set

$$\hat{A}_{\hat{\xi}_1} = \sqrt{\tau^u}\,\frac{\sin(\hat{\theta})}{\sin(\theta)} A_{\xi_1}, \qquad \hat{A}_{\hat{\xi}_2} = \sqrt{\tau^v}\,\frac{\sin(\hat{\theta})}{\sin(\theta)} A_{\xi_2}, \tag{39}$$

and

$$\hat{\psi}^1(\partial_u) := \tan(\hat{\theta})\,\hat{\Lambda}^u - \hat{\theta}_u, \quad \hat{\psi}^1(\partial_v) = -\tan(\hat{\theta})\,\hat{\Lambda}^v, \quad \hat{\psi}^1|_\Gamma = 0.$$

Therefore, as we saw in the direct statement, system (36), together with the projectability of τ^u, τ^v, $\hat{\theta}$, $\hat{\Lambda}^u$ and $\hat{\Lambda}^v$, is equivalent to the fundamental equations for such a second fundamental form and normal connection. Since M^n is simply-connected, the Fundamental Theorem of Submanifolds assures the existence of an isometric immersion $\hat{f} : M^n \to \mathbb{R}^{n+2}$ with second fundamental form and normal connection as above. It is clear by definition that $\mathcal{S}(\hat{f}) = \{\hat{\theta}, \hat{\Lambda}^u, \hat{\Lambda}^v, \hat{\kappa}^u, \hat{\kappa}^v\}$. □

5. The moduli space of deformations

In this section we finally compute the moduli space of deformations of a nowhere flat hyperbolic Euclidean submanifold f by using the six invariants found in the last section. Recall that, by Proposition 4 and Proposition 5, if such a deformation is not hyperbolic on some open subset, then it is somewhere a composition and, in particular, it is not a honest deformation.

Let $f : M^n \to \mathbb{R}^{n+2}$ be a hyperbolic submanifold with main angle θ and main symbols Λ^u and Λ^v, as in (34). Fix $p_0 = (u_0, v_0) \in L^2$, and fix primitives of Λ^u and Λ^v with respect to u and v respectively, that we denote as $\int \Lambda^u du := \int_{u_0}^u \Lambda^u(t, v)dt + \ln(s(u_0, v))/2$, and $\int \Lambda^v dv := \int_{v_0}^v \Lambda^u(u, t)dt + \ln(s(u, v_0))/2$. In terms of a polar surface of f we get from (35) that $\int \Lambda^u du = \ln(s)/2 + \int_{u_0}^u (FG^{-1}\Gamma^u)(t, v)dt$ and $\int \Lambda^v dv = \ln(s)/2 + \int_{v_0}^v (FE^{-1}\Gamma^v)(u, t)dt$.

Let $U = U(u)$ and $V = V(v)$ be a pair of functions of a real variable such that

$$
\begin{aligned}
&U > -e^{-2\int \Lambda^v dv}, \quad V > -e^{-2\int \Lambda^u du}, \\
&(1 + Ue^{2\int \Lambda^v dv})(1 + Ve^{2\int \Lambda^u du}) > \cos^2(\theta).
\end{aligned} \tag{40}
$$

Using these two functions define τ^u and τ^v over L^2 as

$$
\tau^u := 1 + Ve^{2\int \Lambda^u du} > 0, \qquad \tau^v := 1 + Ue^{2\int \Lambda^v dv} > 0. \tag{41}
$$

Finally, define the second-order differential operator $H_{UV} : C^\infty(L^2) \to C^\infty(L^2)$ by

$$
H_{UV}(\rho) = \rho_{uv} + \frac{\cos^2(\theta)}{\tau^u \tau^v} \rho_u \, \rho_v - \frac{\Lambda^v}{\tau^v} \rho_u - \frac{\Lambda^u}{\tau^u} \rho_v. \tag{42}
$$

Our main result can be stated as follows. Observe that, as a consequence, the moduli space of hyperbolic deformations of f only depends on Λ^u, Λ^v and θ, that are functions over the nullity leaf space $L^2 = M^n/\Gamma$. In particular, this moduli space only depends on the metric of the polar surface of f.

Theorem 21. *Let $f : M^n \to \mathbb{R}^{n+2}$ be a nowhere flat hyperbolic submanifold. Then, the moduli space of local hyperbolic isometric immersions of M^n in codimension two can be represented as*

$$
\mathcal{D}_f = \{(U, V) : (40) \text{ holds, and } H_{UV}(\rho_{UV}) = H_{00}(\rho_{00})\},
$$

where $\rho_{UV} = \frac{1}{2} \ln \left(\frac{\tau^u \tau^v}{\cos^2(\theta)} - 1 \right)$.

Proof. Given a hyperbolic deformation $\hat{f} : M^n \to \mathbb{R}^{n+2}$, we have by the invariance of $\mathcal{C}_1^u, \mathcal{C}_1^v$ in Proposition 20 that $\hat{\Lambda}^u = \Lambda^u/\tau^u$ and $\hat{\Lambda}^v = \Lambda^v/\tau^v$. This together with the invariance of $\mathcal{C}_2^u$ and $\mathcal{C}_2^v$ yields that τ^u and τ^v satisfy the uncoupled PDEs

$$
\tau^u_u = 2\Lambda^u(\tau^u - 1), \qquad \tau^v_v = 2\Lambda^v(\tau^v - 1), \tag{43}
$$

which are equivalent to the existence of a pair of functions of one real variable $U(u)$ and $V(v)$ satisfying (41). Furthermore, we have by the invariance of $\mathcal{G}$

that

$$\cos(\hat{\theta}) = \cos(\theta)/\sqrt{\tau^u\tau^v}, \tag{44}$$

(40) holds, and $\hat{\rho} := \ln(|\tan(\hat{\theta})|) = \rho_{UV}$ as in the statement. Thus, the invariance of $\mathcal{R}$ is equivalent to $H_{UV}(\rho_{UV}) = H_{00}(\rho_{00})$, since f itself corresponds to $U = V = 0$.

Conversely, if $(U(u), V(v)) \in \mathcal{D}_f$, then (43) clearly holds for $\tau^u > 0$, $\tau^v > 0$ given by (41) and consequently $\hat{\kappa}^u = \tau^u\kappa^u$, $\hat{\kappa}^v = \tau^v\kappa^v$, $\hat{\Lambda}^u = \Lambda^u/\tau^u$, $\hat{\Lambda}^v = \Lambda^v/\tau^v$ satisfy $\mathcal{C}_1^u = \hat{\kappa}^u\hat{\Lambda}^u$, $\mathcal{C}_1^v = \hat{\kappa}^v\hat{\Lambda}^v$, $\mathcal{C}_2^u = \hat{\kappa}^u_u/\hat{\kappa}^u + 2\hat{\Lambda}^u$, and $\mathcal{C}_2^v = \hat{\kappa}^v_v/\hat{\kappa}^v + 2\hat{\Lambda}^v$. Moreover, $\hat{\theta} = \arccos(\cos(\theta)/\sqrt{\tau^u\tau^v})$ automatically satisfies that $\mathcal{G} = \cos(\hat{\theta})\sqrt{\hat{\kappa}^u\hat{\kappa}^v}$. Lastly, setting $\hat{\rho} = \ln(|\tan(\hat{\theta})|) = \rho_{UV}$, the assumption $H_{UV}(\rho_{UV}) = H_{00}(\rho_{00})$ gives us $\mathcal{R} = \hat{\rho}_{uv} + \cos^2(\hat{\theta})\,\hat{\rho}_u\,\hat{\rho}_v - \hat{\Lambda}^v\hat{\rho}_u - \hat{\Lambda}^u\hat{\rho}_v$. Therefore, we conclude from Proposition 20 that there exists locally an isometric immersion

$$\hat{f} : M^n \to \mathbb{R}^{n+2} \quad \text{with} \;\; \mathcal{S}(\hat{f}) = \{\hat{\theta}, \hat{\Lambda}^u, \hat{\Lambda}^v, \hat{\kappa}^u, \hat{\kappa}^v\}.$$

□

Remark 22. Depending on the problem, it may be useful to change the operator in (42) by composing ρ with a suitable function. For example, using the linear operator $\tilde{H}_{UV}(\rho) = \rho_{uv} - \frac{\Lambda^v}{\tau^v}\rho_u - \frac{\Lambda^u}{\tau^u}\rho_v$, we get in terms of $2\tilde{\rho}_{UV} = \ln(1+\sin(\hat{\theta})) - \ln(1-\sin(\hat{\theta}))$ that $\mathcal{D}_f = \{(U,V) : (40)$ holds, and $\tilde{H}_{UV}(\tilde{\rho}_{UV}) = \tanh(\tilde{\rho}_{UV})\tilde{H}_{00}(\tilde{\rho}_{00})/2\sin(\theta)\}$.

6. Honest and genuine deformations

After classifying in the previous section the substantial rank two deformations of a hyperbolic Euclidean submanifold in codimension two, we now determine when such a deformation is a composition, when it is genuine, and therefore when it is honest.

To describe the type of deformations of a rank two isometric immersion we first need the following result.

Lemma 23. *Let $f, \hat{f} : M^n \to \mathbb{R}^{n+2}$ be a pair of nowhere congruent and nowhere flat rank two isometric immersions neither of which is contained in an affine hyperplane. Then, there is an open dense subset $W \subset M^n$ along which f and $\hat{f}$ have two dimensional first normal spaces, and either one of the following possibilities occur on each connected component of W:*

1) *There are no unit normal vector fields* μ *of* f *and* $\hat{\mu}$ *of* $\hat{f}$ *whose respective shape operators coincide. In this case, the pair* $\{f, \hat{f}\}$ *is genuine;*

2) *There are orthonormal normal frames* $\{\mu, \beta\}$ *of* f *and* $\{\hat{\mu}, \hat{\beta}\}$ *of* $\hat{f}$ *such that* $A_\mu = \hat{A}_{\hat{\mu}}$ *and* $\operatorname{rank} A_\beta = \operatorname{rank} \hat{A}_{\hat{\beta}} = k$*, with* $1 \leq k \leq 2$*. In this case,* $\{f, \hat{f}\}$ *extends isometrically as (regular) Sbrana-Cartan hypersurfaces if* $k = 2$*, or as either flat or (singular) generalized Sbrana-Cartan hypersurfaces if* $k = 1$*. In addition, when* $k = 1$ *and* f *and* $\hat{f}$ *are not mutually ruled, they extend isometrically as flat hypersurfaces if and only if* $\ker A_\beta \subset \ker \psi$*, where* $\psi = \langle \nabla^\perp_\bullet \mu, \beta \rangle$ *is the normal connection form of* $\{\mu, \beta\}$*.*

Proof. The dimension property of the first normal spaces is a consequence of Proposition 5 and the discussion before it. In particular, if the second fundamental forms of f and $\hat{f}$ coincide in an open subset, then by the Codazzi equation also their normal connections agree. Hence, the immersions would be congruent along any simply-connected open subset in U. Thus, the second fundamental forms are almost everywhere different.

Case (1) is obvious, since a normal vector field to the submanifold tangent to the isometric extension gives a normal direction where the shape operators coincide. So assume that there are unit normal vector fields μ and $\hat{\mu}$ such that $A_\mu = \hat{A}_{\hat{\mu}}$, and complete them to orthonormal normal frames $\{\mu, \beta\}$ of f and $\{\hat{\mu}, \hat{\beta}\}$ of $\hat{f}$. In particular, $1 \leq \operatorname{rank} A_\beta = \operatorname{rank} \hat{A}_{\hat{\beta}} \leq 2$ by the Gauss equation. The smoothness of this frame is assured by Lemma 7 in [13].

If $\operatorname{rank} A_\beta = \operatorname{rank} \hat{A}_{\hat{\beta}} = 2$, Lemma 6 in [13] (or Proposition 9 in [7]) says that case (2) holds with regular Sbrana-Cartan hypersurfaces as extensions.

Finally, assume that $\operatorname{rank} A_\beta = \operatorname{rank} \hat{A}_{\hat{\beta}} = 1$. Lemma 9 in [13] assures that the pair extends as generalized Sbrana-Cartan hypersurfaces, unless $\ker A_\beta \subset \ker \psi$, which is equivalent for $A_\mu = \hat{A}_{\hat{\mu}}$ to be a Codazzi tensor by the Codazzi equation for A_μ. Hence, $\ker A_\beta \subset \ker \psi$ is also a necessary condition for f to extend as a flat hypersurface. In particular, $\ker \hat{A}_{\hat{\beta}} \subset \ker \hat{\psi}$ also. We claim that, in this situation, f and $\hat{f}$ extend isometrically as flat hypersurfaces.

To prove the claim, we have to consider two cases:

(i) A_β and $\hat{A}_{\hat{\beta}}$ are pointwise linearly dependent. Define the pair $(\mathcal{T}, D)$, where $D := \ker A_\beta$, and $\mathcal{T}$ is the line bundle isometry that sends μ into $\hat{\mu}$. We conclude from Proposition 9 in [7] applied to $(\mathcal{T}, D)$ that f and $\hat{f}$ extend isometrically as hypersurfaces with common relative nullity of dimension $n = 1 + \dim D$, hence flat.

(ii) A_β and $\hat{A}_{\hat{\beta}}$ are pointwise linearly independent. Notice that this implies also that $\ker A_\beta \subset \ker \psi$ by comparing the Codazzi equations for $A_\mu = \hat{A}_{\hat{\mu}}$. Now, we proceed as in (i) defining $(\mathcal{T}, D)$, but now for $D := \ker A_\beta \cap \ker \hat{A}_{\hat{\beta}} = \Delta$. By dimension reasons, $\mathcal{N}(\phi) \supsetneq D$ for the bilinear form ϕ defined in equation (3) in [7], and again by Proposition 9 in [7] f and $\hat{f}$ extend isometrically by relative nullity. Thus, their extensions have rank one, hence flat. $\square$

Notice that, when $k = 1$ in case (2) above, the condition $\ker A_\beta \subset \ker \psi$ is equivalent for f to be a composition. In fact, if e is an eigenvector field of A_β with $A_\beta e = \lambda e \neq 0$, then $\psi = \gamma\langle e, \cdot\rangle$ for some function γ and it is easy to check that $F(t, x) = f(x) + t(\gamma\lambda^{-1}e + \mu)(x)$ is an immersion with relative nullity distribution $\{\partial_t\} \oplus \ker A_\beta$. In particular F has rank one, so it is flat.

Remark 24. By Proposition 4, Proposition 5 and the above, if M^n is a rank two hyperbolic Riemannian manifold, then an isometric immersion $f : M^n \to \mathbb{R}^{n+2}$ is a composition if and only if either $\dim N_f^1 \neq 2$, or $\nu_f \neq n-2$, or $\ker A_{\xi_i} \subset \ker \psi^i$, for some $i = 1, 2$. Observe that, by (6), (30) and Proposition 12, in terms of a polar surface of f the latter is equivalent to either $\Gamma^u = 0$ or $\Gamma^v = 0$.

Let $f : M^n \to \mathbb{R}^{n+2}$ be a nowhere flat hyperbolic submanifold with main angle θ and main symbols Λ_u, Λ_v as in (34) (or as in (35) in terms of a polar surface g of f). In view of Remark 24 and (9), the condition for f to be a composition is that either $2s\Lambda^u = s_u$, or $2s\Lambda^v = s_v$. In particular, a hyperbolic deformation $\hat{f}$ of f is a composition if and only if either $2\hat{\Lambda}^u = \ln(\sin^2(\hat{\theta}))_u$ or $2\hat{\Lambda}^v = \ln(\sin^2(\hat{\theta}))_v$. But

$$
\begin{aligned}
2\hat{\Lambda}^u - \ln(\sin^2(\hat{\theta}))_u &= \frac{2\Lambda^u}{\tau^u} - \ln\left(1 - \frac{1-s}{\tau^u\tau^v}\right)_u \\
&= -\frac{1}{\tau^u\tau^v - 1 + s}\left(s_u - 2\Lambda^u(\tau^v - 1 + s) + (1-s)\frac{\tau^v_u}{\tau^v}\right),
\end{aligned}
$$

and similarly for $2\hat{\Lambda}^v - \ln(\sin^2(\hat{\theta}))_v$. By (43) we conclude that $\hat{f}$ is a composition if and only if

$$
(45) \qquad \left(\frac{\tau^v - 1 + s}{\tau^v\, e^{2\int \Lambda^u du}}\right)_u = 0 \qquad \text{or} \qquad \left(\frac{\tau^u - 1 + s}{\tau^u\, e^{2\int \Lambda^v dv}}\right)_v = 0.
$$

We have the following criteria to extend isometrically both immersions as flat hypersurfaces.

Proposition 25. *Let $f : M^n \to \mathbb{R}^{n+2}$ be a nowhere flat hyperbolic submanifold and $\hat{f}$ a hyperbolic deformation of f given by $(U, V) \in \mathcal{D}_f$. Then, in terms of a polar surface of f, the pair $\{f, \hat{f}\}$ extends isometrically as flat hypersurfaces if and only if either $\Gamma^u = U = 0$, or $\Gamma^v = V = 0$, or $\Gamma^u = V + 1 = 0$, or $\Gamma^v = U + 1 = 0$. The latter two cases are equivalent to $\tau^u = \cos^2(\theta)$ and $\tau^v = \cos^2(\theta)$, respectively.*

Proof. According to Remark 24 and (45), a necessary condition to extend as flat hypersurfaces is that either $\Gamma^u = 0$ or $\Gamma^v = 0$, and either $\left(\frac{\tau^v - 1 + s}{\tau^v\, e^{2\int \Lambda^u du}}\right)_u = 0$ or $\left(\frac{\tau^u - 1 + s}{\tau^u\, e^{2\int \Lambda^v dv}}\right)_v = 0$. In order for the extensions to be isometric, we also need that the corresponding second fundamental forms agree up to sign, as in Lemma 23 part (2). So we have two possibilities, up to obvious index choices:

(*i*) $A_{\eta_1} = \pm \hat{A}_{\hat{\eta}_1}$, $\Gamma^u = 0$, $\left(\frac{\tau^v - 1 + s}{s\tau^v}\right)_u = 0$. Here, from (7) we easily obtain that the first equation is equivalent to $\tau^v = 1$, that is, $U = 0$, which implies the third equation.

(*ii*) $A_{\eta_1} = \pm \hat{A}_{\hat{\eta}_2}$, $\Gamma^u = 0$, $\left(\frac{\tau^u - 1 + s}{\tau^u\, e^{2\int \Lambda^v dv}}\right)_v = 0$. In this case, the first equation is equivalent to $\tau^u = \cos^2(\theta)$ and then, by the second, $V = -1$ and so the third holds.

In any case, $\Gamma^u = 0$, $\ker A_{\xi_i} \subset \ker \psi^i$ by Remark 24, and the proposition follows from Lemma 23 part (2) since f is not parabolic. □

Our next principal result describes which deformations are genuine and honest, and how the ones that are not extend:

Theorem 26. *Consider two nowhere congruent nowhere flat hyperbolic isometric immersions $f, \hat{f} : M^n \to \mathbb{R}^{n+2}$ which do not extend isometrically as flat hypersurfaces. Then, $\hat{f}$ is locally determined by a pair $(U, V) \in \mathcal{D}_f$, and it holds that:*

1) *If $UV > 0$, then $\{f, \hat{f}\}$ is genuine, and $\hat{f}$ is honest if in addition* (45) *does not hold;*
2) *If $UV = 0$, then the pair $\{f, \hat{f}\}$ extends isometrically in a unique way, and they do so as generalized (singular) Sbrana-Cartan hypersurfaces. In particular, $\{f, \hat{f}\}$ is genuine;*
3) *If $UV < 0$, then the pair $\{f, \hat{f}\}$ extends isometrically in precisely two different ways, and they do so as (regular) Sbrana-Cartan hypersurfaces. In particular, f and $\hat{f}$ are constructed as the intersection of two pairs of isometric Sbrana-Cartan hypersurfaces.*

Moreover, in cases (2) and (3), all the Sbrana-Cartan extensions are of continuous or discrete class.

Proof. Since the immersions are hyperbolic, neither is contained in an affine hyperplane. Moreover, as we already saw, the splitting tensor is intrinsic and we have that $\hat{A}_{\hat{\xi}_i}$ and A_{ξ_i} are linearly dependent for $i = 1, 2$. According to Lemma 23, we have to analyze when there are normal directions μ and $\hat{\mu}$ with the same norm for which the corresponding shape operators coincide. Assume this is the case, i.e.,

$$A_\mu = \hat{A}_{\hat{\mu}}, \tag{46}$$

and set $\mu = a_1\xi_1 + a_2\xi_2 \neq 0$, $\hat{\mu} = \hat{a}_1\hat{\xi}_1 + \hat{a}_2\hat{\xi}_2 \neq 0$, with

$$a_1^2 + a_2^2 + 2a_1a_2\cos(\theta) = \hat{a}_1^2 + \hat{a}_2^2 + 2\hat{a}_1\hat{a}_2\cos(\hat{\theta}). \tag{47}$$

Evaluating (46) in $0 \neq Z_i \in \ker A_{\xi_i} \cap \Delta_f^\perp$, we have

$$a_i\lambda_i = \hat{a}_i\hat{\lambda}_i, \quad i = 1, 2. \tag{48}$$

First observe that, if $\lambda_1 = \hat{\lambda}_1$ and $\lambda_2 = \hat{\lambda}_2$, from (8) we conclude that $\theta = \hat{\theta}$, and thus f and $\hat{f}$ would be congruent. Hence assume that, say, $\lambda_2 \neq \hat{\lambda}_2$, which also implies that $a_1 \neq 0$ and $\hat{a}_1 \neq 0$ in view of (48). Now, dividing (47) by $\hat{a}_1^2$ and using (48) we get for $t := -\hat{a}_2/\hat{a}_1$ and $k := \sin^2(\hat{\theta})/\sin^2(\theta)$ that

$$t^2(k\tau^v - 1) - 2t(k\cos(\theta)\sqrt{\tau^u\tau^v} - \cos(\hat{\theta})) + (k\tau^u - 1) = 0.$$

This is a second degree polynomial since $\lambda_2 \neq \hat{\lambda}_2$. In view of (44) this is equivalent to

$$h(t) := t^2(k\tau^v - 1) - 2t\cos(\hat{\theta})(k\tau^u\tau^v - 1) + (k\tau^u - 1) = 0, \tag{49}$$

whose discriminant with respect to t is $-4k(\tau^u - 1)(\tau^v - 1)$, which by (41) has the same sign as $-UV$. Case (1) is then a consequence of case (1) in Lemma 23.

For case (2), assume that $U = 0$, and hence $\tau^v = 1$. Then $t = \cos(\hat{\theta})^{-1}$ is the only root of (49). Therefore, η_1 and $\hat{\eta}_1$ are the only directions for which the shape operators coincide. But these are precisely the directions orthogonal to ξ_1 and $\hat{\xi}_1$, whose shape operators have rank one. We conclude case (2) from case (2) in Lemma 23 for $k = 1$.

For case (3), we have two different roots in (49), and we claim that neither is equal to $\cos(\hat{\theta})^{-1}$ or $\cos(\hat{\theta})$. To prove this, first we easily check that

$$
\begin{aligned}
h(\cos(\hat{\theta})^{-1}) &= \frac{k\tau^u(\tau^v - 1)}{\cos^2(\theta)}(\tau^v - \cos^2(\theta)),\\
h(\cos(\hat{\theta})) &= \frac{k(\tau^u - 1)}{\tau^u}(\tau^u - \cos^2(\theta)).
\end{aligned}
\tag{50}
$$

Since $UV \neq 0$, then $\tau^u, \tau^v \neq 1$. But if, say, $\tau^v = \cos^2(\theta)$, in view of (35) and (41) the function $e^{-2\int FE^{-1}\Gamma^v dv} = -U$ does not depend on v. Since M^n is nowhere flat, $F \neq 0$ and therefore $\Gamma^v = 0$ and $U = -1$. By Proposition 25 both f and $\hat{f}$ extend isometrically as flat hypersurfaces contradicting our hypothesis, and the claim is proved. The proof of case (3) now follows from the discussion for case (2) and case (2) in Lemma 23 for $k = 2$, since the claim is equivalent to the fact that μ and $\hat{\mu}$ are not collinear with η_i and $\hat{\eta}_i$, $i = 1, 2$.

Now we argue that all the Sbrana-Cartan hypersurfaces that appear are always of continuous or discrete class, that is, the 'interesting' classes III and IV in Theorem 3 in [9]. First, recall that the extensions are by relative nullity, and then the relative nullity of the codimension two hyperbolic submanifold f is contained in the relative nullity of its extension F, which also has rank two. In particular, the splitting tensor $\tilde{C}_T$ of the relative nullity of F for $T \in \Delta_f \subset \Delta_F$ is conjugate to C_T. Since there is $T \in \Delta_f$ such that C_T has two different real eigenvalues, the same holds for $\tilde{C}_T$, and hence, according to Theorem 3 in [9] for $c = 0$, the extension is of continuous or discrete class. □

Remark 27. When $UV < 0$ as in case (3), an interesting and unusual phenomenon occurs. First, notice that, generically, intersections of rank two hypersurfaces only provide rank 4 submanifolds. Yet, f has rank two and is constructed as the transversal intersection of a pair of non-isometric Sbrana-Cartan hypersurfaces $L_1^{n+1}, L_2^{n+1} \subset \mathbb{R}^{n+2}$, while $\hat{f}$ is the transversal intersection of their respective deformations $\hat{L}_1^{n+1}, \hat{L}_2^{n+1} \subset \mathbb{R}^{n+2}$. That is,

$$
f(M^n) = L_1^{n+1} \cap L_2^{n+1} \qquad \text{and} \qquad \hat{f}(M^n) = \hat{L}_1^{n+1} \cap \hat{L}_2^{n+1}. \tag{51}
$$

In particular, although not a honest deformation, this provides examples of interesting Sbrana-Cartan hypersurfaces of the continuous or discrete classes. The lesson we extract from this is not to disregard non genuine deformations, but instead use them to study deformability in lower dimensions. We will see more examples of this kind of phenomena in the last two sections.

Remark 28. As shown in its proof, when the pair $\{f, \hat{f}\}$ extends isometrically as flat hypersurfaces, Theorem 26 still holds except in the following two situations:

- In case (2), they may extend isometrically in a unique way but as flat hypersurfaces instead of singular Sbrana-Cartan hypersurfaces, yet if and only if either $U = \Gamma^u = 0$ or $V = \Gamma^v = 0$, as seen in Proposition 25;
- In case (3), they extend isometrically as flat hypersurfaces only if either $\Gamma^v = 0$ and $U = -1$, or $\Gamma^u = 0$ and $V = -1$, which correspond to $\cos(\hat{\theta})^{-1}$ or $\cos(\hat{\theta})$ to be roots of (49), respectively. But both cannot be roots simultaneously, since otherwise by (50) we would have $\tau^u = \tau^v = \cos^2(\theta)$, which contradicts the third condition in (40). We conclude that $\{f, \hat{f}\}$ extends isometrically also as (regular) Sbrana-Cartan hypersurfaces. In other words, (51) still holds, but with one of the pairs $L_i^{n+1}, \hat{L}_i^{n+1}$ being flat, for some $i = 1, 2$.

As shown in the Examples in [8] page 207, the singular set Σ^n of a hyperbolic Sbrana-Cartan hypersurface $F : N^{n+1} \to \mathbb{R}^{n+2}$ is always a deformable rank two hyperbolic Euclidean submanifold in codimension two, and N^{n+1} itself can be recovered from Σ^n. On the other hand, as a consequence of Theorem 26 and Remark 28, we have that no pair $\{f, \hat{f}\}$ can extend simultaneously both singularly and regularly. This answers positively the natural question that was left open in [13], namely, whether it is actually necessary to consider singular extensions to obtain global genuine rigidity. Indeed, if F' is a deformation of F, for any compact hypersurface $M^n \subset N^{n+1}$, we conclude that $F(M^n) \subset \mathbb{R}^{n+2}$ and $F'(M^n) \subset \mathbb{R}^{n+2}$ are nowhere congruent, yet they can only extend singularly along the interior of $M^n \cap \Sigma^n$. Therefore, we have:

Corollary 29. *The global genuine rigidity for compact Euclidean submanifolds in codimension two as established in [13] does not hold without considering singular extensions.*

7. Hyperbolic submanifolds as hypersurfaces

In the recent paper [11] the moduli space of all (local) isometric immersions $f : M^n \to \mathbb{R}^{n+2}$ of a given Euclidean hypersurface $g : M^n \to \mathbb{R}^{n+1}$ that are not compositions was computed. We can use the machinery built in this work to understand the converse problem: to classify rank two Euclidean submanifolds in codimension two that are also hypersurfaces, and actually classify all their deformations. We will carry out the study for hyperbolic submanifolds since these are the ones that interest us in this paper, but, as we pointed out in the introduction, similar analysis holds for the elliptic ones just by taking complex conjugate coordinates instead of real ones, as done in [9] and [11].

Let $f : M^n \to \mathbb{R}^{n+2}$ be a simply-connected nowhere flat hyperbolic submanifold. In order to find an isometric immersion of M^n as a Euclidean hypersurface we will use the Fundamental Theorem of Submanifolds by constructing a self-adjoint endomorphism A on TM that satisfies the Gauss and Codazzi equations. Since M^n has rank two, so does A, and $\Delta_f = \ker A$. Since A_{ξ_1}, A_{ξ_2} form a basis of the self-adjoint tensors that satisfy (4), we have that $A = a_1 A_{\xi_1} + a_2 A_{\xi_2}$, where we can assume that $a_1 > 0$. By (7) and (8), the Gauss equation for A reduces to $a_1 a_2 = -\cos(\theta)/\sin^2(\theta)$. So defining $\mu = a_1^2$ we have

$$A = \sqrt{\mu}\, A_{\xi_1} - \frac{\cos(\theta)}{\sin^2(\theta)\sqrt{\mu}}\, A_{\xi_2}.$$

Using the notation $DB(X,Y) := \nabla_X BY - \nabla_Y BX - B[X,Y]$ for (1,1) tensors, and $\hat{w}(X,Y) := w(X)Y - w(Y)X$ for 1-forms, we have that the Codazzi equation for A is simply $DA = 0$. So,

$$\frac{1}{2}\, A_{\xi_1}\hat{d\mu} + \mu\, DA_{\xi_1} + A_{\xi_2}\left(\frac{\cos(\theta)}{2\sin^2(\theta)\mu}\hat{d\mu} + \frac{1+\cos^2(\theta)}{\sin^3(\theta)}\hat{d\theta}\right) - \frac{\cos(\theta)}{\sin^2(\theta)}\, DA_{\xi_2} = 0.$$

Recall that the Codazzi equation for A_{ξ_i} is $\sin(\theta) DA_{\xi_i} = (-1)^j (A_{\xi_j} - \cos(\theta) A_{\xi_i})\hat{\psi}^i$, for $1 \le i \ne j \le 2$. Hence, since the images of A_{ξ_1} and A_{ξ_2} are linearly independent we get

$$A_{\xi_1}\left(\frac{1}{2}\hat{d\mu} - \mu\frac{\cos(\theta)}{\sin(\theta)}\hat{\psi}^1 + \frac{\cos(\theta)}{\sin^3(\theta)}\hat{\psi}^2\right) = 0,$$

$$A_{\xi_2}\left(\frac{1}{2\mu}\hat{d\mu} + \mu\frac{\sin(\theta)}{\cos(\theta)}\hat{\psi}^1 - \frac{\cos(\theta)}{\sin(\theta)}\hat{\psi}^2 + \frac{1+\cos^2(\theta)}{\cos(\theta)\sin(\theta)}\hat{d\theta}\right) = 0.$$

Observe first that these equations for one vector in Δ and the other in $\Delta^\perp$ say that μ is projectable, since ψ^1, ψ^2 and θ also are. In view of (9), we obtain that the above two equations are equivalent to the first order system of PDE

$$\mu_u = \mu a - b, \quad \mu_v = \mu(\mu c - d), \tag{52}$$

where, as usual, $s = \sin^2(\theta)$, and

$$a = 2\Lambda^u - \frac{s_u}{s}, \quad b = \frac{2\Lambda^u}{s}, \quad c = \frac{2s\Lambda^v}{1-s}, \quad d = 2\Lambda^v + \frac{s_v}{s(1-s)}.$$

The integrability condition of (52) is therefore

$$P(\mu) := (ac + c_u)\mu^2 - (2bc + d_u + a_v)\mu + (bd + b_v) = 0. \tag{53}$$

We point out for further reference that

$$\begin{aligned} ac + c_u &= \frac{2s}{1-s}\left(\Lambda^v_u + 2\Lambda^u\Lambda^v + \Lambda^v\frac{s_u}{1-s}\right), \\ 2bc + d_u + a_v &= 2\Lambda^v_u + 2\Lambda^u_v + \frac{1}{1-s}\left(8\Lambda^u\Lambda^v + s_{uv} + \frac{s_u s_v}{1-s}\right), \\ bd + b_v &= \frac{2}{s}\left(\Lambda^u_v + 2\Lambda^u\Lambda^v + \Lambda^u\frac{s_v}{1-s}\right). \end{aligned}$$

We conclude that one and only one of the following possibilities, enumerated from the least to the most generic, holds along each connected component of an open dense subset of M^n:

i) $P = 0$, that is, $a = -\ln(|c|)_u, d = -\ln(|b|)_v$ and $2bc = \ln(|bc|)_{uv}$, in which case the manifold admits an isometric immersion as a Sbrana-Cartan hypersurface of the continuous class;

ii) Equation (53) has two positive roots, and both satisfy (52), in which case the manifold admits an isometric immersion as a Sbrana-Cartan hypersurface of the discrete class;

iii) Only one of the positive roots of (53) satisfies (52), in which case the manifold admits an isometric immersion as a rigid hypersurface;

iv) No positive root of (53) satisfies (52), hence the manifold admits no isometric immersion as a hypersurface. In particular, this is the case if $(2bc + d_u + a_v)^2 < 4(ac + c_u)(bd + b_v)$, or $2bc + d_u + a_v \leq 0$, $bd + b_v \geq 0$, $ac + c_u \geq 0$, or $2bc + d_u + a_v \geq 0$, $bd + b_v \leq 0$, $ac + c_u \leq 0$.

8. Deformations preserving the main angle

This and the following section are devoted to give some applications. The purpose in this one is to describe a particularly interesting class of deformations of a hyperbolic submanifold: the ones preserving the main angle θ. Although we will see that these deformations are never honest nor genuine, they provide interesting applications to the Sbrana-Cartan theory of deformable hypersurfaces. This justifies what we pointed out in the introduction: one should not simply ignore the study of non honest and non genuine deformations since they can provide insights for lower codimension rigidity.

So, we study here the implications of Theorem 21 on the structure of a hyperbolic nowhere flat submanifold $f : M^n \to \mathbb{R}^{n+2}$ admitting a hyperbolic deformation $\hat{f}$ with $\hat{\theta} = \theta$. We know that $\hat{f}$ is determined by $(U, V) \in \mathcal{D}_f$, and by (44) the condition which characterizes these deformations is simply that

$$\tau := \tau^u = 1/\tau^v. \tag{54}$$

Observe that $UV < 0$ since $(\tau^u - 1)(\tau^v - 1) = -(\tau - 1)^2/\tau < 0$, and $\tau \neq 1$ since the immersions are not congruent. Hence, in view of Remark 28, Theorem 26 holds and f and $\hat{f}$ extend isometrically as Sbrana-Cartan hypersurfaces in two different ways as in Remark 27, unless $\tau = \cos^2(\theta)$ or $\tau = \cos^{-2}(\theta)$, in which case f and $\hat{f}$ extend isometrically both as flat and Sbrana-Cartan hypersurfaces in a unique way. In any case, $\hat{f}$ is never a genuine deformation and, in fact, $\pm\sqrt{\tau}$ are the two real roots of (49). Moreover, using (54) in (41) we obtain that $V = (\tau - 1)\, e^{-2\int \Lambda^u du}$, and $U = (\tau^{-1} - 1)\, e^{-2\int \Lambda^v dv}$ depend on one variable only. Equivalently,

$$\tau_u = 2\Lambda^u(\tau - 1), \quad \tau_v = 2\Lambda^v \tau(\tau - 1). \tag{55}$$

Remark 30. Observe that system (55) is exactly the system that appears in the Sbrana-Cartan theory, but now for the Euclidean polar surface of f instead of the spherical Gauss map of its extension. Its integrability condition is also given by

$$\tau(\Lambda^v_u + 2\Lambda^u\Lambda^v) = \Lambda^u_v + 2\Lambda^u\Lambda^v, \tag{56}$$

as in the Sbrana-Cartan theory.

To compute $\mathcal{D}_f$ we have $\rho_{00} = \rho_{UV} = \ln(|\tan(\theta)|)$ and thus

$$0 = H_{UV}(\rho_{UV}) - H_{00}(\rho_{00}) = (1-\tau)\Lambda^v(\rho_{00})_u + (1-\tau^{-1})\Lambda^u(\rho_{00})_v.$$

Since $\tau \neq 1$ this is equivalent to

$$\tau\Lambda^v s_u = \Lambda^u s_v. \tag{57}$$

First, consider this equation under the generic condition $\Lambda^u s_v \neq 0$. Hence we also have $\Lambda^v s_u \neq 0$. We conclude that, in this generic situation, f admits at most one deformation $\hat{f}$ preserving the main angle, depending on whether $1 \neq \tau = \Lambda^u s_v/\Lambda^v s_u > 0$ satisfies system (55) or not.

Let us now turn our attention to the non generic case where (57) trivially holds, i.e.,

$$\Lambda^u s_v = \Lambda^v s_u = 0.$$

Thus $H_{UV}(\rho_{UV}) = H_{00}(\rho_{00})$ is automatically satisfied, and three possibilities may occur:

i) Either $s_u = \Lambda^u = 0$, or $s_v = \Lambda^v = 0$;

ii) $\Lambda^v = \Lambda^u = 0$;

iii) $\Lambda^u \neq 0$, $\Lambda^v \neq 0$, and θ is constant.

Case (*i*). Suppose that, say, $s_u = \Lambda^u = 0$. Then, by (55), τ and Λ^v are functions of v only, $V = \tau - 1$ and U is constant. We conclude from Theorem 21 that, in this situation, a deformation of f preserving θ exists if and only if either $s_u = \Lambda^u = \Lambda^v_u = 0$, or $s_v = \Lambda^v = \Lambda^u_v = 0$, in which case there is actually a one-parameter family of such deformations, one for each constant chosen for U or V, respectively.

Observe that, in this case, (9) also gives $\psi^1(\partial_u) = 0$, and hence $\ker A_{\xi_1} \subset \ker\psi^1$, or analogously $\ker A_{\xi_2} \subset \ker\psi^2$. This implies by Remark 24 that f is a composition. Actually, for such an f the polynomial P in (53) vanishes identically. Therefore, M^n admits also a one-parameter family of isometric deformations as a Euclidean hypersurface, so each such an M^n provides an example of a Sbrana-Cartan hypersurface of the continuous class.

Case (*ii*). Here (55) simply says that $1 \neq \tau > 0$ is constant, $U = \tau^{-1} - 1$, $V = \tau - 1$. We conclude that every member of the class of hyperbolic submanifolds satisfying $\Lambda^v = \Lambda^u = 0$ admits a one-parameter family of deformations preserving the main angle.

Moreover, in this case, P in (53) is $P(\mu) = -\mu(s_{uv}/(1-s) + s_u s_v/(1-s)^2)$ and thus M^n admits no isometric immersion as a Euclidean hypersurface unless $(s-1)s_{uv} = s_u s_v$, in which case M^n is also an example of a Sbrana-Cartan hypersurface of the continuous class.

Case (*iii*). Here, we search for a function τ satisfying (55), whose integrability condition is (56). So, we have two subcases:

If $\Lambda^v_u + 2\Lambda^u\Lambda^v \neq 0$ and $\Lambda^u_v + 2\Lambda^u\Lambda^v \neq 0$ are different and have the same sign, then f admits at most one deformation $\hat{f}$ preserving the main angle, depending on whether $1 \neq \tau = (\Lambda^u_v + 2\Lambda^u\Lambda^v)/(\Lambda^v_u + 2\Lambda^u\Lambda^v) > 0$ satisfies system (55) or not.

If, on the contrary, we have $\Lambda^v_u = \Lambda^u_v = -2\Lambda^u\Lambda^v \neq 0$, this easily implies that $\Lambda^v = \tilde{V}'/2(\tilde{U} + \tilde{V})$ and $\Lambda^u = \tilde{U}'/2(\tilde{U} + \tilde{V})$ for some non-constant one-variable functions $\tilde{U} = \tilde{U}(u)$ and $\tilde{V} = \tilde{V}(v)$. Then, it is easy to check that the pairs (U, V) are given by $U = 1/(c - \tilde{U})$ and $V = -1/(c + \tilde{V})$, for $c \in \mathbb{R}$. Therefore, a one-parameter family of deformations preserving θ always exists in this case. Observe that P in (53) is $P(\mu) = 8\mu\Lambda^u\Lambda^v s/(1-s)$, it has no positive roots, and so M^n is not a Euclidean hypersurface.

Summarizing, we have shown:

Theorem 31. *A nowhere flat hyperbolic submanifold $f : M \to \mathbb{R}^{n+2}$ has a hyperbolic deformation $\hat{f}$ preserving the main angle θ if and only if either one of the following occurs:*

1) $\Lambda^u s_v \neq 0, \Lambda^v s_u \neq 0$ *and the function* $1 \neq \tau = \Lambda^u s_v/\Lambda^v s_u > 0$ *satisfies* (55)*;*
2) θ *is constant,* $\Lambda^v_u + 2\Lambda^u\Lambda^v \neq 0$*,* $\Lambda^u_v + 2\Lambda^u\Lambda^v \neq 0$*, and the function* $1 \neq \tau > 0$ *given by* $\tau = (\Lambda^u_v + 2\Lambda^u\Lambda^v)/(\Lambda^v_u + 2\Lambda^u\Lambda^v)$ *satisfies* (55)*;*
3) *Either* $s_u = \Lambda^u = \Lambda^v_u = 0$*, or* $s_v = \Lambda^v = \Lambda^u_v = 0$*, or* $\Lambda^v = \Lambda^u = 0$*, or* θ *is constant and* $\Lambda^v_u = \Lambda^u_v = -2\Lambda^u\Lambda^v$*.*

Moreover, in cases (1) and (2) f has only one noncongruent such deformation, while in case (3) it has precisely a one-parameter family of them.

In any case, $\{f, \hat{f}\}$ *extend isometrically precisely in two different ways as in* (51)*, both as Sbrana-Cartan hypersurfaces if* $\tau \neq 1-s, (1-s)^{-1}$*, or as Sbrana-Cartan hypersurfaces and flat hypersurfaces otherwise. All these Sbrana-Cartan hypersurface extensions are of the continuous or discrete classes.*

9. Sbrana-Cartan hypersurfaces of intersection type

The main results in [9], Theorems 9 and 11, were devoted to the construction of a large family of Sbrana-Cartan hypersurfaces of the discrete class in any dimension by intersecting two flat hypersurfaces in general position $N_i^{n+1} \subset \mathbb{R}^{n+2}$, that is,

$$M^n = N_1^{n+1} \cap N_2^{n+1} \subset \mathbb{R}^{n+2}. \tag{58}$$

These Sbrana-Cartan hypersurfaces $M^n \subset N_i^{n+1} \subset \mathbb{R}^{n+1}$, which we call here *of intersection type*, are characterized by the fact that their Gauss map satisfies $\Gamma_u^1 - \Gamma^1\Gamma^2 + F = 0$; see Lemma 10 in [9]. It is immediate that these, as submanifolds in codimension two $M^n \subset \mathbb{R}^{n+2}$, are nowhere flat and hyperbolic. We proceed now to easily recover these two main results by using the machinery developed in this work. Moreover, we classify all their deformations in codimension two, finding the first known examples of honestly deformable submanifolds in codimension two.

An equivalent way to understand the Sbrana-Cartan hypersurfaces of intersection type is to consider an embedded nowhere flat hyperbolic submanifold $M^n \subset \mathbb{R}^{n+2}$, and ask for it to extend as flat hypersurfaces in two different ways. As we saw in Remark 24, this is equivalent for its polar surface g to satisfy $\Gamma^u = \Gamma^v = 0$ in (29), that is,

$$g(u, v) = \alpha_1(u) + \alpha_2(v)$$

is the sum of two regular curves, with $\alpha_1', \alpha_1'', \alpha_2', \alpha_2''$ pointwise linearly independent. To avoid surface-like submanifolds, we require also for g to have conformal substantial codimension at least 3. We can further assume that α_1 and α_2 are parametrized by arc-length, i.e., $E = G = 1$. Of course, in this situation $F = \cos(\theta) = \langle \alpha_1', \alpha_2' \rangle$ is the sum of $n + 2$ arbitrary functions whose logarithms separate variables. We also have by (35) that

$$2\Lambda^u = s_u/s, \quad 2\Lambda^v = s_v/s.$$

Observe in addition that $\xi_i = \alpha_j'$, and $(-1)^i \sin(\theta)\eta_i = \cos(\theta)\alpha_j' - \alpha_i'$, $1 \leq i \neq j \leq 2$.

We proceed to recover the main results in [9] for which we use the concept of local shared dimension I of a pair of curves defined at the end of Section 1.

Theorem 32. *Let $i : M^n \subset \mathbb{R}^{n+2}$ be a nowhere flat embedded hyperbolic submanifold, and assume its polar surface g separates variables, i.e., $g(u, v) = \alpha_1(u) + \alpha_2(v)$. Then, M^n is the transversal intersection of two flat hypersurfaces as in* (58). *Moreover, as a hypersurface, $M^n \subset \mathbb{R}^{n+1}$ is a Sbrana-Cartan hypersurface of the discrete class if $I(i) := I(\alpha_1, \alpha_2) \geq 2$, and of the continuous class if $I(i) = 1$.*

Conversely, the polar surface of a Sbrana-Cartan hypersurface of intersection type in codimension two separates variables.

Proof. We have already argued for the converse statement. For the direct one, in our situation, the polynomial P in (53) is

$$P(\mu) = \frac{1}{1-s}\left(s_{uv} + \frac{s_u s_v}{1-s}\right)\left(\mu - \frac{1}{s}\right)\left(\mu + 1 - \frac{1}{s}\right). \tag{59}$$

In particular, $P(\mu) = 0$ for $\mu = 1/s$ and $\mu = 1/s - 1$. In other words, A_{η_i} satisfies the Gauss and Codazzi equations for Euclidean hypersurfaces, $i = 1, 2$. We have two possibilities:

$P = 0$. This is the case when $(1-s)^{-2}((1-s)s_{uv} + s_u s_v) = -(\ln(1-s))_{uv} = 0$, or, equivalently, $F = \cos(\theta) = a(u)b(v)$ is the product of two functions of one variable. Hence, by Lemma 10, $I(i) = 1$. Now, the discussion at the end of Section 7 already implies that $M \subset \mathbb{R}^{n+1}$ is a Sbrana-Cartan hypersurface of the continuous class. But here we can do better and actually solve (52): μ is given by $\mu = \lambda(v) + 1/s$, where λ is any solution of the ODE $\lambda' = \lambda(\lambda + 1)b'/b$ for which $\mu > 0$. Notice also that $\lambda = 0$ and $\lambda = -1$ are two solutions of this ODE, which correspond to the two original intersecting flat hypersurfaces.

$P \neq 0$. In this situation (59) has precisely the two positive solutions just described, $\mu = 1/s, 1/s - 1$, and hence $M^n \subset N_i^{n+1} \subset \mathbb{R}^{n+1}$ are the two unique noncongruent isometric immersions of M^n as a Euclidean hypersurface. □

As another application, we now compute all the honest deformations $\hat{i} : M^n \to \mathbb{R}^{n+2}$ of any Sbrana-Cartan hypersurface of intersection type.

Suppose there is such an isometric immersion $\hat{i}$. Since it is not a composition it has rank two and by Theorem 21 it is induced by $(U, V) \in \mathcal{D}_i$. In this case, $\tau^u = 1 + sV(v)$ and $\tau^v = 1 + sU(u)$, and thus the condition (45) for such an $\hat{i}$ not to be a composition turns out to be

$$U' \neq -U(U+1)\ln(\cos^2(\theta))_u \quad \text{and} \quad V' \neq -V(V+1)\ln(\cos^2(\theta))_v.$$

In particular, we assume that $U, V \neq 0, -1$. Moreover, conditions (40) are simply

$$U, V > -1/s \quad \text{and} \quad (U+1)(V+1) > \cos^2(\theta)UV. \tag{60}$$

Define

$$\varphi = \cos^2(\theta)\tilde{U}\tilde{V}, \quad \text{for} \quad \tilde{U} = \frac{U}{U+1}, \quad \tilde{V} = \frac{V}{V+1}. \tag{61}$$

We have that $\hat{i}$ is not a composition if and only if $\varphi_u \neq 0$ and $\varphi_v \neq 0$. It is easy to check that the equation defining $\mathcal{D}_i$ in Theorem 21 now becomes

$$2\varphi(1-\varphi)\varphi_{uv} + (2\varphi - 1)\varphi_u\varphi_v = 0. \tag{62}$$

According to Theorem 32 and Lemma 10, $M^n \subset \mathbb{R}^{n+1}$ is a Sbrana-Cartan hypersurface of the continuous class if and only if

$$0 = \ln(\cos^2(\theta))_{uv} = \ln(\varphi)_{uv} = \varphi^{-2}(\varphi\varphi_{uv} - \varphi_u\varphi_v).$$

But then (62) reduces to $\varphi_u\varphi_v = 0$. We conclude that the only deformations in codimension two of these Sbrana-Cartan hypersurfaces of the continuous class, for which $I(i) = 1$, are compositions.

So let us concentrate on the discrete class, i.e., $I(i) \geq 2$. If we set $\tilde{\varphi} = \arcsin(2\varphi - 1)$ when $\varphi \in (0,1)$ and $\tilde{\varphi} = \ln(|2\varphi - 1 + 2\sqrt{\varphi(\varphi-1)}|)$ otherwise, (62) is just $\tilde{\varphi}_{uv} = 0$. We claim that $I(i) = 2$. Indeed, if $\varphi \in (0,1)$ there are functions $U_0(u), V_0(v)$ such that $2\varphi = \sin(U_0 + V_0) + 1$, and then

$$2\langle\alpha_1', \alpha_2'\rangle^2 = 2\cos^2(\theta) = (\tilde{U}\tilde{V})^{-1}(\sin(U_0 + V_0) + 1) = (\tilde{U}_1\tilde{V}_1 \pm \tilde{U}_2\tilde{V}_2)^2,$$

for $\tilde{U}_i^2 = (1 + (-1)^i\sin(U_0))/|\tilde{U}|$, $\tilde{V}_i^2 = (1 + (-1)^i\cos(V_0))/|\tilde{V}|$, $i = 1, 2$. A similar computation holds for $\varphi \notin (0,1)$, and the claim follows from Lemma 10. In particular, for $I(i) \neq 2$, i is honestly rigid.

Then, assume from now on that $I(i) = 2$ is constant and, for $j = 1, 2$, denote by $\overline{\alpha}_j$ the orthogonal projection of α_j to the shared plane $\mathbb{V}^2$ between α_1 and α_2. First, we claim that (62) holds for $\tilde{U} = t\|\overline{\alpha}_1'\|^{-2}$, $\tilde{V} = t^{-1}\|\overline{\alpha}_2'\|^{-2}$ and $0 \neq t \in \mathbb{R}$. Indeed, in this situation, $\varphi = \langle e_1, e_2\rangle^2 = \cos^2(w)$, where $w = \angle(e_1, e_2)$ and $e_j = \overline{\alpha}_j'/\|\overline{\alpha}_j'\| \in \mathbb{V}^2 \cong \mathbb{C}$, $j = 1, 2$, lies in the unit circle $\mathbb{S}^1 \subset \mathbb{C}$.

Hence, writing $e_j' = k_j i e_j$ for a function $k_j \neq 0$ since $I(i) = 2$, we have

$$\begin{aligned}&2\varphi(1-\varphi)\varphi_{uv} + (2\varphi - 1)\varphi_u\varphi_v\\ &= 4k_1k_2\cos^2(w)\sin^2(w)(\cos(2w) - 2\varphi + 1) = 0,\end{aligned}$$

and the claim is proved. Notice, in particular, that $(\tilde{U}\tilde{V})^{-1} > \cos^2(\theta)$, since

$$\|\overline{\alpha}_1'\|^2\|\overline{\alpha}_2'\|^2 > \langle\overline{\alpha}_1', \overline{\alpha}_2'\rangle^2 = \langle\alpha_1', \alpha_2'\rangle^2 = \cos^2(\theta). \tag{63}$$

We prove next that these are in fact all the solutions.

To determine all $\tilde{U}, \tilde{V}$ in (61) that satisfy (62), write

$$\tilde{U}(u) = \epsilon_1 e^{U_1(u)}\|\overline{\alpha}_1'(u)\|^{-2} \quad \text{and} \quad \tilde{V}(v) = \epsilon_2 e^{V_1(v)}\|\overline{\alpha}_2'(v)\|^{-2},$$

and thus $\varphi = \epsilon_1\epsilon_2 e^{U_1+V_1}\cos^2(w)$, where $\epsilon_1, \epsilon_2 = \pm 1$. Since (62) is independent under change of parametrizations in u and in v, we can assume that $k_1 = k_2 = 1$ and so $w = u - v \neq 0$. Thus (62) is

$$U_1'V_1' + 2\tan(u-v)(U_1' - V_1') + 4(1 - \epsilon_1\epsilon_2 e^{U_1}e^{V_1}) = 0. \tag{64}$$

We claim that U_1 and V_1 are constant, with $U_1 + V_1 = 0$, and $\epsilon_1 = \epsilon_2$. To prove the claim, observe that we can write (64) as $U_1(u)' = a(u,v)e^{U_1(u)} - b(u,v)$, with $a = 4\epsilon_1\epsilon_2 e^{V_1}/(V_1' + 2\tan(u-v))$, $b = (4 - 2\tan(u-v)V_1')/(V_1' + 2\tan(u-v))$, and similarly for V_1'. Then, $e^{U_1} = b_v/a_v$ does not depend on v, and similarly for V_1. A straightforward computation shows that $(b_v/a_v)_v = 0$ if and only if $\tan(u)(A + B\tan(v)) + (B - A\tan(v)) = 0$, where $A = V_1''' + V_1''V_1'$, and $B = V_1'''V_1'/2 - V_1''(V_1'' + 2)$. This happens only when $A = B = 0$ or, equivalently, if either $V_1'' = 0$ or $V_1'' = -V_1'^2/2 - 2$. Similarly, $U_1'' = 0$ or $U_1'' = -U_1'^2/2 - 2$. It is now easy to verify that the only possibility is that $U_1 = -V_1$ is a constant and $\epsilon_1\epsilon_2 = 1$, as wished.

This claim implies that the set of deformations $\hat{\imath} = i_t$ of i that are not compositions is the one-parameter family

$$\left(U_t = (t^{-1}\|\overline{\alpha}_1'\|^2 - 1)^{-1},\ V_t = (t\|\overline{\alpha}_2'\|^2 - 1)^{-1}\right) \in \mathcal{D}_f$$

satisfying (60). In view of (63), conditions (60) are equivalent to

$$\frac{1}{1 - t^{-1}\|\overline{\alpha}_1'\|^2} < \frac{1}{s}, \ \frac{1}{1 - t\|\overline{\alpha}_2'\|^2} < \frac{1}{s}, \ \text{ and } \ (1 - t^{-1}\|\overline{\alpha}_1'\|^2)(1 - t\|\overline{\alpha}_2'\|^2) > 0.$$

These are obviously satisfied for all $t < 0$. For $t > 0$, it cannot happen simultaneously that $t^{-1}\|\overline{\alpha}_1'\|^2 < 1$ and $t\|\overline{\alpha}_2'\|^2 < 1$ since, by the second equation, $\|\overline{\alpha}_1'\|^2 < t < \cos^2(\theta)\|\overline{\alpha}_2'\|^{-2}$, contradicting (63). On the other hand,

$t^{-1}\|\overline{\alpha}_1'\|^2 > 1$ and $t\|\overline{\alpha}_2'\|^2 > 1$ is not possible either because $\|\overline{\alpha}_1'\|^2 > t > \|\overline{\alpha}_2'\|^{-2}$ contradicts the fact that $\|\overline{\alpha}_j'\|^2 \leq \|\alpha_j'\|^2 = 1$, $j = 1, 2$. We conclude that $t < 0$, and therefore $UV > 0$. That is, by Theorem 26 we obtain that the moduli space of hyperbolic deformations is a connected differentiable 1-parameter family of honest deformations, and therefore i is genuinely and honestly deformable.

It is interesting to analyze the boundary of this family, i.e., $t = 0$ and $t = \infty$. We have in this case that $U = 0, V = -1$ and $U = -1, V = 0$, respectively. These do not satisfy the third condition in (40), yet they give rise to a pair of rank two isometric immersions i_0, i_∞ of M^n that lie inside a hyperplane. Indeed, say for $V = 0, U = -1$, we have at the limit that $\tau^u = \cos^2(\theta)$, $\tau^v = 1$, $\hat{\theta} = 0$, $\hat{\xi}_1 = \hat{\xi}_2$, $\hat{\psi}^i = 0$, $\hat{A}_{\hat{\xi}_i} = 0$ by (39), and in view of (7),

$$\begin{aligned} \hat{A}_{\hat{\eta}_i} &= \lim_{t\to 0} \frac{(-1)^i}{\sin(\hat{\theta})}\left(\cos(\hat{\theta})\hat{A}_{\hat{\xi}_i} - \hat{A}_{\hat{\xi}_j}\right) \\ &= \lim_{t\to 0} \frac{(-1)^i}{\sin(\theta)}\left(\cos(\hat{\theta})\sqrt{\tau^i}A_{\xi_i} - \sqrt{\tau^j}A_{\xi_j}\right) = A_{\eta_1}, \end{aligned}$$

that is a Codazzi tensor. Similarly for $U = 0, V = -1$ we get $\hat{A}_{\hat{\eta}_i} = A_{\eta_2}$. In other words, i_0 and i_∞ are precisely the two unique isometric immersions of M^n as a Euclidean hypersurface. In particular, the two pairs $\{i, i_0\}$ and $\{i, i_\infty\}$ are not genuine since each pair extends isometrically in a unique way, and as flat hypersurfaces.

Summarizing, we have shown:

Theorem 33. *A Sbrana-Cartan hypersurface of intersection type $i : M^n \subset \mathbb{R}^{n+2}$ is honestly rigid, unless $I(i) = 2$. In the latter case, the moduli space of local rank two deformations of i is a differentiable compact connected 1-parameter family $\{i_t : -1 \leq t \leq 1\}$. Moreover, the interior members of this family, i_t for $-1 < t < 1$, are honest deformations of i, while the pair of deformations $i_{\pm 1}$ at its boundary are the only two isometric immersions of M^n as a Euclidean hypersurface. In addition, $\{i, i_{\pm 1}\}$ extend isometrically as flat hypersurfaces, and therefore $i_{\pm 1}$ are not genuine deformations of i.*

Remark 34. It was not known until now if a honestly locally deformable Euclidean submanifold of rank two in codimension two existed at all, since, to our surprise, even the highly degenerate elliptic and parabolic Euclidean submanifolds in codimension two were shown to be honestly rigid in Theorems 1 and 4 in [8]. Now Theorem 33 answers affirmatively this question.

Theorem 33 also shows that different kinds of deformations can be glued smoothly in complex ways. Indeed, by taking two curves α_1, α_2 for which $I(\alpha_1, \alpha_2)$ varies from point to point, we can construct a connected submanifold that has an open dense subset such that each connected component deforms in different ways.

References

[1] C. Allendoerfer, *Rigidity for spaces of class greater than one*, Amer. J. Math. **61** (1939), 633–644.

[2] E. Berger, R. Bryant, and P. Griffith, *The Gauss equations and rigidity of isometric embeddings*, Duke Math. J. **50** (1983), 803–892.

[3] M. do Carmo and M. Dajczer, *Conformal Rigidity*, Amer. J. Math. **109** (1987), 963–985.

[4] E. Cartan, *La déformation des hypersurfaces dans l'espace euclidien réel a n dimensions*, Bull. Soc. Math. France **44** (1916), 65–99.

[5] M. Dajczer and L. Florit, *On conformally flat submanifolds.* Comm. Anal. Geometry **4** (1996), 261–284.

[6] M. Dajczer and L. Florit, *A class of austere submanifolds*, Illinois Math. J. **45** (2001), 735–755.

[7] M. Dajczer and L. Florit, *Genuine deformations of submanifolds*, Comm. Anal. Geom. **12** (2004), 1105–1129.

[8] M. Dajczer and L. Florit, *Genuine rigidity of Euclidean submanifolds in codimension two*, Geom. Dedicata **106** (2004), 195–210.

[9] M. Dajczer, L. Florit and R. Tojeiro, *On deformable hypersurfaces in space forms*, Ann. Mat. Pura Appl. **174** (1998), 361–390.

[10] M. Dajczer, L. Florit, and R. Tojeiro, *The vectorial Ribaucour transformation for submanifolds and applications*, Trans. AMS **359** (2007), 4977–4997.

[11] M. Dajczer, L. Florit, and R. Tojeiro, *Euclidean hypersurfaces with genuine deformations in codimension two*, Manuscripta Math. **140** (2013), 621–643.

[12] M. Dajczer and D. Gromoll, *Real Kaehler submanifolds and uniqueness of the Gauss map*, J. Diff. Geometry **22** (1985), 13–28.

[13] M. Dajczer and D. Gromoll, *Isometric deformations of compact Euclidean submanifolds in codimension two*, Duke Math. J. **79** (1995), 605–618.

[14] E. V. Ferapontov, *Surfaces with flat normal bundle: an explicit construction*, Diff. Geom. Appl. **14** (2001), 15–37.

[15] L. Florit and F. Guimarães, *Singular genuine rigidity of submanifolds*, preprint.

[16] L. Florit and R. Tojeiro, *Genuine deformations of submanifolds II: the conformal case*, Comm. Anal. Geom. **18** (2010), 397–419.

[17] L. Florit and W. Ziller, *Manifolds with conullity at most two as graph manifolds*, Preprint, 2016. `arXiv:1611.06572`.

[18] V. Sbrana, *Sulla varietá ad $n-1$ dimensioni deformabili nello spazio euclideo ad n dimensioni*, Rend. Circ. Mat. Palermo **27** (1909), 1–45.

[19] S. Silva, *On isometric and conformal rigidity of submanifolds*, Pacific J. of Math. **199** (2001), 227–247.

IMPA – Estrada Dona Castorina, 110
22460-320, Rio de Janeiro, Brazil
E-mail address: `luis@impa.br`
E-mail address: `gfreitas@impa.br`

Received October 2, 2015