Münster J. of Math. **8** (2015), 1–16
DOI 10.17879/65219682809
urn:nbn:de:hbz:6-65219729719

Münster Journal of Mathematics

Riemannian foliations on contractible manifolds

Luis Florit, Oliver Goertsches, Alexander Lytchak, and Dirk Töben

(Communicated by Burkhard Wilking)

Abstract. We prove that Riemannian foliations on complete contractible manifolds have a closed leaf, and that all leaves are closed if one closed leaf has a finitely generated fundamental group. Under additional topological or geometric assumptions we prove that the foliation is also simple.

1. Introduction

In [6] E. Ghys proved very powerful approximation theorems for Riemannian foliations on compact simply-connected manifolds. As an important application he deduced that for any Riemannian foliation on a rational homology sphere with leaf dimension larger than one, all leaves must be compact rational homology spheres. Unfortunately, his methods (and the powerful methods developed later in [13]) do not apply to Riemannian foliations on complete noncompact Riemannian manifolds, essentially because Tischler's theorem [23] applies to compact manifolds only. But complete noncompact manifolds are very important in Riemannian geometry, most notably the Euclidean space, whose Riemannian foliations have been studied in several papers; see [8, 9].

In this note, we use more abstract topological methods to prove a noncompact analog of the theorem of Ghys on foliations of rational homology spheres mentioned above. In order to formulate the result, we say that a topological space is *rationally contractible* if all of its homotopy groups are torsion groups.

Theorem 1.1. *Let M be a simply-connected, rationally contractible, complete Riemannian manifold and let $\mathcal{F}$ be a Riemannian foliation on M. Then there exists a closed leaf which is rationally contractible. Moreover, if there exists a closed leaf with finitely generated fundamental group, then all leaves are closed and the quotient space $M/\mathcal{F}$ is also rationally contractible.*

The first- and fourth-named authors were partially supported by CNPq, Brazil. The third-named author was partially supported by a Heisenberg grant of the DFG and by the SFB "Groups, Geometry and Actions".

The main examples of rationally contractible spaces, essentially the only class which naturally appears in geometry, are contractible spaces. One might expect that, unlike the case of spheres where Riemannian foliations with exceptional leaves do exist, Riemannian foliations on contractible manifolds are *simple foliations*, i.e., they are given by Riemannian submersions. We can verify this under additional finiteness assumptions (cp. Appendix A for more about H-spaces):

Corollary 1.2. *Let $\mathcal{F}$ be a Riemannian foliation on a complete, contractible Riemannian manifold M. Then there exists at least one closed and rationally contractible leaf L. Its universal cover $\tilde{L}$ is a rationally contractible H-space. Moreover, for any field K, the homology of $\tilde{L}$ with coefficients in K is finitely generated.*

In addition, if the integral homology of $\tilde{L}$ is finitely generated, then the foliation $\mathcal{F}$ is simple, with base $M/\mathcal{F}$ and all leaves being contractible.

The homotopy type of the leaf L in Corollary 1.2 seems to be extremely bizarre if L is not contractible. If either the dimension or the codimension of $\mathcal{F}$ is small, then such objects cannot occur, and the foliation turns out to be simple, even in the general rationally contractible case. The same conclusion can be drawn if $\mathcal{F}$ is *cobounded*, i.e., if the whole manifold M is contained in a tube of finite radius around one leaf. More precisely, we deduce from Theorem 1.1:

Corollary 1.3. *Let $\mathcal{F}$ be a Riemannian foliation on a complete rationally contractible Riemannian manifold M. Then:*
(1) *If $\mathcal{F}$ is cobounded, then $\mathcal{F}$ has only one leaf.*
(2) *If the dimension of the leaves of $\mathcal{F}$ is at most 3, or their codimension is at most 2, then $\mathcal{F}$ is a simple foliation.*

In fact, we expected that non-contractible manifolds with the topological properties described for L in Corollary 1.2 do not exist at all. Unfortunately, this is not true, as the following example due to William Dwyer shows.

Example 1.4. Consider the canonical map $S^3 \to S^3_{\mathbb{Q}}$ from S^3 to its localization at 0 (cp. [15]), and consider the homotopy fiber of this map, the *torsionification* $T(S^3)$. Then, $T(S^3)$ is the loop space of the homotopy fiber of the map of the classifying space $BS^3 = \mathbb{H}P^\infty$ to $\mathbb{H}P^\infty_{\mathbb{Q}}$, its localization at 0. One verifies that $T(S^3)$ satisfies all algebraic conclusions from Corollary 1.2 and is homotopy equivalent to a finite-dimensional manifold.

We have tried to keep the presentation as simple as possible. Using more results from the theory of H-spaces and equivariant cohomology, one can slightly improve the above statements. Namely, one can show that in Corollary 1.2, the universal covering $\tilde{L}$ is a loop space and not only an H-space. In Theorem 1.1 one can conclude that all leaves are rationally contractible, also in the non-finitely generated case. However, new insights are needed to answer the following questions in full generality.

Question 1.5. Under the assumptions of Theorem 1.1, is it true that all leaves are closed, even if the fundamental groups of the leaves are not finitely generated?

Question 1.6. Is it true that every Riemannian foliation on a complete contractible Riemannian manifold is simple?

The finiteness assumption in Corollary 1.2 can be verified if the curvature is nonnegative. More precisely, in this case a variant of the soul construction can be used to reduce the problem to the cobounded case, so that Corollary 1.3 applies:

Theorem 1.7. *Every Riemannian foliation on a complete contractible Riemannian manifold of nonnegative sectional curvature is simple.*

We would like to mention that the above results were not known even for the Euclidean space with its flat Euclidean metric; see [10, p. 148]. Only in this flat case, we have a short direct proof of Theorem 1.7 using [2] instead of Theorem 1.1. We present this proof in Section 4.6, for the sake of completeness.

We have not found a geometric proof of Theorem 1.1 under the natural assumption that M has non-positive (even constant negative!) curvature, the main geometric source of contractible manifolds. Under the additional assumption that $(M, \mathcal{F})$ is invariant under a large group of isometries, a short proof of the simplicity of the foliation is given in [17]. We would like to formulate:

Question 1.8. Is Theorem 1.7 true in non-positive curvature?

Finally, we would like to mention that Theorem 1.7 reduces the classification of Riemannian foliations on Euclidean space to the case of simple foliations, i.e., metric fibrations in terms of [8, 9]. In [8] such fibrations were shown to be homogeneous if the leaves have dimension at most 3. In [9] the homogeneity in all dimensions has been claimed, however, this paper contains a serious gap in Section 3 as Stefan Weil has pointed out in [24]. We could validate the proof only under the additional assumption that the foliation is *substantial* in the terminology of [9]. Thus the following basic question about Riemannian foliations seems to be open in its full generality:

Question 1.9. Is any Riemannian foliation on the Euclidean space homogeneous?

The paper is structured as follows. In Section 2 we recall Molino's construction that describes leaf closures of Riemannian foliations. We obtain a fibration relating the topology of leaf closures to the equivariant cohomology of a natural action of the orthogonal group (Proposition 2.7). Then we derive a simple criterion forcing one leaf or, under a finiteness assumption, all leaves to be closed (Proposition 2.11). In Section 3, we present the proof of Theorem 1.1, modulo topological statements about H-spaces and equivariant cohomology that are proven at the end of the paper. In Section 4, we show how Corollary 1.2, Corollary 1.3 and Theorem 1.7 follow from Theorem 1.1

and provide a short geometric proof of Theorem 1.7 for Euclidean spaces. In Appendix A we recall the theorems of Hopf and Borel about finite H-spaces and present a minor extension to a non-finite situation. In Appendix B we recall some basic facts about equivariant cohomology and prove a small generalization of a well-known result relating ranks of isotropy groups to the Krull dimension of equivariant cohomology rings.

2. General structure of leaf closures

2.1. **Riemannian foliations and leaf closures.** A foliation $\mathcal{F}$ on a Riemannian manifold M is called a Riemannian foliation if any geodesic normal to a leaf remains normal to all leaves it intersects. (Thus, in terms of [20], we always assume that the Riemannian metric is bundle-like with respect to the foliation). If the Riemannian metric is complete, the only case which appears in this paper, this implies (and is essentially equivalent to) the fact that all leaves are equidistant, i.e., the distance function to any leaf is constant on any other leaf. We refer to [20] and will assume some acquaintance with Riemannian foliations.

Let $\mathcal{F}$ be an $(n-k)$-dimensional Riemannian foliation on a complete n-dimensional Riemannian manifold M. For any leaf L of $\mathcal{F}$, the closure $\bar{L}$ is a smooth submanifold and these leaf closures define a decomposition $\bar{\mathcal{F}}$ of M, called a *singular Riemannian foliation.* The quotient $M/\bar{\mathcal{F}}$ is a metric space, which is compact if and only if $\mathcal{F}$ is cobounded.

The restriction of $\mathcal{F}$ to any leaf closure $\bar{L}$ is a Riemannian foliation with dense leaves. Riemannian foliations with dense leaves have been described completely by Haefliger [12]. As a special case of his description we will use the following lemma.

Lemma 2.2. *Let $\bar{L}$ be the closure of the leaf L of the Riemannian foliation $\mathcal{F}$. If $\pi_1(\bar{L})$ is a torsion group, then $\bar{L} = L$. If $\pi_1(\bar{L})$ is abelian, then there is a homomorphism $h : \pi_1(\bar{L}) \to \mathbb{R}^l$ with dense image, where l is the codimension of L in $\bar{L}$. Moreover, the lift of $\mathcal{F}$ to the universal cover of $\bar{L}$ is a simple foliation with quotient space $\mathbb{R}^l$.*

We will use the following easy observation several times along the paper.

Proposition 2.3. *Assume that the foliation $\mathcal{F}$ has a closed contractible leaf L. Then $\mathcal{F}$ is simple.*

Proof. For any leaf L_1 in a neighborhood of L there is a canonical projection $L_1 \to L$ which is a covering map. Since L is simply connected, this covering map must be a diffeomorphism. Hence L is a principal leaf. Since the principal leaf is closed, all other leaves are closed as well and any other leaf L_2 is finitely covered by L. Since L is contractible, L_2 is the classifying space of its fundamental group. But the classifying space of any nontrivial finite group is infinite-dimensional. Hence the fundamental group of L_2 is trivial. Thus all leaves are principal leaves, which just means that $\mathcal{F}$ is a simple foliation. □

2.4. **Molino's construction.** In this subsection, we briefly recall the Molino bundle, but refer to [6, 7, 12, 20] for more details.

Let $\mathcal{F}$ be an $(n-k)$-dimensional Riemannian foliation on a complete n-dimensional Riemannian manifold M. Assume that $\mathcal{F}$ is transversally oriented. We consider the principal $G = SO(k)$-fiber bundle $\pi : \hat{M} \to M$ of oriented transverse orthonormal frames, called the *Molino bundle*. The foliation $\mathcal{F}$ admits a canonical lift to a G-equivariant, $(n-k)$-dimensional Riemannian foliation $\hat{\mathcal{F}}$ on $\hat{M}$. Molino showed that the closures of the leaves of $\hat{\mathcal{F}}$ constitute a G-equivariant simple Riemannian foliation $\bar{\hat{\mathcal{F}}}$, which is therefore given by a Riemannian submersion $\rho : \hat{M} \to W := \hat{M}/\bar{\hat{\mathcal{F}}}$. We denote the typical fiber of this submersion, i.e., the homeomorphism type of the closures of leaves of $\hat{\mathcal{F}}$, by N.

The action of G on $\hat{M}$ descends to an action of G on W in such a way that ρ becomes a G-equivariant fiber bundle. The quotient space W/G is canonically identified with the space of leaf closures of $\mathcal{F}$, i.e., $W/G = M/\bar{\mathcal{F}}$. Let N_0 be a leaf of $\bar{\hat{\mathcal{F}}}$, hence a point in W. Denote by G_{N_0} the isotropy group of the point $\{N_0\} \in W$, which is just the set of all elements in G sending the submanifold $N_0 \subset \hat{M}$ to itself. The action of G_{N_0} on N_0 is free and the quotient space N_0/G_{N_0} is the closure $\bar{L}_0$ of a leaf L_0 of $\mathcal{F}$. The action of G_{N_0} on the submanifold $N_0 \subset \hat{M}$ has orbits transversal to $\hat{\mathcal{F}}$. Therefore we conclude from the preceding considerations:

Lemma 2.5. *With the notations above, if l denotes the codimension of $\hat{\mathcal{F}}$ in N_0, then $\dim(G_{N_0}) \le l$. Equality holds if and only if the projection $\bar{L}_0$ of N_0 is a closed leaf of the original foliation $\mathcal{F}$.*

2.6. **Relation to equivariant cohomology.** For the G-spaces $\hat{M}$ and W as above, we denote their Borel constructions by $\hat{M}_G := \hat{M} \times_G EG$ and $W_G = W \times_G EG$, respectively; see [1] and Appendix B below. By functoriality, we obtain a natural fibration $\rho_G : \hat{M}_G \to W_G$ with typical fiber N. Since G acts freely on $\hat{M}$, we obtain a fibration $EG \to \hat{M}_G \to \hat{M}/G = M$ with contractible fiber EG, which implies that M and $\hat{M}_G$ are homotopy equivalent. Using the above notation, we then have:

Proposition 2.7. *The Borel construction $\hat{M}_G$ of the Molino bundle $\hat{M}$ is homotopy equivalent to M, and the fibration $\hat{M}_G \to W_G$ has fiber N.*

The foliation $\mathcal{F}$ is closed if and only if $\hat{\mathcal{F}}$ is closed. In this case the quotient space $B = M/\mathcal{F}$ is a Riemannian orbifold and $M \to B$ a generalized Seifert fibration. The leaves of $\hat{\mathcal{F}}$ are diffeomorphic via the projection $\pi : \hat{M} \to M$ to the regular leaves L of $\mathcal{F}$. Moreover, the Borel construction W_G in Proposition 2.7 coincides by definition with Haefliger's *classifying space* $\hat{B}$ of B; see [11, §4]. In particular, the projection $W_G = \hat{B} \to B = W/G$ induces an isomorphism on rational (co)homologies.

2.8. **The simply-connected case.** We now assume that the complete Riemannian manifold M is simply connected. Then the Riemannian foliation $\mathcal{F}$

is transversally orientable and we have the whole structure described in the previous two subsections. Moreover, the leaf closures N of $\hat{\mathcal{F}}$ have abelian fundamental group. In addition, the stabilizer G_{N_0} of any point $N_0 \in W$ has trivial adjoint representation. In particular, its identity component is a torus; see [7, Lem. 4.6]. We claim:

Lemma 2.9. *With the notations above, let T be the connected component of G_{N_0}. Then the map $\pi_1(T) \to \pi_1(N_0)$, induced by sending T to an orbit of the action of T on N_0, is an injection.*

Proof. Consider the lift of the action of T on N_0 to the action of the universal cover $\tilde{T}$ on $\tilde{N}_0$. By Lemma 2.2, the foliation $\hat{\mathcal{F}}$ lifts to a simple foliation on $\tilde{N}_0$ with quotient $\mathbb{R}^l$. Since the action of T preserves the foliation and is transversal to it, we get an almost free isometric action of $\tilde{T}$ on $\mathbb{R}^l$. Thus this action of $\tilde{T}$ on $\mathbb{R}^l$ must be free.

Hence the action of $\tilde{T}$ on $\tilde{N}_0$ is free as well. This directly implies that the kernel of the map $\pi_1(T) \to \pi_1(N_0)$ is trivial. □

Let l denote the codimension of $\hat{\mathcal{F}}$ in its closure $\bar{\hat{\mathcal{F}}}$. Because N is the fiber of the fibration $\hat{M}_G \to W_G$ with simply-connected total space, $\pi_1(N)$ is abelian. Due to Lemma 2.2, we have a homomorphism $h : \pi_1(N) \to \mathbb{R}^l$ with a dense image. Hence, $\pi_1(N) \otimes \mathbb{Q}$ is a $\mathbb{Q}$-vector space of dimension at least l. If $\pi_1(N)$ is finitely generated, and the dimension of $\pi_1(N) \otimes \mathbb{Q}$ is l, then the image of h would be a lattice in $\mathbb{R}^l$, hence it could not have a dense image, unless $l = 0$. We have shown:

Lemma 2.10. *We have the inequality $H^1(N, \mathbb{Q}) \geq l$. In case of equality, either $l = 0$ and the foliation $\mathcal{F}$ is closed, or the group $\pi_1(N)$ is not finitely generated and satisfies $\pi_1(N) \otimes \mathbb{Q} = \mathbb{Q}^l$.*

Combining this lemma with Lemma 2.5, we obtain:

Proposition 2.11. *For any $N_0 \in W$ we have $\dim(G_{N_0}) \leq \dim(H^1(N_0, \mathbb{Q}))$. If equality holds, then N_0 projects to a closed leaf of $\mathcal{F}$. If, in addition, $\pi_1(N_0)$ is finitely generated, then the foliation $\mathcal{F}$ is a closed foliation.*

3. The main argument

We now give a proof of Theorem 1.1, modulo some auxiliary topological results which will be proven in the appendices. We continue to use the notation introduced in the previous section, and assume that M is simply connected and rationally contractible.

Consider the fibration $\hat{M}_G \to W_G$ with fiber N. The space $\hat{M}_G$ is simply connected, hence so is W_G and the space N is *abelian*, i.e., its fundamental group is abelian and acts trivially on higher homotopy groups; see [14, p. 419, Ex. 10]. The homotopy fiber of the embedding map $N \to \hat{M}_G$ is the loop space ΩW_G; see [14, p. 409]. Since all homotopy groups of $\hat{M}_G$ are torsion groups, the long exact homotopy sequence of this fibration reveals that the map $p : \Omega W_G \to N$ induces isomorphisms on all rational homotopy groups $p_* : \pi_*(\Omega W_G) \otimes \mathbb{Q} \to \pi_*(N) \otimes \mathbb{Q}$.

Since N and ΩW_G are abelian topological spaces, the map p induces an isomorphism of rational cohomology rings $p^* : H^*(N, \mathbb{Q}) \to H^*(\Omega W_G, \mathbb{Q})$. Thus the rational cohomology of the H-space ΩW_G (see Appendix A for more information on H-spaces) vanishes in degrees larger than the dimension of N. We now use the following result, a slight extension of a classical theorem of Hopf [14, p. 285], whose proof is postponed to Appendix A, and deduce that the rational cohomology ring of N is the cohomology ring of a finite product of spheres.

Proposition 3.1. *Let X be a connected H-space. Let K be a field and assume that $H^*(X, K)$ vanishes in all degrees larger than m. Then $H^*(X, K)$ has dimension at most 2^m over K.*

Moreover, if K has characteristic 0, then $H^*(X, K)$ is isomorphic to the cohomology ring of a product of spheres. Thus $H^*(X, K)$ is the antisymmetric algebra $\Lambda(y_1, \ldots, y_q)$ with $\deg y_i =: e_i$ odd.

Given that $H^*(N, \mathbb{Q}) = \Lambda(y_1, \ldots, y_q)$ with $\deg y_i =: e_i$ odd, and the assumption that $H^*(\hat{M}_G, \mathbb{Q})$ vanishes in positive degrees, the transgression theorem of Borel [3, Thm. 13.1] (see also [19, Thm. VII.2.9]) gives us that the cohomology ring $H^*(W_G, \mathbb{Q})$ is a polynomial ring $\mathbb{Q}[x_1, \ldots, x_q]$, generated by elements x_i of degree $e_i + 1$. In particular, $\dim H^2(W_G, \mathbb{Q}) = \dim H^1(N, \mathbb{Q})$. This dimension is exactly the number of elements y_i of degree 1.

Now, we can apply the following result (essentially [22, Thm. 7.7]), whose proof will be explained in Appendix B, in order to find a lower bound on the ranks and thus also bounds on the dimension of stabilizers of the action of G on W.

Proposition 3.2. *Let G be a compact Lie group that acts smoothly on a manifold X. Assume that the G-equivariant cohomology ring $H^*(X_G)$ is a polynomial ring in q variables. Then there is a point $x \in X$ such that the rank of the stabilizer G_x is at least q.*

Applying this proposition to our action of $G = SO(k)$ on W, we find some point $N_0 \in W$ such that its stabilizer G_{N_0} has rank and thus dimension at least q. On the other hand, $q \geq \dim H^2(W_G, \mathbb{Q}) = \dim H^1(N_0, \mathbb{Q})$. We apply Proposition 2.11 and deduce that $q = \dim H^1(N_0, \mathbb{Q})$ and that N_0 projects to a closed leaf L_0 of $\mathcal{F}$ in M.

Now we are going to analyze the topology of this closed leaf L_0. Since $q = \dim H^1(N_0, \mathbb{Q})$, all generators y_i of the cohomology of N_0 have degree 1 and N_0 has the rational cohomology ring of a q-dimensional torus.

We claim that all higher homotopy groups $\pi_j(N_0)$, $j \geq 2$ of N_0 are torsion groups. Indeed, let $E = \mathbb{S}^1_{\mathbb{Q}}$ be the Eilenberg–MacLane space $K(\mathbb{Q}, 1)$, which is the rationalization of $\mathbb{S}^1$. We have $H^*(E, \mathbb{Q}) = H^*(S^1, \mathbb{Q})$. This space E is the classifying space for $H^1(\cdot, \mathbb{Q})$, hence we find maps $f_1, \ldots, f_q : N_0 \to E$, such that the pullbacks of the canonical generator e of $H^1(E, \mathbb{Q})$ are exactly

$y_i = f_i^*(e)$. The knowledge of the rational cohomology ring of N_0 now tells us that the map $F = (f_1, \dots, f_q) : N_0 \to E^q$ induces an isomorphism on all rational cohomology groups. Since E and N_0 are abelian spaces, the map F induces isomorphisms on all rational homotopy groups. But all higher homotopy groups of E vanish, which proves the claim.

Now we consider the fiber bundle $N_0 \to L_0$ with fiber G_{N_0}. Due to Lemma 2.9, the orbit map $G_{N_0} \to N_0$ induces an injection on π_1. Since $\pi_1(G_{N_0})$ and $\pi_1(N_0)$ tensored with $\mathbb{Q}$ are both isomorphic to $\mathbb{Q}^l$, the cokernel of the injection must be a torsion group. Since $\pi_0(G_{N_0})$ is finite, we deduce that $\pi_1(L_0)$ is a torsion group. Since all higher homotopy groups of N_0 and G_{N_0} are torsion groups, we deduce from the long exact sequence in homotopy that all higher homotopy groups of L_0 are torsion as well. Thus L_0 is a rationally contractible space.

Note further that $\pi_1(L_0)$ contains the abelian image of $\pi_1(N_0)$ as a subgroup of finite index. Considering the same fiber bundle as above, $\pi_1(L_0)$ is finitely generated if and only if $\pi_1(N_0)$ is finitely generated, and this is in turn equivalent to any other leaf closure in M having finitely generated fundamental group. Assuming this in addition, $\pi_1(L_0)$ must be a finite group. The principal leaf L of $\mathcal{F}$ near L_0 admits a canonical covering map to L_0. Since this covering must be finite, due to the finiteness of $\pi_1(L_0)$, we see that L is closed as well. Then all leaves are closed. Since L finitely covers L_0 and any other leaf is finitely covered by L, all leaves are rationally contractible.

Once we know that $\mathcal{F}$ is closed, the quotient $B = M/\mathcal{F}$ is a Riemannian orbifold. Moreover, the Borel construction W_G is Haefliger's classifying space of B, in particular, its rational cohomology groups are nonzero only in finitely many degrees, since they coincide with the rational cohomology groups of B. As we already know that $H^*(W_G, \mathbb{Q})$ is a polynomial ring, this tells us that the rational cohomology groups of W_G vanish in positive degrees. Since W_G is simply connected, W_G is a rationally contractible space. From the long exact homotopy sequence of the fibration $\hat{M}_G \to W_G$, we conclude that N is rationally contractible as well.

Finally, B is simply connected and its rational cohomology groups vanish in positive degrees, since they coincide with the rational cohomology groups of W_G. Therefore B is rationally contractible.

4. Finiteness conditions

We continue to use notations introduced in Section 2.

4.1. **Finite homology.**

Proof of Corollary 1.2. Using Theorem 1.1, we see that there is a closed rationally contractible leaf L. Hence its universal covering is rationally contractible as well.

The manifold N_0 appearing in the proof of Theorem 1.1 is the fiber of the fibration $\hat{M}_G \to W_G$. Since $\hat{M}_G$ is contractible, N is homotopy equivalent to the loop space ΩW_G. In particular, N_0 is an H-space, hence so is its universal covering $\tilde{N}_0$.

We have the fibration $N_0 \to L$ with fiber G_{N_0}. In the course of the proof of Theorem 1.1, we have seen that the induced map $\pi_1(G_{N_0}) \to \pi_1(N_0)$ is injective. Therefore, the fiber of the fibration $\tilde{N}_0 \to \tilde{L}$ between the universal coverings is the universal covering of the connected component of G_{N_0}. Since the connected component of G_{N_0} is a torus, we deduce that $\tilde{L}$ is homotopy equivalent to $\tilde{N}_0$. Thus $\tilde{L}$ is an H-space.

We conclude from Proposition 3.1 that the homology groups of $\tilde{L}$ with coefficients in any field K are finite-dimensional K-vector spaces.

If the homology of $\tilde{L}$ with integral coefficients is finitely generated, then a theorem of Browder [4] (see Corollary A.3 below) tells us that the rationally contractible H-space $\tilde{L}$ must be contractible. Then L is the classifying space of its fundamental group, which is torsion. Thus the fundamental group of L is trivial and L is contractible.

From Proposition 2.3 we get that $\mathcal{F}$ is a simple foliation given by a Riemannian submersion $M \to B$.

Since the fiber L and the total space M are contractible, the base B is contractible as well, due to the exact sequence in homotopy. □

4.2. **Small dimension.**

Proof of Corollary 1.3. Assume first that the quotient $B = M/\bar{\mathcal{F}}$ is compact. Since $B = W/G$, the space W must be compact as well. Hence W has finitely generated integral homology. Applying [15, Lem. 1.9], we see that the space W_G has finitely generated integral homology in each dimension. Therefore, W_G has finitely generated homotopy groups in each dimension. From the fibration $M_G \to W_G$ we deduce that $\pi_1(N)$ is finitely generated. Then the fundamental group of a closed leaf L_0 is finitely generated as well. Using Theorem 1.1, we obtain that $\mathcal{F}$ is closed. But then B is a simply-connected compact orbifold. Hence its rational homology does not vanish in the maximal dimension. Since B is rationally contractible, this implies that $\dim(B) = 0$. Hence $\mathcal{F}$ has only one leaf.

Assume now that the dimension of the leaves of $\mathcal{F}$ is at most 3, and let L be a closed rationally contractible leaf. We claim that L is contractible. If $\dim(L) \leq 2$, this is a direct consequence of the classification of one- and two-dimensional manifolds. Assume $\dim(L) = 3$. Then the universal covering $\tilde{L}$ is still rationally contractible. In particular, $\tilde{L}$ is nonclosed, hence $H_3(\tilde{L}) = 0$. By assumption, $\pi_1(\tilde{L}) = 0$. From Poincaré duality we know that $H_2(\tilde{L})$ is torsion-free. Since $\pi_2(\tilde{L}) = H_2(\tilde{L})$, and since $\tilde{L}$ is rationally contractible, we obtain $H_2(\tilde{L}) = 0$. Therefore, $\tilde{L}$ is contractible. Thus L is the classifying space of its fundamental group $\pi_1(L)$, which is a torsion group. We conclude that $L = \tilde{L}$. Now the simplicity of the foliation follows from Proposition 2.3.

Let us assume now that the codimension of the leaves is at most 2. Assume first that the foliation $\mathcal{F}$ is closed. The quotient $B = M/\mathcal{F}$ is an orbifold with trivial orbifold fundamental group (which is just the fundamental group of W_G). The classification of one- and two-dimensional orbifolds (cp. [16, §2.3]) shows that either B has no singularities and is diffeomorphic to the Euclidean space, or its underlying topological space is the two-sphere. But the second case cannot occur, since we already know that B cannot be compact. Thus B has no singularities. Hence $\mathcal{F}$ has no exceptional leaves and $\mathcal{F}$ is simple.

Assume now that $\mathcal{F}$ is nonclosed. Then $M/\bar{\mathcal{F}}$ has dimension less than two. Due to Lemma 2.2, $\mathcal{F}$ cannot have dense leaves in M, unless $\mathcal{F}$ has only one leaf, hence we may assume that $\mathcal{F}$ has codimension 2 and $\bar{\mathcal{F}}$ has codimension 1. The closed leaves of $\mathcal{F}$ are exactly the singular leaves of $\bar{\mathcal{F}}$. Since at least one such leaf exists, the quotient $M/\bar{\mathcal{F}}$ is either an interval or a ray. If the quotient is a ray, then M retracts to the closed leaf of $\mathcal{F}$, which has an infinitely generated fundamental group by Theorem 1.1. This is impossible, since M is simply connected. Hence the quotient B is a compact interval. But B cannot be compact, unless it is a point. □

4.3. **Nonnegative curvature.**

Proof of Theorem 1.7. Due to Proposition 2.3, it suffices to find a closed contractible leaf. Following the ideas of [8] and [2, §2.1], we are going to find a *totally convex* leaf. Recall that a subset X' of M is called totally convex, if it contains any (not necessarily minimal) geodesic connecting any pair of its points. A closed totally convex subset is a topological manifold with boundary, whose set of inner points is a totally geodesic submanifold. If a closed totally convex subset X is a manifold (without boundary), then X is homotopy equivalent to M.

Assume now that we can find a closed totally convex $\mathcal{F}$-saturated submanifold M' of M, such that the restriction of $\mathcal{F}$ to M' is cobounded. Then M' is contractible and by Corollary 1.3, M' must be a leaf of $\mathcal{F}$. Hence, Theorem 1.7 follows directly from the following general geometric observation, whose proof, essentially contained in [2], we shortly recall below. □

Lemma 4.4. *Let M be a (not necessarily contractible) complete Riemannian manifold of nonnegative curvature with a Riemannian foliation $\mathcal{F}$. Then there exists a closed, totally convex $\mathcal{F}$-saturated submanifold M' of M such that the restriction of $\mathcal{F}$ to M' is cobounded.*

Proof. Let $\mathcal{C}$ be the set of all closed, totally convex, $\mathcal{F}$-saturated subsets of M. The set $\mathcal{C}$ is nonempty, since it contains M, and it is closed under intersections. Let us fix some set $X \in \mathcal{C}$ of smallest dimension in $\mathcal{C}$. Fix some $x \in X$ and let L be the leaf through x. We may assume that X is the smallest element of $\mathcal{C}$ which contains L.

We claim that X is a submanifold and that $\mathcal{F}$ is cobounded on X. First, assume that $\mathcal{F}$ is not cobounded. Hence we find leaves $L_i \subset X$ running away from x, i.e., such that $l_i = d(x, L_i)$ converges to infinity. Consider the normalized distance functions $f_i : M \to \mathbb{R}$ to the subset L_i given by $f_i(y) = d(y, L_i) - l_i$. The functions f_i vanish at x, are 1-Lipschitz and $\mathcal{F}$-basic, i.e., constant on leaves of $\mathcal{F}$. Consider a pointwise limit f of a subsequence of f_i. Since the sets L_i run to infinity, the function f is concave by Toponogov's theorem (see the usual proof of the soul theorem). Moreover, f is non-constant on X since it has velocity 1 on a ray starting at x and being a limit of shortest geodesics from x to $\bar{L}_i$. Hence the superlevel set $f^{-1}([0,\infty)) \cap X$ is a closed totally convex subset of M, which is $\mathcal{F}$-saturated, contains L and is properly contained in X. This contradicts the minimality of X and thus we may assume that $\mathcal{F}$ is cobounded on X.

Assume now that X is a manifold with nonempty boundary ∂X. The boundary ∂X must be $\mathcal{F}$-saturated as well, as we immediately see in the local picture (see [2] for a slightly more difficult argument in the case of singular foliations). The distance function $d_{\partial X}$ to the boundary is a concave function on X (see the usual soul construction), hence its superlevel sets are closed totally convex subsets of X. Now, by the coboundedness, the distance function $d_{\partial X}$ assumes a maximum on X. And the set M' of points where this maximum is attained is a convex set of dimension less than the dimension of X, contradicting to the choice of X. □

Remark 4.5. We mention that as in [2], the above proof applies without changes to singular Riemannian foliations (and more generally transnormal decompositions) on nonnegatively curved manifolds.

4.6. **The Euclidean case.** Here we present a short geometric proof of Theorem 1.7 for the Euclidean space.

Namely, consider the closed singular Riemannian foliation $\bar{\mathcal{F}}$ on $M = \mathbb{R}^n$. Due to [2, Thm. 2.1] (cp. [17, Cor. 1.3] for a simplification of the argument, starting from Lemma 4.4 above), there is at least one leaf $\bar{L}$ of $\bar{\mathcal{F}}$ which is an affine subspace of $\mathbb{R}^n$. Since an affine subspace is contractible, this leaf closure $\bar{L}$ must be in fact a leaf $L = \bar{L}$ of $\mathcal{F}$ due to Lemma 2.2. The result now follows from Proposition 2.3.

Appendix A. Preliminaries on H-spaces

We refer to [14, pp. 281–292] for this section, in which we assume that all spaces have the homotopy type of a CW complex. Recall that an *H-space* is a topological space together with a "multiplication" $\mu : X \times X \to X$ that has a "canonical unit element". Any H-space is an abelian topological space (cp. [14, Ex. 4A.3]) and its universal covering is again an H-space. In our applications, the space X will be the loop space ΩZ of some other space Z, equipped with the multiplication given by concatenation of loops.

For a ring R, a *Hopf algebra* over R is a graded R-algebra A (always associative and graded-commutative) together with a graded homomorphism $\mu : A \to A \otimes_R A$, called the comultiplication, that satisfies an algebraic equivalent of the unity axiom for H-spaces; see [14, p. 283]. In the sequel, all Hopf algebras will be *connected*, i.e., satisfy $A_0 = R$.

The structure of finite-dimensional Hopf algebras over fields is essentially known due to the following theorem proved by Hopf in characteristic 0 and by Borel in positive characteristics; see [14, p. 285] or [18, Thm. 7.11].

Theorem A.1. *Let A be a finite-dimensional Hopf algebra over a perfect field K. Then A is a tensor product of Hopf algebras A_i over K, each generated by one element. Moreover, if K has characteristic 0, each A_i has dimension 2, and A is isomorphic to the cohomology ring with coefficients in K of a product of odd-dimensional spheres.*

As a consequence we have:

Corollary A.2. *Let A be a finite-dimensional Hopf algebra over a perfect field K. Then the dimension of A is bounded from above by 2^d, where d is the maximal degree in which the graded algebra A is not 0.*

Proof. The result clearly holds for algebras generated by one element. Since the bound is preserved under tensor products, the statement follows from the above theorem of Borel and Hopf. □

If X is an H-space with multiplication μ and if the cohomology ring $A = H^*(X, K)$ with coefficients in a field K is finite-dimensional, then μ induces a comultiplication on A that makes it into a Hopf algebra. Using this, the above theorem of Borel and Hopf and the Bockstein sequence, it is not difficult to get the following result of Browder [4, Cor. 7.2].

Corollary A.3. *Let X be a connected H-space, with finitely generated integer homology $H_*(X, \mathbb{Z})$. Then $H_d(X, \mathbb{Z}) = \mathbb{Z}$, where d is the maximal degree, in which the homology does not vanish. In particular, if X is rationally contractible, it is in fact contractible.*

The H-spaces X arising in the proof of Theorem 1.1, a priori, do not satisfy the assumption of the last corollary. Instead, their cohomology is the cohomology of noncompact finite-dimensional manifolds, in particular, their cohomology groups vanish in all but finitely many degrees.

We are going to prove Proposition 3.1 now, showing that in this more general situation the cohomologies with coefficients in a field do not show anomalies and behave like in the finite case.

Proof of Proposition 3.1. By the universal coefficient theorem, it suffices to prove the result for $K = \mathbb{Q}$ and $K = \mathbb{F}_p$, for all prime numbers p. Thus let K be one of these fields and let us consider homologies and cohomologies with coefficients in K until the end of the proof. Hence the field K is a perfect field and we may use Theorem A.1 and Corollary A.2.

We denote by A the cohomology ring $H^*(X, K)$. Since any finite-dimensional Hopf algebra contained in A has dimension at most 2^m by Corollary A.2, and since any Hopf algebra can be exhausted by finite-dimensional subalgebras (cp. [18]), we conclude that any Hopf subalgebra of A has finite dimension, bounded above by 2^m.

It remains to prove that A is a Hopf algebra. Unfortunately, the homology is not finitely generated, and the assumption that X is an H-space does not provide A automatically with the structure of a Hopf algebra. Instead we argue as follows: The map μ defines a homomorphism $\mu^* : H^*(X) \to H^*(X \times X)$. We consider the Künneth formula with coefficients in a field (see [14, Cor. 3B.7])

$$\begin{aligned} H^k(X \times X) &= \mathrm{Hom}_K(H_k(X \times X), K) \\ &= \bigoplus_{i=0}^{k} \mathrm{Hom}_K(H_i(X) \otimes H_{k-i}(X), K). \end{aligned}$$

The i-th summand in this decomposition contains $H^i(X) \otimes H^{k-i}(X)$ as a subspace in a canonical way. Moreover, the i-th summand coincides with this tensor product if (and only if) one of the factors is finite-dimensional. In particular, this is always the case for $i = 0$ and $i = k$. With respect to this decomposition, for any $\alpha \in H^k(X)$ we have

$$\mu^*(\alpha) = 1 \otimes \alpha + \alpha \otimes 1 + r(\alpha),$$

where $r(\alpha)$, with respect to the above decomposition, has nonzero coordinates only in summands with $0 < i < k$; see [14, p. 283].

Let us assume that A is not finite-dimensional. Let d be the smallest integer such that $H^d(X)$ is not finite-dimensional. Consider the subalgebra C of A that is generated by all elements of degree at most d. For each $\alpha \in H^k(X)$, with $k \leq d$, we have

$$r(\alpha) \in \bigoplus_{i=0}^{k} \mathrm{Hom}_K(H_i(X) \otimes H_{k-i}(X), K) = \bigoplus_{i=1}^{k-1} H^i(X) \otimes H^{k-i}(X),$$

by the assumption about finite-dimensionality of all $H^i(X)$, with $i < d$.

In particular, $\mu^*(\alpha)$ is contained in the subalgebra $C \otimes C \subset A \otimes A \subset H^*(X \times X)$. Since μ is a homomorphism, and C is by assumption generated by such elements α of degree at most d, we conclude that $\mu^*(C) \subset C \otimes C$. Thus C is a Hopf algebra. As we have observed, C is finite-dimensional. This is in contradiction to our assumption on $H^d(X)$. □

Appendix B. Equivariant cohomology and dimension

We are going to prove Proposition 3.2 in this section. First, we recall some notations and basics about equivariant cohomology. We refer the reader to [1] for more details. In this section all cohomologies are considered with $\mathbb{Q}$-coefficients.

For a compact Lie group G let BG denote the classifying space of G with the classifying principal G-bundle $EG \to BG$, such that EG is contractible.

Let X be a topological space with a G-action. The *Borel construction* of the G-action on X is the space $X_G := (X \times EG)/G$, where G acts diagonally. The G-equivariant cohomology of the space X is by definition $H^*_G(X) := H^*(X_G)$.

Let T be a maximal torus of G and let $N(T)$ be its normalizer in G. Then we have a canonical isomorphism $H^*_G(X) = H^*_{N(T)}(X)$; see [1, pp. 205–206]. In fact, in [1] this result is explained only in the case of a connected group G (the only case we need), but the same proof applies to disconnected groups as well.

We recall the notion of *Krull dimension* (cp. [5, §8]) which we will only use in the case of commutative finitely generated $\mathbb{Q}$-algebras. Namely, the Krull dimension of such an algebra A is the maximal number s such that A contains a polynomial algebra on s variables (the usual definition is equivalent to this one due to *Noether's normalization lemma*). The Krull dimension does not change if A is replaced by a larger algebra A', such that A' is a finite module over A. In particular, for a finite group Γ acting on A by homomorphisms, the set of fixed points A^Γ is a finitely generated $\mathbb{Q}$-algebra of the same Krull dimension as A (this is a theorem of Noether [21], cp. [5, Ex. 13.2 and 13.3]).

Now, Proposition 3.2 is a direct consequence of the following more general result, which is an analog of [22, Thm. 7.7] for $p = 0$.

Proposition B.1. *Let X be a smooth manifold with a smooth action of a compact Lie group G. Assume that $H^*_G(X)$ is a finitely generated $\mathbb{Q}$-algebra. If the Krull dimension of the even-dimensional part of $H^*_G(X)$ is q, then there is a point $x \in X$ whose stabilizer has rank q.*

Proof. We have seen that, by replacing G by the normalizer of the maximal torus, we may assume that the connected component G^0 of G is a torus. The algebra $H^*_G(X)$ is the set of fixed points of $H^*_{G^0}(X)$ under the canonical action of the finite group $G/G^0 = \pi_0(G)$. Hence, $H^{\mathrm{even}}_{G^0}(X)$ is a finitely generated $\mathbb{Q}$-algebra of Krull dimension q. Thus, we may assume that G is a torus.

For any fixed degree $*$, the space $H^*_G(X)$ is finite-dimensional. From the principal fibration $X \times EG \to X_G$ we deduce that in each degree $*$, the cohomology $H^*(X \times EG) = H^*(X)$ is finite-dimensional (see [15, Lem. 1.9] for the Serre class $\mathcal{C}$ of abelian groups, whose tensor product with $\mathbb{Q}$ has finite rank over $\mathbb{Q}$). Since X is a finite-dimensional space, the total cohomology ring $H^*(X)$ is finite-dimensional.

Thus, G is a torus and $H^*(X)$ is finite-dimensional. In this case the claim is well known and appears for instance in [1, p. 257] under the additional assumption that the set of connected components of isotropy groups of the action is finite. (Note that the Krull dimension of $H^{\mathrm{even}}_G(X)$ coincides with the Krull dimension of $H^*_G(X)$ used in [1], since $H^{\mathrm{even}}_G(X)$ is a finite algebra over the ring $H^*(BG)/\mathrm{ann}(H^*_G(X))$, which is used there to define the dimension.) This assumption is verified due to Lemma B.2 below. □

Lemma B.2. *Let G be a torus and let X be a smooth manifold with a smooth action of G. Assume that $H^k_G(X)$ is finite-dimensional for each k. Then the set of connected components of isotropy groups of the action of G on X is finite.*

Proof. Assume the contrary. Then we find infinitely many subtori T_i of T, together with some connected components $X_i \neq \varnothing$ of the set of fixed points X^{T_i}, such that no point in X_i is fixed by a torus larger than T_i; the T_i are not necessarily distinct, but the X_i can be chosen not to intersect. Then, in a small neighborhood U_i of X_i the set of connected components of isotropy groups is finite. The equivariant Thom class u_i of X_i is a nonzero cohomology class in $H^*_G(X)$ of degree equal to the codimension of X_i in M, and with support on X_i, thus its restriction to $X \setminus U_i$ is zero. Therefore, the elements u_i are linearly independent. Hence, there exists $k \leq \dim X$ such that $H^k_G(M)$ is infinite-dimensional, contradicting the finiteness assumption. □

Acknowledgements. We thank William Dwyer for providing Example 1.4, and Jesús Álvarez López, Stephan Wiesendorf, and Burkhard Wilking for helpful comments.

References

[1] C. Allday and V. Puppe, *Cohomological methods in transformation groups*, Cambridge Studies in Advanced Mathematics, 32, Cambridge Univ. Press, Cambridge, 1993. MR1236839

[2] C. Boltner, *On the structure of equidistant foliations of* $\mathbb{R}^n$, PhD thesis, Augsburg, 2007. arXiv:0712.0245v1 [math.DG] (2007)

[3] A. Borel, Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts, Ann. of Math. (2) **57** (1953), 115–207. MR0051508

[4] W. Browder, Torsion in H-spaces, Ann. of Math. (2) **74** (1961), 24–51. MR0124891

[5] D. Eisenbud, *Commutative algebra*, Graduate Texts in Mathematics, 150, Springer, New York, 1995. MR1322960

[6] É. Ghys, Feuilletages riemanniens sur les variétés simplement connexes, Ann. Inst. Fourier (Grenoble) **34** (1984), no. 4, 203–223. MR0766280

[7] O. Goertsches and D. Töben, Equivariant basic cohomology of Riemannian foliations, J. Reine Angew. Math. (2016), DOI 10.1515/crelle-2015-0102

[8] D. Gromoll and G. Walschap, Metric fibrations in Euclidean space, Asian J. Math. **1** (1997), no. 4, 716–728. MR1621572

[9] D. Gromoll and G. Walschap, The metric fibrations of Euclidean space, J. Differential Geom. **57** (2001), no. 2, 233–238. MR1879226

[10] D. Gromoll and G. Walschap, *Metric foliations and curvature*, Progress in Mathematics, 268, Birkhäuser, Basel, 2009. MR2500106

[11] A. Haefliger, Groupoïdes d'holonomie et classifiants, Astérisque 116 (1984), 70–97. MR0755163

[12] A. Haefliger, Leaf closures in Riemannian foliations, in *A fête of topology*, 3–32, Academic Press, Boston, MA, 1988. MR0928394

[13] A. Haefliger and É. Salem, Riemannian foliations on simply connected manifolds and actions of tori on orbifolds, Illinois J. Math. **34** (1990), no. 4, 706–730. MR1062771

[14] A. Hatcher, *Algebraic topology*, Cambridge Univ. Press, Cambridge, 2002. MR1867354

[15] A. Hatcher, *Spectral sequences in algebraic topology*, unfinished book available at www.math.cornell.edu/~hatcher/SSAT/SSATpage.html

[16] B. Kleiner and J. Lott, Geometrization of three-dimensional orbifolds via Ricci flow, Astérisque 365 (2014), 101–177. MR3244330

[17] A. Lytchak, Singular Riemannian foliations on spaces without conjugate points, in *Differential geometry*, 75–82, World Sci. Publ., Hackensack, NJ, 2009. MR2523492

[18] J. W. Milnor and J. C. Moore, On the structure of Hopf algebras, Ann. of Math. (2) **81** (1965), 211–264. MR0174052

[19] M. Mimura and H. Toda, *Topology of Lie groups. I, II*, translated from the 1978 Japanese edition by the authors, Translations of Mathematical Monographs, 91, Amer. Math. Soc., Providence, RI, 1991. MR1122592

[20] P. Molino, *Riemannian foliations*, translated from the French by Grant Cairns, Progress in Mathematics, 73, Birkhäuser Boston, Boston, MA, 1988. MR0932463

[21] E. Noether, *Der Endlichkeitssatz der Invarianten endlicher linearer Gruppen der Charakteristik p*, Nachr. Ges. Wiss. Göttingen 1926 (1926), 28–35.

[22] D. Quillen, The spectrum of an equivariant cohomology ring. I, II, Ann. of Math. (2) **94** (1971), 549–572; ibid. (2) **94** (1971), 573–602. MR0298694

[23] D. Tischler, On fibering certain foliated manifolds over S^1, Topology **9** (1970), 153–154. MR0256413

[24] S. Weil, *Metric foliations of space forms of nonnegative sectional curvature*, Master thesis, Bonn, 2010.

Received October 1, 2013; accepted June 15, 2014

Luis A. Florit
IMPA,
Est. Dona Castorina, 110 22460-320, Rio de Janeiro, RJ, Brazil
E-mail: luis@impa.br

Oliver Goertsches
Fachbereich Mathematik und Informatik, Philipps-Universität Marburg,
Hans-Meerwein-Str. 6, 35032 Marburg, Germany
E-mail: oliver.goertsches@math.uni-hamburg.de

Alexander Lytchak
Mathematisches Institut, Universität Köln,
Weyertal 86-90, 50931 Köln, Germany
E-mail: alytchak@math.uni-koeln.de

Dirk Töben
Departamento de Matemática, Universidade Federal São Carlos,
Rod. Washington Luiz, 13565-905, São Carlos, SP, Brazil
E-mail: dirktoben@dm.ufscar.br