On Chen's basic equality

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Given an isometric immersion $f\colon M^n \to \mathbb{Q}_c^{n+p}$ of a riemannian manifold into a space of constant sectional curvature c, it was shown by B. Y. Chen ([**Ch**$_1$]) that the inequality

$$\delta_M \leq \frac{n-2}{2}\left\{\frac{n^2}{(n-1)}\|H\|^2 + (n+1)c\right\} \tag{1}$$

holds pointwise. Here, H denotes the mean curvature vector of f and δ_M stands for the intrinsic invariant defined as

$$\delta_M(x) = s(x) - \inf\{K(\sigma) : \sigma \subset T_xM\},$$

where K and s denote, respectively, sectional and not normalized scalar curvature of M^n.

It is then natural to try to understand all submanifolds for which equality in (1) holds everywhere. In euclidean space, Chen showed that the *trivial* examples satisfying his *basic equality* are either affine subspaces or rotation hypersurfaces obtained by rotating a straight line, that is, cones and cylinders. Nontrivial examples for $n \geq 4$ divide in two classes, namely, *any* minimal submanifolds of rank two, which we completely describe in [**DF**], and a certain class of nonminimal submanifolds foliated by totally umbilic spheres of codimension two.

In this paper, we show that connected elements in Chen's second nontrivial class have the simplest possible geometric structure among submanifolds foliated by totally umbilic spheres, namely, they are rotation submanifolds over surfaces. This means that M^n is isometric to an open subset of a warped product $L^2 \times_\varphi \mathbb{S}_1^{n-2}$, $\varphi \in C^\infty(L)$ positive, and

$$f(x,y) = (h(x), \varphi(x)\, y) \tag{2}$$

being $h\colon L^2 \to \mathbb{R}^{p+1}$ a surface and $\mathbb{S}_1$ denotes a unit sphere. The surface $k := (h, \varphi)\colon L^2 \to \mathbb{R}^{p+2}$ is the profile of f.

The paper is organized as follows. First, we discuss the general problem whether a submanifold foliated by totally umbilic spheres of codimension two is rotational, and present necessary and sufficient conditions for this to occur. Then, we see that submanifolds satisfying the basic equality are either minimal or fall under those conditions. Finally, we present the restrictions for f in (2) in order to satisfy the basic equality. In particular, we show that rotational hypersurfaces over surfaces satisfying the basic equality are in correspondence with solutions of the second order quasilinear elliptical partial differential equation in the plane

$$\varphi \operatorname{tr}(R \operatorname{Hess}_\varphi) + 1 = 0, \quad \text{where } \; R = \mathrm{I} - (1 + \|\nabla\varphi\|^2)^{-1}\langle \nabla\varphi, * \rangle \nabla\varphi.$$

Rotation (n-2)-umbilic submanifolds.

We say that a submanifold $f\colon M^n \to \mathbb{Q}_c^{n+p}$ is *k-umbilic* when it carries a maximal k-dimensional totally umbilic distribution $\mathcal{U}$. This means that there exist a smooth vector field $\eta \in T_f^\perp M$ of unit length and a positive function $\mu \in C^\infty(M)$ so that

$$\mathcal{U} = \{X \in TM\colon \alpha_f(X,Y) - \mu\langle X, Y\rangle \eta = 0, \; \forall\, Y \in TM\}$$

where $\alpha_f\colon TM \times TM \to T_f^\perp M$ denotes the second fundamental form of f. It is then well-known that $\mathcal{U}$ is an integrable distribution whose leaves are totally umbilic submanifolds of M^n and $\mathbb{Q}_c^{n+p}$ along which μ is constant and η is parallel in the normal connection; cf. [**Re**] for details.

We call a k-umbilic submanifold f *generic* if it satisfies

$$\dim\{\ker(A_\eta - \mu\,\mathrm{I})(x)\} = k, \quad \forall\, x \in M^n,$$

where A_η denotes the second fundamental form of f in direction η. Any k-umbilic hypersurface is trivially generic and can be parametrized, when in euclidean space, as an $(n-k)$-parameter envelope of spheres; see [**AD**] for details. We discuss next an useful extension to higher codimension of this parametrization.

Given a submanifold $g\colon L^{n-k} \to \mathbb{R}^{n+p}$, an orthogonal smooth splitting of its normal bundle $T_g^\perp L = \Lambda^{k+1} \oplus \Lambda^\perp$, a positive function $r \in C^\infty(L)$

and a smooth section $\xi \in \Lambda^\perp$ so that $\|\nabla r\|^2 + \|\xi\|^2 < 1$, we define a map $\phi\colon \Lambda_1 \to \mathbb{R}^{n+p}$ by

$$\phi(w) = g - r\eta, \qquad \eta = \nabla r + \xi + \Omega\, w, \tag{3}$$

where $\Omega := (1 - \|\nabla r\|^2 - \|\xi\|^2)^{1/2}$ and Λ_1 stands for the unit bundle of Λ^{k+1}. Although at regular points ϕ parametrizes a submanifold foliated by k-dimensional spheres, it is *not* a k-umbilic submanifold in general. Nevertheless, we have the following basic fact.

Proposition 1. *Any generic k-umbilic submanifold $f\colon M^n \to \mathbb{R}^{n+p}$ admits locally a parametrization* (3)*.*

Proof: First observe that the map

$$g := f + r\eta, \qquad r = 1/\mu, \tag{4}$$

is constant along the leaves of $\mathcal{U}$ and, being f generic, has constant rank $n-k$. Hence, we may also consider g and r as smooth maps on the submanifold $L^{n-k} = g(M^n)$ endowed with the induced metric. Being η normal to f, we have that

$$0 = \langle \eta, f_* X\rangle = \langle \eta, g_* X\rangle - X(r), \quad \forall\, X \in TM.$$

Hence, the TL–component of η is ∇r. The proof follows now from the fact that the leaves of $\mathcal{U}$ are spheres in $\mathbb{R}^{n+p}$ and that $\|\eta\| = 1$. ∎

The following is our main result in this paper.

Theorem 2. *Let $f\colon M^n \to \mathbb{R}^{n+p}$, $n \geq 4$, be a generic $(n-2)$-umbilic submanifold and assume that $\operatorname{tr} A_\eta \neq n\mu$. Then f is a rotation submanifold over a surface if and only if $\operatorname{tr} A_\eta$ is constant along the leaves of $\mathcal{U}$.*

Proof: The direct statement is trivial. For the converse we use Proposition 1. It suffices to show that Λ^{n-1} is constant in ambient space. Then g reduces codimension to $p+1$, and the result follows.

From (4), we have

$$f_* X = g_* X - X(r)\, \eta - r\, \eta_* X, \quad \forall\, X \in TM. \tag{5}$$

Denote by P_M and P_L the orthogonal projections on TM and TL, respectively. Hence,

$$r P_M \eta_* X = P_M g_* X - f_* X \tag{6}$$

and

$$P_L f_* X = (S - rQ_w) g_* X, \tag{7}$$

being $S, Q_w : TL \to TL$ the tensors on L^2 given by

$$S = \mathrm{I} - \langle \nabla r, * \rangle \nabla r$$

and

$$Q_w = \mathrm{Hess}_r - B_\xi - \Omega B_w, \quad w \in \Lambda^{n-1},$$

where B_τ denotes the second fundamental form of L^2 relative to τ.

We claim that $T = P_L P_M|_{TL}$ is a well defined tensor on L^2. From

$$g_* X = (\mathrm{I} - rA_\eta) f_* X + \nabla^\perp_X r\eta, \tag{8}$$

we get

$$\begin{aligned} Tg_*X &= g_*X - P_L(\mathrm{I} - P_M) g_*X = g_*X - P_L(\nabla^\perp_X r\eta) \\ &= (S - P_L P_{\langle\eta\rangle^\perp}) g_* X \end{aligned}$$

where $T_f^\perp M = \langle \eta \rangle \oplus \langle \eta \rangle^\perp$. The claim follows from the fact, easy to check, that the subbundle $\langle \eta \rangle^\perp$ is constant in $\mathbb{R}^{n+p}$ along leaves of $\mathcal{U}$.

Fix a point $x \in M^n$, and let λ_1, λ_2 be the eigenvalues of A_η different from μ corresponding to the eigenvectors X_1, X_2. We want to compute $\lambda_1 + \lambda_2$ in terms of g and r. Taking the TL-component of

$$-P_M \, \eta_* X_i = \lambda_i f_* X_i$$

and using (6) and (7), we get

$$Tg_*X_i = (S - rQ_w)(1 - r\lambda_i) g_* X_i, \quad 1 \le i \le 2,$$

Now observe that $T > 0$. In fact, this is equivalent to $T_f^\perp M \cap T_g L = 0$, which follows from (8) and $\lambda_j \neq \mu$. We conclude that $S - rQ_w$ is not singular.

Our assumption yields

$$0 \neq \theta := (2\mu - \lambda_1 - \lambda_2) r = \mathrm{tr}\,(S - rQ_w)^{-1} T = \mathrm{tr}\,(P + \nu B_w)^{-1} T$$

is independent of w. Here,

$$P = S - r\mathrm{Hess}_r + rB_\xi \quad \text{and} \quad \nu = r\Omega.$$

For a pair C, D of 2×2 matrices, we have

$$\mathrm{tr}\,(C^{-1}D)\ \det C = \mathrm{tr}\,C\,\mathrm{tr}\,D - \mathrm{tr}\,(CD) = \det(C+D) - \det C - \det D,$$

where we assume that C is not singular only for the first equality. Therefore,

$$\theta\ \det(P+\nu B_w) = \mathrm{tr}\,T\,\mathrm{tr}\,(P+\nu B_w) - \mathrm{tr}\,(T(P+\nu B_w)).$$

Thus,

$$\begin{aligned}\theta\nu^2 \det B_w = \nu\,\mathrm{tr}\,(T-\theta P)\mathrm{tr}\,B_w - \nu\,\mathrm{tr}\,((T-\theta P)B_w) + \mathrm{tr}\,T\,\mathrm{tr}\,P\\ -\mathrm{tr}\,(TP) - \theta\det P, \qquad \forall\, w\in\Lambda_1.\end{aligned}$$

Since $\dim\Lambda^{n-1}\geq 3$, we easily obtain

$$\det B_w = 0,$$

$$\mathrm{tr}\,(T-\theta P)\mathrm{tr}\,B_w = \mathrm{tr}\,((T-\theta P)B_w), \tag{9}$$

and

$$\det(T-\theta P) = \det T > 0. \tag{10}$$

Suppose that $B_{w_0}\neq 0$ for some $w_0\in\Lambda^{n-1}$. Then (9) yields $\langle (T-\theta P)v, v\rangle = 0$ for $0\neq v\in\ker B_{w_0}$, which is in contradiction with (10) and proves that

$$B_w = 0, \quad \forall\, w\in\Lambda^{n-1}. \tag{11}$$

Since leaves of $\mathcal{U}$ are the images of Λ_1 under parametrization (3), we have

$$\langle\eta\rangle^\perp \subset T_{g(x)}L\oplus\Lambda^\perp_{g(x)}, \quad \forall\, x\in M^n. \tag{12}$$

Observe that $T^\perp_{f(x)}M\cap T_{g(x)}L = 0$ implies that $\langle\eta\rangle^\perp\cap T_{g(x)}L = 0$. Hence, the orthogonal projection

$$\pi(x)\colon \langle\eta(x)\rangle^\perp \subset T^\perp_x M \to \Lambda^\perp_{g(x)} \subset T^\perp_{g(x)}L$$

is an isomorphism. On the other hand, we have using (5) that

$$\langle g_*Y - r\tilde{\nabla}_Y(\nabla r + \xi + \Omega\, w), \delta\rangle = 0, \quad \forall\, Y\in TL,\ w\in\Lambda^{n-1} \text{ and } \delta\in\langle\eta\rangle^\perp.$$

It follows from (11), (12) and that $\langle\eta\rangle^\perp$ is constant along the leaves that

$$\langle\tilde{\nabla}_Y w, \delta\rangle = 0, \quad \forall\, Y\in TL,\ w\in\Lambda^{n-1} \text{ and } \delta\in\langle\eta\rangle^\perp.$$

Being π an isomorphism, we conclude from (11) that Λ^{n-1} is constant and this proves the theorem. ∎

Remark 3. The assumption that $\operatorname{tr} A_\eta \neq n\mu$ in Theorem 2 is essential. Otherwise, from $\theta = 0$ in the proof we have that $\operatorname{tr} T \operatorname{tr} B_w = \operatorname{tr}(TB_w)$ and $\operatorname{tr} T \operatorname{tr} P = \operatorname{tr}(TP)$. This alone does *not* imply that f is rotational. For instance, in the hypersurface case, we conclude that f is $(n-2)$-umbilic with $\operatorname{tr} A_\eta = n\mu$ if and only if $\{g, r\}$ in (3) satisfies $\operatorname{tr}(S^{-1}B_w) = 0$ for all $w \in T_g^\perp L$ and $r \operatorname{tr}(S^{-1}\mathrm{Hess}_r) = 2$.

Theorem 4. *Let* $f\colon M^n \to \mathbb{S}_1^{n+p}$, $n \geq 4$, *be a generic* $(n-2)$*-umbilic submanifold and assume that* $\operatorname{tr} A_\eta \neq n\mu$. *Then* $\operatorname{tr} A_\eta$ *is constant along the leaves of* $\mathcal{U}$ *if and only if there exists a surface* $h\colon L^2 \to \mathbb{R}^{p+2}$, $\|h\| < 1$, *such that* $f\colon M^n \subset L^2 \times_{\sqrt{1-\|h(x)\|^2}} \mathbb{S}_1^{n-2} \to \mathbb{S}_1^{n+p} \subset \mathbb{R}^{n+p+1}$ *is a rotation submanifold.*

Proof: It suffices to show that the composition $\hat{f}$ of f with the inclusion of $\mathbb{S}_1^{n+p}$ into $\mathbb{R}^{n+p+1}$ satisfies the conditions in Theorem 2. In fact, the principal curvatures for the umbilic direction $\hat{\eta} = 1/\sqrt{1+\mu^2}\,(\mu\eta - f)$ for $\hat{f}$ are

$$\hat{\lambda}_j = \frac{1}{\sqrt{1+\mu^2}}\,(\lambda_j\mu - 1),\ j = 1, 2, \qquad \hat{\mu} = \frac{1}{\sqrt{1+\mu^2}}\,(\mu^2 - 1),$$

and the proof follows. ∎

Next, we analyze nongeneric $(n-2)$-umbilic submanifolds.

Theorem 5. *Assume that* $f\colon M^n \to \mathbb{R}^{n+p}$ *is a* $(n-2)$*-umbilic submanifold with*

$$\dim\{\ker(A_\eta - \mu\, I)(x)\} = n - 1, \quad \forall\, x \in M^n.$$

Then f *is a rotation submanifold over a surface if and only if the mean curvature vector is parallel in the normal connection along the leaves of* $\mathcal{U}$.

Proof: The direct statement is trivial. For the converse, let $X, Y \in \mathcal{U}^\perp$ be orthonormal eigenvectors for A_η with eigenvalues λ, μ, respectively. By assumption, there is a smooth field of unit length $\xi \in T_f^\perp M$, $\xi \perp \eta$, parallel along $\mathcal{U}$ with $A_\xi Y \neq 0$ and $\operatorname{tr} A_\xi$ constant along $\mathcal{U}$. Taking the $\mathcal{U}^\perp$-component of the Codazzi equation for (X, T, η), i.e.,

$$(\nabla_X A_\eta)T - A_{\nabla_X^\perp \eta} T = (\nabla_T A_\eta)X - A_{\nabla_T^\perp \eta} X, \quad T \in \mathcal{U},$$

we get

$$\nabla_X^v X = 0, \tag{13}$$

where Z^v (respectively, Z^h) denotes taking the $\mathcal{U}$ (respectively, $\mathcal{U}^\perp$) component of Z. Similarly, the X-component of the Codazzi equation for (Y, T, η) yields

$$\nabla^v_Y X = 0. \tag{14}$$

Now, a straightforward computation of the Codazzi equations for (X, T, ξ) and (Y, T, ξ) gives

$$\langle \nabla_Y Y, T\rangle\langle A_\xi Y, Y\rangle + \langle \nabla_X Y, T\rangle\langle A_\xi Y, X\rangle = 0$$

and

$$\langle \nabla_X Y, T\rangle\langle A_\xi Y, Y\rangle - \langle \nabla_Y Y, T\rangle\langle A_\xi Y, X\rangle = 0,$$

from which we conclude that

$$\nabla^v_Y Y = 0 = \nabla^v_X Y. \tag{15}$$

Equations (13), (14) and (15) say that the distribution $\mathcal{U}^\perp$ is totally geodesic (autoparallel) in M^n. The following observation (cf. [**DT**]) concludes the proof. ∎

Lemma 6. *Let $f\colon M^n \to \mathbb{R}^{n+p}$ be k-umbilic. Assume that the distribution $\mathcal{U}^\perp$ is totally geodesic (autoparallel) in M^n. Then f is a rotation submanifold.*

Proof: Let γ denote the mean curvature vector of the leaves of $\mathcal{U}$ in M^n, i.e.,

$$\nabla^h_S T = \langle S, T\rangle\gamma, \quad \forall\, S, T \in \mathcal{U}.$$

Take $X \in \mathcal{U}^\perp$ and $T \in \mathcal{U}$ of unit length. We have,

$$\nabla^h_T \nabla_X T = R^h(T, X)T + \nabla^h_X \nabla^v_T T + \nabla^h_X \gamma + \nabla^h_{[T,X]^v} T + \nabla^h_{[T,X]^h} T.$$

A straightforward computation using that $\mathcal{U}^\perp$ is totally geodesic and the Gauss equation yields

$$\nabla_X \gamma = \mu A_\eta X + \langle X, \gamma\rangle\gamma. \tag{16}$$

We claim that the mean curvature vector $\sigma = \gamma + \mu\eta$ of the leaves of $\mathcal{U}$ in euclidean space satisfies

$$\widetilde{\nabla}_X \sigma = \langle X, \gamma\rangle\sigma, \quad \forall\, X \in \mathcal{U}^\perp. \tag{17}$$

In fact, the T-component of the Codazzi equation for (X,T,η) gives

$$X(\mu)=\mu\langle X,\gamma\rangle-\langle A_\eta X,\gamma\rangle. \tag{18}$$

On the other hand, the Codazzi equation for A_ξ yields

$$\langle A_\xi X,\gamma\rangle+\mu\langle\nabla^\perp_X\eta,\xi\rangle=0. \tag{19}$$

To obtain the claim, compute $\tilde\nabla_X\sigma$ and use (16), (18) and (19).

From the claim,

$$X(\|\sigma\|^2)=2\langle X,\gamma\rangle\|\sigma\|^2. \tag{20}$$

Set

$$\Gamma=f+\|\sigma\|^{-2}\sigma.$$

Using (17) and (20), we get

$$\tilde\nabla_T\Gamma=0\quad\text{and}\quad\tilde\nabla_X\Gamma=X-\|\sigma\|^{-2}\langle X,\gamma\rangle\sigma,\qquad\forall\,T\in\mathcal{U},\ X\in\mathcal{U}^\perp.$$

From $\tilde\nabla_X T=\nabla_X T\in\mathcal{U}$ and (17), the subspaces $L=\mathcal{U}\oplus\operatorname{span}\{\sigma\}$ containing the leaves of $\mathcal{U}$ are parallel in ambient space. Since Γ_*X is orthogonal to $\mathcal{U}$ and σ for all $X\in\mathcal{U}$, we conclude that Γ is contained in an affine subspace orthogonal to L, and the proof follows. ∎

The basic equality.

In this section we deal with nonminimal submanifolds satisfying the basic equality.

Theorem 7. *Let $f\colon M^n\to\mathbb{R}^{n+p}$, $n\geq 4$, be a connected submanifold satisfying everywhere the basic equality which is nowhere trivial or minimal. Then f is any rotation submanifold with profile $k\colon L^2\to\mathbb{R}^{p+2}$ whose mean curvature vector H satisfies the condition*

$$e=\nabla\varphi-2\varphi H, \tag{21}$$

where $\varphi=\langle k,e\rangle$ is the high function of k with respect to a constant vector $e\in\mathbb{R}^{p+2}$ of unit length and L^2 is endowed with the metric induced by k.

First, we give the general analytic conditions for a rotation submanifold to satisfy the basic equality.

Proposition 8. *Let $f\colon M^n = L^2 \times_\varphi \mathbb{S}_1^{n-2} \to \mathbb{R}^{n+p}$ be a rotation submanifold. Then $f(x,y) = (h(x), \varphi(x)\, y)$ satisfies the basic equality if and only if φ is a solution on L^2 of the second order quasilinear elliptical differential equation*

$$\varphi\, tr\,(R\, Hess_\varphi) + 1 = 0, \tag{22}$$

and the second fundamental form of $h\colon L^2 \to \mathbb{R}^{p+1}$ satisfies

$$tr\;(R\, B^h_\xi) = 0, \quad \forall\, \xi \in T^\perp_h L. \tag{23}$$

Proof: We have,

$$T^\perp_f M = T^\perp_h L \oplus^\perp \langle \eta \rangle, \qquad \eta(x,y) = \frac{1}{\sqrt{1+\|\nabla\varphi\|^2}}(\nabla\varphi, -y).$$

Moreover,

$$f_*(X,0) = (X, \langle \nabla\varphi, X\rangle y) \;\text{ and }\; f_*(0,v) = \varphi(0,v). \tag{24}$$

In particular,

$$\mu^{-1} = \varphi\sqrt{1+\|\nabla\varphi\|^2}. \tag{25}$$

Now take $(\xi, 0) \in T^\perp_f M$ where $\xi \in T^\perp_h L$. Then, $\xi_*(X,0) = (-B^h_\xi X + \nabla^\perp_X \xi, 0)$. We have,

$$\begin{aligned} A^f_\xi X &= -\xi_* X + (\xi_* X)_{T^\perp_h L} + \langle \xi_* X, \eta\rangle \eta \\ &= B^h_\xi X - \frac{1}{1+\|\nabla\varphi\|^2}\langle B^h_\xi X, \nabla\varphi\rangle(\nabla\varphi, -y) \\ &= (R\, B^h_\xi X, \,*\,). \end{aligned}$$

From (24), we have that A^f_ξ and $R\, B^h_\xi$ have the same eigenvalues. Thus,

$$\operatorname{tr} A^f_\xi = 0 \iff \operatorname{tr} R\, B^h_\xi = 0.$$

Also,

$$\begin{aligned} A^f_\eta X &= -\eta_* X + (\eta_* X)_{T^\perp_h L} = (-(\eta_* X)_{TL}, \,*\,) \\ &= \left(\frac{\langle \mathrm{Hess}_\varphi X, \nabla\varphi\rangle \nabla\varphi}{(1+\|\nabla\varphi\|^2)^{3/2}} - \frac{\mathrm{Hess}_\varphi X}{\sqrt{1+\|\nabla\varphi\|^2}}, \,*\right) \\ &= \frac{-1}{\sqrt{1+\|\nabla\varphi\|^2}}\,(R\,\mathrm{Hess}_\varphi X, \,*\,)\,. \end{aligned}$$

From (24) and (25), it follows that

$$\begin{aligned}\operatorname{tr} A^f_\eta = (n-1)\mu \quad &\Longleftrightarrow \quad \frac{-1}{\sqrt{1+\|\nabla\varphi\|^2}} \operatorname{tr} R\,\mathrm{Hess}_\varphi = \mu \\ &\Longleftrightarrow \quad \operatorname{tr} R\,\mathrm{Hess}_\varphi = -1/\varphi,\end{aligned}$$

and this concludes the proof. ∎

Proof of Theorem 7: By Lemmas 3.2 and 3.3 of [$\mathbf{Ch}_1$] and our assumptions there are two possibilities along each connected component of an open dense subset. Namely, f is either $(n-1)$-umbilic or is $(n-2)$-umbilic. Moreover, in both situations η is in the direction of the mean curvature vector and $\operatorname{tr} A_\eta = (n-1)\mu$. Then f is trivial in the first case and, in the second case, it follows from Theorems 2 and 5 that f is a rotation submanifold.

We use Proposition 8 to conclude the proof. A straightforward computation of the mean curvature vector of the profile k yields that condition (21) is equivalent to equations (22) and (23). ∎

We now extend a result in [**CY**] to arbitrary codimension.

Corollary 9. *Let $f\colon M^n \to \mathbb{R}^{n+p}$, $n \geq 4$, be a connected submanifold with constant mean curvature satisfying the basic equality. Then f is either a minimal submanifold or an open subset of a riemannian product $\mathbb{R}\times\mathbb{S}^{n-1}_c \subset \mathbb{R}^{n+1}$.*

Proof: Suppose that f is a rotation submanifold over a surface. By assumption and (25),

$$\varphi^2(1+\|\nabla\varphi\|^2) = r > 0 \tag{26}$$

is constant in M^n. Let $\{X_1, X_2\}$ be a local orthonormal frame such that $X_2(\varphi) = 0$. Notice that φ cannot be constant on an open subset by (22). Taking the derivative of (26) in direction X_2 we get $X_2X_1(\varphi) = 0$. From $X_1X_2(\varphi) = 0$ it follows that

$$[X_1, X_2] \in \langle X_2\rangle. \tag{27}$$

Hence, there exists $\lambda \in C^\infty(L^2)$ so that $\{X_1, \lambda X_2\}$ are the coordinate fields of a coordinate system (u, v) and $\varphi = \varphi(u)$. The derivative of (26) gives

$$\varphi'' = -r\varphi^{-3}. \tag{28}$$

On the other hand, a straightforward computation of (22) yields

$$\varphi^2\varphi'' + r\varphi\langle\nabla_{X_2}X_1, X_2\rangle + r\varphi^{-1} = 0. \tag{29}$$

From (28) and (29) we obtain that $\nabla_{X_2}X_1 = 0$. We conclude from this and (27) that L^2 is flat. Notice that $\{X_1, X_2\}$ are coordinates fields for a euclidean system of coordinates. We have,

$$RX_1 = (1 + (\varphi')^2)^{-1}X_1 \quad \text{and} \quad RX_2 = X_2.$$

We easily obtain from this, (23) and the Gauss equation that h is totally geodesic. On the other hand, we have from (26) that $\varphi(u) = \sqrt{r - u^2}$, and this concludes the proof. ∎

Remarks 10. 1) Notice that Theorem 7 and Corollary 9 hold for submanifolds in the sphere.
2) For hypersurfaces, the second condition in Proposition 8 is trivially satisfied since g parametrizes an affine plane.
3) Theorems 3.1 and 3.2 in [$\mathbf{Ch}_2$] and Theorems 3 and 4 in [**CY**] for $n \geq 4$ follow immediately from our results.

References

[**AD**] Asperti, A. and Dajczer, M., *N-dimensional submanifolds of* R^{N+1} *and* S^{N+2}. Illinois J. of Math. **28** (1984), 621–645.

[**Ch**$_1$] Chen, B. Y., *Some pinching and classification theorems for minimal submanifolds*, Arch. Math. **60** (1993) 568–578.

[**Ch**$_2$] Chen, B. Y., *Tubular hypersurface satisfying a basic equality*, Soochow J. Math. **20** (1994) 569–586.

[**CY**] Chen, B. Y. and Yang, J., *Elliptic functions, theta function and hypersurfaces satisfying a basic equality*, Preprint.

[**DF**] Dajczer, M. and Florit, L., *A class of austere submanifolds*, Preprint.

[**DT**] Dajczer, M. and Tojeiro, R., *On Cartan's conformally deformable hypersurfaces*, Preprint.

[**Re**] Reckziegel, H., *Completeness of curvature surfaces of an isometric immersion*, J. Diff. Geometry **14** (1979), 7–20.

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