A class of austere submanifolds

Marcos Dajczer & Luis A. Florit

To Prof. Detlef Gromoll on his 60^{th} *birthday.*

Austerity is a pointwise algebraic condition on the second fundamental form of a submanifold in euclidean space. It requires that the nonzero principal curvatures in any normal direction occur in oppositely signed pairs. Introduced by Harvey and Lawson ([**HL**]) in the context of special Lagrangian submanifolds, the austerity condition is, aside from surfaces, much stronger than minimality. Immediate examples of austere submanifolds are holomorphic submanifolds and cones of minimal spherical surfaces. A large class of non-holomorphic ones are the minimal real Kaehler submanifolds; see [$\mathbf{DG}_2$] and [$\mathbf{DG}_4$].

Among other results, R. Bryant ([**Br**]; see also [**Bo**]) described parametrically the austere submanifolds of dimension three locally. These are submanifolds of "rank two", i.e., the Gauss map has rank two, or equivalently, the kernel of the second fundamental form has constant codimension two. Observe that under this condition austerity and minimality are equivalent.

Our main result is an extension of Bryant's description to rank two austere submanifolds of arbitrary dimension. Bryant himself remarked the similarity between his parametrization and the Gauss parametrization from [$\mathbf{DG}_1$] when dealing with hypersurfaces. In this paper we provide two alternative "dual" classifications. One is the *polar parametrization*, an extension of the Gauss parametrization for hypersurfaces of rank two. It performs better for submanifolds in low codimension. The other one reduces to Bryant's in the three–dimensional case and we call it the *bipolar parametrization*. It is much easier to compute in most situations.

In this paper we proceed as follows. We first observe that austere submanifolds of rank two belong to a much broader class of rank two submanifolds which we call *elliptic*. Then we construct the above pair of parametrizations for all elements in this class. Roughly speaking, we prove that locally an elliptic submanifold is parametrically determined by a (euclidean or spherical) associated *polar* or *bipolar* elliptic surface and a function on the surface which satisfies a certain elliptic PDE. Classically, euclidean elliptic surfaces are contained in the larger class of surfaces called *nets* and studied by Eisenhart ([**Ei**]) in local coordinates. The defining condition is that all coordinate functions satisfy the same differential equation

$$A\frac{\partial^2}{\partial x^2} + 2B\frac{\partial^2}{\partial x \partial y} + C\frac{\partial^2}{\partial y^2} + D\frac{\partial}{\partial x} + E\frac{\partial}{\partial y} = 0,$$

where $A, \ldots, E$ are smooth functions defined on an open subset of the plane. Ellipticity of the surface means, of course, that $AC - B^2 > 0$.

Extending a well-known construction from the theory of minimal surfaces, one may associate to any elliptic surface a sequence of ellipses of curvature. It turns out that an elliptic submanifold is austere if and only if the ellipse of curvature of a certain order of the associated (polar or bipolar) elliptic surface is a circle.

We should point out that our classification problem is essentially of a local nature thus making the parametric approach satisfactory. In fact, we prove that, up to euclidean factor, complete elliptic submanifolds may have dimension at most three, and provide an explicit three dimensional irreducible example. In higher dimensions, we show that the set of singular points admits a Whitney stratification by elliptic submanifolds with dimensions decreasing by two.

In their paper *Calibrated Geometries*, Harvey and Lawson proved that the canonical Lagrangian immersion in $\mathbb{C}^N$ of the normal bundle of a submanifold in $\mathbb{R}^N$ is special Lagrangian if and only if the submanifold is austere. Special Lagrangian submanifolds are of interest because they are not only minimal but absolutely area minimizing. Here we construct new special Lagrangian submanifolds generalizing those of [**HL**]. They are *not* normal bundles over austere submanifolds in general, and have quite interesting singularities.

We finish the paper with the study of rank two euclidean submanifolds which are Kaehler manifolds. We first show that nonflat irreducible real Kaehler submanifolds of rank two other than surfaces or hypersurfaces (classified in [$\mathbf{DG}_2$]) are austere submanifolds. This result is somewhat unexpected since the hypersurface situation is quite different. Our main result in this topic is a complete description of the rank two real Kaehler submanifolds by means of a Weierstrass–type representation originated from our bipolar parametrization. The parametrization of the holomorphic ones is rather simple, and goes as follows.

Take a holomorphic curve $g\colon U\subset\mathbb{C}\to\mathbb{R}^{2m}\cong\mathbb{C}^m$ defined on a simply connected domain, and let $\Psi\colon U\times\mathbb{C}^{n-1}\to\mathbb{R}^{2m}, n+1\leq 2m$, be given as

$$\Psi(z,w)=\mathrm{Re}\left\{\int^z\psi\frac{dg}{dz}dz+\sum_{j=1}^{n-1}w_j\frac{d^jg}{dz^j}(z)\right\},$$

where ψ is a holomorphic function on U. Then Ψ parametrizes a holomorphic Kaehler submanifold of rank two and, conversely, any such submanifold can be parametrized this way at least locally.

We conclude this introduction pointing out that minimal submanifolds of rank two are also interesting in a quite different context. B. Y. Chen ([**Cb**]) has shown that any minimal euclidean submanifold M^n satisfies pointwise the inequality $2\inf K\geq n(n-1)s$, where K and s denote the sectional and the scalar curvature of M^n respectively. Equality, an intrinsic condition, holds if and only if the minimal submanifold either has rank two or is totally geodesic; see also [**DF**].

§1 Elliptic submanifolds.

After some preliminaries, we introduce the concept of elliptic submanifold and analyze in detail the consequences that ellipticity has on the structure of the normal bundle. Then, we turn our attention to the special case of elliptic surfaces and other related tools in the construction of our parametrizations.

Throughout this paper, we denote by $f\colon M^n\to\mathbb{Q}_\epsilon^N$, $\epsilon=0,1$, a submanifold of either euclidean space $\mathbb{R}^N$ $(\epsilon=0)$ or unit euclidean sphere $\mathbb{S}^N$ $(\epsilon=1)$ with substantial codimension $N-n$. The k^{th}*-normal space* $N_k^f(x)$ of f at $x\in M^n$ is defined as

$$N_k^f(x)=\mathrm{span}\{\alpha_f^{k+1}(X_1,\ldots,X_{k+1}):\ \forall\, X_1,\ldots,X_{k+1}\in T_xM\},$$

where $\alpha_f^\ell\colon TM\times\cdots\times TM\to T_f^\perp M$, $\ell\geq 2$, is the symmetric tensor called the ℓ^{th}*-fundamental form* and given by

$$\alpha_f^\ell(X_1,\ldots,X_\ell)=\pi^{\ell-1}\left(\nabla^\perp_{X_\ell}\ldots\nabla^\perp_{X_3}\alpha_f(X_2,X_1)\right).$$

Here, $\pi^1 = I$ and π^ℓ stands for the projection onto $(N_1^f \oplus \ldots \oplus N_{\ell-1}^f)^\perp \cap T_f^\perp M$. We denote $\alpha_f^2 = \alpha_f$, and agree that $\alpha_f^1\colon TM \to TM$ is $\alpha_f^1 = I$. Whenever necessary, we admit that all N_k^f's form subbundles of the normal bundle. Clearly, this condition is verified along connected components of an open dense subset of M^n.

From now on, we assume that $f\colon M^n \to \mathbb{Q}_\epsilon^N$ has constant rank 2. This means that the relative nullity subspaces $\Delta(x) \subset T_xM$, defined as

$$\Delta(x) = \{X:\ \alpha_f(X,Y) = 0,\ \forall\, Y \in T_xM\},$$

form a tangent subbundle of codimension two. Recall that the leaves of the integrable relative nullity distribution are totally geodesic submanifolds in the ambient $\mathbb{Q}_\epsilon^N$.

The cone $Cf\colon M^n \times \mathbb{R}_+ \to \mathbb{R}^{N+1}$ of a submanifold $f\colon M^n \to \mathbb{S}^N$ of rank two has the same rank since the relative nullity leaves of Cf are the cones of the relative nullity leaves of f. Moreover, one has that $N_k^{Cf} = N_k^f$, $k \geq 1$, up to parallel transport in $\mathbb{R}^{N+1}$. Thus, it suffices to consider the euclidean case since we had restricted ourselves to submanifolds of $\mathbb{R}^N$ and $\mathbb{S}^N$.

The rank condition and the symmetry of the second fundamental form imply that the first normal spaces of f satisfy $\dim N_1^f \leq 3$. Theorem 1 in [**DT**] says that f is a hypersurface in substantial codimension when $\dim N_1^f = 1$. On the other hand, one can show that a submanifold with $\dim N_1^f = 3$ is either a euclidean surface or the cone over a spherical surface up to euclidean factor. In the remaining case $\dim N_1^f = 2$, at a point either there exist linearly independent "conjugate directions" $X_1, X_2 \in \Delta^\perp$, i.e., $\alpha_f(X_1,X_1) \pm \alpha_f(X_2,X_2) = 0$, or f admits an "asymptotic direction" $0 \neq X \in \Delta^\perp$, i.e., $\alpha_f(X,X) = 0$.

Proposition 1. *If $f\colon M^n \to \mathbb{Q}_\epsilon^N$ satisfies $\dim N_1^f = 2$, then $\dim N_k^f \leq 2$, for all $k \geq 1$.*

Proof: If there exists a pair of conjugate directions, we thus have

$$\begin{aligned}\alpha_f^{k+1}&(X_1,X_1,Y_1,\ldots,Y_{k-1}) \pm \alpha_f^{k+1}(X_2,X_2,Y_1,\ldots,Y_{k-1})\\ &= \pi^k\left(\nabla^\perp_{Y_{k-1}}\ldots\nabla^\perp_{Y_1}(\alpha_f(X_1,X_1) \pm \alpha_f(X_2,X_2))\right) = 0,\end{aligned}$$

and the proof follows easily. The argument in the case of an asymptotic direction is similar. ∎

Given a submanifold $f\colon M^n \to \mathbb{Q}_\epsilon^N$ with $\dim N_1^f = 2$, we analyze the case of conjugate $X_1, X_2 \in \Delta^\perp$ so that $\alpha_f(X_1,X_1) + \alpha_f(X_2,X_2) = 0$ everywhere. The pairs $aX_1 + bX_2$, $aX_2 \mp bX_1$ also satisfy the condition and, up to signs, there are no others. Thus, the almost complex structure $J\colon \Delta^\perp \to \Delta^\perp$ ($J^2 = -\mathrm{I}$) given by $JX_1 = X_2$ and $JX_2 = -X_1$ is locally well defined up to sign. Notice that J is orthogonal only when f is minimal.

Definition 2. We call *elliptic* a submanifold $f\colon M^n \to \mathbb{Q}_\epsilon^N$ in codimension $N - n \geq 2$ if it has rank 2 and there is a (necessarily unique up to sign) almost complex structure $J\colon \Delta^\perp \to \Delta^\perp$ such that

$$\alpha_f(Z,Z) + \alpha_f(JZ,JZ) = 0, \quad \forall\, Z \in \Delta^\perp. \tag{1}$$

Notice that cones of elliptic spherical submanifolds are trivially elliptic. Moreover, if $\tau = \tau_f$ denotes the index of the "last" of the normal subbundles of f, i.e.,

$$T_f^\perp M = N_1^f \oplus \cdots \oplus N_\tau^f, \tag{2}$$

then $\sum_{i=1}^\tau \dim N_i^f = N - n$ since f is by assumption substantial. Set,

$$\tau^* = \begin{cases} \tau & \text{if } N-n \text{ is even} \\ \tau - 1 & \text{if } N-n \text{ is odd.} \end{cases}$$

Definition 3. Given an elliptic submanifold $f\colon M^n \to \mathbb{Q}_\epsilon^N \subseteq \mathbb{R}^{N+\epsilon}$, we call an element $\beta \in C^\infty(M^n, \mathbb{R}^{N+\epsilon})$ an *s–cross section* to f, $1 \leq s \leq \tau^*$, when

$$d\beta\,(TM) \subset N_s^f \oplus \cdots \oplus N_\tau^f,$$

at each point, up to parallel transport in $\mathbb{R}^{N+\epsilon}$.

For the sake of simplicity, we now argue with the help of the pair of normal vector fields $\xi_1^k, \xi_2^k \in N_k^f$ defined as

$$\xi_1^k = \alpha_f^{k+1}(\mathcal{Z}, \overset{k+1}{\overbrace{\ldots}}, \mathcal{Z}), \quad \xi_2^k = \alpha_f^{k+1}(J\mathcal{Z}, \mathcal{Z}, \overset{k}{\overbrace{\ldots}}, \mathcal{Z}), \quad k \geq 0.$$

Here, $\mathcal{Z} \in N_0^f := \Delta^\perp$ stands for a fixed arbitrary local vector field which does not vanish at any point. Let $\mathcal{V}_s \subset N_s^f \times N_s^f$, $0 \leq s \leq \tau$, be the subspace defined as

$$\mathcal{V}_s = \{(\mu_1, \mu_2) \in N_s^f \times N_s^f\colon \langle \mu_1, \xi_1^s\rangle + \langle \mu_2, \xi_2^s\rangle = 0 = \langle \mu_2, \xi_1^s\rangle - \langle \mu_1, \xi_2^s\rangle\}, \tag{3}$$

and let $\mathcal{P}_s\colon C^\infty(M^n, \mathbb{R}^{N+\epsilon}) \to N_s^f \times N_s^f$ be given by

$$\mathcal{P}_s(\beta) = \left((\widetilde{\nabla}_{\mathcal{Z}}\beta)_{N_s^f}, (\widetilde{\nabla}_{J\mathcal{Z}}\beta)_{N_s^f} \right).$$

Lemma 4. *With the above notations, we have:*

i) Any nonzero element in $\mathcal{V}_s$ is a basis of N_s^f. Moreover, $\dim \mathcal{V}_s = 2$ if and only if $\dim N_s^f = 2$, and $\mathcal{V}_s = 0$ if and only if $\dim N_s^f = 1$.

ii) $\mathcal{P}_s(\beta) \in \mathcal{V}_s$ for any s–cross section β to f. In particular, the tensor $\mathcal{P}_s|_{N_{s+1}^f}\colon N_{s+1}^f \to \mathcal{V}_s$ is injective when $s \leq \tau - 1$, thus an isomorphism for $s \leq \tau^* - 1$.

Proof: From the proof of Proposition 1, we get $N_k^f = \operatorname{span}\{\xi_1^k,\ \xi_2^k\}$, $k \geq 0$, and the proof of part $i)$ follows immediately.

By definition,

$$\xi_1^{k+1} = \left(\widetilde{\nabla}_{\mathcal{Z}}\xi_1^k\right)_{N_{k+1}^f} \quad \text{and} \quad \xi_2^{k+1} = \left(\widetilde{\nabla}_{\mathcal{Z}}\xi_2^k\right)_{N_{k+1}^f}, \quad k \geq 0.$$

Let us show that

$$\xi_1^{k+1} = -\left(\widetilde{\nabla}_{J\mathcal{Z}}\xi_2^k\right)_{N_{k+1}^f} \quad \text{and} \quad \xi_2^{k+1} = \left(\widetilde{\nabla}_{J\mathcal{Z}}\xi_1^k\right)_{N_{k+1}^f}, \quad k \geq 0.$$

We only prove the first equation since the second one follows by a similar argument. We compute,

$$\begin{aligned}
\xi_1^{k+1} &= \alpha_f^{k+2}(\mathcal{Z}, \ldots, \mathcal{Z}) = \pi^{k+1}(\nabla_{\mathcal{Z}}^\perp \ldots \nabla_{\mathcal{Z}}^\perp \alpha_f(\mathcal{Z}, \mathcal{Z})) \\
&= -\pi^{k+1}(\nabla_{\mathcal{Z}}^\perp \ldots \nabla_{\mathcal{Z}}^\perp \alpha_f(J\mathcal{Z}, J\mathcal{Z})) = -\alpha_f^{k+2}(J\mathcal{Z}, J\mathcal{Z}, \mathcal{Z}, \ldots, \mathcal{Z}) \\
&= -\left(\widetilde{\nabla}_{J\mathcal{Z}}\alpha_f^{k+1}(J\mathcal{Z}, \mathcal{Z}, \ldots, \mathcal{Z})\right)_{N_{k+1}^f},
\end{aligned}$$

and the claim follows.

To prove part $ii)$ we first verify the conditions in (3). We have,

$$\begin{aligned}
\langle \widetilde{\nabla}_{\mathcal{Z}}\beta, \xi_1^s \rangle &= -\langle \widetilde{\nabla}_{\mathcal{Z}}\beta, \widetilde{\nabla}_{J\mathcal{Z}}\xi_2^{s-1} \rangle = \langle \widetilde{\nabla}_{J\mathcal{Z}}\widetilde{\nabla}_{\mathcal{Z}}\beta, \xi_2^{s-1} \rangle = \langle \widetilde{\nabla}_{\mathcal{Z}}\widetilde{\nabla}_{J\mathcal{Z}}\beta, \xi_2^{s-1} \rangle \\
&= -\langle \widetilde{\nabla}_{J\mathcal{Z}}\beta, \widetilde{\nabla}_{\mathcal{Z}}\xi_2^{s-1} \rangle = -\langle \widetilde{\nabla}_{J\mathcal{Z}}\beta, \xi_2^s \rangle.
\end{aligned}$$

Similarly, $\langle \widetilde{\nabla}_{J\mathcal{Z}}\beta, \xi_1^s \rangle = \langle \widetilde{\nabla}_{\mathcal{Z}}\beta, \xi_2^s \rangle$. To conclude the proof observe that $\mathcal{P}_s|_{N_{s+1}^f}$ is injective by the definition of the N_k^f's. ∎

The following result contains several basic facts which will be very useful throughout the paper.

Proposition 5. *With the above notations and for* $1 \le s \le \tau^*$, *we have:*

i) $\dim N_s^f = 2$ *and* $\dim N_\tau^f \le 2$*; hence* $\tau^* = [(N-n)/2]$.

ii) The almost complex structure $J_0 = J$ *on* $N_0^f = \Delta^\perp$ *induces an almost complex structure* J_s *on each* N_s^f *such that*

$$J_s(\widetilde{\nabla}_X\xi)_{N_s^f} = (\widetilde{\nabla}_X J_{s-1}\xi)_{N_s^f} = (\widetilde{\nabla}_{JX}\xi)_{N_s^f}, \quad \forall\, \xi \in N_{s-1}^f,\ X \in \Delta^\perp$$

and

$$J_{s-1}^t(\widetilde{\nabla}_X\xi)_{N_{s-1}^f} = (\widetilde{\nabla}_X J_s^t\xi)_{N_{s-1}^f} = (\widetilde{\nabla}_{JX}\xi)_{N_{s-1}^f}, \quad \forall\, \xi \in N_s^f,\ X \in \Delta^\perp.$$

iii) If $\beta\colon M^n \to \mathbb{R}^{N+\epsilon}$ *is an* s*–cross section to* f*, then*

$$J_s^t(\beta_* X)_{N_s^f} = (\beta_* JX)_{N_s^f}, \quad \forall\, X \in \Delta^\perp.$$

Proof: Part i) follows from Lemma 4. For part ii), define J_s on N_s^f by

$$J_s\alpha_f^{s+1}(X_1,\ldots,X_{s+1}) = \alpha_f^{s+1}(JX_1,\ldots,X_{s+1}). \tag{4}$$

A simple way to see that J_s is well defined is to make use of the formula

$$\alpha_f^k(\mathcal{Z}^{\varphi_1},\ldots,\mathcal{Z}^{\varphi_k}) = \cos(\Sigma\varphi_j)\xi_1^{k-1} + \sin(\Sigma\varphi_j)\xi_2^{k-1}, \tag{5}$$

where $\mathcal{Z}^\varphi = \cos\varphi\mathcal{Z} + \sin\varphi J\mathcal{Z}$. The remaining of the argument is straightforward.

Finally, to prove iii) observe that $\mathcal{V}_s = \{(\mu, J_s^t\mu)\colon \mu \in N_s^f\}$ and use that $\mathcal{P}_s(\beta) \in \mathcal{V}_s$ by Lemma 4. ∎

We now examine the important two–dimensional case. Take $X \in TL$ and $\lambda \in C^\infty(L^2)$ on an oriented riemannian manifold L^2. It is easy to see that the spherical or euclidean surface $f\colon L^2 \to \mathbb{Q}_\epsilon^N \subseteq \mathbb{R}^{N+\epsilon}$, $N \ge 4$, whose coordinate functions are any $N+\epsilon$ linearly independent solutions (with length one if $\epsilon = 1$) of the linear elliptic differential equation

$$\Delta u + X(u) + \epsilon\lambda u = 0, \tag{6}$$

is elliptic (except possibly at isolated points) with respect to the complex structure in L^2. Conversely, if one considers on a given elliptic surface

$f\colon L^2 \to \mathbb{Q}_\epsilon^N$ a metric $\langle\ ,\ \rangle_J$ which makes its almost complex structure J orthogonal, condition (1) means that all coordinate functions are solutions of (6). Now $X \in TL$ and $\lambda \in C^\infty(L^2)$ are, respectively, the constriction of the symmetric tensors $T = {}^J\nabla - \nabla$ and $\langle\ ,\ \rangle$ with respect to the metric $\langle\ ,\ \rangle_J$, i.e.,

$$X = T(e,e) + T(Je,Je) \quad \text{and} \quad \lambda = \|e\|^2 + \|Je\|^2, \quad \|e\|_J = 1. \tag{7}$$

For f minimal and taking $\langle\ ,\ \rangle_J = \langle\ ,\ \rangle$, we get $X = 0$ and $\lambda = 2$.

Even though s–cross sections have been defined for submanifolds of arbitrary dimension, our interest is restricted to surfaces. In this case, a complete characterization is achieved as a consequence of the following considerations.

Given an elliptic $g\colon L^2 \to \mathbb{Q}_\epsilon^N$, we denote by Σ the vector space of classes of functions $\varphi \in C^\infty(L^2)$ satisfying (6) where two functions which differ by a constant are considered to be equivalent only when $\epsilon = 0$. A straightforward computation shows that (6) takes the form

$$(\mathrm{Hess}_\varphi + \epsilon\varphi\,\mathrm{I})\,J = J^t\,(\mathrm{Hess}_\varphi + \epsilon\varphi\,\mathrm{I}) \tag{8}$$

with respect to the metric induced by g.

Now let $\mathcal{T}_r$, $1 \le r \le \tau_g^*$, stand for the vector space of classes of r–cross sections where two maps are equivalent when, up to a constant, differ by a section of $N_{r+1}^g \oplus \cdots \oplus N_{\tau_g}^g$. Given $[h] \in \mathcal{T}_r$, $1 \le r < s \le \tau_g^*$, it follows easily from $ii)$ in Lemma 4 that there exist unique sections $\gamma_j \in N_j^g$, $r+1 \le j \le s$, such that

$$\bar{h} = h + \gamma_{r+1} + \cdots + \gamma_s \tag{9}$$

satisfies $[\bar{h}] \in \mathcal{T}_s$. We show next that all $\mathcal{T}_r$'s are canonically isomorphic to Σ.

Given $[h] \in \mathcal{T}_r$, set $h = \epsilon\varphi g + Z + \delta$ where $\varphi \in C^\infty(L^2)$, $Z \in T_gL$ and $\delta \in T_g^\perp L$. The vanishing of the T_gL–component of h_*Y, $Y \in TL$, says that $\epsilon\varphi Y + \nabla_Y Z - A_\delta^g Y = 0$. In particular, the map $(Y,X) \mapsto \langle \nabla_Y Z, X\rangle$ has to be symmetric. An easy argument, which for $\epsilon = 1$ uses that the span$\{g\}$–component of h_*Y also vanishes, gives that $Z = \nabla\varphi$ and

$$\mathrm{Hess}_\varphi + \epsilon\varphi\mathrm{I} = A_\delta^g. \tag{10}$$

Ellipticity of g yields $A_\delta^g J = J^t A_\delta^g$. We conclude from (8) and (10) that φ satisfies (6).

Now define a linear map $\Upsilon\colon \mathcal{T}_r \to \Sigma$ by $\Upsilon([h]) = [\varphi]$. That $\Upsilon([h]) = 0$ is equivalent to $(h)_{T_gL} = \nabla\varphi = 0$. It follows from (10) that $A^g_\delta = 0$, hence $(h)_{N^g_1} = 0$. In turn, Lemma 4 yields $h \in N^g_{r+1} \oplus \cdots \oplus N^g_{\tau_g}$. Hence, Υ is injective.

Given $[\varphi] \in \Sigma$, there exists a unique $\gamma_1 \in N^g_1$ such that $A^g_{\gamma_1} = \mathrm{Hess}_\varphi + \epsilon\varphi \mathrm{I}$. This follows easily from $\dim N^g_1 = 2$ and (8). Therefore, $h^1 = \epsilon\varphi g + \nabla\varphi + \gamma_1$ satisfies $[h_1] \in \mathcal{T}_1$. We conclude from (9) that Υ is an isomorphism. In particular, we have the following recursive procedure for the construction of the r–cross sections to an elliptic surface.

Proposition 6. *Let $g\colon L^2 \to \mathbb{Q}^N_\epsilon$ be an elliptic surface. Then any r–cross section, $1 \le r \le \tau^*_g$, can be given as*

$$h_\varphi = \epsilon\varphi g + \nabla\varphi + \gamma_0 + \gamma_1 + \ldots + \gamma_r, \tag{11}$$

where φ satisfies (6) and is unique (up to a constant for $\epsilon = 0$), γ_0 is any section of $N^g_{r+1} \oplus \cdots \oplus N^g_{\tau_g}$, $\gamma_1 \in N^g_1$ is the unique solution of $A^g_{\gamma_1} = Hess_\varphi + \epsilon\varphi I$ and $\gamma_j \in N^g_j$, $2 \le j \le r$, are the unique sections given by (9). Conversely, any h_φ as in (11) is an r–cross section.

§2 Polar surfaces.

By a polar surface to an elliptic submanifold $f\colon M^n \to \mathbb{Q}^{N-\epsilon}_\epsilon \subseteq \mathbb{R}^N$ we will understand, roughly speaking, a surface whose Gauss map in the Grassmannian $G(2, N)$ coincides with the last two dimensional subbundle in the splitting (2) of the normal bundle. We first prove that any elliptic submanifold carries a polar surface. Then we show that polar surfaces are elliptic with respect to an almost complex structure naturally induced by f.

Since our work is of local nature, we may assume that an elliptic submanifold f is the saturation of a fixed cross section $L^2 \subset M^n$ to the relative nullity foliation. The almost complex structure J on $\Delta^\perp$ induces an almost complex structure $\widetilde{J}$ on TL defined by

$$P\widetilde{J} = JP, \tag{12}$$

where $P\colon TL \to \Delta^\perp$ denotes the orthogonal projection.

We claim that all subbundles in the orthogonal sum decomposition (2) are parallel in the normal connection (thus parallel in $\mathbb{Q}_\epsilon^{N-\epsilon}$) along Δ. Consequently, each N_k^f will be seen as a plane bundle along L^2. The claim for N_1^f follows using the Codazzi equation. We have,

$$(\nabla_T^\perp \alpha_f(X,Y))_{(N_1^f)^\perp} = (\nabla_X^\perp \alpha_f(T,Y))_{(N_1^f)^\perp} = 0, \quad \forall\, T \in \Delta.$$

A similar use of the Codazzi equations of higher order (see [**Sp**]) yields the same conclusion for the remaining normal subbundles.

Definition 7. A *polar surface* to an elliptic $f\colon M^n \to \mathbb{Q}_\epsilon^{N-\epsilon} \subseteq \mathbb{R}^N$ is an immersion of a cross section L^2 (as above) defined as follows:

i) When $N-n-\epsilon$ is odd, then $g\colon L^2 \to \mathbb{S}^{N-1}$ is the spherical image of a unit normal field spanning the one dimensional last normal bundle, i.e.,

$$\operatorname{span}\{g(x)\} = N_\tau^f(x). \tag{13}$$

ii) When $N-n-\epsilon$ is even, then $g\colon L^2 \to \mathbb{R}^N$ is any surface such that

$$T_{g(x)}L = N_\tau^f(x) \tag{14}$$

up to parallel identification in $\mathbb{R}^N$.

Proposition 8. *Any elliptic submanifold f admits locally a polar surface. Moreover, in substantial codimension any polar surface g to f is elliptic with respect to $\widetilde{J}$ and, up to parallel identification,*

$$N_s^g = N_{\tau_f^*-s}^f \quad \textit{and} \quad \widetilde{J}_s = J_{\tau_f^*-s}^t, \quad \forall\, 0 \le s \le \tau_f^*. \tag{15}$$

In particular, g is substantial if and only if f has no euclidean factor.

Proof: In the odd codimensional situation the existence of a polar surface is by definition. When $N-n$ is even, endow L^2 with the orientation and a riemannian metric which makes $\widetilde{J}$ orientation preserving and orthogonal. Take a nowhere vanishing smooth local section $\xi \in N_{\tau_f}^f$ which is constant along Δ. To prove the first statement, it suffices to show that there exist linearly independent 1-forms θ, ψ so that the differential equation

$$dg = \theta\xi + \psi J_{\tau_f}^t \xi \tag{16}$$

has solution.

Let v and w be dual to θ and ψ, respectively. The integrability condition for (16) is

$$d\theta\,\xi + d\psi\, J^t_{\tau_f}\xi - (\widetilde{\nabla}_{\tilde{J}v}\xi + \widetilde{\nabla}_{\tilde{J}w} J^t_{\tau_f}\xi)\; dV = 0, \tag{17}$$

where dV stands for the volume element of L^2. From $ii)$ in Proposition 5 we easily obtain using (12) that the vanishing of the $N^f_{\tau_f-1}$–component of (17) is equivalent to $w = \tilde{J}v$, i.e., $\psi = -\theta\circ\tilde{J}$. In particular, θ and ψ are linearly independent when $\theta\neq 0$. Take $a, b\in C^\infty(L^2)$ and a 1–form θ_0 such that

$$\widetilde{\nabla}_{\tilde{J}v}\xi - \widetilde{\nabla}_v J^t_{\tau_f}\xi = a\xi + bJ^t_{\tau_f}\xi \quad\text{and}\quad d\theta_0 = a\,dV.$$

The $N^f_{\tau_f}$–component of (17) yields $\theta = \theta_0 + d\varphi$ where φ is any solution of the elliptic equation $\Delta\varphi = \operatorname{div}\theta_0 - b$, and the first claim has been proved.

For the remaining of the proof we use Proposition 5 several times. From (13) and (14) it follows that $N^f_{\tau^*_f-1} = N^g_1$. Thinking g as a τ^*_f–cross section to f constant along Δ, and using that also $N^f_{\tau^*_f}$ is constant along Δ, we easily get

$$(\widetilde{\nabla}_{\tilde{J}Y} g_*\tilde{J}Y)_{N^g_1} = (\widetilde{\nabla}_{JPY} g_*JPY)_{N^f_{\tau^*_f-1}} = (\widetilde{\nabla}_{JPY} J^t_{\tau^*_f} g_*PY)_{N^f_{\tau^*_f-1}} = -(\widetilde{\nabla}_Y g_*Y)_{N^g_1}.$$

This shows that g is elliptic. The equality between normal spaces is now clear. In addition,

$$(\widetilde{\nabla}_X J^t_{\tau^*_f-s}\xi)_{N^f_{\tau^*_f-s-1}} = (\widetilde{\nabla}_{JX}\xi)_{N^g_{s+1}} = (\widetilde{\nabla}_X \tilde{J}_s\xi)_{N^g_{s+1}}, \quad \xi\in N^f_{\tau^*_f-s} = N^g_s,$$

and

$$J^t_{\tau^*_f-s}(\widetilde{\nabla}_X\varphi)_{N^f_{\tau^*_f-s}} = (\widetilde{\nabla}_{JX}\varphi)_{N^g_s} = \tilde{J}_s(\widetilde{\nabla}_X\varphi)_{N^g_s}, \quad \varphi\in N^f_{\tau^*_f-s+1} = N^g_{s-1},$$

so (15) follows for all possible values of s. ∎

Remark 9. Notice that Proposition 6 gives an alternative proof for the existence of polar surfaces to elliptic surfaces.

§3 The parametrizations.

In this section, we describe parametrically elliptic submanifolds by means of two alternative representations, namely, the polar and bipolar parametrizations, each of which is determined by an elliptic surface and a solution of a certain elliptic differential equation.

An interesting feature in the case of the *polar parametrization*, the one we describe first, is that the differential equation mentioned above is the same which defines the elliptic surface.

Theorem 10. *Given an elliptic surface $g\colon L^2 \to \mathbb{Q}^{N-\epsilon}$ and $1 \leq s \leq \tau_g^*$, consider the smooth map $\Psi\colon \Lambda_s \to \mathbb{R}^N$ defined as*

$$\Psi(\delta) = h(x) + \delta, \quad \delta \in \Lambda_s(x), \tag{18}$$

where $\Lambda_s := N_{s+1}^g \oplus \cdots \oplus N_{\tau_g}^g$ and h is any s–cross section to g. Then, at regular points, $M^n = \Psi(\Lambda_s)$ is an elliptic submanifold with polar surface g. Conversely, any elliptic submanifold $f\colon M^n \to \mathbb{R}^N$ without local euclidean factor admits a local parametrization (18) being g a polar surface to f.

Proof: We prove the direct statement. Since h is an s–cross section to g, it follows that $T_{\xi(x)}M = \Lambda_{s-1}(x)$ and that $\Delta_{\Psi(\xi(x))} = \Lambda_s(x)$. It remains to show that Ψ is elliptic. For any s–cross section β to g and $X \in TL$, we have by Proposition 5 that

$$\begin{aligned}\left(\widetilde{\nabla}_{JX}\widetilde{\nabla}_{JX}\beta\right)_{N_{s-1}^g} &= \left(\widetilde{\nabla}_{JX}\left(\widetilde{\nabla}_{JX}\beta\right)_{N_s^g}\right)_{N_{s-1}^g} = \left(\widetilde{\nabla}_{JX}J_s^t\left(\widetilde{\nabla}_X\beta\right)_{N_s^g}\right)_{N_{s-1}^g}\\ &= -\left(\widetilde{\nabla}_X\widetilde{\nabla}_X\beta\right)_{N_{s-1}^g}.\end{aligned}$$

For a local section $\xi \in \Lambda_s$ and $Y \in T_xL$, set $Z = (\widetilde{\nabla}_Y(h+\xi))_{N_s^g(x)} \in T_{\xi(x)}M$. Since $h+\xi$ is an s–cross section to g, we have

$$\begin{aligned}\alpha_\Psi(Z,Z)(\xi_x) &= \left(\widetilde{\nabla}_Y\widetilde{\nabla}_Y(h+\xi)\right)_{N_{s-1}^g(x)} = -\left(\widetilde{\nabla}_{JY}\widetilde{\nabla}_{JY}(h+\xi)\right)_{N_{s-1}^g(x)}\\ &= -\alpha_\Psi(J_s^tZ, J_s^tZ)(\xi_x),\end{aligned}$$

and the ellipticity of Ψ follows.

For the converse, take a polar surface $g\colon L^2 \to \mathbb{Q}_\epsilon^{N-\epsilon}$ to f. Since f has no euclidean factor, then g is substantial, hence elliptic. From Proposition 8, $\Delta_f = \Lambda_{\tau_f^*}$ and $TM = \Lambda_{\tau_f^*-1}$ along L^2. Then, the cross section $h := f|_{L^2}$ is a τ_f^*–cross section to g. ∎

Observe that the effect of picking a different γ_0 in (11) is only to get a reparametrization of $\Psi(\Lambda_s)$. Hence, it looks convenient to take $\gamma_0 = 0$ when using to generate s–cross sections the recursive procedure from Proposition 6. By doing that one can see why the polar parametrization may be more effective for submanifolds in low codimension. For instance, in codimension two it suffices to take 1–cross sections of the form $h_\varphi = \nabla\varphi + \gamma$, where $\gamma \in N_1^g$ is unique satisfying $A_\gamma^g = \mathrm{Hess}_\varphi$ for a given solution φ of (6).

Our next goal is to introduce the *bipolar parametrization*, but first we discuss two additional concepts.

Definition 11. We define a *bipolar surface* to an elliptic submanifold f to be any polar surface to a polar surface to f.

Notice that the only bipolar surface to an elliptic spherical surface is the surface itself. When the elliptic surface is euclidean, we get as bipolar surfaces all surfaces with the same Gauss map.

Definition 12. Given an elliptic surface $g\colon L^2 \to \mathbb{Q}_\epsilon^N$ and $0 \le s \le \tau_g^* - 1$, we call *dual s–cross section* to g any element $\hat{h} \in C^\infty(L^2, \mathbb{R}^{N+\epsilon})$ satisfying at each point
$$d\hat{h}\,(TL) \subset \epsilon\,\mathrm{span}\{g\} \oplus N_0^g \oplus \cdots \oplus N_s^g.$$

Notice that a dual 0–cross section to an elliptic surface in euclidean space is just a bipolar surface whose nature we discussed above. The terminology is justified by the following observation.

Proposition 13. *Let $g\colon L^2 \to \mathbb{Q}_\epsilon^N$ be an elliptic surface with polar surface $\hat{g}$. A dual s–cross section to g is just a $([N/2]-s-1)$–cross section to $\hat{g}$.*

Proof: From $i)$ in Proposition 5 we have $\tau_g^* = \tau_{\hat{g}}^* = [N/2] - 1$, and the proof follows using Proposition 8. ∎

The exact dual to the polar parametrization goes as follows.

Theorem 10′. *Given an elliptic surface* $g\colon L^2 \to \mathbb{Q}_\epsilon^{N-\epsilon}$ *and* $0 \le s \le \tau_g^* - 1$, *consider the smooth map* $\widehat{\Psi}\colon \widehat{\Lambda}_s \to \mathbb{R}^N$ *defined as*

$$\widehat{\Psi}(\delta) = \hat{h}(x) + \hat{\delta}, \quad \hat{\delta} \in \widehat{\Lambda}_s(x), \tag{19}$$

where $\widehat{\Lambda}_s := \epsilon\, span\{g\} \oplus N_0^g \oplus \cdots \oplus N_{s-1}^g$ *and* $\hat{h}$ *is any dual* s*–cross section to* g. *Then, at regular points,* $M = \Psi(\widehat{\Lambda}_s)$ *is an elliptic submanifold with bipolar surface* g. *Conversely, any elliptic submanifold* $f\colon M^n \to \mathbb{R}^N$ *without local euclidean factor admits a local parametrization (19) being* g *a bipolar surface to* f.

Proof: The proof follows from Theorem 10 and Propositions 8 and 13. ∎

Making use of the above result we obtain a rather simple quite computable parametrization. In particular, there is no need whatsoever to go through complicate recursive procedures in order to determine cross sections to the elliptic surface or subbundles in the decomposition of its normal bundle.

Endow a simply connected elliptic $g\colon L^2 \to \mathbb{Q}_\epsilon^{N-\epsilon}$ with a metric $\langle\, ,\, \rangle_J$ which makes J orthogonal. Now, consider the linear second order elliptic operator

$$L(\varphi) := \Delta\varphi - X(\varphi) + (\epsilon\lambda - \operatorname{div} X)\varphi, \tag{20}$$

where $X \in TL$, $\lambda \in C^\infty(L^2)$ are as in (7), and let $\varphi \in C^\infty(L^2)$ satisfy $L(\varphi) = 0$. If $\epsilon = 0$, take $\theta \in C^\infty(L^2)$ such that $d\theta = (d\varphi - \varphi X^*) \circ J$. Then,

$$dh = \begin{cases} dg \circ (\theta I + \varphi J) & \text{if } \ \epsilon = 0, \quad \textit{or} \\[2ex] ((d\varphi - \varphi X^*)g + \varphi dg) \circ J & \text{if } \ \epsilon = 1, \end{cases} \tag{21}$$

is a completely integrable first order system of PDEs.

Theorem 14. *Consider a simply connected elliptic surface* $g\colon L^2 \to \mathbb{Q}_\epsilon^{N-\epsilon}$ *and* $\varphi \in C^\infty(L^2)$ *satisfying* $L(\varphi) = 0$. *Let* $h\colon L^2 \to \mathbb{R}^N$ *be the solution of (21). Then, at regular points, the map* $\Psi\colon L^2 \times \mathbb{R}^{2s+\epsilon} \to \mathbb{R}^N$ *defined as*

$$\Psi(x,t) = h(x) + \epsilon\, t_0\, g(x) + \sum_{j=1}^{s} \left\{ t_{2j-1} \frac{\partial^j g}{\partial v \partial u^{j-1}}(x) + t_{2j} \frac{\partial^j g}{\partial u^j}(x) \right\}$$

for $0 \le s \le [(N-\epsilon)/2] - 2$ *and any coordinate system* (u,v) *for* L^2, *parametrizes an elliptic submanifold. Conversely, any elliptic submanifold without local euclidean factor can be parametrized this way locally.*

Proof: From Lemma 4 we see easily that the vectors

$$(\partial^{j+1} g/\partial u^j \partial v)_{N_j^g},\ (\partial^{j+1} g/\partial u^{j+1})_{N_j^g}, \quad 0 \le j \le \tau_g^*$$

form a basis of N_j^g for any coordinate system. On the other hand, in (19) we may take $\hat{h}$ to be a dual 0–cross section without loss of generality. In fact, from (9) and Proposition 13 we have that any given dual s–cross section to g differs from an associated (and essentially unique) dual 0–cross section to g by an element $\gamma_0 \in \widehat{\Lambda}_s$.

It remains to show that any dual 0–cross section to g is locally of the form $h + \epsilon\mu g$ where h is a solution of (21) and $\mu \in C^\infty(L^2)$. In fact, one must have a 1-form ψ and a section $S \in \mathrm{End}\,(TL)$ such that

$$dh = \epsilon\,\psi\, g + dg \circ S.$$

The integrability condition reduces to the equations

$$\alpha(Y, SZ) = \alpha(SY, Z),$$

$$(\nabla_Y S)Z - (\nabla_Z S)Y = \epsilon\,(\psi(Y)Z - \psi(Z)Y),$$

and an additional equation for $\epsilon = 1$,

$$d\psi(Y, Z) = \langle SZ, Y\rangle - \langle SY, Z\rangle, \quad \forall\, Y, Z \in TL.$$

The first equation is equivalent to $S = \theta I + \varphi J$ for some $\theta, \varphi \in C^\infty(L^2)$. It is now easy to see that the other equations become

$$d\theta = (d\varphi - \varphi X^*) \circ J + \epsilon\psi, \tag{22}$$

and when $\epsilon = 1$,

$$\mathrm{div}\ \psi \circ J + \varphi\lambda = 0. \tag{23}$$

The integrability condition for (22) when $\epsilon = 0$ is (20). On the other hand, if $\epsilon = 1$ we can take $\theta = 0$ by replacing h by $h - \theta g$. Now (20) follows from (22) and (23). ∎

Remark 15. The Gauss parametrization for hypersurfaces is due to Sbrana ([**Sb**]) and was rediscovered in [$\mathbf{DG}_1$]. On the other hand, the parametrization used by Bryant and Borisenko goes back to Schur and Bianchi ([$\mathbf{Bi}_1$]) when considered for hypersurfaces $M^3 \subset \mathbb{R}^4$.

§4 The singularities.

In the present section, we first show that the classification of complete elliptic submanifolds reduces to the three dimensional case, and provide a complete example in this dimension. Then, we describe the structure of the singular set of elliptic submanifolds of higher dimensions.

Theorem 16. *Let $f\colon M^n \to \mathbb{R}^N$ be a complete submanifold elliptic on a dense subset of M^n. Then, each connected component of an open dense subset of M^n is isometric to $L^3 \times \mathbb{R}^{n-3}$ and f splits accordingly. Moreover, the splitting is global if M^n is simply connected and does not contain an open subset $L^2 \times \mathbb{R}^{n-2}$.*

Proof: The minimum of the dimensions of the relative nullity subspaces of f is $\nu_0 = \ n-2$. Moreover, $\dim N_1^f \leq 2$ everywhere. It follows that the open subsets $\mathcal{U}_0 = \{x \in M^n : \nu(x) = \nu_0\}$ and $\mathcal{U}_1 = \{x \in M^n : \dim N_1^f(x) = 2\}$ are also dense. This clearly implies that $\mathcal{U}_2 = \{x \in M^n : f \text{ satisfies } (1)\}$ is open. Hence, the dense subset $\widetilde{M}$ of M^n where f is elliptic satisfies that $\widetilde{M} = \mathcal{U}_0 \cap \mathcal{U}_1 \cap \mathcal{U}_2$ is open.

It is a standard fact that the leaves of minimum relative nullity are complete when M^n is complete. We recall next some basic facts about the intrinsic *splitting tensor* $C\colon \Delta \times \Delta^\perp \to \Delta^\perp$ which is defined as

$$C_T X = -(\nabla_X T)_{\Delta^\perp}.$$

From the Codazzi equation, we get

$$\nabla_T A_\xi = A_\xi C_T + A_{\nabla_T^\perp \xi}, \quad \forall\, T \in \Delta,\ \xi \in T_f^\perp M.$$

In particular,

$$A_\xi C_T = C_T^t A_\xi. \tag{24}$$

Moreover, the Codazzi equation also yields

$$\nabla_S C_R = C_R C_S + C_{\nabla_S R}, \quad \forall\, S, R \in \Delta. \tag{25}$$

Lemma 17 ([**DG**$_3$]). *The following facts hold along $\mathcal{U}_0$:*

i) The codimension of ker C in Δ satisfies codim ker $C \leq 1$.

ii) *For any $S \in \Delta(x)$ the only possible real eigenvalue of C_S is 0, and $\ker C_S$ is parallel along the velocity field S of the line $x + tS$.*

iii) *Let T be a unit vector field perpendicular to $\ker C$ on the subset $\mathcal{U} \subset \mathcal{U}_0$ defined as $\mathcal{U} = \{x \in \mathcal{U}_0 \colon C(x) \neq 0\}$. If C_T is invertible and the leaves of Δ are complete along $\mathcal{U}$, then $\mathcal{U} = L^3 \times \mathbb{R}^{n-3}$ and f splits.*

Going back to the proof of the theorem, we first show that

$$C_S \in \operatorname{span}\{I, J\}, \quad \forall\, S \in \Delta. \tag{26}$$

To see this, observe that condition (1) may be stated as $A_\xi J = J^t A_\xi$, for all $\xi \in T_f^\perp M$. We easily get (26) using (24) and that $\dim N_1^f = 2$.

We now follow closely the arguments in the proof of Proposition 2.1 in [**DG**$_3$]. Consider the disjoint union $\mathcal{U}_0 = M_0 \cup M_1 \cup M_2$, where M_0 is the closed subset where $C = 0$ and M_2 where C_T is invertible. By *ii)* in Lemma 17, each M_j is a union of complete leaves of Δ. Take $x \in \widetilde{M} \cap \mathcal{U}$. From *ii)* in Lemma 17 and (26), it follows that $C_T(x)$ has no real eigenvalues, i.e., $\widetilde{M} \subset M_0 \cup M_2$. Hence, $\operatorname{int}(M_0) \cup M_2$ is dense since $\widetilde{M}$ is open. By the de Rham decomposition theorem each connected component of $\operatorname{int}(M_0)$ is a product $L^2 \times \mathbb{R}^{n-2}$ where f splits. Moreover, by *iii)* in Lemma 17 each component of M_2 is a product $L^3 \times \mathbb{R}^{n-3}$ on which f splits, and this concludes the proof. ∎

Corollary 18. *Let $f \colon M^n \to \mathbb{R}^N$ be a complete elliptic submanifold. Then $M^n = L^3 \times \mathbb{R}^{n-3}$ and f splits accordingly.*

Proof: Consider the open subsets $U_1 \subset M^n$ where f splits a $\mathbb{R}^{n-2}$ factor and $U_2 \subset M^n$ along which it splits a $\mathbb{R}^{n-3}$ factor but nowhere a $\mathbb{R}^{n-2}$ factor. Then, a polar surface to f has substantial codimension $N - n + 2$ on U_1 and $N - n + 3$ on U_2. Since the zeroes of a solution of an elliptic equation are isolated, it follows that U_1, U_2 cannot have a common boundary point, and this concludes the proof. ∎

Example 19. The following example due to F. Zheng (private communication) is a complete irreducible 3-dimensional submanifold which is elliptic everywhere. Consider the graph $f \colon \mathbb{R}^3 \to \mathbb{R}^5$ given by

$$f(x, y, z) = \left(x, y, z, \frac{2xy - zx^2 + zy^2}{1 + z^2}, \frac{2zxy + x^2 - y^2}{1 + z^2} \right).$$

It is easy to verify that

$$(-y+xz)f_x + (x+yz)f_y + (1+z^2)f_z \in \Delta(x,y,z).$$

We get from $f_{xx} = -f_{yy} \not\in T_f\mathbb{R}^3$ that $\alpha_f(f_x,f_x)+\alpha_f(f_y,f_y) = 0$ and that the sectional curvature satisfies $K(f_x,f_y) < 0$. In particular, f has rank 2 at all points. Finally, from $T_f\mathbb{R}^3 \oplus \mathrm{span}\{f_{xx}, f_{xy}\} = \mathbb{R}^5$ everywhere, we obtain that $\dim N_1^f = 2$.

In Theorem 10, we may restrict ourselves to take h to be a τ_g^*–cross section without loss of generality. This is by an argument already given in the proof of Theorem 14. By doing that, the singular set of Ψ is $\Lambda_{s+1} \subset \Lambda_s$. In fact, from *ii*) in Lemma 4 we have $\mathrm{Im}\Psi_*(\delta_x) = \Lambda_{s-1}(x)$ for any $\delta_x \in \Lambda_s \setminus \Lambda_{s+1}$ and $\mathrm{Im}\Psi_*(\delta_x) = \Lambda_s(x)$ for $\delta_x \in \Lambda_{s+1}$. We thus get a Whitney stratification

$$\Lambda_s \supset \Lambda_{s+1} \supset \Lambda_{s+2} \supset \cdots \supset \Lambda_{\tau_g^*} \tag{27}$$

of the singular set of Ψ, and each image $\Psi(\Lambda_j), s+1 \leq j \leq \tau_g^*$, is also an elliptic submanifold.

Given an elliptic submanifold $f\colon M^n \to \mathbb{R}^N,\ n \geq 4$, without euclidean factor, let $\widetilde{M}^n$ be the extension of $f(M^n)$ in $\mathbb{R}^N$ which consists in extending each leaf of relative nullity of f to a complete affine euclidean $\mathbb{R}^{n-2}$. Locally, this extension is obtained in an obvious way in terms of a polar (or bipolar) parametrization. From our next result, we conclude that the singular set of $\widetilde{M}^n$ is an elliptic submanifold in $\mathbb{R}^N$ of dimension $n-2$ with similar singularities.

Proposition 20. *Let $\Psi\colon \Lambda_s \to \mathbb{R}^N$ be an elliptic submanifold of dimension $n \geq 4$ given in terms of the polar parametrization by the use of a τ_g^*–cross section to a polar surface g. Then, $\Psi(\Lambda_{s+1})$ is the singular set of $\Psi(\Lambda_s)$.*

Proof: Since f has no local euclidean factor and $n \geq 4$, we obtain that $\dim N^g_{[(N-n+2)/2]} = 2$. This is equivalently to codim ker $C = 2$. We conclude from (26) that

$$\mathrm{span}\{C_T : T \in \Delta\} = \mathrm{span}\{I, J\}. \tag{28}$$

Hence, $D(x) = \{S \in \Delta(x) : C_S(x) = I\}$ is a codimension 2 affine subspace of $\Delta(x)$ at any $x \in L^2$. By (25), the operator $C_S(t)$ for $S \in D(x)$ satisfies the Ricatti equation $\nabla_S C_S = C_S^2$ along the line $x + tS$. Hence, $C_S(t) =$

$C_S(0)(I - tC_S(0))^{-1}$ is singular, precisely, at $t = 1$. Thus, the submanifold is singular at $x + S$. We conclude from (27) that the set of singular points forms an affine codimension 2 subbundle of the nullity bundle. ∎

§5 Austere and special Lagrangian submanifolds.

This section is devoted to the description of the austere elliptic submanifolds. In particular, this leads to the construction of a new family of special Lagrangian submanifolds with interesting singularities.

Definition 21. *Given an elliptic submanifold $f\colon M^n \to \mathbb{Q}_\epsilon^N$, we define at $x \in M^n$ the k^{th}-order curvature ellipse $\mathcal{E}_k^f(x) \subset N_k^f(x)$, $0 \le k \le \tau_f^*$, as*

$$\mathcal{E}_k^f(x) = \{\alpha_f^{k+1}(Z^\varphi, \ldots, Z^\varphi) \colon Z^\varphi = \cos\varphi Z + \sin\varphi JZ \text{ and } \varphi \in [0, 2\pi)\},$$

where $Z \in \Delta^\perp(x)$ has unit length and satisfies that $\langle Z, JZ \rangle = 0$.

It follows from (5) that $\mathcal{E}_k^f(x)$ is, in fact, an ellipse. Notice that $\mathcal{E}_k^f(x)$ is the same for different points in a leaf of relative nullity.

Theorem 22. *Let $f\colon M^n \to \mathbb{R}^N$ be an elliptic submanifold with polar surface g and bipolar surface $\hat{g}$. Then,*

$$f \text{ is austere} \iff \mathcal{E}^g_{[(N-n)/2]} \text{ is a circle} \iff \mathcal{E}^{\hat{g}}_{[(n-2)/2]} \text{ is a circle.}$$

Proof: Observe first that f is minimal if and only if $\mathcal{E}_0^f$ is a circle. On the other hand, from (4) and (5) we have

$$\mathcal{E}_k^f(x) \text{ is a circle} \iff J_k \text{ is orthogonal} \tag{29}$$

for all k, and the proof follows from Proposition 8. ∎

The bipolar parametrization in the minimal case extends Bryant's ([**Br**]) to higher dimensions. Observe that his three dimensional situation is quite special in the sense that the bipolar surface has to be minimal.

Remarks 23. 1) In next section we discuss an explicit recursive procedure which yields the (necessarily minimal) euclidean surfaces whose ellipses of curvature are all circles up to an arbitrary order. In particular, the polar

surface to such a surface has circular curvature ellipses from some order on.
2) It was shown in [**DG**$_1$] that any simply connected minimal submanifold of rank 2 admits a 1–parameter *associated family* of isometric deformations which are also minimal.

It is easy to see that the canonical immersion into $\mathbb{C}^N \cong \mathbb{R}^N \oplus \mathbb{R}^N$ of the normal bundle of a submanifold $f\colon M^n \to \mathbb{R}^N$ given by

$$F(\delta_x) = (f(x), \delta_x), \quad \delta_x \in T^\perp_{f(x)}M,$$

is Lagrangian with respect to the complex structure $J(X,Y) = (-Y,X)$. Moreover, it was proved in [**HL**] that F is special Lagrangian if and only if f is austere. We parametrize the special Lagrangian immersions associated to our austere submanifolds using previous results and notations.

Given an elliptic surface g with $\mathcal{E}^g_s$ a circle, set $X^N_s = (N^g_s)^\perp = \Lambda_s \oplus \widehat{\Lambda}_s$, and define maps Φ, $\widehat{\Phi}\colon X^N_s \to \mathbb{C}^N$ as

$$\Phi(\delta_x + \hat{\delta}_x) = (h(x) + \delta_x, \hat{\delta}_x) \quad \text{and} \quad \widehat{\Phi}(\delta_x + \hat{\delta}_x) = (\delta_x, \hat{h}(x) + \hat{\delta}_x),$$

where h and $\hat{h}$ are, respectively, a τ^*_g–cross section and dual 0–cross section to g. These are special Lagrangian submanifolds which generalize those of [**HL**] and [**Bo**]. In fact, they belong to a more general class of special Lagrangian immersions, to be discussed next, which are not normal subbundles over austere submanifolds in general. Moreover, they have rank 4 and are ruled by euclidean spaces of codimension 2.

Theorem 24. *With the above notations, the map $\widetilde{\Phi}\colon X^N_s \to \mathbb{C}^N$ given by*

$$\widetilde{\Phi}(\delta_x + \hat{\delta}_x) = (h(x) + \delta_x, \hat{h}(x) + \hat{\delta}_x) \tag{30}$$

is special Lagrangian at regular points. Moreover, the set of singular points of $\widetilde{\Phi}$ is $\Lambda_{s+1} \oplus \widehat{\Lambda}_{s-1}$, which has a Whitney stratification

$$X^N_s \supset \Lambda_{s+1} \oplus \widehat{\Lambda}_{s-1} \supset \Lambda_{s+2} \oplus \widehat{\Lambda}_{s-2} \supset \cdots$$

Proof: Being special Lagrangian is a condition on the Gauss map only; see [**HL**]. Since Φ and $\widetilde{\Phi}$ have trivially the same Gauss map, the first statement follows. The remaining of the proof is straightforward. ∎

§6 Elliptic real Kaehler submanifolds.

In this section, we first show that rank two euclidean isometric immersions of nonflat irreducible Kaehler manifolds, other than surfaces, are either hypersurfaces or austere submanifolds. Then, we completely describe the later by means of a Weierstrass–type representation.

Theorem 25. *Let $f\colon M^{2n} \to \mathbb{R}^N$, $n \geq 2$ and $N - 2n \geq 2$, be a locally substantial rank two isometric immersion of a nowhere flat Kaehler manifold without local euclidean factor. Then f is austere.*

Proof: Let R and J' denote the curvature tensor and the Kaehler structure of M^{2n}. By our rank assumption, the relative nullity Δ of f coincides with the nullity of R. From $J' \circ R(X,Y) = R(X,Y) \circ J'$ and the Gauss equation, we obtain that Δ and $\Delta^\perp$ are J'–invariant. We only need to show that M^{2n} is elliptic with respect to the Kaehler structure $J'|_{\Delta^\perp}$ on a dense subset of M^{2n}. We have,

$$C_{J'T} = J'C_T, \quad \forall\, T \in \Delta. \tag{31}$$

In fact, $C_{J'T}X = -(\nabla_X J'T)_{\Delta^\perp} = -J'(\nabla_X T)_{\Delta^\perp} = J'C_T X$, as we wished.

Let $U \subset M^{2n}$ be an open subset where N_1^f has constant dimension. If $\dim N_1^f = 1$, we obtain from Theorem 1 in [**DT**] that $f(U)$ is a hypersurface in substantial codimension, which has been ruled out. Suppose now that $\dim N_1^f = 3$. From (24), we easily get $\mathrm{span}\{C_T : T \in \Delta\} \subset \mathrm{span}\{I\}$. This and (31) yield $C = 0$, a contradiction with the assumption on euclidean factors. Thus, $\dim N_1^f = 2$ on an open dense subset of M^{2n}. In particular, from $C \neq 0$, (24) and (31), we easily see that $\mathrm{span}\{C_T : T \in \Delta\}$ is a plane in the vector space of 2×2 real matrices at each point. Again from $\dim N_1^f = 2$, we get easily that there is $T \in \Delta$ such that $C_T = I$. Hence, $C_{J'T} = J'|_{\Delta^\perp}$ by (31). We conclude the proof using (24). ∎

It was shown in [**DR**] that any minimal immersion of a Kaehler manifold in euclidean space is pluriharmonic. If already non-holomorphic, then it can be made the real part of a holomorphic isometric immersion, its *holomorphic representative*, and admits an associated 1–parameter family of noncongruent isometric deformations; see [$\mathbf{DG}_2$]. There exist many hypersurfaces of rank 2 and sectional curvature $K \leq 0$ which are Kaehler manifolds but are not minimal; cf. [**DG**$_2$]. This is possible because (28) does not necessarily hold when first normal spaces are one-dimensional.

Following [**DG**$_4$], we call an elliptic surface *m–isotropic* when the ellipses of curvature up to order m are circles. Holomorphic curves in $\mathbb{C}^p$ are precisely $(p-1)$–isotropic surfaces in $\mathbb{R}^{2p}$; cf. [**La**] or [**Cc**]. We have the following characterization.

Proposition 26. *Let $f\colon M^{2n} \to \mathbb{R}^N$, $n \geq 2$, be an elliptic submanifold without local euclidean factor. Then, M^{2n} is Kaehler if and only if a bipolar surface $\hat{g}$ to f is $(n-1)$–isotropic. Moreover, f is holomorphic if and only if $\hat{g}$ is a holomorphic curve.*

Proof: To prove the converse in the first statement we argue for a polar surface $g\colon L^2 \to \mathbb{R}^N$ to f. For each $x \in L^2$, set $\Omega_x = N^g_{\tau^*_f}(x)\oplus\cdots\oplus N^g_{\tau_g}(x)$. Hence, $\Omega_x = T_{f(x)}M$ up to parallel transport along $\mathbb{R}^N$. Define $J' \in \mathrm{End}(\Omega_x)$ by

$$J' = \tilde{J}_{\tau^*_f} \oplus \cdots \oplus \tilde{J}_{\tau_g}.$$

Because tangents spaces to $f(M)$ are constant along the relative nullity leaves, we may extend J' to the whole M^{2n} by parallel transport. We have that $J'^2 = -\mathrm{I}$ and, by the hypothesis on the curvature ellipses and (29), that J' is orthogonal. Take $\xi \in N^g_k$ and $X \in TL$. Using Proposition 5 and twice that J' is orthogonal, we get

$$\nabla_X J'\xi = -(\widetilde{\nabla}_X \tilde{J}^t_k\xi)_{N^g_{k-1}} + \tilde{J}_k(\widetilde{\nabla}_X\xi)_{N^g_k} + \tilde{J}_{k+1}(\widetilde{\nabla}_X\xi)_{N^g_{k+1}} = J'\nabla_X\xi.$$

Since J' was extended to M^n by parallel transport, it is easy to conclude that $\nabla J' = 0$, i.e., (M^{2n}, J') is Kaehler.

We now prove the direct statement. At each point, define

$$\Delta_{k+1} = \{(\nabla_Z X)_{(\Delta^\perp\oplus\cdots\oplus\Delta_k)^\perp}\colon X \in \Delta_k, Z \in \Delta^\perp\},\quad k \geq 0.$$

The identification $\Delta^\perp = N^g_{\tau^*_f}$ from Proposition 8 easily yields

$$\Delta_k = N^g_{\tau^*_f+k},\quad 0 \leq k \leq n-1.$$

Since $J = \pm J'|_{\Delta^\perp}$ by Theorem 25 and f has no euclidean factor, using Proposition 5 and the parallelism of J', we easily see that $\pm J' = \tilde{J}_{\tau^*_f}\oplus\cdots\oplus\tilde{J}_{\tau_g}$, and the proof of the first part has been completed. The second statement in the proposition follows from similar arguments. ∎

A complete description of m–isotropic euclidean surfaces was given in [**DG**$_4$] based in results due to C. C. Chen ([**Cc**]), and goes at follows. On a simply connected domain $U \subset \mathbb{C}$, a minimal surface $\hat{g}: U \to \mathbb{R}^N$ has the Weierstrass representation

$$\hat{g} = \mathrm{Re} \int^z \gamma dz, \tag{32}$$

where the Gauss map $\gamma: U \to \mathbb{C}^N$ of $\hat{g}$ has an expression

$$\gamma = \frac{\beta}{2}\left(1 - \phi^2, i(1 + \phi^2), 2\phi\right),$$

being β holomorphic and $\phi: U \to \mathbb{C}^{N-2}$ meromorphic; see [**HO**] for details. From [**Cc**], we have that $\hat{g}$ is m–isotropic if and only if

$$(\phi', \phi') = \cdots = (\phi^m, \phi^m) = 0,$$

where $(\ ,\ )$ stands for the standard symmetric inner product in $\mathbb{C}^{N-2}$. Hence, to construct any m–isotropic surface start with a nonzero holomorphic

$$\alpha_0: U \to \mathbb{C}^{N-2(m+1)}.$$

Assuming that $\alpha_r: U \to \mathbb{C}^{2r+p}$, $0 \le r \le m$, has been defined already, set

$$\alpha_{r+1} = \beta_{r+1}\left(1 - \phi_r^2, i(1 + \phi_r^2), 2\phi_r\right),$$

where $\phi_r = \int^z \alpha_r dz$ and $\beta_{r+1} \neq 0$ is any holomorphic function. Then, the elliptic surface with Gauss map $\gamma = \alpha_m$, i.e., $\hat{g} = \mathrm{Re}\ \alpha_{m+1}$, is m–isotropic.

Given a minimal surface $\hat{g}: U \to \mathbb{R}^N$ with Gauss map γ, it is immediate that the non-constant dual 0–cross sections to $\hat{g}$ are the minimal surfaces which can be represented as

$$h = \mathrm{Re} \int^z \psi\gamma dz, \tag{33}$$

where $\psi \neq 0$ is an arbitrary holomorphic function on U. We have the following result.

Theorem 27. *Consider a* $(n-1)$*–isotropic surface* $\hat{g}\colon U \to \mathbb{R}^N$ *with Gauss map* γ *defined on a simply connected domain* $U \subset \mathbb{C}$*, and let* ψ *be a holomorphic function on* U*. Then,* $\Psi\colon U \times \mathbb{C}^{n-1} \to \mathbb{R}^N$ *given by*

$$\Psi(z,w) = Re\left\{\int^z \psi\gamma dz + \sum_{j=0}^{n-2} w_{j+1}\frac{d^j\gamma}{dz^j}(z)\right\} \tag{34}$$

is, at regular points, a Kaehler austere submanifold of rank two with bipolar surface $\hat{g}$*. Conversely, any real Kaehler submanifold* $f\colon M^{2n} \to \mathbb{R}^N$ *of rank two has locally a Weierstrass representation (34).*

Proof: Follows from Theorem 14, Proposition 26 and (33). ∎

Remarks 28. 1) Elements in the Whitney stratification (27) are now elliptic Kaehler submanifolds.
2) Parametrization (34) when starting with just a minimal surface yields a large family of elliptic submanifolds.

References

[**Bi**$_1$] L. Bianchi, *Sulle varietà a tre dimensioni deformabili entro lo spazio euclideo a quattro dimensioni.* Memorie di Matematica e di Fisica della Società Italiana delle Scienze, serie III, t. XIII, 261–323 (1905).

[**Br**] R. Bryant, *Some remarks on the geometry of austere manifolds.* Bol. Soc. Brasil. Mat. **21**, 122–157 (1991).

[**Bo**] A. Borisenko, *Ruled special Lagrangian surfaces.* Advances in Soviet Mathematics **15**, 269–285 (1993).

[**Cb**] B. Y. Chen, *Some pinching and classification theorems for minimal submanifolds.* Arch. Math. **60**, 568–578 (1993).

[**Cc**] C. C. Chen, *The generalized curvature ellipses and minimal surfaces.* Bull. Acad. Sinica **11**, 329–336 (1983).

[**DF**] M. Dajczer and L. Florit, *On Chen's Basic Equality.* Illinois J. of Math. **42**, 97–106 (1998).

[**DG**$_1$] M. Dajczer and D. Gromoll, *Gauss Parametrizations and Rigidity Aspects of Submanifolds.* J. Diff. Geometry **22**, 1–12 (1985).

[**DG**$_2$] M. Dajczer and D. Gromoll, *Real Kaehler submanifolds and uniqueness of the Gauss map.* J. Diff. Geometry **22**, 13–28 (1985).

[**DG**$_3$] M. Dajczer and D. Gromoll, *Rigidity of complete euclidean hypersurfaces.* J. Diff. Geometry **31**, 401–416 (1990).

[**DG**$_4$] M. Dajczer and D. Gromoll, *The Weierstrass representation for complete minimal real Kaehler submanifolds.* Invent. Math. **119**, 235–242 (1995).

[**DR**] M Dajczer and L. Rodríguez, *Rigidity of real Kaehler submanifolds.* Duke Math. J. **53**, 211–220 (1986).

[**DT**] M. Dajczer and R. Tojeiro, *Submanifolds with nonparallel first normal bundle.* Canad. Math. Bull. **37**, 330–337 (1994).

[**Ei**] L. Eisenhart, "Transformations of surfaces". Chelsea, New York, 1923.

[**HL**] R. Harvey and B. Lawson, *Calibrated Geometries.* Acta Mathematica **148**, 47–157 (1982).

[**HO**] D. Hoffman and R. Osserman, *The geometry of the generalized Gauss map.* Memoirs Am. Math. Soc. **28** Number 236, (1980).

[**La**] H. Lawson, *The Riemannian geometry of holomorphic curves.* Bol. Soc. Brasil. Mat. **2**, 45–62 (1971).

[**Sb**] V. Sbrana, *Sulla varietá ad $n-1$ dimensioni deformabili nello spazio euclideo ad n dimensioni.* Rend. Circ. Mat. Palermo **27**, 1–45 (1909).

[**Sp**] M. Spivak, "A comprehensive introduction to differential geometry". Vol IV. Publish or Perish Inc. Berkeley, 1979.

IMPA – Estrada Dona Castorina, 110
22460-320 — Rio de Janeiro — Brazil
marcos@impa.br —— luis@impa.br