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Geometric graph manifolds with non-negative scalar curvature

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Abstract

We classify n-dimensional geometric graph manifolds with non-negative scalar curvature by first showing that if $n > 3$, the universal cover splits off a codimension 3-Euclidean factor. We then proceed with the classification of the 3-dimensional case, where the condition is equivalent to the eigenvalues of the Ricci tensor being $(\lambda, \lambda, 0)$ with $\lambda \geqslant 0$. In this case we prove that such a manifold is either a lens space or a prism manifold with a very rigid metric. This allows us to also classify the moduli space of such metrics: it has infinitely many connected components for lens spaces, while it is connected for prism manifolds.

A geometric graph manifold M^n is a Riemannian manifold which is the union of twisted cylinders $C^n = (L^2 \times \mathbb{R}^{n-2})/G$, where $G \subset \mathrm{Iso}(L^2 \times \mathbb{R}^{n-2})$ acts properly discontinuously and freely on the Riemannian product of a connected surface L^2 with the Euclidean space In addition, the boundary of each twisted cylinder is a union of compact totally geodesic flathypersurfaces, each of which is isometric to a boundary component of another twisted cylinder. In its simplest form, as first discussed in [5], they are the union of building blocks of the form $L^2 \times S^1$, where L^2 is a surface, not diffeomorphic to a disk or an annulus, whose boundary is a union of closed geodesics. The building blocks are glued along common boundary totally geodesic flat tori by switching the role of the circles. Such graph manifolds have been studied frequently in the context of manifolds with non-positive sectional curvature. In fact, they were the first examples of such metrics with geometric rank one. Furthermore, in [13] it was shown that if a complete 3-manifold with non-positive sectional curvature has the fundamental group of a graph manifold, then it is isometric to a geometric graph manifold.

One of the most basic features of geometric graph manifolds is that their curvature tensor has nullity space of dimension at least $n - 2$ everywhere. This property by itself already guarantees that each finite volume connected component of the set of non-flat points is a twisted cylinder, and under some further weak assumptions, the manifold is isometric to a geometric graph manifold in the above sense; see [4]. See also [1] and references therein for extensive literature on manifolds with nullity equal to $n - 2$.

In dimension 3, the nullity condition is equivalent to saying that the eigenvalues of the Ricci tensor are $(\lambda, \lambda, 0)$, or to the assumption, called cvc(0), that every tangent vector is contained in a flat plane; see [12]. Notice that this is in fact the only choice for the eigenvalues of the Ricci tensor where the metric is allowed to be locally reducible.

This nullity condition also arose in a different context. In [3] it was shown that a compact immersed submanifold $M^n \subset \mathbb{R}^{n+2}$ with non-negative sectional curvature is either diffeomorphic to the sphere $\mathbb{S}^n$, isometric to a product of two convex hypersurfaces $\mathbb{S}^k \times \mathbb{S}^{n-k} \subset \mathbb{R}^{k+1} \times \mathbb{R}^{n-k+1}$, isometric to $(\mathbb{S}^{n-1} \times \mathbb{R})/\mathbb{Z}$, or diffeomorphic to a lens space $\mathbb{S}^3/\mathbb{Z}_p \subset \mathbb{R}^5$. In the latter case it was shown that each connected component of the set of non-flat points

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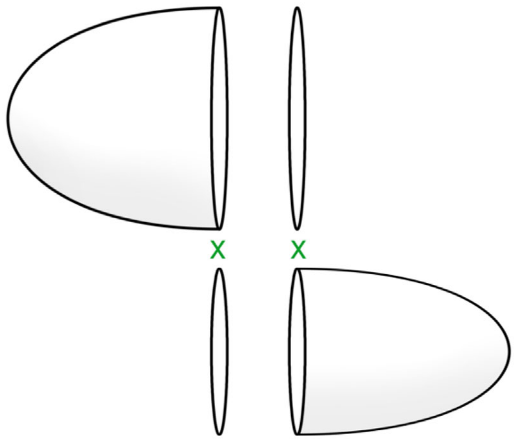

FIGURE 1 (colour online). $\mathbb{S}^3 \subset \mathbb{R}^5$ *with non-negative curvature.*

is a twisted cylinder. The present paper arose out of an attempt to understand the intrinsic geometry of such metrics. We thus want to classify all compact geometric graph manifolds with non-negative sectional curvature, or equivalently, with non-negative scalar curvature. Notice that under this curvature assumption compactness is equivalent to finite volume.

We first show that their study can be reduced to dimension 3.

THEOREM A. *Let M^n, $n \geqslant 4$, be a compact geometric graph manifold with non-negative scalar curvature. Then, the universal cover $\tilde{M}^n$ of M^n splits off an $(n-3)$-dimensional Euclidean factor isometrically, that is, $\tilde{M}^n = N^3 \times \mathbb{R}^{n-3}$. Moreover, either M^n is flat, or $N^3 = \mathbb{S}^2 \times \mathbb{R}$ splits isometrically, or $N^3 = \mathbb{S}^3$ with a geometric graph manifold metric.*

By the splitting theorem, the curvature condition by itself already implies that $\tilde{M}^n$ is isometric to a product $Q^k \times \mathbb{R}^{n-k}$ with Q^k compact and simply connected, but it is surprisingly delicate to show that $k \leqslant 3$.

In dimension 3, the simplest non-trivial example of a geometric graph manifold with non-negative scalar curvature is the usual description of $\mathbb{S}^3$ as the union of two solid tori $D^2 \times S^1$ endowed with a product metric, see Figure 1. If this product metric is invariant under $\mathrm{SO}(2) \times \mathrm{SO}(2)$, we can also take a quotient by the cyclic group generated by $R_p \times R_p^q$ to obtain a geometric graph manifold metric on any lens space $L(p,q) = \mathbb{S}^3/\mathbb{Z}_p$. Here $R_p \in \mathrm{SO}(2)$ denotes the rotation of angle $2\pi/p$.

There is a further family whose members also admit geometric graph manifold metrics with non-negative scalar curvature: the so-called *prism manifolds*$P(m,n) := \mathbb{S}^3/G_{m,n}$, which depend on two relatively prime positive integers m, n. Such a metric on $P(m,n)$ can be constructed as a quotient of the metric on $\mathbb{S}^3$ as above by the group $G_{m,n}$ generated by $R_{2n} \times R_{2n}^{-1}$ and $(R_m \times R_m) \circ J$, where J is a fixed point free isometry switching the two isometric solid tori. Topologically $P(m,n)$ is thus a single solid torus whose boundary is identified to be a Klein bottle. Its fundamental group $G_{m,n}$ is abelian if and only if $m = 1$, and in fact $P(1,n)$ is diffeomorphic to $L(4n, 2n-1)$; see Section 1. Unlike in the case of lens spaces, the diffeomorphism type of a prism manifold is determined by its fundamental group.

Our main purpose is to show that these are the only 3-dimensional compact geometric graph manifolds with non-negative scalar curvature, and to classify the moduli space of such metrics. We will see that the twisted cylinders in this case are of the form $C = (D \times \mathbb{R})/\mathbb{Z}$, where D is the interior of a 2-disk of non-negative Gaussian curvature, whose boundary ∂D is a closed geodesic along which the curvature vanishes to infinite order. We fix once and for all such a metric $\langle\, ,\rangle_0$ on a 2-disc D_0, whose boundary has length 1 and which is rotationally symmetric. We call a geometric graph manifold metric on a 3-manifold *standard* if the generating disk D of a twisted cylinder C as above is isometric to the interior of D_0 with metric $r^2\langle\, ,\rangle_0$ for some constant $r > 0$. Observe that the projection of $\partial D \times \{s\}$ for $s \in \mathbb{R}$ is a parallel foliation by closed geodesics of the flat totally geodesic 2-torus $(\partial D \times \mathbb{R})/\mathbb{Z}$.

We provide the following classification.

THEOREM B. *Let M^3 be a compact geometric graph manifold with non-negative scalar curvature and irreducible universal cover. Then M^3 is diffeomorphic to a lens space or a prism manifold. Moreover, we have either:*

a) M^3 is a lens space and $M^3 = C_1 \sqcup T^2 \sqcup C_2$, that is, M^3 is isometrically the union of two twisted cylinders $C_i = (D_i \times \mathbb{R})/\mathbb{Z}$ over disks D_i glued together along their common totally geodesic flat torus boundary T^2. Conversely, any flat torus endowed with two parallel foliations by closed geodesics uniquely defines a standard geometric graph manifold metric on a lens space;
b) M^3 is a prism manifold and $M^3 = C \sqcup K^2$, that is, M^3 is isometrically the closure of a single twisted cylinder $C = (D \times \mathbb{R})/\mathbb{Z}$ over a disk D, whose totally geodesic flat interior boundary is isometric to a rectangular torus T^2, and $K^2 = T^2/\mathbb{Z}_2$ is a Klein bottle. Conversely, any rectangular flat torus endowed with a parallel foliation by closed geodesics uniquely defines a standard geometric graph manifold metric on a prism manifold.

In addition, any geometric graph manifold metric with non-negative scalar curvature on M^3 is isotopic, through geometric graph manifold metrics with nonnegative scalar curvature, to a standard one.

We call T^2, respectively, K^2, the *core* of the geometric graph manifold and will see that it is in fact an isometry invariant.

Observe that a twisted cylinder with generating surface a disc is diffeomorphic to a solid torus. In topology one constructs a lens space by gluing two such solid tori along their boundary by an element of $GL(2, \mathbb{Z})$. In order to make this gluing into an isometry, we twist the local product structure. An alternate way to view this construction is as follows. Start with an arbitrary twisted cylinder C_1 and regard the flat boundary torus as the quotient of $\mathbb{R}^2$ with respect to a lattice. We can then choose a second twisted cylinder C_2 whose boundary is a different fundamental domain of the same lattice, and hence the two twisted cylinders can be glued with an isometry of the boundary tori. We note that in principle, a twisted cylinder can also be flat, but we will see that in that case it can be absorbed by one of the non-flat twisted cylinders.

The diffeomorphism type of M^3 in Theorem B is determined by the (algebraic) oriented slope between the parallel foliations of T^2 by closed geodesics. As we will see, this is also an isometry invariant $\mathcal{S}(M^3, \mathfrak{o}) \in \mathbb{Q}$ of M^3 which we call its *slope*, once orientations $\mathfrak{o}$ of M^3 and its core are chosen; see Section 3 for the precise definition.

THEOREM C. *Let M^3 be a compact geometric graph manifold of non-negative scalar curvature with irreducible universal cover and slope $\mathcal{S}(M^3, \mathfrak{o}) = q/p \in \mathbb{Q}$. Then, in case (a) of Theorem B, M^3 is diffeomorphic to the lens space $L(p, q)$, while in case (b) it is diffeomorphic to the prism manifold $P(q, p)$.*

This result can be used to classify the moduli space of geometric graph manifold metrics. We first deform any such metric in Theorem B to be standard, preserving the metric on the torus T^2, and then deform T^2 to be the unit square $S^1 \times S^1$, while preserving also the sign of the scalar curvature in the process. In case (a), we can also make one of the foliations equal to $S^1 \times \{w\}$. The metric is then determined by the remaining parallel foliation of the unit square by closed geodesics. Since the diffeomorphism type of a lens space $L(p, q)$ is determined by $\pm q^{\pm 1} \bmod p$, we conclude:

COROLLARY. *The moduli space of geometric graph manifold metrics with non-negative scalar curvature on a lens space has infinitely many connected components, whereas on a prism manifold $P(q, p)$ with $q > 1$ it is connected.*

We will see that the moduli space for the lens space $L(4p, 2p - 1)$ has a special component arising from the fact that it is diffeomorphic to $P(1, p)$.

Finally, we apply our results, combined with those in [4], to the class of compact 3-dimensional manifolds M^3 with Ricci eigenvalues $(\lambda, \lambda, 0)$ for $\lambda \geqslant 0$. Theorem A in [4] implies that any connected component of the set M' of non-flat points of M^3 is isometric to a twisted cylinder. The basic geometric feature of M' is that it admits a parallel foliation by complete geodesics tangent to the kernel of the Ricci tensor. If there exists a larger open set $M'' \supset M'$ which admits a parallel foliation by complete geodesics extending that of M', then any connected component of M'' is still isometric to a twisted cylinder. Such an extension M'' is called *full* if it is dense in M^3 and if its collection of twisted cylinders is locally finite. From the second theorem in [4], we thus conclude the following.

COROLLARY. *Let M^3 be a compact Riemannian manifold with Ricci eigenvalues $(\lambda, \lambda, 0)$ for some function $\lambda \geqslant 0$. Then M^3 is isometric to one of the manifolds in Theorem B if and only if its set of non-flat points admits a full extension.*

This applies of course if M' is already dense, as long as it satisfies the mild regularity assumption that its collection of twisted cylinders is locally finite. Although in [4] we built an explicit example where M' admits no full extension, we conjecture that it always admits a full extension when $\lambda \geqslant 0$.

The paper is organized as follows. In Section 1 we recall some facts about geometric graph manifolds. In Section 2 we prove Theorem A by showing that the manifold is a union of one or two twisted cylinders over disks, while in Section 3 we classify their metrics and prove Theorems B and C.

1. *Preliminaries*

Let us begin with the definition of twisted cylinders and geometric graph manifolds. Unless otherwise stated, manifolds have no boundary.

Consider the cylinder $L^2 \times \mathbb{R}^{n-2}$ with its natural product metric, where L^2 is a connected surface. We call the quotient

$$C^n = (L^2 \times \mathbb{R}^{n-2})/G$$

a *twisted cylinder*, where $G \subset \mathrm{Iso}(L^2 \times \mathbb{R}^{n-2})$ acts properly discontinuously and freely on $L^2 \times \mathbb{R}^{n-2}$, and L^2 the *generating surface* of C^n. We also say that C^n is a twisted cylinder *over* L^2. The Euclidean factor induces a foliation Γ on C^n whose leaves will be called the *nullity leaves* of C^n. These leaves are complete flat totally geodesic and locally parallel of codimension 2. Such twisted cylinders are the building blocks of geometric graph manifolds:

DEFINITION. A complete connected Riemannian manifold M^n, $n \geqslant 3$, is called a *geometric graph manifold* if M^n is a locally finite disjoint union of twisted cylinders C_i glued together along disjoint compact connected totally geodesic flat hypersurfaces H_λ of M^n. That is,

$$M^n \setminus W = \bigsqcup_\lambda H_\lambda, \quad \text{where} \quad W := \bigsqcup_i C_i.$$

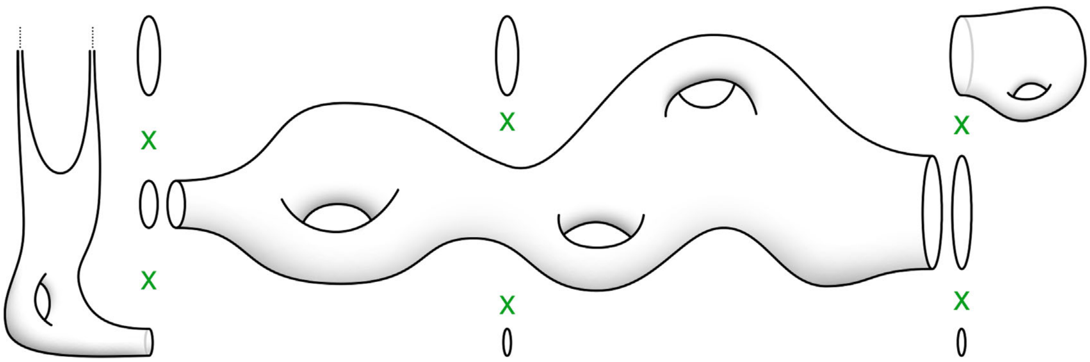

FIGURE 2 (colour online). *An irreducible 4-dimensional geometric graph manifold with three cylinders and two (finite volume) ends.*

See Figure 2 for a typical (4-dimensional) example, where each twisted cylinder is just the isometric product $L^2 \times S^1 \times S^1$ of a surface L^2 and a flat torus.

We first make some general remarks about this definition.

1. We allow the possibility that the hypersurfaces H_λ are one-sided, even when M^n is orientable.
2. The locally finiteness condition is equivalent to the assumption that each H_λ is a common boundary component of two twisted cylinders C_i and C_j that may even be globally the same. When H_λ is one-sided, it is a boundary component of only one twisted cylinder.
3. As shown in [4], the foliations Γ_i and Γ_j of C_i and C_j induce two totally geodesic foliations on H_λ. When they agree, C_i, C_j, and H_λ can be considered as a single twisted cylinder. Thus, without loss of generality, *we assume from now on that they are different*. This implies that the generating surface L^2 of each twisted cylinder C is the interior of a surface with boundary consisting of complete geodesics along which the Gaussian curvature vanishes to infinite order. We refer to these geodesics as boundary geodesics of L^2 itself.
4. These boundary geodesics of L^2 do not have to be closed, even when C is compact.
5. The complement of W is contained in the set of flat points of M^n, but we do not require that the generating surfaces have non-vanishing Gaussian curvature.
6. In principle, we could ask for the hypersurfaces H_λ to be complete instead of compact. However, compactness follows when M^n has finite volume; see [4].
7. If none of the generating surfaces in a geometric graph manifold are discs, it also admits a metric with non-positive sectional curvature. On the other hand, if all of the generating surfaces are discs, we will see that it admits a metric with non-negative sectional curvature.

In [4] we gave a characterization of geometric graph manifolds with finite volume in terms of the nullity of the curvature tensor. But since a complete non-compact manifold with non-negative Ricci curvature has linear volume growth by [16], we will assume from now on that M^n is compact.

We now recall some properties of 3-dimensional lens spaces and prism manifolds that will be needed later on; see, for example, [7, 8, 11 14] for details.

One way of defining a lens space is as the quotient $L(p,q) = \mathbb{S}^3/\mathbb{Z}_p$, where $g \in \mathbb{Z}_p \subset S^1 \subset \mathbb{C}$ acts as $g\cdot(z,w) = (gz, g^q w)$ for $(z,w) \in \mathbb{S}^3 \subset \mathbb{R}^4 = \mathbb{C}^2$ for coprime integers p, q with $p \neq 0$. It is a well-known fact that two lens spaces $L(p,q)$ and $L(p,q')$ are diffeomorphic if and only if $q' = \pm q^{\pm 1}$ mod p. An alternative description we will use is as the union of two solid tori

$D_i \times S^1$, with boundary identified such that $\partial D_1 \times \{p_0\} \in \pi_1(\partial D_1 \times S^1)$ is taken into $(q,p) \in \mathbb{Z} \oplus \mathbb{Z} = \pi_1(\partial D_2 \times S^1)$ with respect to its natural basis.

A prism manifold can also be described in two different ways. The first one is to define it as the quotient $\mathbb{S}^3/(H_1 \times H_2) = H_1 \backslash \mathbb{S}^3/H_2$, where $H_1 \subset \mathrm{Sp}(1)$ is a cyclic group acting as left translations on $\mathbb{S}^3 \simeq \mathrm{Sp}(1)$ and $H_2 \subset \mathrm{Sp}(1)$ a binary dihedral group acting as right translations. A more useful description for our purposes is as the union of a solid torus $C = D \times S^1$ with the 3-manifold

$$N^3 = (S^1 \times S^1 \times I)/\langle (j, -Id)\rangle, \qquad \text{where} \quad j(z,w) = (-z, \bar{w}). \tag{1.1}$$

Notice that N^3 is a bundle over the Klein bottle $K = T^2/\langle j \rangle$ with fiber an interval $I = [-\epsilon, \epsilon]$ and orientable total space. Thus ∂N^3 is the torus $S^1 \times S^1$, and we glue the two boundaries via a diffeomorphism. Here $\pi_1(N^3) = \pi_1(K) = \{a, b \mid bab^{-1} = a^{-1}\}$ and $\pi_1(\partial N^3) = \mathbb{Z} \oplus \mathbb{Z}$, with generators a, b^2, where a represents the first circle and b^2 the second one. Then $P(m,n)$ is defined as gluing ∂C to ∂N^3 by sending $\partial D \times \{p_0\}$ to $a^m b^{2n} \in \pi_1(\partial N^3)$. We can again assume that $m, n > 0$ with $\gcd(m,n) = 1$. Furthermore,

$$\pi_1(P(m,n)) = G_{m,n} = \{a, b \mid bab^{-1} = a^{-1},\ a^m b^{2n} = 1\}.$$

This group has order $4mn$ and its abelianization has order $4n$. Thus the fundamental group determines and is determined by the ordered pair (m,n). In addition, $G_{m,n}$ is abelian if and only if $m = 1$ in which case $P(m,n)$ is diffeomorphic to the lens space $L(4n, 2n-1)$. Unlike in the case of lens spaces, the diffeomorphism type of $P(m,n)$ is uniquely determined by (m,n). Prism manifolds can also be characterized as the 3-dimensional spherical space forms which contain a Klein bottle, which for $m > 1$ is also incompressible. Observe in addition that in N^3 we can shrink the length of the interval I in (1.1) down to 0, and hence, $P(m,n)$ can also be viewed as a single solid torus whose rectangular flat torus boundary has been identified to a Klein bottle, as in part (b) of Theorem B.

2. *A dichotomy and the proof of Theorem* A

In this section we provide the general structure of geometric graph manifolds with non-negative scalar curvature by showing a dichotomy: they are built from either one or two twisted cylinders over 2-disks. This will then be used to prove Theorem A.

Let M^n be a compact non-flat geometric graph manifold with non-negative scalar curvature. We will furthermore assume that M^n is not itself a twisted cylinder since in this case the universal cover of M^n is isometric to $\mathbb{S}^2 \times \mathbb{R}^{n-2}$, where $\mathbb{S}^2$ is endowed with a metric of non-negative Gaussian curvature. Recall that we also assume that the nullity foliations of two twisted cylinders glued along a hypersurface H induce two different foliations on H, which in turn implies that the Gaussian curvature of the two generating surfaces vanishes to infinite order along their boundary geodesic.

By assumption, there exists a collection of compact flat totally geodesic hypersurfaces in M^n whose complement is a disjoint union of (open) twisted cylinders C_i. Let $C = (L^2 \times \mathbb{R}^{n-2})/G$ be one of these cylinders whose boundary in M^n is a disjoint union of compact flat totally geodesic hypersurfaces. There is also an *interior boundary* $\partial_i C$ of C, which we also denote for convenience as ∂C by abuse of notation. This boundary can be defined as the set of equivalence classes of Cauchy sequences $\{p_n\} \subset C$ in the interior distance function d_C of C, where $\{p_n\} \sim \{p'_n\}$ if $\lim_{n\to\infty} d_C(p_n, p'_n) = 0$. Since M^n is compact, such a Cauchy sequence $\{p_n\}$ converges in M^n, and we have a natural map $\sigma : \partial C \to M$ that sends $[\{p_n\}]$ to $\lim_{n\to\infty} p_n \in M^n$. This map is, on each component of ∂C, either an isometry or a locally isometric twofold covering map since $H = \sigma(\partial C)$ consists of disjoint smooth hypersurfaces which are two-sided in the former case, and one-sided in the latter. Therefore, ∂C is smooth as well and $C \sqcup \partial C$ is a closed twisted cylinder with totally geodesic flat compact interior boundary, that by abuse of notation we still denote by C. Similarly, L^2 is a smooth surface with geodesic interior boundary components along which the Gaussian curvature vanishes to infinite order.

We first determine the generating surfaces of the twisted cylinders.

PROPOSITION 2.1. *Let $C = (L^2 \times \mathbb{R}^{n-2})/G$ be a compact twisted cylinder with non-negative curvature as above. Then one of the following holds:*

i) The surface L^2 is isometric to a 2-disk D with non-negative Gaussian curvature, whose boundary is a closed geodesic along which the curvature of D vanishes to infinite order.

ii) C is flat and there exists a compact flat hypersurface S such that C is isometric to either $[-s_0, s_0] \times S$, or to $([-s_0, s_0] \times S)/\{(s, x) \sim (-s, \tau(x))\}$ for some involution τ of S.

Proof. Since C is compact and the boundary is totally geodesic, we can apply the soul theorem to C, see [**2**, Theorem 1.9] and [**10**, Theorem 4.1]. Thus there exists a compact totally geodesic submanifold $S \subset C$ and C is diffeomorphic to the disc bundle $D_\epsilon(S) = \{v \in T_pC \mid v \perp T_pS,\ |v| \leqslant \epsilon\}$ for some $\epsilon > 0$. Recall that S is constructed as follows. Let $C^s = \{p \in C \mid d(p, \partial C) \geqslant s\}$. Then C^s is convex, and the set of points C^{s_0} at maximal distance s_0 from ∂C is a totally geodesic submanifold, possibly with boundary. Repeating the process if necessary, one obtains the soul S. In our situation, let $q = [(p, v)] \in C^{s_0}$, and γ a minimal geodesic from q to ∂C. Since it meets $\partial C = ((\partial L^2) \times \mathbb{R}^{n-2})/G$ perpendicularly, we have $\gamma = [(\alpha, v)]$ where α is a geodesic in the leaf $L^2_v = [L^2 \times \{v\}]$ meeting ∂L^2_v perpendicularly. So, for every $w \in \mathbb{R}^{n-2}$, the geodesic $[(\alpha, w)]$ is also minimizing, $[(p, w)] \in C^{s_0}$ lies at maximal distance s_0 to ∂C, and hence $C^{s_0} = (T \times \mathbb{R}^{n-2})/G$ where $T \subset L^2$ is a segment, a complete geodesic or a point. Therefore $S = (T' \times \mathbb{R}^{n-2})/G$, where T' is a point or a complete geodesic (possibly closed).

We first consider the case where T' is a point, and hence, the soul is a single nullity leaf. Recall, that in order to show that C is diffeomorphic to the disc bundle $D_\epsilon(S)$, one constructs a gradient like vector field X by observing that the distance function to the soul has no critical points. In our case, the initial vector to all minimal geodesics from $[(p, v)] \in C$ to S lies in the leaf L^2_v and hence we can construct X such that X is tangent to L^2_v for all v. The diffeomorphism between C and $D_\epsilon(S)$ is obtained via the flow of X, which now preserves the leaves L^2_v and therefore L^2 is diffeomorphic to a disc.

If T' is a complete geodesic, the soul S is flat and has codimension 1. If X is a unit vector field in L^2 along T' and orthogonal to T', it is necessarily parallel and its image under the normal exponential map of S determines a flat surface by Perelman's solution to the soul conjecture, see [**9**]. This surface lies in L^2, and every point $q \in L^2$ is contained in such a surface since we can connect q to S by a minimal geodesic, which is contained in some L_v, and is orthogonal to T'. Thus L^2 is flat and hence either $L^2 = T' \times [-s_0, s_0]$, and hence $C = [-s_0, s_0] \times S$, or L^2 is a Moebius strip and hence $C = ([-s_0, s_0] \times S)/\{(s, x) \sim (-s, \tau(x))\}$ for some involution τ of S. $\square$

REMARK 2.2. A flat twisted cylinder as in (*ii*) can be absorbed by any cylinder C' attached to one of its boundary components by either attaching $[-s_0, s_0]$ to the generating surface of C' in the first case, or attaching $(0, s_0]$ in the second, in which case $\{0\} \times (S/\tau) = S/\tau$ becomes a one sided boundary component of C'. *We will therefore assume from now on that the generating surfaces of all twisted cylinders are 2-discs.*

REMARK 2.3. The properties at the boundary γ of a disk D as in Proposition 2.1 are easily seen to be equivalent to the fact that the natural gluing $D \sqcup (\gamma \times (-\epsilon, 0])$, $\gamma \cong \gamma \times \{0\}$, is smooth when we consider on $\gamma \times (-\epsilon, 0]$ the flat product metric. In fact, in Fermi coordinates $(s \geqslant 0, t)$ along γ, the metric is given by $ds^2 + f(t, s)dt^2$. The fact that γ is a (unparameterized) geodesic is equivalent to $\partial_s f(0,t) = 0$, while the curvature condition is equivalent to $\partial_s^k f(0,t) = 0$ for all t and $k \geqslant 2$. Therefore, $f(s,t)$ can be extended smoothly as $f(0,t)$ for $-\epsilon < s < 0$, which gives the smooth isometric attachment of the flat cylinder $\gamma \times (-\epsilon, 0]$ to D.

As a consequence of Proposition 2.1, and the assumption that there are no flat cylinders, $\partial C = (\gamma \times \mathbb{R}^{n-2})/G$ is connected, and so is $H = \sigma(\partial C)$. In particular, M^n contains at most two twisted cylinders with non-negative curvature glued along H. We call such a connected compact flat totally geodesic hypersurface H a *core* of M^n. We conclude:

COROLLARY 2.4. *If M^n is not flat and not itself a twisted cylinder, then $M^n = W \sqcup H$ with core H, and either:*

a) H is two-sided, σ is an isometry, and $W = C \sqcup C'$ is the disjoint union of two open non-flat twisted cylinders as above attached via an isometry $\partial C \simeq H \simeq \partial C'$; or

b) H is one-sided, σ is a locally isometric twofold covering map, $W = C$ is a single open non-flat twisted cylinder as above, and $M^n = C \sqcup H = C \sqcup (\partial C/\mathbb{Z}_2)$.

Furthermore, in case (a), if $H' \subset M^n$ is an embedded compact flat totally geodesic hypersurface, then there exists an isometric product $H \times [0,a] \subset M^n$, with $H = H \times \{0\}$ and $H' = H \times \{a\}$. In particular, any such H' is a core of M^n, and hence the core is unique up to isometry. On the other hand, in case (b) the core H is already unique.

Proof. We only need to prove the uniqueness of the cores. In order to do this, any limit of nullity leaves of C at its boundary in M^n will be called a boundary nullity leaf, or BNL for short.

For case (a), first assume that $H \cap H' \neq \emptyset$ and take $p \in H \cap H'$. Then a BNL of C in H at p is contained in H'. Indeed if not, the product structure of the universal cover $\pi : \tilde{C} = L^2 \times \mathbb{R}^{n-2} \to C$, together with the fact that H' is flat totally geodesic and complete and intersects H transversely, would imply that L^2, and hence C, is flat since by dimension reasons the projection of $\pi^{-1}(H' \cap C)$ onto L^2 would be a surjective submersion. Analogously, the (distinct) BNL of C' at p lies in H', and since H is the unique hypersurface containing both BNLs, we have that $H = H'$. If, on the other hand, $H \cap H' = \emptyset$, we can assume $H' \subset C = (L^2 \times \mathbb{R}^{n-2})/G$. Again, by the product structure of $\tilde{C}$ and the fact that H' is embedded, we see that $H' = (\gamma' \times \mathbb{R}^{n-2})/G'$ where $\gamma' \subset L^2$ is a simple closed geodesic and $G' \subset G$ the subgroup preserving γ'. Since the boundary γ of L^2 is also a closed geodesic and L^2 is a 2-disk with non-negative Gaussian curvature, by Gauss–Bonnet there is a closed interval $I = [0,a] \subset \mathbb{R}$ such that the flat strip $\gamma \times I$ is contained in L^2, with $\gamma = \gamma \times \{0\}$ and $\gamma' = \gamma \times \{a\}$. Thus G' acts trivially on I, which implies our claim.

In case (b) we have that $H \cap H' = \emptyset$ as in case (a) since at any point $p \in H$ we have two different BNLs at $\sigma^{-1}(p)$. Hence as before $H' = (\gamma' \times \mathbb{R}^{n-2})/G' \subset C$ and $H \times [0,a] \subset M^n$, with $H = H \times \{0\}$ and $H' = H \times \{a\}$. But then the normal bundle of H' is trivial, contradicting the fact that H is one-sided. $\square$

REMARK 2.5. Any manifold in case (b) admits a twofold cover whose covering metric is as in case (a). Indeed, we can attach to C another copy of C along its interior boundary $\partial_i C$ using the involution that generates $\mathbb{Z}_2$. Switching the two cylinders induces the twofold cover of M^n.

We proceed by showing that our geometric graph manifolds are essentially 3-dimensional. Observe that we only use here that $M^n \setminus W$ is connected, with no curvature assumptions. In fact, the same proof shows that if $M^n \setminus W$ has k-connected components, then M^n splits off an $(n-k-2)$-dimensional Euclidean factor.

Claim. *If $n > 3$, the universal cover of M^n splits off an $(n-3)$-dimensional Euclidean factor.*

Proof. Assume first that M^n is the union of two cylinders C and C' with common boundary H. Consider the nullity distributions Γ and Γ' on the interior of C and C', which extend uniquely to parallel codimension 1 distributions F and F' on H, respectively. Recall that $F \neq F'$ since otherwise the universal cover is an isometric product $N^2 \times \mathbb{R}^{n-2}$. So $J := F \cap F'$ is a codimension 2 parallel distribution on H. We claim that J extends to a parallel distribution on the interior of both C and C'.

To see this, we only need to argue for C, so lift the distributions J and F to the cover $S^1 \times \mathbb{R}^{n-2}$ of H under the projection $\pi\colon L^2 \times \mathbb{R}^{n-2} \to C = (L^2 \times \mathbb{R}^{n-2})/G$, and denote these lifts by $\hat{J}$ and $\hat{F}$. They are again parallel distributions whose leaves project to those of J and F under π. At a point $(x_0, v_0) \in S^1 \times \mathbb{R}^{n-2}$ a leaf of $\hat{F}$ is given by $\{x_0\} \times \mathbb{R}^{n-2}$ and hence a leaf of $\hat{J}$ by $\{x_0\} \times W$ for some affine hyperplane $W \subset \mathbb{R}^{n-2}$. Since $\hat{J}$ is parallel, any other leaf is given by $\{x\} \times W$ for $x \in S^1$. Since G permutes the leaves of $\hat{F}$, W is invariant under the projection of G into $\mathrm{Iso}(\mathbb{R}^{n-2})$. Therefore $\pi(\{x\} \times W)$ for $x \in L^2$ are the leaves of a parallel distribution on the interior of C, restricting to J on its boundary.

Therefore, we have a global flat parallel distribution J of codimension 3 on M^n, which implies that the universal cover splits isometrically as $N^3 \times \mathbb{R}^{n-3}$.

Now, if M^n consists of only one open cylinder C and its one-sided boundary, by Remark 2.5 there is a twofold cover $\hat{M}^n$ of M^n which is the union of two cylinders as above and whose universal cover splits an $(n-3)$-dimensional Euclidean factor. □

We can now finish the proof of Theorem A. Since M^n is compact with non-negative curvature, the splitting theorem implies that the universal cover splits isometrically as $\tilde{M}^n = Q^k \times \mathbb{R}^{n-k}$ with Q^k compact and simply connected. According to the above claim, $k = 2$ and hence $Q^2 \simeq \mathbb{S}^2$, or $k = 3$ and by [**6**, Theorem 1.2] we have $Q^3 \simeq \mathbb{S}^3$. In the latter case, we claim that the metric on $\mathbb{S}^3$ is again a geometric graph manifold metric. Indeed, if $\sigma\colon \mathbb{S}^3 \times \mathbb{R}^{n-3} \to M^n$ is the covering map, and $C \subset M^n$ a twisted cylinder, then in $C' = \sigma^{-1}(C)$ the codimension 2 nullity leaves contain the $\mathbb{R}^{n-3}$ factor. Since the universal cover of C' has the form $L^2 \times \mathbb{R}^{n-2}$, the metric on $\mathbb{S}^3$ must be a geometric graph manifold metric.

3. *Geometric graph 3-manifolds with non-negative curvature*

In this section we classify 3-dimensional geometric graph manifolds with non-negative scalar curvature, giving an explicit construction of all of them. As a consequence, we show that, for each lens space, the number of connected components of the moduli space of such metrics is infinite, while for each prism manifold, the moduli space is connected. Recall that we assume that M^3 itself is not a single twisted cylinder. Furthermore, none of the twisted cylinders are flat, hence their generating surfaces are discs and M^3 is the union of one or two twisted cylinders according to the dichotomy in Corollary 2.4.

Let M^3 be such a compact geometric graph manifold with non-negative scalar curvature. We first observe that M^3 is orientable. Indeed, by Theorem A, $M^3 = \mathbb{S}^3/\Pi$ for some finite group Π acting freely. Moreover, if an element $g \in \Pi$ reverses orientation, the Lefschetz fixed point theorem implies that g has a fixed point. Thus every cylinder $C = (D \times \mathbb{R})/G$ is orientable as well, that is, the action of G preserves orientation.

For $g \in G$, we write $g = (g_1, g_2) \in \mathrm{Iso}(D \times \mathbb{R})$. Thus g_1 preserves the closed geodesic ∂D and fixes the soul point $x_0 \in D$. If $g \neq e$ and g_1 reverses orientation, then so does g_2 and hence g would have a fixed point. Thus g_2 preserves orientation and is a translation which is non-trivial since g_1 has a fixed point. This easily implies that $G = \mathbb{Z}$. Altogether, the twisted cylinders are

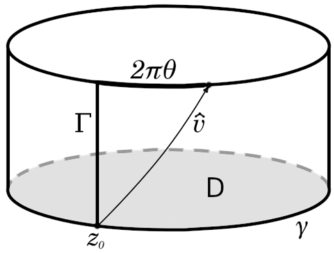

Figure 3. *A twisted cylinder.*

of the form $C = (D \times \mathbb{R})/\mathbb{Z}$ with $\mathbb{Z}$ generated by some $g = (g_1, g_2)$. If g_1 is non-trivial, then g_1 is determined by its derivative at x_0. After orienting D, $d(g_1)_{x_0}$ is a rotation R_θ of angle $2\pi\theta$, $0 \leqslant \theta < 1$. We simply say that g_1 acts as a rotation R_θ on D. Thus g acts via

$$g(x, s) = (R_\theta(x), s + h) \in \mathrm{Iso}(D \times \mathbb{R}), \tag{3.1}$$

for a certain constant $h > 0$, after orienting the nullity distribution $\Gamma \cong T^\perp D$. We can regard θ as the twist of the cylinder and h as its height; see Figure 3. These, together with the length of ∂D, are the geometric invariants that characterize the twisted cylinder up to isometry. Moreover, C has a parallel foliation by the nullity lines, that is, the images of $\{p_0\} \times \mathbb{R}, p_0 \in D$, which are closed if and only if θ is rational. The interior boundary of C is a flat 2-torus and the limits of the nullity lines induce a parallel foliation on $\partial_i C$. Observe that $\partial_i C$ also has a parallel foliation by closed geodesics given by the projection of $\partial D \times \{s_0\}$, $s_0 \in \mathbb{R}$, which will be denoted $\mathcal{F}(C)$.

Notice that the action of $\mathbb{Z}$ can be changed differentiably until $\theta = 0$, and hence, C is diffeomorphic to a solid torus $D \times S^1$. According to Corollary 2.4, M^3 is thus either the union of two solid tori glued along their boundary, and hence diffeomorphic to a lens space, or it is a solid torus whose boundary is identified via an involution to form a Klein bottle, and therefore diffeomorphic to a prism manifold.

Remark 3.2. Let us clarify the role of the orientations in our description of C in (3.1). Take a twisted cylinder C with non-negative scalar curvature, and D a maximal leaf of $\Gamma^\perp$. Orienting Γ is then equivalent to orienting $T^\perp D$, which in turn is equivalent to choosing one of the two generators of $\mathbb{Z}$. On the other hand, orienting D is equivalent to choosing between the oriented angle θ above or $1 - \theta$. In particular, these orientations are unrelated to the metric on C, that is, changing orientations give isometric cylinders.

Next, we show that the geometric graph manifold metric on M^3 is isotopic to a standard one. In order to do this, fix once and for all a metric $\langle\ ,\ \rangle_0$ on the disc $D_0 = \{x \in \mathbb{R}^2 : |x| \leqslant 1\}$ which is rotationally symmetric, has positive Gaussian curvature on the interior of D_0, and whose boundary is a closed geodesic of length 1 along which the Gaussian curvature vanishes to infinite order. We call the metric on M^3 *standard*, if for each twisted cylinder $C = (D \times \mathbb{R})/\mathbb{Z}$ in the complement of a core of M^3, the metric on D is isometric to $r^2\langle\ ,\ \rangle_0$ for some constant $r > 0$. Notice that such a metric on M^3 is unique up to isometry. For this we first show the following.

Lemma 3.3. *Let $\langle\ ,\ \rangle$ be a metric on a disc D with non-negative Gaussian curvature. Assume that its boundary is a closed geodesic along which the curvature vanishes to infinite order, and that the metric is invariant under a group of isometries K. Then, given a constant $r > 0$, there exists a smooth path of metrics on D, $\langle\ ,\ \rangle_s$, $1 \leqslant s \leqslant 2$, satisfying the same assumptions for all*

s, such that $\langle\ ,\ \rangle_1 = \langle\ ,\ \rangle$ and $\langle\ ,\ \rangle_2 = r^2\langle\ ,\ \rangle_0$, where $\langle\ ,\ \rangle_0$ is the fixed rotationally symmetric metric on D_0.

Proof. Let $\langle\ ,\ \rangle'$ be the standard flat metric on D_0. By the uniformization theorem we can write $\langle\ ,\ \rangle = f_1^*(e^{2v}\langle\ ,\ \rangle')$ for some diffeomorphism $f_1 : D \to D_0$ and a smooth function v on D_0. The metric $e^{2v}\langle\ ,\ \rangle'$ is thus invariant under $C_{f_1}(K) = \{f_1 \circ g \circ f_1^{-1} : g \in K\}$ which fixes $f_1(x_0)$, where $x_0 \in D$ is the fixed point of the action of K. Equivalently, $h \in C_{f_1}(K)$ is a conformal transformation of $(D_0, \langle\ ,\ \rangle')$ with conformal factor $e^{2v-2v\circ h}$. Recall that the conformal transformations of $\langle\ ,\ \rangle'$ on the interior of D_0 can be viewed as the isometry group of the hyperbolic disc model. Hence there exists a conformal transformation j of D_0 with $j(f_1(x_0)) = 0$ and conformal factor $e^{2\tau}$. We can thus also write $\langle\ ,\ \rangle = f^*(e^{2u}\langle\ ,\ \rangle')$, where $f = j \circ f_1 : D \to D_0$ and $u := (v - \tau) \circ j$. Now the metric $e^{2u}\langle\ ,\ \rangle'$ is invariant under $C_f(K)$, which this time fixes the origin of D_0. So $k \in C_f(K)$ is a conformal transformation of $\langle\ ,\ \rangle'$ fixing the origin, with conformal factor $e^{2u-2u\circ k}$. But an isometry of the hyperbolic disc model, fixing the origin, is also an isometry of $\langle\ ,\ \rangle'$. Hence $e^{2u} = e^{2u\circ k}$, that is, u is invariant under k. Altogether, $C_f(K) \subset \mathrm{SO}(2) \subset \mathrm{Iso}(D_0, \langle\ ,\ \rangle')$ and u is $C_f(K)$-invariant. Analogously, $r^2\langle\ ,\ \rangle_0 = f_0^*(e^{2u_0}\langle\ ,\ \rangle')$ with $f_0 \in \mathrm{Diff}(D_0)$ satisfying $f_0(0) = 0$ and u_0 being $\mathrm{SO}(2)$-invariant. In particular, u_0 is also $C_f(K)$-invariant.

We now consider the two metrics $e^{2u}\langle\ ,\ \rangle'$ and $e^{2u_0}\langle\ ,\ \rangle'$ on D_0. They both have the property that the boundary is a closed geodesic along which the curvature vanishes to infinite order. An easy computation shows that the assumption that the boundary is a closed geodesic, up to parametrization, is equivalent to the condition that the normal derivatives of u and u_0, with respect to a unit normal vector in $\langle\ ,\ \rangle'$, are equal to 1. Furthermore, since the curvature G of a metric $e^{2w}\langle\ ,\ \rangle'$ is given by $Ge^{2w} = -\Delta w$, G vanishes to infinite order if and only if Δw does. For each $0 \leqslant s \leqslant 1$, consider the $C_f(K)$-invariant metric on D_0 given by $\langle\ ,\ \rangle^s = e^{2(1-s)u_0+2su+a(s)}\langle\ ,\ \rangle'$, where $a(s)$ is the function that makes the boundary to have length r for all s. Clearly, for each s, the boundary is again a closed geodesic up to parametrization and G^s vanishes at the boundary to infinite order. Furthermore, since $G^s e^{2(1-s)u_0+2su+a(s)} = -(1-s)\Delta u_0 - s\Delta u$ and $\Delta u_0 < 0$, $\Delta u \leqslant 0$, the curvature of $\langle\ ,\ \rangle^s$ is non-negative and positive on the interior of D_0. Thus $\langle\ ,\ \rangle_s = f^*\langle\ ,\ \rangle^s$ is the desired family of metrics on D. □

We can now apply this to deform the metric on M^3:

PROPOSITION 3.4. *A geometric graph manifold metric with non-negative scalar curvature is isotopic, through geometric graph manifold metrics with non-negative scalar curvature, to a standard one.*

Proof. We define the isotopy separately on each cylinder $C = (D \times \mathbb{R})/\mathbb{Z}$, such that the isometry type of the core $H = \partial C$, and the foliation of H induced by the nullity leaves of C, stays fixed. The metric on D is invariant under the group of isometries $K = \{g_1 \mid (g_1, g_2) \in \mathbb{Z}\}$ and we apply Lemma 3.3 to obtain a family of metrics $\langle\ ,\ \rangle_s + dt^2$ on $D \times \mathbb{R}$, which is invariant under the action of $\mathbb{Z}$. We now glue the induced metrics on $(D \times \mathbb{R})/\mathbb{Z}$ to the core H and choose r such that the arc length parametrization of ∂C and nullity leaves in H match. Performing this process on each cylinder, we obtain the desired deformation of the metric on M^3. □

We now discuss how C induces a natural marking on its interior boundary $\partial_i C$. For this, let us first recall some elementary facts about lattices $\Lambda \subset \mathbb{R}^2$, where we assume that the orientation on $\mathbb{R}^2$ is fixed.

DEFINITION 3.5. A *marking* of the lattice Λ is a choice of an oriented basis $\{v, \hat{v}\}$ of Λ, and we say that the marking is *normalized* if

$$\langle v, \hat{v}\rangle/\|v\|^2 \in [0,1).$$

Notice that for any primitive $v \in \Lambda$, that is, $tv \notin \Lambda$ for $0 < t < 1$, there exists a unique oriented normalized marking $\{v, \hat{v}\}$ of Λ. Indeed, if $\{v, w\}$ is some oriented basis of Λ, then $\langle v, w + nv\rangle/\|v\|^2 = \langle v, w\rangle/\|v\|^2 + n$ and hence there exists a unique $n \in \mathbb{Z}$ such that $\{v, \hat{v}\}$ with $\hat{v} = w + nv$ is normalized.

If T^2 is an oriented flat torus and $z_0 \in T^2$ a base point, then $T^2 = T_{z_0}T^2/\Lambda$ where Λ is the lattice given by $\Lambda = \{w \in T_{z_0}T^2 : exp_{z_0}(w) = z_0\}$. A (normalized) marking of T^2 is a (normalized) marking of its lattice Λ.

Now consider an oriented twisted cylinder $C = (D \times \mathbb{R})/\mathbb{Z}$ with its standard metric, where the action of $\mathbb{Z}$ is given by (3.1) for some θ and h. The totally geodesic flat torus $T^2 = \partial_i C$, which inherits an orientation from C, has a natural marking based at $z_0 = [(p_0, s_0)]$. For this, denote by $\gamma : [0,1] \to \partial D$ the simple closed geodesic with $\gamma(0) = p_0$ which follows the orientation of $D = [D \times \{s_0\}] \subset C$. Then, since $\theta \in [0,1)$, we have that

$$\mathcal{B}(\gamma) := \{v, \hat{v}\}, \quad \text{where} \quad v = \gamma'(0) \quad \text{and} \quad \hat{v} = \theta v + h\partial/\partial s,$$

is a normalized marking of T^2 based at z_0; see Figure 3. Notice that the geodesic $\sigma(s) = \exp(s\hat{v}), 0 \leqslant s \leqslant 1$, is simple and closed with length $\|\hat{v}\|$. Recall that $\mathcal{F}(C)$ denotes the foliation of T^2 by parallel closed geodesics $[\gamma \times \{s\}]$, $s \in [0, h)$.

It is important for us that the above process can be reversed for standard metrics.

Proposition 3.6. *Let T^2 be a flat oriented torus and $\mathcal{F}$ an oriented foliation of T^2 by parallel closed simple geodesics. Then there exists an oriented twisted cylinder $C_{\mathcal{F}} = (D \times \mathbb{R})/\mathbb{Z}$ over a standard oriented disk D, unique up to isometry, such that $\partial_i C_{\mathcal{F}} = T^2$ and $\mathcal{F}(C_{\mathcal{F}}) = \mathcal{F}$. Moreover, different orientations induce isometric metrics.*

Proof. Choose $\gamma \in \mathcal{F}$, and set $z_0 = \gamma(0)$ and $v = \gamma'(0)$. By the above, there exists a unique vector $\hat{v}$ such that $\mathcal{B}(\gamma) = \{v, \hat{v}\}$ is a normalized marking of T^2 based at z_0. Set $r = \|v\|$, $\theta = \langle v, \hat{v}\rangle/\|v\|^2$ and $h = \|\hat{v} - \theta v\|$. With respect to the oriented orthonormal basis $e_1 = v/r$, $e_2 = (\hat{v} - \theta v)/h$ of $T_{z_0}T^2$ we have

$$T^2 = \mathbb{R}^2/\Lambda = (\mathbb{R} \oplus \mathbb{R})/(\mathbb{Z}v \oplus \mathbb{Z}\hat{v}) = (S^1_r \times \mathbb{R})/\mathbb{Z}\hat{v},$$

where S^1_r is the oriented circle of length r. Since $v = re_1$ and $\hat{v} = \theta v + he_2$, we can also write $T^2 = (S^1_r \times \mathbb{R})/\langle g\rangle$ where $g(p, s) = (R_\theta(p), s + h)$. Now we simply attach $(D_0, r^2\langle\ ,\ \rangle_0)$ to S^1_r preserving orientations to build $C = (D_0 \times \mathbb{R})/\langle g\rangle$. Notice that any two base points of T^2 are taken to each other by an orientation preserving isometry of C, restricted to $\partial C = T^2$. Thus the construction is independent of the choice of z_0 and the choice of $\gamma \in \mathcal{F}$. By Remark 3.2, different choices of orientation induce the same metric on C, and hence $C_{\mathcal{F}}$ is unique up to isometry. □

Remark 3.7. If we do not assume that the metric on C is standard, then the construction of $C_{\mathcal{F}}$ depends on the choice of base point, and one has to assume that the metric on D is invariant under R_θ, where θ is the angle determined by the marking of T^2 induced by $\mathcal{F}$.

We can now easily classify standard geometric graph manifold metrics with two-sided core, proving case (a) of Theorem B.

Theorem 3.8. *Let M^3 be a compact geometric graph manifold of non-negative scalar curvature with irreducible universal cover, and assume that its core T^2 is two-sided. Then, $M^3 = C_1 \sqcup T^2 \sqcup C_2$, where $C_i = (D_i \times \mathbb{R})/\mathbb{Z}$ are twisted cylinders over 2-disks that induce two different foliations $\mathcal{F}_i = \mathcal{F}(C_i)$ of T^2 by parallel closed geodesics, $i = 1, 2$.*

Conversely, given a flat 2-torus T^2 with two different foliations $\mathcal{F}_i$ by parallel closed geodesics, there exists a standard geometric graph manifold $M^3 = C_1 \sqcup T^2 \sqcup C_2$ with irreducible universal cover whose core is T^2 and $C_i = C_{\mathcal{F}_i}$. Moreover, this data determine the standard metric up to isometries, that is, if $h : T^2 \to \hat{T}^2$ is an isometry between flat tori, then $\hat{M}^3 = \hat{C}_1 \sqcup \hat{T}^2 \sqcup \hat{C}_2$ is isometric to M^3, where $\hat{C}_i = C_{h(\mathcal{F}_i)}$.

Proof. We only need to prove uniqueness. The core of a standard metric is unique since, by the choice of the metric on D_0, the set of non-flat points is dense. It is clear then that an isometry between standard geometric graph manifolds will send the core to the core, and the parallel foliations to the parallel foliations. Hence the core and the parallel foliations are determined by the isometry class of M^3.

Conversely, by uniqueness in Proposition 3.6 the standard twisted cylinders $C_{\mathcal{F}_i}$ and $C_{h(\mathcal{F}_i)}$ are isometric, which in turn induces an isometry between M^3 and $\hat{M}^3$. The only ambiguity is on which side of the torus to attach each of the twisted cylinders, but this simply gives an orientation reversing isometry fixing the core. □

Now, let us consider the one-sided core case. Here we know that $M^3 = C \sqcup K$ and that K is a non-orientable quotient of the flat torus $\partial_i C$ and hence a flat Klein bottle. It is easy to see that, if a flat torus admits an orientation reversing fixed-point free isometric involution j, then T^2 has to be isometric to a rectangular torus $S^1_r \times S^1_s$ on which j acts as in (1.1), that is, $j(z, w) = (-z, \bar{w})$. Thus, since the universal cover of M^3 is irreducible, $\mathcal{F}(C)$ does not to coincide with one of the two invariant parallel foliations $\{S^1_r \times \{w\} : w \in S^1_s\}$ and $\{\{z\} \times S^1_s : z \in S^1_r\}$. We denote the first one by $\mathcal{F}(j)$.

As in the proof of Theorem 3.8, we conclude the next which implies case (b) of Theorem B.

THEOREM 3.9. *Let M^3 be a compact geometric graph manifold of non-negative scalar curvature with irreducible universal cover, and assume that its core K is one-sided. Then $M^3 = C \sqcup K$, where $C = (D \times \mathbb{R})/\mathbb{Z}$ is a twisted cylinder over a 2-disk with $\partial_i C = T^2$ isometric to a rectangular torus, and $\partial C = K = T^2/\mathbb{Z}_2$ a flat totally geodesic Klein bottle.*

Conversely, a rectangular flat torus $T^2 = S^1_r \times S^1_s$ and a foliation $\mathcal{F}$ of T^2 by parallel closed geodesics different from $S^1_r \times \{p\}$ or $\{p\} \times S^1_s$ define a standard geometric graph manifold with irreducible universal cover $M^3 = C_{\mathcal{F}} \sqcup K$ whose core K is one-sided. Moreover, T^2 and $\mathcal{F}$ determine M^3 up to isometry.

We now introduce an isometric invariant of a geometric graph manifold. As we will see, this invariant determines the diffeomorphism type of the manifold.

For this purpose, we start by defining the slope $\mathcal{S}(\mathcal{F}_1, \mathcal{F}_2)$ of a foliation $\mathcal{F}_2$ by closed simple geodesics of an oriented flat torus T^2 with respect to another such foliation $\mathcal{F}_1$. In order to do this, we first assume that the foliations are oriented. Fix $z_0 \in T^2$, and take $\gamma_i \in \mathcal{F}_i$ parametrized over [0,1] such that $\gamma_1(0) = \gamma_2(0) = z_0$. Then v_i is primitive, and as observed above, there exists a unique $\hat{v}_i$ such that $\mathcal{B}(\gamma_i) = \{v_i, \hat{v}_i\}$ are two normalized markings of T^2 based at z_0. Since $\mathrm{SL}(2, \mathbb{Z})$ acts transitively on the set of oriented bases of a given lattice, there exist coprime integers p, q and a, b with $bq - ap = 1$ such that

$$v_2 = qv_1 + p\hat{v}_1, \quad \hat{v}_2 = av_1 + b\hat{v}_1. \tag{3.10}$$

We also have $p \neq 0$ since $v_1 \neq \pm v_2$. Notice that, since v_2 determines $\hat{v}_2$, the integers p and q determine a and b. Observe that $q/p \in \mathbb{Q}$ is independent of the choice of z_0 since the foliations are parallel. It does not depend on the orientations of the foliations either, since $\{-v, -\hat{v}\}$ is the oriented marking associated to $-\gamma$. We call

$$\mathcal{S}(\mathcal{F}_1, \mathcal{F}_2) := q/p,$$

the *slope*of $\mathcal{F}_2$ with respect to $\mathcal{F}_1$. Note though that reversing the orientation of the torus changes the sign of the slope, since this corresponds to replacing $\hat{v}_i$ with $-\hat{v}_i$. Moreover, since $v_1 = bv_2 - p\hat{v}_2$, we have that $\mathcal{S}(\mathcal{F}_2, \mathcal{F}_1) = -b/p$.

If $M^3 = C_1 \sqcup T^2 \sqcup C_2$ has a two-sided core, a choice of orientations $\mathfrak{o} = (\mathfrak{o}_M, \mathfrak{o}_T)$ of both M^3 and its core T^2 orients the normal bundle of T^2. We can thus choose the order of the two twisted cylinders (C_1, C_2) by letting C_1 be the cylinder containing the positive direction of the normal bundle. We thus define the *slope* of the lens space as

$$\mathcal{S}(M^3, \mathfrak{o}) = \mathcal{S}(M^3, (\mathfrak{o}_M, \mathfrak{o}_T)) := \mathcal{S}(\mathcal{F}(C_1), \mathcal{F}(C_2)) \in \mathbb{Q}.$$

Notice that $\mathcal{S}(M^3, (\mathfrak{o}_M, -\mathfrak{o}_T)) = -q/p$ and $\mathcal{S}(M^3, (-\mathfrak{o}_M, \mathfrak{o}_T)) = -b/p$ where b is defined in (3.10). Since $b = q^{-1} \mod p$, this is consistent with the fact that $L(p, q)$ and $L(p, q')$ are diffeomorphic if and only if $q' = \pm q^{\pm 1} \mod p$.

Analogously, if $M^3 = C \sqcup K$ has a one-sided core $K = \partial_i C/\langle j \rangle$, a choice of an orientation $\mathfrak{o} = \mathfrak{o}_M$ induces an orientation of the torus $\partial_i C$. We call $\mathcal{S}(M^3, \mathfrak{o}) := \mathcal{S}(\mathcal{F}(j), \mathcal{F}(C))$ the *slope* of the prism manifold, recalling that $\mathcal{F}(j) = \{S^1 \times \{w\} : w \in S^1\}$. Here we have $\mathcal{S}(M^3, -\mathfrak{o}) = -\mathcal{S}(M^3, \mathfrak{o})$.

Notice that, in either case, the slope of M^3 is well defined even when the geometric graph manifold metric is not standard.

We now observe the following.

Proposition 3.11. *The slope $\mathcal{S}(M^3, \mathfrak{o}) = q/p$ is an oriented isometry invariant of a geometric graph manifold . Furthermore, the slopes $-q/p$ and $\pm b/p$ are achieved by changing the orientation on M^3 or the core T^2. Conversely, any rational number is the slope of a geometric graph manifold , both on a lens space and on a prism manifold.*

Proof. First, assume that $M^3 = C_1 \sqcup T^2 \sqcup C_2$ is a lens space and let $f\colon M \to M'$ be an orientation preserving isometry. By Corollary 2.4 the core H is unique up to isometry, that is, there exists a maximal isometric product $\bar{H} \times [0, a] \subset M^n$, such that any $\bar{H} \times \{s\}$ for $0 \leqslant s \leqslant a$ can be regarded as a core, and any core is of this form. If we choose $H = \bar{H} \times \{a/2\}$, and similarly H' for M', then f takes H to H' and by Theorem 3.8, the isometry $f|_H$ takes the boundary nullity foliations of H into those of H'. Since we also assume that $f|_H$ is orientation preserving, the slopes of M and M' are the same. We can argue similarly for a prism manifold, in which case the core is even unique.

To achieve any slope q/p, we can choose the standard basis e_1, e_2 of a product torus $T^2 = S^1 \times S^1$ and let $v = qe_1 + pe_2$. Then there exists a unique $\hat{v}$ such that $\{v, \hat{v}\}$ is a normalized marking of the torus. This gives rise to two parallel foliations of T^2 with slope q/p and by Theorem 3.8 they can be realized by a geometric graph manifold metric on a lens space. The same data also gives rise to a prism manifold by Theorem 3.9. □

We are now in position to prove Theorem C in the introduction, which states that $\mathcal{S}(M^3, \mathfrak{o})$ determines the diffeomorphism type of the manifold.

Proof of Theorem C. Recall that the twisted cylinders C_i with invariants θ_i, h_i as in (3.1) are diffeomorphic to $D_i \times S^1$ by deforming θ_i continuously to 0. For a two-sided core T^2, choose $\gamma_i \in \mathcal{F}_i$, and let $\mathcal{B}(\gamma_i) = \{v_i, \hat{v}_i\}$ be the normalized markings of T^2 defined by C_i. Then the natural generators of $\pi_1(\partial(D_i \times S^1)) = \mathbb{Z} \oplus \mathbb{Z}$ are represented by the simple closed geodesics γ_i and $\sigma_i(t) = \exp(t\hat{v}_i)$, $0 \leqslant t \leqslant 1$, since the marking $\{v_i, \hat{v}_i\}$ is normalized. According to the definition of slope, $v_2 = qv_1 + p\hat{v}_1$ which implies that under the diffeomorphism from $\partial D_2 \times S^1 \simeq \partial C_2$ to $\partial C_1 \simeq \partial D_1 \times S^1$, the element $(1, 0) \in \pi_1(\partial(D_2 \times S^1))$ is taken to $(q, p) \in \pi_1(\partial(D_1 \times S^1))$. By definition this is the lens space $L(p, q)$; see Section 1.

To determine the topological type in the one-sided case, we view M^3 as the union of C with the flat twisted cylinder N^3 defined in (1.1). Then $\partial N^3 = T^2$ is a rectangular torus which we glue to $\partial_i C$. Taking $\epsilon \to 0$ (or considering $T^2 \times (0, \epsilon]$ as part of C instead), we obtain M^3. We can now use our second description of prism manifolds in Section 1 and the proof finishes as in the previous case. $\square$

We finally classify the moduli space of metrics.

Proposition 3.12. *On a lens space $(L(p, q), \mathfrak{o})$ the connected components of the moduli space of geometric graph manifold metrics with non-negative scalar curvature are parametrized by its slope $q/p \in \mathbb{Q}$, and therefore, it has infinitely many components. On the other hand, on a prism manifold $P(q, p)$ with $q > 1$ the moduli space is connected.*

Proof. In Proposition 3.4 we saw that we can deform any geometric graph manifold metric into one which is standard. According to Theorem 3.8, the standard geometric graph manifold metric on a lens space can equivalently be uniquely defined by the triple $(T^2, \mathcal{F}_1, \mathcal{F}_2)$. Thus, we can deform the flat metric on the torus, carrying along the foliations $\mathcal{F}_i$, which induces a deformation of the original metric by standard metrics. In the proof of Proposition 3.6 we saw that, after choosing orientations, for $\gamma_i \in \mathcal{F}_i$ with $v_i = \gamma_i'(0)$ we have the normalized markings $\mathcal{B}(\gamma_i) = \{v_i, \hat{v}_i\}$ which represents a fundamental domain of the lattice defined by T^2. We can thus deform the flat torus to a unit square torus such that the first marking is given by $v_1 = (1, 0),\ \hat{v}_1 = (0, 1)$. Then $v_2 = (q, p) = qv_1 + p\hat{v}_1$, which in turn determines $\hat{v}_2$, and q/p is the slope of $\mathcal{F}_2$ with respect to $\mathcal{F}_1$. Metrics with different slope can clearly not be deformed into each other since the invariant is a rational number. Since the diffeomorphism type of the lens space only depends on $\pm q^{\pm 1}$ mod p, we obtain infinitely many components.

For a prism manifold, we similarly deform the metric to be standard and the rectangular torus into a unit square. But then the absolute value of its slope already uniquely determines its diffeomorphism type. $\square$

Remarks. *a*) For a lens space $L(p, q) = \mathbb{S}^3/\mathbb{Z}_p$ one can assume that $p, q > 0$, $\gcd(p, q) = 1$ and $q \leqslant p$ since the action of $\mathbb{Z}_p$ is determined by q mod p. Then the slopes $q'/p + n$ for $n \in \mathbb{N} \cup \{0\}$, and $q' = \pm q^{\pm 1}$ mod p with $0 < q' \leqslant p$, parametrize the infinitely many distinct connected components of geometric graph manifold metrics of non-negative curvature in $L(p, q)$. Yet, the lens space $L(4p, 2p - 1)$ has one further component since it is diffeomorphic to $P(1, p)$. This component is distinct from the others since the core is one sided.

b) One easily sees that the angle α between the nullity foliations of a lens space, that is, the angle between v_1 and v_2, is given by $\cos(\alpha) = (q + p\theta_1)r_1/r_2 = (b - p\theta_2)r_2/r_1$, where $r_i = |v_i|$ and θ_i are the twists of the two cylinders. One can thus make the nullity leaves orthogonal if and only if $0 \leqslant -q/p < 1$ and in that case $r_2 = ph_1,\ h_2 = r_1/p$ and $\theta_1 = -q/p,\ \theta_2 = b/p$. This determines the metric on the lens space described in the introduction as a quotient of Figure 1, and is thus the only component containing a metric with orthogonal nullity leaves.

c) We can explicitly describe the geometric graph manifold metrics on $\mathbb{S}^3 = L(1, 1)$ up to deformation. We assume that the core is a unit square and that the first foliation is parallel to (1,0), that is, the first cylinder is a product cylinder. Then the second marking is given by $v_2 = (q, 1),\ \hat{v}_2 = (q - 1, 1)$. By choosing the orientations appropriately, we can assume $q \geqslant 0$. According to the proof of Proposition 3.4, the marking $\{v, \hat{v}\}$ corresponds to a twisted cylinder as in (3.1) with $r = \|v\|$, $\theta = \langle v, \hat{v}\rangle/\|v\|^2$ and $h = \|\hat{v} - \theta v\|$. Thus in our case the second cylinder is given by $r = 1/h = \sqrt{1 + q^2}$, and $\theta = (1 + q^2 - q)(1 + q^2)$. The slope is q, and the standard metric in Figure 1 corresponds to $q = 0$.

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