Riemannian Geometry: a class guide

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Prerequisites: Basics about manifolds and tensors, at least up to page 18 here. Fundamental group and covering maps.

Bliography: [**CE**], [**dC**], [**Me**], [**ON**], [**Pe**], [**Sp**], [**KN**], [**Es**], [**Ri**].

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§1. Notations

Top. manifolds: Hausdorff + countable basis. Partitions of unity.
n-dimensional differentiable manifolds: M^n. Everything is C^∞.
$\mathcal{F}(M) := C^\infty(M, \mathbb{R})$; $\mathcal{F}(M, N) := C^\infty(M, N)$.
(x, U) chart $\Rightarrow$ coordinate vector fields $= \partial_i := \partial/\partial x_i \in \mathfrak{X}(U)$.
Tangent bundle TM, vector fields $\mathfrak{X}(M) := \Gamma(TM) \cong \mathcal{D}(M)$.
Submersions, immersions, embeddings, local diffeomorphisms.
Vector bundles, trivializing charts, transition functions, sections.
Tensor fields $\mathfrak{X}^{r,s}(M)$, k-forms $\Omega^k(M)$, orientation, integration.
Pull-back of a vector bundle $\pi : E \to N$ over N: $f^*(E)$.
Vector fields along a map $f : M \to N \Rightarrow \mathfrak{X}_f \cong \Gamma(f^*(TN))$.
f-related vector fields: closed by Lie bracket.

Example: Lie Groups G, L_g, R_g; $\mathfrak{g} := T_e G$ is an algebra;
Integral curve γ of $X \in \mathfrak{g}$ through e is a homomorphism $\Rightarrow$
$exp^G : \mathfrak{g} \to G$, $exp^G(X) := \gamma(1) \Rightarrow exp^G(tX) = \gamma(t)$.

Exercise. If φ_t is the flux of X as above, then $L_g \circ \varphi_t = \varphi_t \circ L_g \ \forall g \in G$ and $\varphi_t = R_{\gamma(t)}$.

§2. Geometry = Measurement of the Earth

Geography: Protagoras (481BC - 411BC): Earth should be somehow curved, since boats "sank" at the horizon. Anaximander (610BC - 546BC): Imagined Earth as a "column" floating in the center of the universe, "without resting on anything, but without falling". Pythagoras (570 BC - 495 BC): Believed a spherical Earth, and so Aristotles. By 350BC, every illustrated Greek believed in a spherical Earth. Eratosthenes (276BC - 194BC), measured the Earth circumference in 'stadia'. He computed the angle as "a fiftieth of a circle." Total error $< 16.3\%$. Columbus knew Eratosthenes measurement (!!!) But cited Strabo (63BC - 23BC) and Ptolomy (100AC - 170AC), who wrongly computed 29000km instead of 40000km. Eratosthenes also measured the angle of the Earth axis with respect to the ecliptic, and its distance to the Sun. Watch Cosmos video.

§3. Riemannian metrics

Gauss, 1827: $M^2 \subset \mathbb{R}^3 \Rightarrow \langle\,,\rangle|_{M^2},\ K_M = K_M(\langle\,,\rangle)$, distances, areas, volumes... Non-Euclidean geometries.
Riemann, 1854: $\langle\,,\rangle \Rightarrow K_M$ (relations proved decades later).
Slow development. General Relativity pushed up!
Riemannian metric, Riemannian manifold: $(M^n, \langle\,,\rangle) = M^n$.
$g_{ij} := \langle \partial_i, \partial_j \rangle \in \mathcal{F}(U) \Rightarrow (g_{ij}) \in C^\infty(U, S(n,\mathbb{R}) \cap Gl(n,\mathbb{R}))$.
Isometries, local isometries, isometric immersions.
Product metric. $T_p\mathbb{V} \cong \mathbb{V},\ T\mathbb{V} \cong \mathbb{V} \times \mathbb{V}$.

Examples: $(\mathbb{R}^n, \langle\,,\rangle_{can})$, Euclidean submanifolds. Nash.

Example: (bi-)invariant metrics on Lie groups.

Proposition 1. *Every differentiable manifold admits a Riemannian metric.*

Angles between vectors at a point. Norm.
It always exists <u>local</u> orthonormal frames: $\{e_1, \ldots, e_n\}$. $\Rightarrow$

Proposition 2. *Given an <u>oriented</u> Riemannian manifold M^n, there exists a unique volume form $dvol \in \Omega^n(M^n)$ such* *that* $dvol(e_1, \ldots, e_n) = 1$ *for any positively oriented orthonormal basis* $\{e_1, \ldots, e_n\}$ *at any point.*

If $\partial_i = \sum_j C_{ij} e_j \Rightarrow (g_{ij}) = CC^t \Rightarrow dvol(\partial_1, \cdots, \partial_n) = \det(C) \Rightarrow$

$$dvol|_U = \sqrt{\det(g_{ij})}\, dx_1 \wedge \cdots \wedge dx_n.$$

So, we can "integrate functions". Volume of (compact) sets.
Riemannian vector bundles: $(E, \langle\, , \rangle)$.

§4. Distance

Length of a piecewise differentiable curve $\Rightarrow$ Riem. distance d.
The topology of d coincides with the original one on M.

§5. Linear connections

If $M^n = \mathbb{R}^n$, or even if $M^n \subset \mathbb{R}^N$, there is a natural way to differentiate vector fields. And this depends only on $\langle\, , \rangle$.

Def.: An *affine connection* or a *linear connection* or a *covariant derivative* on M is a map

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

with $\nabla_X Y$ being $\mathbb{R}$-bilinear, tensorial in X and a derivation in Y.

Tensoriality in $X \Rightarrow (\nabla_X Y)(p) = \nabla_{X(p)} Y$ makes sense.
In particular, $(\nabla_X Y) \circ f = \nabla_{X \circ f} Y$ for every $f : N \to M$.
Local oper.: $Y|_U = 0 \Rightarrow (\nabla_X Y)|_U = 0 \Rightarrow \exists!$ affine connection ∇^U in U (chart) such that $(\nabla_X Z)|_U = \nabla^U_{X|_U}(Z|_U), \forall X, Z \in \mathfrak{X}(M)$.
$\Rightarrow$ The *Christoffel symbols* Γ^k_{ij} of ∇ in a coordinate system $\Rightarrow$

Christoffel symbols completely determine the connection: all that is needed is to have local basis of sections $\Rightarrow$
Connections on vector bundles: formally exactly the same.
The above property on U is a particular case of the following:

Proposition 3. *(or "Everything I know about connections.") Let ∇ be a (linear) connection on a vector bundle E over M. Then, for every $f\colon N \to M$, there exists a unique connection $\nabla^f : \mathfrak{X}(N) \times \Gamma(f^*(E)) \to \Gamma(f^*(E))$ on $f^*(E)$ such that*

$$\nabla^f_Y(\xi \circ f) = \nabla_{f_* Y}\xi, \quad \forall\, Y \in \mathfrak{X}(N),\ \xi \in \Gamma(E).$$

Exercise. Give meaning and prove that $g^*(f^*(E)) = (f \circ g)^*(E)$ and $\left(\nabla^f\right)^g = \nabla^{f\circ g}$.

Exercise. Let $p \in M$, $i(q) = (p,q) \in M \times N$, $\tilde{f} : M \times N \to (\tilde{M}, \nabla)$ and $f = \tilde{f} \circ i$. Show that $(\nabla^{\tilde{f}}_{\tilde{X}} \tilde{f}_* \tilde{Y}) \circ i = \nabla^f_X f_* Y$, $\forall\, X \stackrel{i}{\sim} \tilde{X}$, $Y \stackrel{i}{\sim} \tilde{Y}$.

Exercise. If $f\colon M \to (\tilde{M}, \nabla)$ $\Rightarrow \nabla^f_X f_* Y = f^*(\nabla_{\tilde{X}} \tilde{Y})$, $\forall\, X \stackrel{f}{\sim} \tilde{X}$, $Y \stackrel{f}{\sim} \tilde{Y}$. More generally: If $f\colon M \to \tilde{M}$, and $(E, \nabla) \to \tilde{M}$ $\Rightarrow \nabla^f_X f^*\xi = f^*(\nabla_{\tilde{X}} \xi)$, $\forall\, X \stackrel{f}{\sim} \tilde{X}$, $\xi \in \Gamma(E)$.

We will omit the superindex f in ∇^f.
In particular, Proposition 3 holds for any smooth curve $\alpha(t) = \alpha : I \subset \mathbb{R} \to M$, and if $V \in \mathfrak{X}_\alpha$ we denote $V' := \nabla_{\partial_t} V \in \mathfrak{X}_\alpha$.
So, if $\alpha'(0) = v$, $\nabla_v Y = (Y \circ \alpha)'(0)$. But beware of "$\nabla_{\alpha'}\alpha'$"!!

Def.: $V \in \mathfrak{X}_\alpha$ is *parallel* if $V' = 0$. We denote by $\mathfrak{X}''_\alpha$ the set of parallel vector fields along α.

Proposition 4. *Let $\alpha : I \subset \mathbb{R} \to M$ be a piecewise smooth curve, and $t_0 \in I$. Then, for each $v \in T_{\alpha(t_0)}M$, there exists a unique parallel vector field $V_v \in \mathfrak{X}_\alpha$ such that $V_v(t_0) = v$.*

The map $v \mapsto V_v$ is an isomorphism between $T_{\alpha(t_0)}M$ and $\mathfrak{X}''_\alpha$, and the map $(v,t) \mapsto V_v(t)$ is smooth when α is smooth $\Rightarrow$

Def.: The *parallel transport of* $v \in T_{\alpha(t)}M$ *along* α *between* t *and* s is the map $P^{\alpha}_{ts} : T_{\alpha(t)}M \to T_{\alpha(s)}M$ given by $P^{\alpha}_{ts}(v) = V_v(s)$.

Notice that $\mathcal{F}(M) = \mathfrak{X}^0(M) = \mathfrak{X}^{0,0}(M)$ and $\mathfrak{X}(M) = \mathfrak{X}^{0,1}(M)$. Covariant differentiation of 1-forms and tensors: $\forall r, s \geq 0$,

$$\nabla \Rightarrow \begin{cases} \nabla : \mathfrak{X}^r(M) \to \mathfrak{X}^{r+1}(M); \\ \nabla : \mathfrak{X}^{r,s}(M) \to \mathfrak{X}^{r+1,s}(M); \\ \nabla : \mathfrak{X}^{r,s}(E, \hat{\nabla}) \to \mathfrak{X}^{r+1,s}(E, \hat{\nabla}); \end{cases}$$

for any affine vector bundle $(E, \hat{\nabla})$ (in partic., for $E = (TM, \nabla)$).

§6. The Levi-Civita connection !

Def.: A linear connection ∇ on a Riemannian manifold $(M, \langle\,,\rangle)$ is said to be *compatible with* $\langle\,,\rangle$ if, for all $X, Y, Z \in \mathfrak{X}(M)$,

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle.$$

Exercise. ∇ is compatible with $\langle\,,\rangle \iff \forall\, V, W \in \mathfrak{X}_\alpha$, $\langle V, W\rangle' = \langle V', W\rangle + \langle V, W'\rangle \iff \forall\, V, W \in \mathfrak{X}''_\alpha$, $\langle V, W\rangle$ is constant $\iff P^{\alpha}_{ts}$ is an isometry, $\forall\, \alpha, t, s \iff \nabla\langle\,,\rangle = 0$.

Def.: The tensor $T_\nabla(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y]$ is called the *torsion* of ∇. We say that ∇ is *symmetric* if $T_\nabla = 0$.

Miracle: *Every Riemannian manifold* $(M, \langle\,,\rangle)$ *has a unique linear connection that is symmetric and compatible with* $\langle\,,\rangle$, *called the Levi-Civita connection of* $(M, \langle\,,\rangle)$.

This is a consequence of the *Koszul formula*: $\forall\, X, Y, Z \in \mathfrak{X}(M)$,

$$2\langle \nabla_X Y, Z\rangle = X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle - \langle X, [Y, Z]\rangle - \langle Y, [X, Z]\rangle + \langle Z, [X, Y]\rangle.$$

Exercise. Verify that this formula defines a linear connection with the desired properties.

This is the only connection that we will work with. In coordinates, if $(g^{ij}) := (g_{ij})^{-1}$,

$$\Gamma_{ij}^k = \frac{1}{2}\sum_r \left(\frac{\partial g_{ir}}{\partial x_j} + \frac{\partial g_{jr}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_r}\right) g^{rk} \ .$$

Exercise. Show that, for $(\mathbb{R}^n, \langle\, ,\rangle_{can})$, $\Gamma_{ij}^k = 0$ and ∇ is the usual vector field derivative.

Exercise. Show that the Levi-Civita connection of a bi-invariant metric of a Lie Group satisfies, and is characterized if symmetric, by the property that $\nabla_X X = 0 \ \forall X \in \mathfrak{g}$. Sug: Use Koszul formula and Exercise 1.

Lemma 5. *(Symmetry and Compatibility Lemma)*
Let N be any manifold, and $f : N \to M$ a smooth map into a Riemannian manifold M. Then:

- *∇^f is symmetric, that is, $\nabla^f_X f_*Y - \nabla^f_Y f_*X = f_*[X,Y]$, $\forall\ X, Y \in \mathfrak{X}(N)$;*
- *∇^f is compatible with the natural metric on $f^*(TM)$. In fact, this holds for any Riemannian vector bundle.*

Exercise. If $T_{\nabla^f}(X,Y) := \nabla^f_X f_*Y - \nabla^f_Y f_*X - f_*[X,Y], X, Y \in \mathfrak{X}(N)$, then $T_{\nabla^f} = f^*T_\nabla$.

Example: $f\colon N \to M$ an isometric immersion $\Rightarrow f^*(TM) = f_*(TN) \oplus^\perp T_f^\perp N \ \Rightarrow \forall\ Z \in \mathfrak{X}_f, Z = Z^\top + Z^\perp \ \Rightarrow$ the relation between the Levi-Civita connections is $f_*\nabla^N_X Y = (\nabla^f_X f_*Y)^\top$.

Remark 6. $f : N \to M \Rightarrow \mathfrak{X}_f = T_f(\mathcal{F}(N, M))$ *(check only for $f(N) \subset$ chart of M).*

§7. Geodesics !!

When do we have minimizing curves? What are those curves? The Brachistochrone problem and the Calculus of Variations.

Galileo, 1638: wrong solution (circle) in the *Discorsi*. Johann Bernoulli posed the problem in 1696 and gave 6 months to solve it (he already knew the solution was a cycloid). Leibniz asked for more time for 'foreign mathematicians' to attack the problem. They tempted Newton, who didn't like to be teased 'by foreigners', but solved the problem in less than half a day. The Royal Society published Newton's solution anonymously, but there is a legend of Johann Bernoulli claiming in awe with the solution in his hands: *"I recognize the lion by his paw."*

Critical points of the arc-length funct. $L : \Omega_{p,q} \to \mathbb{R}$: *geodesics*:

$$\gamma'' := \nabla_{\frac{d}{dt}} \gamma' = 0.$$

Geodesics = second order nonlinear nice ODE $\Rightarrow$

Proposition 7. *$\forall\ v \in TM$, $\exists\ \epsilon > 0$ and a unique geodesic $\gamma_v : (-\epsilon, \epsilon) \to M$ such that $\gamma_v'(0) = v$ ($\Rightarrow \gamma_v(0) = \pi(v)$).*

γ a geodesic $\Rightarrow \|\gamma'\| =$ constant.
γ and $\gamma \circ r$ nonconstant geodesics $\Rightarrow r(t) = at + b$, $a, b \in \mathbb{R} \Rightarrow$ $\gamma_v(at) = \gamma_{av}(t)$; $\gamma_v(t+s) = \gamma_{\gamma_v'(s)}(t) \Rightarrow$ *geodesic field* G of M:

Proposition 8. *There is a unique vector field $G \in \mathfrak{X}(TM)$ such that its trajectories are of the form γ', where γ is a geodesic of M.*

Proof: Just define $G(v) = (\gamma_v')'(0)$. ∎

The local flux of G is called the *geodesic flow* of M. In particular:

Corollary 9. *For each $p \in M$, there is a neighborhood $U_p \subset M$ of p and positive real numbers $\delta, \epsilon > 0$ such that the map*

$$\gamma : T_\epsilon U_p \times (-\delta, \delta) \to M, \quad \gamma(v,t) = \gamma_v(t),$$

is differentiable, where $T_\epsilon U_p := \{v \in TU_p \colon \|v\| < \epsilon\}$*.*

Since $\gamma_v(at) = \gamma_{av}(t)$, changing ϵ by $\epsilon\delta/2$ we can assume $\delta = 2 \Rightarrow$ We have the *exponential map* of M (terminology from $O(n)$):

$$\exp : T_\epsilon U_p \to M, \quad \exp(v) = \gamma_v(1).$$

$\Rightarrow \exp(tv) = \gamma_v(t) \Rightarrow \exp_p = \exp|_{T_pM} \colon B_\epsilon(0_p) \subset T_pM \to M \Rightarrow$

Proposition 10. *For every* $p \in M$ *there is* $\epsilon > 0$ *such that* $B_\epsilon(p) := \exp_p(B_\epsilon(0_p)) \subset M$ *is open and* $\exp_p \colon B_\epsilon(0_p) \to B_\epsilon(p)$ *is a diffeomorphism.*

An open set $p \in V \subset M$ onto which $\exp_p$ is a diffeomorphism as above is called a *normal neighborhood* of p, and when $V = B_\epsilon(p)$ it is called a *normal* or *geodesic ball* centered at p.

Proposition 10 $\Rightarrow \left(\exp_p|_{B_\epsilon(0_p)}\right)^{-1}$ is a chart of M in $B_\epsilon(p) \Rightarrow$ We always have (local!) *polar coordinates* for any $(M, \langle\, ,\rangle)$:

$$\varphi : (0, \epsilon) \times \mathbb{S}^{n-1} \to B_\epsilon(p)\backslash\{p\}, \qquad \varphi(s, v) = \gamma_v(s), \qquad (1)$$

where $\mathbb{S}^{n-1} = \{v \in T_pM : \|v\| = 1\}$ is the unit sphere in T_pM.

Examples: $(\mathbb{R}^n, can)$; (T^n, can); $(\mathbb{S}^n, can)$.

Exercise. Show that for a bi-invariant metric on a Lie Group, it holds that $exp_e = exp^G$.

Exercise. Show that if ∇ is any connection on M, then there is another torsion free connection on M which has the same geodesics as ∇ (up to reparamtrizations).

Exercise. ∇ and $\bar{\nabla}$ are two torsion free connections on M with the same geodesics (up to rep.) $\iff$ there exists a 1-form ω such that $\nabla_X Y - \bar{\nabla}_X Y = \omega(X)Y + \omega(Y)X$ for all X, Y.

§8. Geodesics are (local) arc-length minimizers

Lemma 11. *(Gauss' Lemma) Let $p \in M$ and $v \in T_pM$ such that $\gamma_v(s)$ is defined up to time $s = 1$. Then,*

$$\langle (\exp_p)_{*v}(v), (\exp_p)_{*v}(w) \rangle = \langle v, w \rangle, \quad \forall\ w \in T_pM.$$

Proof: If $f(s,t) := \gamma_{v+tw}(s) = \exp_p(s(v+tw))$ then, for $t = 0$, $f_s = (\exp_p)_{*sv}(v)$, $f_t = (\exp_p)_{*sv}(sw)$ and $\langle f_s, f_t \rangle_s = \langle v, w \rangle$. ∎

Gauss' Lemma $\Rightarrow \mathbb{S}_\epsilon(p) := \partial B_\epsilon(p) \subset M$ is a regular hypersurface of M orthogonal to the geodesics emanating from p, called the *geodesic sphere* of radius ϵ centered at p.
Now, $B_\epsilon(p) := \exp_p(B_\epsilon(0_p)) \subset M$ as in Proposition 10 agrees with the metric ball of (M, d) !!!!! More precisely:

Proposition 12. *Let $B_\epsilon(p) \subset U$ a normal ball centered at $p \in M$. Let $\gamma : [0, a] \to B_\epsilon(p)$ be the geodesic segment with $\gamma(0) = p$, $\gamma(a) = q$. If $c : [0, b] \to M$ is another piecewise differentiable curve joining p and q, then $l(\gamma) \leq l(c)$. Moreover, if equality holds, then c is a monotone reparametrization of γ.*

Proof: In polar coordinates, $c(t) = \exp_p(s(t)v(t))$ in $B_\epsilon(p) \backslash \{p\}$, and if $f(s,t) := \exp_p(sv(t)) = \gamma_{v(t)}(s)$, we have that $c' = s' f_s + f_t$. Now, use that $f_s \perp f_t$, by Gauss' Lemma. ∎

Corollary 13. *d is a distance on M, $d_p := d(p, \cdot)$ is differentiable in $B_\epsilon(p) \backslash \{p\}$, and d_p^2 is differentiable in $B_\epsilon(p)$.*

Exercise. Compute $\|\nabla d_p\|$ and the integral curves of ∇d_p inside $B_\epsilon(p) \backslash \{p\}$.

Remark 14. Proposition 12 is LOCAL ONLY, and $\epsilon = \epsilon(p)$: $\mathbb{R}^n$; T^n; $\mathbb{S}^n$; $\mathbb{R}^n \setminus \{0\}$.

§9. Geodesics: convex neighborhoods

Problem: a normal ball $B_\epsilon(p)$ may not be a *convex set*, like in $\mathbb{S}^n$. But it is a *strongly convex set* for ϵ small enough!

Proposition 15. *For each $p \in M$, there is an open neighborhood W of p and $\delta > 0$ such that, for all $q \in W$, $B_\delta(q)$ is a normal ball around q and $W \subset B_\delta(q)$ (e.g., $W = B_{\delta/2}(p)$). That is, W is a normal neighborhood of all of its points.*

Proof: Following the notations in Corollary 9, consider $F : T_\epsilon U_p \to M \times M$, $F(v) = (\pi(v), \exp(v))$ for the usual bundle projection $\pi : TM \to M \Rightarrow F_{*0_p} = \begin{pmatrix} I & 0 \\ I & I \end{pmatrix} \Rightarrow F : T_\delta U'_p \to \mathcal{W}$ is a diffeo, with $p \in U'_p$ and $F(0_p) = (p,p) \in \mathcal{W} \subset M \times M$. Now take any $W \subset M$ with $(p,p) \in W \times W \subset \mathcal{W}$. ∎

W as Proposition 15 is called a <u>*totally normal neighborhood*</u>.

Remark 16. The proof shows that, $\forall q, q' \in W, \exists!$ geodesic γ_v joining q and q' with $l(\gamma_v) < \delta$. Moreover, $v = v(q,q')$ is a differentiable function, so γ_v depends differentiably of q and q'.

Corollary 17. *If a piecewise differentiable curve $c : [a,b] \to M$ p.b.a.l. realizes the distance between $c(a)$ and $c(b)$, then c is a geodesic. In particular, c is regular (see Proposition 12).*

Corollary 18. *Given $f : N \to M$ with $f(N)$ precompact, then $\mathfrak{X}_f = T_f\mathcal{F}(N,M)$. Same holds for proper variations.*

Proof: Put any Riemannian metric in M, take $\epsilon > 0$ such that $B_\epsilon(p)$ is a normal ball for every $p \in f(N)$. If $V \in \mathfrak{X}_f$, then $f_t(p) = \exp_p(tV(p))$ has V as its variational vector field. ∎

Lemma 19. *Given $p \in M$, there exists an $\epsilon' > 0$ such that, for all $0 < r < \epsilon'$, every geodesic γ tangent to $\mathbb{S}_r(p)$ at $\gamma(0)$ stays outside of $B_r(p)$ around 0.*

Proof: Let W and δ as in Proposition 15, and consider $\gamma : (-\delta, \delta) \times T_1 W \to M$, $\gamma(t, v) = \gamma_v(t)$. If $w(t, v) := \exp_p^{-1}(\gamma_v(t))$, then $F(t, v) := \|w(t, v)\|^2 = d^2(p, \gamma_v(t))$ for $|t| < \delta$. Observe that for $q = p$, $\partial^2 F/\partial t^2(0, v) = 2$, and hence $\partial^2 F/\partial t^2(0, v) > 0$ for $q \in W$ close to p and all unit $v \in T_q M$. But for $B_s(p) \subset W$ and $v \in T_q(\mathbb{S}_s(p))$, by Gauss Lemma $\partial F/\partial t(0, v) = 0$. Therefore, $t = 0$ is a local minimum of $F(\cdot, v)$ for $v \in T_q(\mathbb{S}_s(p))$. ∎

Proposition 20. *For every $p \in M$, there is $\delta > 0$ such that $B_\delta(p)$ is strongly convex.*

Proof: Take $\delta < \epsilon'/2$ for ϵ' as in Lemma 19 in such a way that $B_{\epsilon'}(p) \subset W$ for any W as in Proposition 15. ∎

What we have shown can be summarized as follows:

Theorem 21. *For all $p \in M$, there is $\epsilon_0 > 0$ such that, for every $0 < \epsilon < \epsilon_0$, $B_\epsilon(p)$ is a totally normal and strongly convex neighborhood. In particular, for every $q \neq q' \in \overline{B_\epsilon(p)}$, there exists a unique minimizing (p.b.a.l.) piecewise differentiable curve joining q and q', which is a smooth geodesic segment (whose interior is) contained in $B_\epsilon(p)$, and that depends differentiably on q and q'.*

Examples: $\mathbb{R}^n$; T^n; $\mathbb{S}^n$.

§10. Curvature !!

Gauss: $K(M^2 \subset \mathbb{R}^3) = K(\langle\, ,\rangle)$. Riemann: $K(\sigma) = K_p(\exp_p(\sigma))$.

Def.: The *curvature tensor* or *Riemann tensor* of M is (sign!)

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

We also call R the (4,0) tensor given by

$$R(X,Y,Z,W) = \langle R(X,Y)Z, W\rangle.$$

• Similarly, any affine vector bundle $(E, \hat{\nabla}) \to M$ has a curvature tensor $R_{\hat{\nabla}} : TM \times TM \to \mathrm{End}(E)$.

Proposition 22. *For all $X, Y, Z, W \in \mathfrak{X}(M)$, it holds that:*

- *R is a tensor;*
- *$R(X,Y,Z,W)$ is skew-symmetric in X, Y and in Z, W;*
- *$R(X,Y,Z,W) = R(Z,W,X,Y)$;*
- *$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$ (first Bianchi id.);*
- *$R_{ijk}^s = \sum_l \Gamma_{ik}^l \Gamma_{jl}^s - \sum_l \Gamma_{jk}^l \Gamma_{il}^s + \partial_j \Gamma_{ik}^s - \partial_i \Gamma_{jk}^s$ ($\Rightarrow R \cong \partial^2 \langle\, ,\rangle$).*

Proof: Exercise. ∎

$\langle\, ,\rangle \Rightarrow \mathfrak{X}(M) \cong \Omega^1(M)$ and $\langle\, ,\rangle$ extends to the tensor algebra $\Rightarrow$ the *curvature operator* $R : \Omega^2(M) \to \Omega^2(M)$ is self-adjoint.

Def.: If $\sigma \subset T_pM$ is a plane, then the *sectional curvature* of M in σ is given by

$$K(\sigma) := \frac{R(u,v,v,u)}{\|u\|^2\|v\|^2 - \langle u,v\rangle^2}, \qquad \sigma = \mathrm{span}\{u,v\}.$$

Proposition 23. *If R and R' are tensors with the symmetries of the curvature tensor + Bianchi such that $R(u,v,v,u) = R'(u,v,v,u)$ for all u, v, then $R = R'$ ($\Rightarrow K$ determines R).*

Corollary 24. *If M has constant sectional curvature $c \in \mathbb{R}$, then $R(X, Y, Z, W) = c(\langle X, W\rangle\langle Y, Z\rangle - \langle X, Z\rangle\langle Y, W\rangle)$.*

Def.: The *Ricci tensor* is the symmetric (2,0) tensor given by

$$Ric(X, Y) := \frac{1}{n-1}\operatorname{trace} R(X, \cdot, \cdot, Y),$$

and the *Ricci curvature* is $Ric(X) = Ric(X, X)$ for $\|X\| = 1$.

Example: $\mathbb{CP}^n$ as $\mathbb{S}^{2n+1}/\mathbb{S}^1$, $n \geq 2$, has $K(X, Y) = 1 + 3\langle JX, Y\rangle^2$ and $Ric \equiv (n+2)/(n-1)$.

Def.: The *scalar curvature* of M is $\frac{1}{n}\operatorname{trace} Ric$.

Lemma 25. *(Compare with Lemma 5) Let $f : U \subset \mathbb{R}^2 \to M$ be a map into a Riemannian manifold and $V \in \mathfrak{X}_f$. Then,*

$$\nabla_{\partial_t}\nabla_{\partial_s}V - \nabla_{\partial_s}\nabla_{\partial_t}V = R(f_*\partial_t, f_*\partial_s)V.$$

More generally, $R_{\hat{\nabla}^f} = f^*(R_{\hat{\nabla}})$ for any $f : N \to M$ and any affine vector bundle $(E, \hat{\nabla}) \to M$.

Proof: Since $R_{\hat{\nabla}^f}$ is a tensor, it is enough to check for $\xi \circ f$ with $\xi \in \Gamma(E)$ using Proposition 3: Take $X, Y \in \mathfrak{X}(N)$ and $\bar{X}, \bar{Y} \in \mathfrak{X}(M)$ f-related to them near a point $p \in N$. Now compute $\hat{\nabla}^f_X \hat{\nabla}^f_Y (\xi \circ f)$. But this prove fails if f is not an immersion! Instead write in a chart of M, $f_*Z = \sum_i Z(f_i)\partial_i \circ f$ and perform the same computation. And don't use charts of N! ∎

Corollary 26. *Let (E^k, ∇) be an affine vector bundle over a simply connected manifold M. Then, (E^k, ∇) is trivial and affinely isomorphic to $(M \times \mathbb{R}^k, can) \iff R_\nabla = 0$.*

Proof: The hypothesis of the direct statement is equivalent to the existence of a globally parallel frame $\{\xi_1, \ldots, \xi_k\}$ of E^k, which obviously implies that $R_\nabla(\cdot, \cdot)\xi_i = 0$. For the converse, take $\eta \in E^k_p$, two smooth curves c_0, c_1 in M between p and q, and a (smooth) proper homotopy $f(s,t) = c_s(t)$ between them. Define $\xi_s(t) = \xi(s,t) \in \Gamma(f^*E)$ as the parallel transport of η along c_s. Then, $\nabla_{\partial_t}\nabla_{\partial_s}\xi = \nabla_{\partial_s}\nabla_{\partial_t}\xi = 0$. So $(\nabla_{\partial_s}\xi)(s, \cdot)$ is parallel along c_s. But $(\nabla_{\partial_s}\xi)(s,0) = 0$, and therefore $\frac{d}{ds}(\xi(s,1)) = (\nabla_{\partial_s}\xi)(s,1) = 0$. ∎

Exercise. Let $B_\epsilon(p)$ be a closed normal ball in a Riemannian surface L^2. If $\measuredangle_\epsilon(p)$ is the variation of the angle of the parallel translation around $\partial B_\epsilon(p)$, then $K(p) = \lim_{\epsilon \to 0} \measuredangle_\epsilon(p)/\mathrm{Vol}(B_\epsilon(p)) = \lim_{\epsilon \to 0} \measuredangle_\epsilon(p)/\pi\epsilon^2$. (Suggestion: Let $e \in \mathfrak{X}(B_\epsilon(p))$, $\|e\| = 1$, $w := \langle \nabla_\bullet e, e^\perp \rangle$, and use Stokes.)

§11. Jacobi fields

There's a strong relationship between geodesics and curvature, since curvature measures how fast geodesics come apart. The same tool to prove this is used also to understand the singularities of the exponential map: the Jacobi fields.
Given a variation of a geodesic γ by geodesics, the variational vector field $J \in \mathfrak{X}_\gamma$ satisfies the *Jacobi equation*, i.e.,

$$J'' + R(J, \gamma')\gamma' = 0.$$

A vector field along a geodesic γ satisfying the Jacobi equation above is called a *Jacobi field*: $\mathfrak{X}^J_\gamma = \{J \in \mathfrak{X}_\gamma \colon J'' = R(\gamma', J)\gamma'\}$.

The Jacobi equation is a second order linear ODE (take a parallel frame if not convinced) $\Rightarrow \forall$ geodesic γ and every $u, v \in T_{\gamma(t_0)}M$, there exists a unique $J \in \mathfrak{X}_\gamma^J$ such that $J(t_0) = u,\ J'(t_0) = v$.

Remark 27. $\gamma'(t), t\gamma'(t) \in \mathfrak{X}_\gamma^J$, $\langle J, \gamma' \rangle'' = 0\ \Rightarrow$ WLG, $J \perp \gamma$.

Proposition 28. *Let $\gamma(s)$ a geodesic, $v = \gamma'(0) \in T_pM$, and $J \in \mathfrak{X}_\gamma^J$ with $J(0) = 0, J'(0) = w \Rightarrow J(t) = d(\exp_p)_{*tv}(tw)$, and there is a variation ξ of γ by geodesics such that $J = \xi_t(0, \cdot)$.*

Example: If $K = c =$ constant $\Rightarrow J(t) = s_c(t)w(t)$, where $w \in \mathfrak{X}_\gamma''$ and $s_c(t) = \sin(t), t, \sinh(t)$ according to $c = 1, 0, -1$.

Proposition 29. *With the notations of Proposition 28,*

$$\|J(t)\|^2 = t^2\|w\|^2 - \frac{1}{3}\langle R(w,v)v, w\rangle t^4 + O(t^4).$$

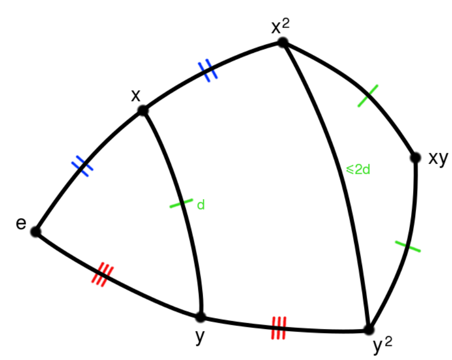
Exercise. Show that $d(\gamma_v(t), \gamma_w(t)) = \|v - w\|t - \frac{1}{6}\frac{\langle R(w,v)v,w\rangle}{\|v-w\|}t^3 + O(t^4)$; see eq.(9) in [**Me**].

Corollary 30. *If in addition $v \perp w$, $\|v\| = \|w\| = 1$, then*

$$\|J(t)\| = t - \frac{1}{6}K(v,w)t^3 + O(t^3).$$

OBS: Geometric relation between geodesics and curvature!!!

Exercise. Prove that a bi-invariant metric on a Lie group has $K \geq 0$ justifying the following diagram:



§12. Conjugate points

Conjugate points and their multiplicity = singularities of $\exp_p$. $C(p)$ = locus of the *first conjugate points to p*.

Example: $\mathbb{S}^n$.

NCP manifolds.

Proposition 31. *If $p' = \gamma(t_0)$ is not conjugate to $p = \gamma(0)$ along $\gamma \Rightarrow \forall\ v \in T_pM$, $\forall\ v' \in T_{p'}M$, there exists a unique $J \in \mathfrak{X}^J_\gamma$ such that $J(0) = v$ and $J(t_0) = v'$. In particular, if $\{J_1, \ldots, J_{n-1}\}$ is a basis of the space of Jacobi fields orthogonal to γ vanishing at 0, then $\{J_1(t_0), \ldots, J_{n-1}(t_0)\}$ is a basis of $\gamma'(t_0)^\perp \subset T_{p'}M$.*

This is useful to construct special bases of vector fields along geodesics.

§13. Isometric immersions

As we have seen in the Example in page 5, if $f : M \to N$ is an isometric immersion $\Rightarrow f^*(TN) = f_*(TM) \oplus^\perp T_f^\perp M$, and $\nabla^M_X Y = (\nabla^f_X f_* Y)^\top$, $\forall X, Y \in TM$. Moreover, we have that

$$\alpha(X,Y) := \left(\nabla^f_X f_* Y\right)^\perp$$

is a symmetric tensor, called the *second fundamental form of f*. In addition, $\nabla^\perp : TM \times \Gamma(T_f^\perp M) \to \Gamma(T_f^\perp M)$ given by

$$\nabla^\perp_X \eta = \left(\nabla^f_X \eta\right)^\perp$$

is a connection in $T_f^\perp M$, called the *normal connection of* f.
Identifications.

Exercise. Show that $\nabla^\perp$ is a connection, and is compatible with the induced metric on $T_f^\perp M$.

$\alpha(p)$ is the quadratic approximation of $f(M) \subset N$ at $p \in M$.
Picture!
$\eta \in T_{f(p)}^\perp M \Rightarrow$ (self-adjoint!) *shape operator* $A_\eta : T_pM \to T_pM$.
Hypersurfaces: Principal curvatures and directions; mean curvature; Gauss-Kronecker curvature; Gauss map.
The Fundamental Equations. Particular case: $K =$ constant $\Rightarrow$ the *Fundamental Theorem of Submanifolds* (give proof!).
Gauss equation $\Leftrightarrow K(\sigma) = \overline{K}(\sigma) + \langle \alpha(u,u), \alpha(v,v)\rangle - \|\alpha(u,v)\|^2$
$\Rightarrow$ Riemann notion of sectional curvature agrees with ours.

Example: $\mathbb{S}^{n-1}(r) \subset \mathbb{R}^n \Rightarrow K \equiv 1/r^2$ (it *had* to be constant!).

Model of the hyperbolic space $\mathbb{H}^n$ as a submanifold of $\mathbb{L}^{n+1}$.

§14. An interesting example: the geodesic spheres

If γ is a unit geodesic, $p = \gamma(0)$, we consider the shape operator $A(s) = -A_{\gamma'(s)} \in \mathrm{End}(T_{\gamma(s)}M)$ with respect to the unit inward normal at $\gamma(s)$ of a small geodesic sphere of radius s centered at p, then $AJ = J'$ for any $J \in \mathfrak{X}_\gamma^J$ with $J(0) = 0$ and $J \perp \gamma$.
In particular: $K \equiv 0 \Rightarrow A(s) = s^{-1}I$; $K \equiv 1 \Rightarrow A(s) = \cot(s)I$.

Exercise. Show that $A = -\mathrm{Hess}_{d_p}|_{\gamma^\perp}$, and $\lim_{s\to 0} sA(s) = Id$ (Sug: use normal coordinates).

If $R_X := R(\cdot, X)X$, then $AJ = J' \Rightarrow$

$$A' + A^2 + R_{\gamma'} = 0 \tag{2}$$

This is known as the *Riccati equation*, and has the same information as the Jacobi equation. Moreover, it implies that: *if we can compare the curvature of two manifolds, we can also compare the shape of geodesic balls* (like $s^{-1}I < \cot(s)I$ above). We will see this in Section 25 and Section 29.

Global Riemannian Geometry

§15. Completeness and the Hopf-Rinow Theorem

Until now, only local stuff. We have problems: Geodesics not defined in $\mathbb{R}$; domain of the exponential map may be strange; far away points may not have geodesics joining them; even if they do, may not be minimizing; the manifolds may have "holes"; (M, d) may not be complete... All these problems have the same solution!

Def.: M is (geod.) *complete* if all geodesics are defined in $\mathbb{R}$.

Proposition 32. *M complete $\Rightarrow$ M is non-extendible.*

Lemma 33. *If $q \notin B_\epsilon(p)$ normal $\Rightarrow d(q, \partial B_\epsilon(p)) = d(q,p) - \epsilon$.*

Theorem 34. *(Hopf-Rinow) Let $(M, \langle\,,\rangle)$ be a connected Riemannian manifold, and $p_0 \in M$. The following assertions are equivalent:*

a) $\exp_{p_0}$ *is defined in $T_{p_0}M$;*

b) Closed bounded subsets of M are compact;

c) (M, d) is a complete metric space;

d) $(M, \langle\,,\rangle)$ is (geodesically) complete;

e) There is a sequence of compact sets $K_n \subset K_{n+1} \subset M$, $\cup_n K_n = M$ such that if $p_n \notin K_n$ $\forall n \Rightarrow \lim_{n\to+\infty} d(p_0, p_n) = +\infty$.

In addition, any of these implies the following:

f) $\forall p, q \in M$, there is a minimizing geodesic joining p and q.

Corollary 35. *M compact $\Rightarrow (M, \langle\,,\rangle)$ is complete $\forall \langle\,,\rangle$.*

Corollary 36. *If $S \subset M$ is a closed embedded submanifold of a complete Riemannian manifold M, then S is complete.*

Embedding above is necessary, as shown by $(a, b) \mapsto$ "8" $\subset \mathbb{R}^2$.

§16. Quick review of covering spaces (see [**Ha**])

Group actions, proper discontinuous group actions, quotients.

Def.: A *covering map* is a surjective local diffeo $\pi : \tilde{M} \to M$ such that $\forall p \in M,\ \exists\, p \in U_p \subset M$ for which $\pi^{-1}(U_p) = \cup_\lambda V_\lambda$, where each $\pi|_{V_\lambda} : V_\lambda \to U_p$ is a diffeomorphism.

Example: $\pi(\theta) = e^{2\pi i\theta}$ is a covering map from $\mathbb{R}$ to $\mathbb{S}^1 \subset \mathbb{C}$, but $\pi|_{(-1,1)}$ is not.

Proposition 37. *A surjective local diffeomorphism π is a covering map $\Leftrightarrow$ π lifts curves: $\forall p' \in \pi^{-1}(p), \forall c : I \to M$ with $c(0) = p$, $\exists!\ \tilde{c} : I \to \tilde{M}$ such that $\tilde{c}(0) = p'$ and $\pi \circ \tilde{c} = c$.*

Def.: *Homotopic loops* at $p_0 \in M$.

Def.: $\pi_1(M) = \pi_1(M, p_0) =$ *fundamental group* of M.

Def.: M is *simply connected* if $\pi_1(M) = 0$.

Proposition 38. *If $\sigma_1, \sigma_2 : I \to M$ are homotopic, then $\tilde{\sigma}_1(1) = \tilde{\sigma}_2(1)$. The converse holds if $\tilde{M}$ is simply connected.*

Def.: $\mathrm{Deck}(\pi) := \{g \in \mathrm{Diff}(\tilde{M}) : \pi \circ g = \pi\}$, the *deck group*.

$\mathrm{Deck}(\pi)$ acts properly discontinuously on $\tilde{M}$, transitively on the fibers if $\pi_1(\tilde{M}) = 0$, and $\tilde{M}/\mathrm{Deck}(\pi) \cong M$.

Corollary 39. *$\tilde{M}$ simply connected $\Rightarrow j : \pi_1(M) \to \mathrm{Deck}(\pi)$ given by $j([\sigma]) = g$ where $g(\tilde{\sigma}(0)) = \tilde{\sigma}(1)$ is an isomorphism.*

Proposition 40. *For any manifold M there exists a unique (up to diffeomorphisms) simply connected manifold $\tilde{M}$ covering M, called the* <u>*universal cover*</u> *of M.*

<u>*Exercise*</u>. $\forall G \subset \pi_1(M)$ subgroup $\Rightarrow \exists\ \pi' : \tilde{M} \to M'$ with $\pi_1(M') = G$. Particular case: $G = \{g \in \mathrm{Deck}(\pi) : g$ preserves orientation$\}$ has index 2 $\Rightarrow$ oriented double covering.

Proposition 41. *If M is compact and $f : M \to M'$ is a surjective local diffeomorphism, then f is a covering map.*

<u>*Exercise*</u>. Give a counterexample to Proposition 41 when M is only complete.

§17. Hadamard manifolds

Lemma 42. *M complete, $f : M \to N$ local diffeo such that $\|f_* v\| \geq \epsilon > 0\ \forall\, v \in T_1 M \Rightarrow f$ is a covering map ($\Rightarrow$ Prop.39)*

Proof: f has the curve lifting property ($\Rightarrow f$ is surjective). ∎

Def.: A point $p \in M$ is called a *pole* if $C(p) = \emptyset$.

Theorem 43. *(Hadamard) M complete simply connected with a pole $p \Rightarrow \exp_p$ is a diffeomorphism ($\Rightarrow M \cong \mathbb{R}^n$!!).*

Lemma 44. *$K \leq 0 \Rightarrow C(p) = \emptyset\ \forall\, p \in M$ (M is said NCP).*

Proof: $\|J\|^{2''} \geq 0$ for $0 \neq J \in \mathfrak{X}_\gamma^J$ with $J(0) = 0$. ∎

Def.: M is a *Hadamard manifold* if it is complete, simply connected and $K \leq 0$.

Corollary 45. *(Hadamard) M Hadamard $\Rightarrow \exp_p$ is a diffeomorphism, $\forall p \in M$.*

Remark 46. M compact has $NCP \not\Rightarrow K \leq 0$. But is there *some* metric on M with $K \leq 0$?? This is a deep open problem!

§18. Manifolds with constant sectional curvature

These are the "simplest" spaces: lots of (local) isometries; congruencies; rigid motions: *geometric postulates*.
We can always assume that $K \equiv -1, 0, 1$: $\mathbb{Q}^n_c = \mathbb{S}^n, \mathbb{R}^n, \mathbb{H}^n$ are complete, connected and simply connected. And they are unique!

Any isometry is locally constructed as i, ϕ, f like in the following:

Theorem 47. *(Cartan) Given $p \in M^n$ and $\hat{p} \in \hat{M}^n$, let $i\colon T_pM \to T_{\hat{p}}\hat{M}$ be a linear isometry. Let V_p a star shaped normal neighborhood of p such that $\exp_{\hat{p}}$ is defined in $\hat{V}_{\hat{p}} := i(\exp_p^{-1}(V_p))$. Define*

$$f = \exp_{\hat{p}} \circ i \circ \exp_p^{-1}|_{V_p} : V_p \to \hat{V}_{\hat{p}}.$$

Let $\phi : TV_p \to TV_{\hat{p}}$ be the natural bundle isometry defined using radial parallel transports and i, that is,

$$\phi(P^{0,t}_{\gamma_v}(w)) = P^{0,t}_{\hat{\gamma}_{iv}}(iw), \quad \forall v, w \in T_pM.$$

If $\phi^*\hat{R} = R$, then f is a local isometry with $f_{*p} = i$ and $f_* = \phi$.

Proof: Observe that $f_*J = \hat{J}$ for Jacobi fields along corresponding radial geodesics γ_v and $\hat{\gamma}_{iv}$ such that $J(0) = 0$, $\hat{J}(0) = 0$,

$\hat{J}'(0) = iJ'(0)$. Since ϕ is parallel in radial directions, ϕJ is Jacobi: $(\phi J)'' = \phi J'' = -\phi R_{\gamma_v'} J = -\hat{R}_{\hat{\gamma}_{iv}'}(\phi J)$. Since $\phi|_{T_pM} = i$, then $\hat{J} = \phi J$ and the result follows since ϕ is a bundle isometry. ∎

Remark 48. $\phi^* \hat{R} = R \Leftrightarrow K(\gamma_v', \cdot) = \hat{K}(\hat{\gamma}_{iv}', \phi(\cdot))\ \forall v \in T_pM$.

Corollary 49. *If M^n and $\hat{M}^n$ have the same constant sectional curvature, then $\forall p \in M$, $\forall \hat{p} \in \hat{M}$, $\forall i \in \mathrm{Iso}(T_pM, T_{\hat{p}}\hat{M})$ there exists an isometry $f\colon V_p \to \hat{V}_{\hat{p}}$ with $f(p) = \hat{p}$ and $f_{*p} = i$.*

Remark 50. This holds in particular for $\hat{M} = M$: *spaces of constant curvature are rich (the richest!) in local isometries.*

Let $\pi : \tilde{M} \to M$ be a covering map. Given a metric $\langle\,,\rangle$ in M, $\pi^*\langle\,,\rangle$ is called the *covering metric* on $\tilde{M} \Rightarrow \mathrm{Deck}(\pi) \subset \mathrm{Iso}(\tilde{M})$. Conversely, given a metric in $\tilde{M}$, if $\Gamma \subset \mathrm{Iso}(\tilde{M})$ acts properly discontinuous (called a *crystallographic group* when $\tilde{M} = \mathbb{R}^n$), $M := \tilde{M}/\Gamma$ is naturally a Riemannian manifold and the projection π is a local isometry. Moreover, $\tilde{M}$ is complete or has constant $K \Leftrightarrow$ same for M. In particular, $\mathbb{Q}_c^n/\Gamma$ is a space form: connected complete with constant sectional curvature $K \equiv c$.

Theorem 51. *(Hopf-Killing) If M^n is a space form, then its universal cover (with the covering metric) is isometric to $\mathbb{Q}_c^n$, and M^n is isometric to $\mathbb{Q}_c^n/\Gamma$, with $\pi_1(M) \cong \Gamma \subset \mathrm{Iso}(\mathbb{Q}_c^n)$.*

Therefore, the classification of space forms is purely an algebraic problem (solved for $c > 0$ in the 60's, well understood for $c = 0$, wide open for $c < 0$).

Corollary 52. *M^{2n} complete with $K \equiv 1 \Rightarrow M^{2n}$ isometric to $\mathbb{S}^{2n}$ or $\mathbb{RP}^{2n}$.*

Remark 53. $\mathbb{R}^n/\mathbb{Z}^n$ is not isometric to $\mathbb{R}^n/2\mathbb{Z}^n$, and two 3-dimensional *lens spaces* $L^3(p,q)$ and $L^3(p,q')$ are not even homeomorphic if $q \neq \pm q'^{\pm 1} \bmod (p)$. In particular, the isomorphism type of $\pi_1(M)$ does <u>not</u> determine the space form. However, it does if $c < 0$, $n \geq 3$ and M^n has finite volume (Mostow rigidity theorem), or if $c > 0$, $n = 3$, and $\pi_1(M^3)$ is not cyclic.

Remark 54. *Does the curvature determine the metric?* More precisely: If f is a diffeo with $f^*\hat{K} = K$, is f an isometry? This is false if $n = 2$ (just take the flow of a generic vector field orthogonal to the gradient of the curvature), or if M^n contains an open subset with constant curvature. However, we have:
If M^n has nowhere constant sectional curvature and $n \geq 4$, then any curvature preserving diffeomorphism is an isometry. For $n = 3$ it is true if M^3 is compact. (Kulkarni-Yau). See here.

<u>*Exercise*</u>. Read from the book the classification of $\mathrm{Iso}(\mathbb{H}^n)$.

§19. Geodesics as minimizers: Variations of energy

We already know that geodesics are the critical points of the arc-length functional $L(c)$ when restricted to piecewise differentiable (p.d. from now on) curves $c : [0,a] \to M$ p.b.a.l.. To understand when a geodesic is an actual minimizer, we will take second derivatives. But it is easier to work with the *energy functional*:

$$E(c) := \frac{1}{2}\int_0^a \|c'(t)\|^2 dt.$$

Cauchy-Schwarz $\Rightarrow L(c)^2 \leq 2aE(c)$, with $= \Leftrightarrow c$ is p.b.a.l.

Def.: $\Omega_{p,q} = \Omega^a_{p,q} := \{c : [0,a] \to M \ p.d. : c(0) = p, c(a) = q\}$.

Proposition 55. *If $\gamma : [0,a] \to M$ is a minimizing geodesic between $p = \gamma(0)$ and $q = \gamma(a)$, then $E(\gamma) \leq E(c)$ for every $c \in \Omega_{p,q}$, with equality $\Leftrightarrow$ c is a minimizing geodesic.*

Proof: $2aE(\gamma) = L(\gamma)^2 \leq L(c)^2 \leq 2aE(c)$. ∎

That is, E is not only easier to work with than L, but it also takes into account the parametrization. So let's try to minimize E.

Def.: *Variation* $c(s,t)$ of a curve $c = c(0,\cdot)$: $c(s,t) \in C^0$ and there is a partition $0 = t_0 < t_1 < \cdots < t_{m+1} = a$ of $[0,a]$ such that $c|_{(-\epsilon,\epsilon)\times[t_i,t_{i+1}]} \in C^\infty$ ($\Rightarrow V = c_s(0,\cdot) \in C^0$ and $c_{ss}(0,\cdot) \in C^0$).

Let $c = c_0 : [0,a] \to M$ be a p.d. curve, $V \in \mathfrak{X}_c$ ($\Rightarrow V \in C^0$), and $c(s,\cdot)$ a variation of c with variational vector field V. For $E(s) = E(c(s,\cdot))$ we have:

Proposition 56. *(Formula for the first variation of energy)*

$$E'(0) = -\int_0^a \langle V(t), c''(t)\rangle dt \ + \ \langle V, c'\rangle|_0^a \ + \ \sum_{i=1}^m \langle V(t_i), c'(t_i^-) - c'(t_i^+)\rangle.$$

Corollary 57. *c is a geodesic $\Leftrightarrow$ c is a critical point of E for proper variations (i.e., for $E|_{\Omega_{c(0),c(a)}}$).*

<u>*Exercise*</u>. Given N and N' two compact submanifolds of a complete Riemannian manifold $\Rightarrow$ there exists a minimizing geodesic γ between N and N'. For such a γ, $\gamma \perp N$ and $\gamma \perp N'$.

Proposition 58. *(Formula for the second variation of E) If $\gamma(t)$ is a geodesic and $f(s,t)$ a variation of γ with variational vector field V, then* (*recall that $R_v := R(\cdot,v)v$*)

$$\begin{aligned} E''(0) &= -\int_0^a \langle V, V'' + R_{\gamma'}V\rangle dt + \sum_{i=1}^m \langle V(t_i), V'(t_i^-) - V'(t_i^+)\rangle + \langle V, V'\rangle|_0^a + \langle \gamma', \nabla_{\partial_s} f_s(0,\cdot)\rangle|_0^a \\ &= I_a(V,V) + \langle \gamma', \nabla_{\partial_s} f_s(0,\cdot)\rangle|_0^a \ , \end{aligned}$$

where $I_a(V,W) := \int_0^a(\langle V',W'\rangle - \langle R_{\gamma'}V,W\rangle)dt$ *is the* *index form* *of* γ.

Corollary 59. *(Jacobi) If a geodesic* γ *has a conjugate point* $\gamma(b)$ *to* $\gamma(0)$ $\Rightarrow I_{b+\delta} \not\geq 0$ $\Rightarrow \gamma$ *does not minimize after* b.

Proof: Let $0 \neq J \in \mathfrak{X}_\gamma^J$, $J(0) = 0$, $J(b) = 0$, $\delta > 0$ and choose any $Z \in \mathfrak{X}_\gamma$ with $Z|_{[0,b-\delta]} = 0$, $Z(b+\delta) = 0$ and $\langle Z(b), J'(b)\rangle < 0$. Define $V_\epsilon \in \mathfrak{X}_\gamma$ as $V_\epsilon = J + \epsilon Z$ in $[0,b]$ and $V_\epsilon = \epsilon Z$ in $[b, b+\delta]$. Then, $I_{b+\delta}(V_\epsilon, V_\epsilon) = 2\epsilon I_b(J,Z) + \epsilon^2 I_{b+\delta}(Z,Z) = 2\epsilon\langle Z(b), J'(b)\rangle + \epsilon^2 I_{b+\delta}(Z,Z) < 0$ for $\epsilon > 0$ small enough. ∎

Remark 60. If the variation is proper, $E''(0) = I_a(V,V)$ only depends on V, and hence I_a is actually the Hessian of $E|_{\Omega^a_{\gamma(0),\gamma(a)}}$ at its critical point γ ($\forall f\colon M \to N \Rightarrow T_f(\mathcal{F}(M,N)) = \mathfrak{X}_f$).

§20. Application: The Bonnet-Myers Theorem

Theorem 61. *If* M *is complete with Ric* $\geq 1/k^2 > 0$*, then* M *is compact, and* $\operatorname{diam}(M) \leq \pi k$*. In particular, its universal cover is compact and hence* $\#\pi_1(M) < \infty$.

Remark 62. This is false for $K > 0$ (paraboloid). But the curvature bound can be relaxed asking for slow decay at infinity.

Remark 63. The estimate in diam is sharp: $\mathbb{S}^n_k$. And there's *rigidity* (!!): *If* $\operatorname{diam}(M) = \pi k$*, then* $M^n = \mathbb{S}^n_k$ (Corollary 99).

§21. Application: The Synge-Weinstein Theorem

Theorem 64. (*Weinstein*) M^n *compact and oriented with* $K > 0$*. If* $f \in \operatorname{Iso}(M^n)$ *preserves (resp.reverses) the orientation of* M^n *if* n *is even (resp.odd), then* f *has a fixed point.*

Proof: Let $g(x) := d(x, f(x))^2$ and assume $g(p) = \min g > 0$. If γ is a unit minimizing geodesic between p and $f(p)$, then $f(\gamma) = \gamma$. So, $(P^\gamma)^{-1} \circ f_{*p}$ fixes some vector $v \in \gamma'(0)^\perp \Rightarrow f \circ \gamma_v = \gamma_{P^\gamma(v)}$. Now the second variation for $c_s(t) = \exp_{\gamma(t)}(sP^\gamma_{0t}v)$ says that 0 is a strict maximum of $E(s) \Rightarrow g(\gamma_v(s))^2 \leq L(c_s)^2 \leq 2g(p)E(c_s) < 2g(p)E(\gamma) = L(\gamma)^2 = g(p)^2$, a contradiction. ∎

Remark 65. Weinstein Theorem 64 is still true for conformal diffeomorphisms, but it is not known for diffeomorphisms. If this were also true, then $\mathbb{S}^2 \times \mathbb{S}^2$ would not admit a metric with $K > 0$ ($f = (-Id, -Id)$): this is the well known *Hopf conjecture*, one of the most important open conjectures in Riemannian geometry!

Corollary 66. *(Synge) If M^n is compact with $K > 0$, then:*

a) If n is even, then $\pi_1(M^n) = 0$ if M^n orientable, while $\pi_1(M^n) = \mathbb{Z}_2$ if M^n is nonorientable (see Corollary 52);

b) If n is odd, then M^n is orientable.

Remark 67. $\mathbb{RP}^2$ and $\mathbb{RP}^3$ show that the 3 hypothesis in Corollary 66 (a) and (b) are necessary. Yet, compactness is not since for noncompact M^n the soul of its universal cover is a unique point, hence fixed by Deck(π).

Remark 68. B-M and S-W theorems are quite deep:
- Compact manifolds with $K \geq 0$ abound: products of compact manifolds with $K \geq 0$; compact Lie groups G with bi-invariant metrics; homogeneous spaces G/H; biquotients $G//H$; etc.
- OTOH, very few examples are know with $K > 0$: aside from CROSSES ($\mathbb{S}^n, \mathbb{RP}^n, \mathbb{CP}^n, \mathbb{HP}^n, Ca^2$), Eschenburg spaces E^7_p and

Bazaikin spaces B_q^{13} for infinite many $p, q \in \mathbb{Z}^5$, only a handful of examples are known, and only in dimensions 6, 7, 12 and 24.
• However, very few obstructions are known for $K > 0$ that do not hold already for $K \geq 0$ and Theorem 61 and Theorem 64 are the most important. In fact: *there is no known obstruction that distinguishes the class of compact simply connected manifolds which admit $K \geq 0$ from the ones that admit $K > 0$* !!

§22. The Index Lemma

We show next that *Jacobi fields are the unique minimizers of the index form* (up to the first conjugate point):

Lemma 69. *(Index lemma). Let $\gamma : [0, a] \to M$ be a geodesic without conjugate points to $\gamma(0)$. Let $V \in \mathfrak{X}_\gamma$ p.d. with $V \perp \gamma'$ and $V(0) = 0$. Consider $t_0 \in (0, a]$ and $J \in \mathfrak{X}_\gamma^J$ the unique Jacobi field such that $J(0) = 0$ and $J(t_0) = V(t_0)$. Then, $I_{t_0}(J, J) \leq I_{t_0}(V, V)$, and equality holds $\Leftrightarrow V = J$ in $[0, t_0]$.*

Proof: $\{J_1, \ldots, J_{n-1}\}$ basis of $\{J \in \mathfrak{X}_\gamma^J : J \perp \gamma, J(0) = 0\}$, and write $V = \sum f_i J_i$ on $(0, t_0]$.
Claim: $\{f_i\}$ extend C^∞ to 0: If $J_i(t) = tA_i(t) \Rightarrow A_i(0) = J_i'(0)$ are L.I. $\Rightarrow V = \sum g_i A_i$ with g_i p.d. on $[0, t_0]$ and $g_i(0) = 0$ $\Rightarrow g_i(t) = th_i(t)$ where $h_i(t) = \int_0^1 g_i'(ts)ds \Rightarrow f_i = h_i|_{(0,t_0]}$.
But $\langle V', V'\rangle - \langle R_{\gamma'} V, V\rangle = \| \sum f_i' J_i\|^2 + \langle \sum f_i J_i, \sum f_i J_i'\rangle'$ since $\langle J_i, J_j'\rangle = \langle J_i', J_j\rangle$, so $I_{t_0}(V, V) = I_{t_0}(J, J) + \int_0^{t_0} \| \sum f_i' J_i\|^2$. ∎

§23. The Rauch comparison Theorem

Two goals: refine the idea of Bonnet-Myers, and make a global version of Proposition 29: compare Jacobi fields when there is comparison of curvature (can only expect this for NCP geodesics). *Motivation.* Recall a classical ODE result (used in Theorem 95):

Theorem 70. *(Sturm) Let* $K, \tilde{K}, f, \tilde{f}\colon [0,a] \to \mathbb{R}$ *satisfying* $f'' + Kf = 0$ *and* $\tilde{f}'' + \tilde{K}\tilde{f} = 0$*, with* $f(0) = \tilde{f}(0) = 0$ *and* $f'(0) = \tilde{f}'(0) > 0$*. If* $\tilde{f} \neq 0$ *in* $(0,a]$ *and* $K \leq \tilde{K}$*, then* $f/\tilde{f}$ *is nondecreasing (in particular* $f \geq \tilde{f}$*). Moreover, if* $f(r) = \tilde{f}(r)$ *for some* $r \in (0,a]$*, then* $\tilde{K} = K$ *and* $f = \tilde{f}$ *in* $[0,r]$*.*

Proof: Since $(f'\tilde{f} - f\tilde{f}')(t) = \int_0^t (\tilde{K} - K) f\tilde{f} \;\Rightarrow (f/\tilde{f})' \geq 0$ where $f \geq 0 \;\Rightarrow f \geq \tilde{f} > 0$ in $(0,a] \;\Rightarrow f/\tilde{f}$ is nondecreasing. ∎

Theorem 71. *(Rauch Comparison) Let* $\gamma\colon [0,a] \to M^n$*,* $\tilde{\gamma}\colon [0,a] \to \tilde{M}^{n+p}$ *be geodesics, and* $J \in \mathfrak{X}^J_\gamma$ *and* $\tilde{J} \in \mathfrak{X}^J_{\tilde{\gamma}}$ *with comparable initial conditions, i.e.,* $\|\gamma'\| = \|\tilde{\gamma}'\|$*,* $J(0) = 0$*,* $\tilde{J}(0) = 0$*,* $\langle J'(0), \gamma'(0)\rangle = \langle \tilde{J}'(0), \tilde{\gamma}'(0)\rangle$*, and* $\|J'(0)\| = \|\tilde{J}'(0)\|$*. Assume that* $\tilde{\gamma}$ *has no conjugate points and that, on* $(0,a]$*,* $K(\gamma', J) \leq \tilde{K}(\tilde{\gamma}', \bullet)$*. Then,* $\|J\|/\|\tilde{J}\|$ *is non-decreasing and, in particular,* $\|J\| \geq \|\tilde{J}\|$*. Moreover, if* $\|\tilde{J}(r)\| = \|J(r)\|$ *for some* $r \in (0,a]$*, then* $K(\gamma', J) = \tilde{K}(\tilde{\gamma}', \tilde{J})$ *on* $(0,r]$*.*

Proof: We may assume $0 \neq J \perp \gamma'$, $0 \neq \tilde{J} \perp \tilde{\gamma}'$. If $f := \|J\|^2$ and $\tilde{f} := \|\tilde{J}\|^2$, $g := f/\tilde{f}$ is well defined in $(0,a]$ and $g(0^+) = 1$. So it is enough to see that $g' \geq 0$, or, equivalently, $\tilde{f}'(r)/\tilde{f}(r) \leq f'(r)/f(r)$ when $f(r) \neq 0$. Since $U := J/\sqrt{f(r)}$ and $\tilde{U} := \tilde{J}/\sqrt{\tilde{f}(r)}$ are Jacobi fields, by the hypothesis on the curvature and the Index Lemma 69, $\tilde{f}'(r)/\tilde{f}(r) = 2\tilde{I}_r(\tilde{U},\tilde{U}) \leq 2\tilde{I}_r(\phi U, \phi U) \leq 2I_r(U,U) = f'(r)/f(r)$, where $\phi : \mathfrak{X}_\gamma \to \mathfrak{X}_{\tilde{\gamma}}$ is any parallel isometry (with the image) with $\phi(\gamma') = \tilde{\gamma}'$ and $\phi(U(r)) = \tilde{U}(r)$. Equality $\Rightarrow$ on $(0,r]$: $g \equiv 1$, $\tilde{I}_r(\phi U, \phi U) = I_r(U,U)$, $\tilde{U} = \phi U$, and so $K(\gamma', J) = \tilde{K}(\tilde{\gamma}', \tilde{J})$. ∎

Corollary 72. *If $K \geq 1/k^2$ (resp. $K \leq 1/k^2$) for some $k > 0$, then the distance d between two consecutive conjugate points along any geodesic satisfies that $d \leq \pi k$ (resp. $d \geq \pi k$).*

Remark 73. According to Section 14, $AJ = J'$ along geodesics without conjugate points, so the inequality $\tilde{f}'/\tilde{f} \leq f'/f$ in the proof above is equivalent to $\tilde{A} \leq A$. In fact, Rauch Theorem 71 is equivalent to a Sturm-type comparison for the general Riccati equation (2); see Theorem 3.1 pg.12 due to J. Eschenburg here.

Exercise. Prove the Sturm comparison Theorem using Rauch Theorem 71.

§24. An application to submanifold theory

Theorem 74. *(Moore) Let M^n be a compact submanifold of a Hadamard manifold $\tilde{M}^{n+p}$ with $K \leq \tilde{K} + c \leq 0$ for certain $c \geq 0$. Then, $p \geq n$.*

Proof: Fix $\tilde{q}_0 \notin M$, $q \in M$ realizing the maximum distance to $\tilde{q}_0$, γ a unit minimizing geodesic between $\tilde{q}_0 = \gamma(0)$ and $q = \gamma(\ell)$, $v \in T_qM$ unitary and $\hat{c}(s)$ a curve in M with $\hat{c}'(0) = v$. If $c(s) = \exp_{\tilde{q}_0}^{-1}(\hat{c}(s))$, for the variation $\gamma_{c'(s)}(t)$ of γ we have that $0 \geq E''(0) = I_\ell(J,J) + \langle \alpha(v,v), \gamma'(\ell)\rangle$, with $J(\ell) = v$. Comparing $\tilde{M}$ with $\mathbb{Q}^{n+p}_{-c}$ we have $I_\ell(J,J) \geq \tilde{I}_\ell(\tilde{J},\tilde{J}) > \sqrt{c} \Rightarrow \|\alpha(v,v)\|^2 \geq \langle \alpha(v,v), \gamma'(\ell)\rangle^2 > c$. Now apply Otsuki's Lemma. ∎

Remark 75. Simply connectedness of $\tilde{M}$ is essential ($T^n \subset T^{n+1}$), as well as compactness of M (catenoid in $\mathbb{R}^3$; even bounded minimal surfaces exist), but $\mathbb{H}^2 \not\subset \mathbb{R}^3$ (Hilbert). The nonexistence of an is.im. $\mathbb{H}^n \subset \mathbb{R}^{2n-1}$ is a famous century old open conjecture!

§25. Applications: comparing geometries!! :o))

As in Cartan's Theorem 47, take $p \in M^n$, $\tilde{p} \in \tilde{M}^n$, $i\colon T_pM \to T_{\tilde{p}}\tilde{M}$ a linear isometry and $r > 0$ such that $B_r(p) \subset M$ is a normal ball and $\exp_{\tilde{p}}$ is non-singular in $B_r(0_{\tilde{p}}) \subset T_{\tilde{p}}\tilde{M}$. For the map $f := \exp_{\tilde{p}} \circ i \circ \exp_p^{-1}|_{B_r(p)}\colon B_r(p) \subset M \to B_r(\tilde{p}) \subset \tilde{M}$ we have:

Proposition 76. *If $K(\gamma_v'(t), \cdot) \leq \tilde{K}(\tilde{\gamma}_{iv}'(t), \cdot)$ $\forall\, v \in T_pM$, $\|v\| = 1, |t| < r \Rightarrow f$ is a <u>contraction</u>: $\|f_*\| \leq 1$. In particular, if $c : I \to B_r(p)$ is any p.d. curve, then $L(f \circ c) \leq L(c)$, and, if $B_r(p)$ is convex, then f is also a metric contraction, i.e.,*

$$\tilde{d}(f(x), f(y)) \leq d(x,y) \quad \forall\, x, y \in B_r(p).$$

<u>*Exercise*</u>. Check that Corollary 49 follows immediately from Proposition 76.

Corollary 77. *If $K(\gamma_v'(t), \cdot) = k$ is constant $\forall\, v \in T_{p_0}M$, $\|v\| = 1$, $|t| < r \Rightarrow K \equiv k$ in $B_r(p_0)$ (see Remark 48).*

Remark 78. Proposition 76 implies the local version of Toponogov's Theorem 102.

§26. Index Lemma and Rauch Thm for focal points

Focal points are generalizations of conjugate points: given $p \in N \subset M$, a normal variation by geodesics of a geodesic γ

emanating orthogonally from p gives rise to $J \in \mathfrak{X}^J_\gamma$ such that

$$J(0) \in T_pN \quad \text{and} \quad J'(0) + A_{\gamma'(0)}J(0) \in T^\perp_p N, \qquad (3)$$

and conversely, by considering $\gamma_s(t) = \exp_{\alpha(s)}(t\eta(s))$, where $\eta \in T^\perp_\alpha N$, $\alpha'(0) = J(0)$, $\eta(0) = \gamma'(0)$ and $\eta'(0) = J'(0)$.

Exercise. See the details in the book.

Def.: $q \in M$ is a *focal point* of a submanifold $N \subset M$ if there is a geodesic γ orthogonal to N at $\gamma(0) \in N$ with $q = \gamma(r)$, and $0 \neq J \in \mathfrak{X}^J_\gamma$ as in (3) such that $J(r) = 0$. The *focal set* $F(N)$ *of* N is the union of its focal points.

Examples: $\mathbb{S}^n \subset \mathbb{S}^{n+1}, F(\mathbb{S}^n) = \pm N$. $\mathbb{S}^n \subset \mathbb{R}^{n+1}, F(\mathbb{S}^n) = \{0\}$.

Def.: The *normal exponential map* of N is $\exp^\perp : T^\perp N \to M$.

Proposition 79. *The focal points of $N \subset M$ are precisely the singularities of* $\exp^\perp : T^\perp N \to M$*.*

Exercise. See the details in the book.

Exercise. Compute the focal points of $N^n \subset \mathbb{R}^{n+1}$ in terms of its principal curvatures.

Analogously to Theorem 43, the following holds: *If M is complete and $N \subset M$ is closed and without focal points, then* $\exp^\perp : T^\perp N \to M$ *is a covering map.* (Hermann).

Def.: A geodesic $\gamma : [0, a] \to M$ is *free of focal points* if $N_\epsilon = \exp_{\gamma(0)}(B_\epsilon(0_p) \cap \gamma'(0)^\perp)$ has no focal points along γ (equivalently, $0 \neq J \in \mathfrak{X}^J_\gamma$ with $J \perp \gamma$ and $J'(0) = 0 \Rightarrow J(t) \neq 0 \ \forall t \in [0, a]$).

Making slight modifications in their proofs, we have:

Both the Index Lemma 69 and Rauch Theorem 71 hold for geodesics free of focal points.

<u>*Exercise*</u>. Prove the last assertion without looking at the book.

Def.: We say that M *has no focal points* (NFP) if no embedded geodesic $\gamma(-\epsilon, \epsilon) \subset M$ has focal points (as a submanifold).

Proposition 80. *$K \leq 0 \Rightarrow NFP \Rightarrow NCP$. In fact:*

i) $K \leq 0 \Leftrightarrow \|J\|^{2''} \geq 0,\ \forall J \in \mathfrak{X}^J_\gamma$*;*

ii) NFP $\Leftrightarrow \|J(t)\|^{2'} > 0,\ \forall t > 0, 0 \neq J \in \mathfrak{X}^J_\gamma$ *with* $J(0) = 0$*;*

iii) NCP $\Leftrightarrow \|J(t)\|^2 > 0,\ \forall t > 0, 0 \neq J \in \mathfrak{X}^J_\gamma$ *with* $J(0) = 0$*;*

Remark 81. $NCP \not\Rightarrow NFP \not\Rightarrow K \leq 0$ for complete metrics. But what about plain differentiable manifolds *admitting* such metrics? Two important open problems: it is not known if $\mathcal{M}^n_C \subset \mathcal{M}^n_F$, or if $\mathcal{M}^n_F \subset \mathcal{M}^n_0$, for $\mathcal{M}^n_0 = \{M^n \colon \exists \langle\, ,\rangle \text{ with } K \leq 0\}$, $\mathcal{M}^n_F = \{M^n \colon \exists\ NFP \langle\, ,\rangle\}$ and $\mathcal{M}^n_C = \{M^n \colon \exists\ NCP \langle\, ,\rangle\}$.

§27. The Morse Index Theorem

Given a geodesic $\gamma\colon [0, a] \to M$, consider $\mathcal{V}_a$ the set of p.d. vector fields along γ that vanish at 0 and a (i.e., $\mathcal{V}_a = T_\gamma \Omega_{\gamma(0),\gamma(a)}$). For proper variations of γ, $\text{Hess}_E = I_a$ where $I_a\colon \mathcal{V}_a \times \mathcal{V}_a \to \mathbb{R}$.

Def.: The *nullity* of I_a is $\nu(I_a) := \dim \text{Ker}\,(I_a)$, while its *index* is $i(I_a) := \max\{\dim L\colon I_a|_{L\times L} < 0\}$. ($\gamma$ minimizing $\Rightarrow i(I_a) = 0$).

The purpose now is to show that $i(I_a) = \#$ of conjugate points along γ. We will reduce the problem to a finite dimensional one.

Proposition 82. Ker $(I_a) = \mathcal{V}_a \cap \mathfrak{X}^J_\gamma$. *I.e., I_a is degenerate $\Leftrightarrow \gamma(a)$ is conjugate to $\gamma(0)$ along γ, with $\nu(I_a)$ as multiplicity.*

Proof: Immediate from the two expressions in Proposition 58. ∎

Let $0 = t_0 < t_1 < \cdots < t_k = a$ be a *normal subdivision* of $[0, a]$ ($\gamma([t_i, t_{i+1}])$ is contained in a totally normal neighborhood). Define

$$\mathcal{V}_a^+ := \{V \in \mathcal{V}_a : V(t_i) = 0, i = 0, \dots, k\},$$

$$\mathcal{V}_a^- := \{V \in \mathcal{V}_a : V|_{[t_i,t_{i+1}]} \text{ is Jacobi}\} \Rightarrow \dim \mathcal{V}_a^- = n^{k-1} < +\infty.$$

Proposition 83. $\mathcal{V}_a = \mathcal{V}_a^+ \oplus \mathcal{V}_a^-$, $I_a|_{\mathcal{V}_a^+ \times \mathcal{V}_a^-} = 0$, $I_a|_{\mathcal{V}_a^+ \times \mathcal{V}_a^+} > 0$.

Proof: Proposition 58 + $\gamma|_{[t_i,t_{i+1}]}$ minimizing + Proposition 82. ∎

Corollary 84. $i(I_a) = i(I_a|_{\mathcal{V}_a^- \times \mathcal{V}_a^-}) < +\infty$, $\nu(I_a) = \nu(I_a|_{\mathcal{V}_a^- \times \mathcal{V}_a^-})$.

Theorem 85. *(Morse) $i(I_a) < +\infty$ is equal to the number of conjugate points (with multiplicities) to $\gamma(0)$ along γ in $[0, a)$.*

Proof: Take $t \in (0, a)$ and choose the normal partition such that $t \in (t_i, t_{i+1})$. Consider $\varphi_t : S := T_{\gamma(t_1)}M \times \cdots \times T_{\gamma(t_i)}M \to \mathcal{V}_t^-$, $\varphi_t^{-1}(V) = (V(t_1), \dots, V(t_i))$, and work with $\hat{I}_t = \varphi_t^* I_t : S \times S \to \mathbb{R}$, that also depends continuously on t (since the vector $\left(d(\exp_{\gamma(t)})_{-(t-t_i)\gamma'(t)}\right)^{-1}(v_0/(t - t_i))$ depends continuously on t as long as no conjugate points appear). Set $i(t) := i(\hat{I}_t)$ and $\nu(t) := \nu(\hat{I}_t)$. By continuity, $i(t+\epsilon) \leq i(t) + \nu(t)$ for all $|\epsilon|$ small enough. But by the Index Lemma 69 we have that $\hat{I}_t \geq \hat{I}_{t+\epsilon}$ with $\hat{I}_t > \hat{I}_{t+\epsilon}$ over Ker $(I_t) \times$ Ker (I_t), and then $i(t+\epsilon) \geq i(t) + \nu(t)$ if $\epsilon > 0$. Then, $i(t)$ is increasing and $i(t + \epsilon) = i(t) + \nu(t)$. ∎

Corollary 86. *(Jacobi) Let* $\gamma\colon [0,a] \to M$ *be a geodesic such that* $q = \gamma(a)$ *is not conjugate to* $p = \gamma(0)$ *along* γ*. Then,* γ *has no conjugate points* $\Leftrightarrow$ γ *is a strict local minimum of* $E|_{\Omega^a_{p,q}}$*. In particular,* γ *minimizing* $\Rightarrow$ γ *has no conjugate points (compare with Corollary 59).*

Corollary 87. *The set of conjugate points to* $\gamma(0)$ *along* γ *is discrete.*

Example. Every nontrivial closed geodesic in a flat torus is of course not minimizing, but they do minimize locally by Corollary 86. And we can say more: by Theorem 110 below, they are actually *the* global minimizers in their homotopy class (up to isometries).

§28. The cut locus

Given M complete, $p \in M$ and $v \in \mathbb{S}^{n-1}(0_p) \subset T_pM$, define $\rho(v) = \rho_p(v) := \sup\{t > 0 : d(p, \gamma_v(t)) = t\} \in (0, +\infty]$. If $\rho(v) < +\infty$, $\gamma_v(\rho(v))$ is called the *cut point of* p *along* γ. The *cut locus* $C_m(p)$ *of* p is the union of its cut points.
$i(p) := d(p, C_m(p)) \in (0, +\infty]$ is the *injectivity radius at* p.
$i(M) := \inf_{p\in M} i(p) \in [0, +\infty]$ is the *injectivity radius of* M.
This proposition justifies the name:

Proposition 88. *Let* γ *be a minimizing geodesic between* p *and* q*. Then,* q *is the cut point of* p *along* γ *if and only if either* q *is the first conjugate point of* p *along* γ*, or there exists another minimizing geodesic between* p *and* q*.*

Corollary 89. $q \in C_m(p) \Leftrightarrow p \in C_m(q)$.

Corollary 90. *$q \in M \setminus C_m(p) \Rightarrow$ there exists a unique minimizing geodesic between p and q.*

Examples: $C(p)$ and $C_m(p)$: $\mathbb{S}^n, \mathbb{RP}^n, \mathbb{S}^1 \times \mathbb{S}^1, \mathbb{S}^1 \times \mathbb{R}$, ellipsoid.

Proposition 91. *$\rho : T_1M \to (0, +\infty]$ is continuous.*

Proof: Continuity of d + Proposition 88 using the funcion F in Proposition 15, since $F_{*v} = \begin{pmatrix} I & 0 \\ * & d(exp_p)_v \end{pmatrix}$ for $p = \pi(v)$. ∎

Corollary 92. *$C_m(p)$ is closed.*

Corollary 93. *M is compact $\Leftrightarrow \rho$ is bounded.*

Corollary 94. *$M \setminus C_m(p)$ is a normal neighborhood of p that is homeomorphic to a ball, open, dense and star-shaped. In particular, $d^2(p, \cdot) = \| \exp_p^{-1}(\cdot)\|^2$ is smooth in $M \setminus C_m(p)$.*

Exercise. Show that $C_m(p)$ has measure 0 (Sug.: show that $C_m(p) \cap B_r(p)$ has measure 0). In fact, $C(p)$ and $C_m(p)$, and even $C(N)$ and $C_m(N)$, are Lipschitz submanifolds; see [**IT**].

§29. Bishop-Gromov volume comparison, I ([Pe])

Consider a normal ball $B_{r_0}(p) \subset M^n$, $r < r_0$ (but the same computation works for normal neighborhoods) and set $\mathbb{S} = \mathbb{S}^{n-1} = \mathbb{S}_1^{n-1}(0_p) \subset T_pM$. Let $v \in \mathbb{S}$, $\gamma = \gamma_v$, $\{e_i\}$ an o.n. basis of $v^\perp \subset T_pM$ and $J_i(t) = t(d\exp_p)_{tv}(e_i) \in \mathfrak{X}_\gamma^J$. Then,

$$\mathrm{Vol}(\mathbb{S}_r^{n-1}(p)) = \int_{\mathbb{S}} \det\left((d\exp_p)_{rv}\right) r^{n-1} dv = \int_{\mathbb{S}} j_v(r)^{n-1} dv,$$

where $j_v^{n-1} = \|J_1 \wedge \cdots \wedge J_{n-1}\|$ is the volume in $\gamma'^\perp$ of the parallelepiped spanned by $\{J_i\}$. Therefore, $j_v' = h_v j_v$, where

$h_v(r) = \frac{1}{n-1}\text{trace}(A(r))$ is the mean curvature and $A(r) = A_v(r)$ is the second fundamental form of $\mathbb{S}_r^{n-1}(p)$ at $\gamma_v(r)$ as seen in Section 14. Writing $A = h_v Id + A_0$ with A_0 symmetric and traceless, by (2),

$$h_v' + h_v^2 + \mathcal{R}_v = 0, \quad \text{with} \;\; \mathcal{R}_v := Ric(\gamma') + \frac{\|A_0\|^2}{n-1} \geq Ric(\gamma').$$

So, $j_v' = h_v j_v \Rightarrow j_v'' + \mathcal{R}_v j_v = 0$, with $j_v(0) = 0$ and $j_v'(0) = 1$. In particular, for $M^n = \mathbb{Q}_k^n$, we have $\overline{j}'' + k\overline{j} = 0$ (indep. of v !!).

Now assume that $Ric \geq k \Rightarrow$ by Sturm Theorem 70, $j_v/\overline{j}$ is decreasing $\Rightarrow q_v := (j_v/\overline{j})^{n-1}$ is decreasing $\Rightarrow$

$$\textit{the map} \;\; r \mapsto \text{Vol}(\mathbb{S}_r^{n-1}(p))/\text{Vol}(\mathbb{S}_{r,k}^{n-1}) \;\; \textit{is decreasing} \text{ !!}$$

where $B_{r,k}^n$ is a ball of radius r in $\mathbb{Q}_k^n$ and $\mathbb{S}_{r,k}^{n-1}$ its geodesic sphere. Moreover, setting $V_r(p) := \text{Vol}(B_r(p))$ and $V_r^k := \text{Vol}(B_{r,k}^n)$, by Gauss Lemma $V_r(p)/V_r^k = \text{Vol}(\mathbb{S})^{-1}\int_{\mathbb{S}} m_v(r)dv$, where $m_v(r) := \int_0^r q_v \overline{j}^{n-1} / \int_0^r \overline{j}^{n-1}$ is the ($\overline{j}^{n-1}$-weighted) average of q_v. Since q_v is decreasing, so is m_v, and we conclude:

Theorem 95. *(Bishop–Gromov, local: for normal balls).* *If $Ric_M \geq k$, the function $r \mapsto V_r(p)/V_r^k$ is non-increasing, $0 \leq r \leq i(p)$. If, in addition, $V_s(p)/V_s^k = V_r(p)/V_r^k$ for some $0 < s < r \leq diam(M)$, then $B_r(p)$ is isometric to $B_{r,k}^n$.*

Proof: We already proved the first part, so we only need to check the equality case. But in this case by monotonicity of m_v we get $m_v(s) = m_v(r)$ $\forall v \in \mathbb{S}$. By monotonicity of q_v this implies that $q_v \equiv 1$ on $[0, r]$ $\forall v$. By the equality in Sturm Theorem 70, $\mathcal{R}_v \equiv k \Rightarrow Ric(\gamma') \equiv k$ and $A_0 \equiv 0 \Rightarrow A$ agrees to the one for

$\mathbb{Q}_k^n \Rightarrow$ the Jacobi fields along γ are $sn_k(t)e(t)$ with $e(t)$ parallel (as for $\mathbb{Q}_k^n$) $\Rightarrow f$ in Proposition 76 is an isometry. ∎

Remark 96. The corresponding version of [B-G] Theorem 95 for $Ric \leq k$ does not hold because of A_0, but the non-decreasing statement works for $K \leq k$ using the same idea of the proof of Rauch Theorem 71. (exercise)

§30. Bishop-Gromov volume comparison, II ([Pe])

Theorem 97. *(Bishop-Gromov) If M is complete, Theorem 95 holds for all $r \geq 0$ (i.e., no restriction $r \leq i(p)$).*

Proof: Since all the arguments in Section 29 need only for $\exp_p$ to be a chart, we can repeat everything on $M \setminus C_m(p)$ using Corollary 94. Hence, $\mathrm{Vol}(B_p(r)) = \int_{\mathbb{S}} \int_0^r j_v(t)^{n-1} dt dv$ still holds once we set $j_v(t)$ as 0 for $t > \rho(v)$. Indeed, all that is needed is that the functions $q_v = j_v/\overline{j}$ are still decreasing. ∎

Remark 98. Bishop proved in 1963 the weaker inequality $\mathrm{Vol}(B_r(p)) \leq \mathrm{Vol}(B^n_{r,k})$ and Gromov the full statement in 1981.

Corollary 99. *(Cheng, 1975) If $diam(M^n) = \pi k$ in Bonnet-Myers Theorem 61, then M^n is isometric to $\mathbb{S}^n(k) = \mathbb{Q}^n_{1/k^2}$.*

Proof: WLG $k = 1$, and take $p_1, p_2 \in M^n$ with $d(p_1, p_2) = \pi$. Then, we have that $M^n = \overline{B_\pi(p_i)}$, and $B_{\frac{\pi}{2}}(p_1) \cap B_{\frac{\pi}{2}}(p_2) = \emptyset$. But $\mathrm{Vol}(M^n)/\mathrm{Vol}(B_{\frac{\pi}{2}}(p_i)) = V_\pi(p_i)/V_{\frac{\pi}{2}}(p_i) \leq V^1_\pi / V^1_{\frac{\pi}{2}} = 2$. So, $\mathrm{Vol}(M^n) \leq \mathrm{Vol}(B_{\frac{\pi}{2}}(p_1) \cup B_{\frac{\pi}{2}}(p_2)) \leq \mathrm{Vol}(M^n) \Rightarrow V_\pi(p_i)/V_{\frac{\pi}{2}}(p_i) = 2$ $\Rightarrow$ by the equality case in Theorem 97 $B_\pi(p_i)$ and $B^n_{\pi,1} = \mathbb{S}^n \setminus \{N\}$ are isometric and $\overline{B_\pi(p_i)} = M^n \Rightarrow M^n = \mathbb{S}^n$. ∎

Corollary 100. *(Calabi-Yau, 1975) M^n complete noncompact $Ric_M \geq 0 \Rightarrow \mathrm{Vol}(B_r(p)) \geq r\frac{\mathrm{Vol}(B_{r_0}(p))}{2^{n+3}r_0}$ if $r \geq 6r_0$, i.e., $\mathrm{Vol}(B_{r_0}(p)$ grows at least linearly in r (notice that it grows linearly in $\mathbb{S}^n \times \mathbb{R}$).*

Proof: $V_t = V_t(p) = \mathrm{Vol}(B_t(p))$, $\hat{V}_t = t^n w_{n-1}$ in $\mathbb{R}^n$. For a ray γ at p, $t \geq 2r_0$, and $q = \gamma(t + r_0)$ we have $V_{3t} \geq V_t(q) \geq \frac{V_{t+2r_0}(q)-V_t(q)}{\hat{V}_{t+2r_0}-\hat{V}_t}\hat{V}_t \geq \frac{V_{r_0}\ t^n}{(t+2r_0)^n-t^n} = \frac{V_{r_0}\ t}{2r_0\sum_{i=1}^n\binom{n}{i}(2r_0/t)^{i-1}} \geq t\frac{V_{r_0}}{2r_0(2^n-1)}$. ∎

Corollary 101. *If M is complete with finite volume and $Ric \geq 0$ (in particular, if M is flat), then M is compact.*

§31. The Toponogov Theorem ([Me])

A global generalization of Rauch Theorem 71 is the following.

Theorem 102. *(Toponogov, hinge version) M complete with $K \geq k$, and γ_1, γ_2 normalized geodesics arcs with $\gamma_1(0) = \gamma_2(0)$. Assume γ_1 is minimizing and, if $k > 0$, that $L(\gamma_2) \leq \pi/\sqrt{k}$. Let $\hat{\gamma}_1, \hat{\gamma}_2$ be the corresponding hinge in $\mathbb{Q}_k^2$, that is, $L(\hat{\gamma}_i) = L(\gamma_i)$ and $\angle(\hat{\gamma}_1'(0), \hat{\gamma}_2'(0)) = \angle(\gamma_1'(0), \gamma_2'(0))$. Then, $d(\gamma_1(\ell_1), \gamma_2(\ell_2)) \leq \hat{d}(\hat{\gamma}_1(\ell_1), \hat{\gamma}_2(\ell_2))$.*

Remark 103. Theorem 102 is immediate from Proposition 76 when γ_1 and γ_2 are contained in a metric ball centered at p onto which $\exp_p$ is nonsingular, and $L(\gamma_i) \leq \pi/\sqrt{4k}$, $i = 1, 2$, when $k > 0$.

There are several versions of Toponogov Theorem 102, each one useful in different circumstances. Some of them do not need anything but distances!! For example, the next version follows imediately from Theorem 102 using the fact that in $\mathbb{Q}_k^2$ the length of a closing edge in a hinge with minimal geodesics and the hinge angle are in a monotone relation, but they are actually equivalent:

Theorem 104. *Let M be complete with $K \geq k$. If $\{\gamma_j\}$ is a minimizing geodesic triangle in M, then there is a unique minimizing geodesic triangle $\{\hat{\gamma}_j\}$ in $\mathbb{Q}^2_k$ with $L(\hat{\gamma}_j) = L(\gamma_j)$, $j = 0, 1, 2$, and satisfies $d(\gamma_1(t_1), \gamma_2(t_2)) \geq \hat{d}(\hat{\gamma}_1(t_1), \hat{\gamma}_2(t_2))$ $\forall t_i \in [0, L(\gamma_i)]$.*

Another equivalent version:

Theorem 105. *Let M be complete with $K \geq k$. If $\{\gamma_j\}$ is a minimizing geodesic triangle in M, then there is a unique minimizing geodesic triangle $\{\hat{\gamma}_j\}$ in $\mathbb{Q}^2_k$ with $L(\hat{\gamma}_j) = L(\gamma_j)$, $j = 0, 1, 2$, and satisfies $d(o, \gamma_0(t)) \geq \hat{d}(\hat{o}, \hat{\gamma}_0(t))$ $\forall t \in [0, L(\gamma_0)]$.*

Theorem 102 follows easily from Theorem 106 below (which in turn is slightly more general than Theorem 105) using the Exercise in Section 11; see [**Me**], page 16 Remarks 3 and 5. However, they are actually equivalent. Hence, we will prove:

Theorem 106. *(Toponogov, metric version) M complete, $p_1 \neq o \neq p_2 \in M$, γ_i a minimizing geodesic between o and p_i, $i = 1, 2$, and γ_0 a non-constant geodesic between p_1 and p_2 satisfying $L(\gamma_0) \leq L(\gamma_1) + L(\gamma_2)$, all p.b.a.l.. If $K \geq k$, and $L(\gamma_0) \leq \pi/\sqrt{k}$ when $k > 0$, then there is a minimizing geodesic triangle $\{\hat{\gamma}_j\}$ in $\mathbb{Q}^2_k$ with $L(\hat{\gamma}_j) = L(\gamma_j)$, $j = 0, 1, 2$, and it satisfies that $d(o, \gamma_0(t)) \geq \hat{d}(\hat{o}, \hat{\gamma}_0(t))$ $\forall t \in [0, L(\gamma_0)]$.*

Proof: Let $\rho = d(o, \cdot)$, $\hat{\rho} = \hat{d}(\hat{o}, \cdot)$. If $A = \mathrm{Hess}_\rho|_{\nabla\rho^\perp}$ is the second fundamental form of (pieces of) geodesic spheres centered at o, Rauch says that $A \leq \hat{A} = \frac{s'}{s} I$, where s is the solution of $s''+ks = 0, s(0) = 0, s'(0) = 1$ (see Remark 73). To get a uniform Hessian estimate (not just on $\nabla\rho^{\perp}$), take f such that $f' = s$. Then, $f'' + kf = C =$ constant. So, if $\sigma := f \circ \rho$ and $\hat{\sigma} := f \circ \hat{\rho}$ we have $\mathrm{Hess}_{\sigma} = (f'' \circ \rho) d\rho \otimes d\rho + (f' \circ \rho)\mathrm{Hess}_{\rho}$ and therefore $\mathrm{Hess}_{\sigma} \leq (-k\sigma + C)I$ on $M \setminus C_m(o)$ and $\mathrm{Hess}_{\hat{\sigma}} = (-k\hat{\sigma} + C)I$.

If $k > 0$, assume first that $L(\gamma_0)+L(\gamma_1)+L(\gamma_2)<2\pi/\sqrt{k}$, so the corresponding minimizing geodesic triangle exists in $\mathbb{Q}^2_k$ and it is not a great circle. In particular, $\ell := L(\gamma_0) < \pi/\sqrt{k}$.

Consider now $\delta := \sigma \circ \gamma_0 - \hat{\sigma} \circ \hat{\gamma}_0$ on $[0, \ell]$. Since $\mathrm{diam}(M) \leq \pi/\sqrt{k}$ if $k > 0$ by Bonnet-Myers Theorem 61, in any case f is monotonous increasing and we only have to see that $\delta \geq 0$. Observing that $\delta(0) = \delta(\ell) = 0$, assume that $m := \min \delta < 0$. If $k > 0$, comparing with a sphere of curvature $k - \epsilon$ for $\epsilon \to 0$, we may assume that $\mathrm{diam}(M) < \pi/\sqrt{k}$ (or use Theorem 99!). Hence, there exist $k' > k$ and $\tau > 0$ such that $\ell < \pi/\sqrt{k'} - \tau$. In any case, it is easy to find a function a_0 such that $a_0''+k'a_0 = 0$, $a_0(-\tau) = 0$ and $a_0|_{[0,\ell]} \leq m$. Thus, there is $\lambda > 0$ such that the function $a = \lambda a_0$ satisfies $a'' + k'a = 0$, $a \leq \delta$, and $a(t_0) = \delta(t_0) < 0$ for some $t_0 \in (0, \ell)$. (make a picture!)

Case 1. $x := \gamma_0(t_0) \notin C_m(o)$. Then δ is smooth in a neighborhood of t_0, and $\delta'' = \langle \mathrm{Hess}_{\sigma}\gamma_0', \gamma_0' \rangle - \langle \mathrm{Hess}_{\hat{\sigma}}\hat{\gamma}_0', \hat{\gamma}_0' \rangle \leq -k\delta$. Hence, $(\delta - a)''(t_0) \leq (k' - k)\delta(t_0) < 0$, which contradicts the fact that t_0 is a minimum of $\delta - a$.

Case 2. $x \in C_m(o)$. Let β be a minimizing geodesic from o to x, $o_\epsilon := \beta(\epsilon)$, and replace ρ by $\rho_\epsilon = d(o, o_\epsilon) + d(o_\epsilon, \cdot)$. By the triangle inequality, $\rho_\epsilon \geq \rho$ with equality at x, i.e., ρ_ϵ is an *upper support function (USF) of* ρ *at* x. Moreover, $x \notin C_m(o_\epsilon)$, and so ρ_ϵ is smooth at x. Since f is monotonously increasing, $\sigma_\epsilon := f \circ \rho_\epsilon$ is then an USF of σ at x. Thus $\delta_\epsilon - a$ is also an USF of $\delta - a$ at t_0, and therefore it also attains its minimum at t_0. Since we get the same estimates as in Case 1 up to a small error, $\delta_\epsilon'' \leq -k\delta_\epsilon + O(\epsilon)$ (exercise), we have $(\delta_\epsilon - a)''(t_0) \leq (k' - k)\delta(t_0) + O(\epsilon) < 0$ for ϵ small enough, again a contradiction.

Finally, we need to argue for $L(\gamma_0) + L(\gamma_1) + L(\gamma_2) \geq 2\pi/\sqrt{k}$ if $k > 0$. The "=" case follows from the "<" case with a limit argument in $k - \epsilon$ as we did with the diameter. For the ">" case, take $r < k$ given by $L(\gamma_0) + L(\gamma_1) + L(\gamma_2) = 2\pi/\sqrt{r}$ and use the "=" case comparing with $\mathbb{Q}^2_r$: the comparison triangle in $\mathbb{Q}^2_r$ has to be a great circle, so $-\hat{o} = \hat{\gamma}_0(s_0)$ and therefore $\pi/\sqrt{r} = \hat{d}(\hat{o}, -\hat{o}) \leq d(o, \gamma_0(s_0)) \leq \pi/\sqrt{k} < \pi/\sqrt{r}$, a contradiction. ∎

<u>*Application.*</u> For noncompact M, $\pi_1(M)$ may not be finitely generated (exercise). However, this does not happen if $K \geq 0$; in fact, there is an *a-priori* bound on the number of generators:

Theorem 107. *(Gromov)* M^n *complete with* $K \geq 0 \Rightarrow \pi_1(M^n)$ *can be generated by less than* 3^n *elements.*

Proof: Fix $x \in \tilde{M}$, and for $f \in \Gamma = \mathrm{Deck}(\pi)$ define $\|f\| = d(x, f(x))$. Notice that $\{g \in \Gamma\colon \|g\| \leq r\}$ is finite for all $r > 0$. So choose $f_1 \in \Gamma$ such that $\|f_1\| = \min\{\|f\|\colon f \in \Gamma\}$, and $f_k \in \Gamma$ with $\|f_k\| = \min\{\|f\|\colon f \in \Gamma \backslash < f_1, \ldots, f_{k-1} >\}$. Setting $l_i := \|f_i\|$ and $l_{ij} := d(f_i(x), f_j(x))$, we have for $i < j$ that $l_{ij} = d(x, f_i^{-1} f_j(x)) \geq l_j \geq l_i$ since $f_i^{-1} f_j \notin < f_1, \ldots, f_{j-1} >$.

Now choose a minimizing geodesic γ_i from x to $f_i(x)$ of length l_i, and for $i < j$ a minimizing geodesic γ_{ij} from $f_i(x)$ to $f_j(x)$ of length l_{ij}. Take $\alpha_{ij} = \langle \gamma_i'(0), \gamma_j'(0) \rangle$ that is bounded from below by the angle $\tilde{\alpha}_{ij}$ of the corresponding minimizing triangle in $\mathbb{R}^2$ by Toponogov's Theorem 102. The cosine law says that $\cos \tilde{\alpha}_{ij} = (l_i^2 + l_j^2 - l_{ij}^2)/2l_il_j \leq (l_i^2 + l_j^2 - l_j^2)/2l_i^2 = 1/2$. Hence, $\alpha_{ij} \geq \tilde{\alpha}_{ij} \geq \pi/3$, and so the balls $B^n_{1/2}(\gamma_i'(0))$ are disjoint in $B^n_{3/2}(0) \subset T_x\tilde{M}$. The estimate now follows easily by comparing volumes. ∎

Remark 108. Essentially the same proof shows that if M^n is complete with K bounded from below, $K \geq -\lambda^2$, and bounded diameter, $\mathrm{diam}(M^n) \leq D$, then $\pi_1(M^n)$ is generated by less than $\sqrt{n\pi/2}\,(2+2\cosh(2\lambda D))^{\frac{n-1}{2}}$ elements (see Theorem 3.1 in [**Me**]).

To estimate the maximum number of balls of a fixed radius r that fit in the unit n-sphere is an old subject. For $\pi/6$ an exponential known bound is 1.321^n ([**CZ**]). But we have a natural:

Open problem: Is there a linear (or polynomial, or even subexponential) bound in n for Theorem 107?

§32. Drops of Alexandrov Spaces ([BBI])

Toponogov's Theorem 106 (or even Proposition 76) gives rise to curvature notions for metric (length) spaces(!):

Def.: (E, d) a metric space $\Rightarrow d_i = \inf\{L(c)\}$ (may be $+\infty$) is called the *interior distance*. If $d_i = d$, (E, d) is called a *length space* (actually, $d_{ii} = d_i$).

Hopf-Rinow Theorem 34 holds for locally compact length spaces: *If a locally compact length space* (E, d) *is complete, then any two points in* E *can be connected by a minimizing geodesic.*

Def.: A length space (E, d) is called an *Alexandrov space with curvature* $\geq c$ if for all $x \in E$ there exists a neighborhood U_x of x such that, for every triangle pqr in U_x, $q' \in \overline{pr}$ and $p' \in \overline{qr}$, it holds that $d(p', q') \geq \hat{d}(\hat{p}', \hat{q}')$, where $\hat{p}'$ and $\hat{q}'$ are the corresponding points on the comparison triangle $\hat{p}\hat{q}\hat{r}$ in $\mathbb{Q}_c^2$.

Remark 109. In the same way that the local Proposition 76 gives rise to its global version Toponogov Theorem 106 for complete manifolds, the previous local definition implies the corresponding global theorem for complete Alexandrov spaces, a result due to Burago, Gromov and Perelman (for a proof, see [**LS**]).

Alexandrov spaces appear as *limits of manifolds:*

Given two compact metric spaces X, Y we define the *Gromov-Hausdorff distance* $d_{GH}(X, Y) = \inf\{d_H(f(X), g(Y))\}$ where the infimum is taken over all metric spaces Z and all distance preserving maps $f\colon X \to Z$, $g\colon Y \to Z$, and d_H is the *Hausdorff distance* given by $d_H(R, S) = \inf\{\epsilon \geq 0\colon R \subseteq B_\epsilon(S), S \subseteq B_\epsilon(R)\}$. With d_{GH} the isometry classes of compact metric spaces $\mathcal{C}$ is itself a metric space(!) and we can talk about convergence of compact metric spaces(!!). A celebrated result by M. Gromov states that

$$\mathcal{M}(n, c, D) = \{M^n \text{ compact} : Ric \geq c,\ \operatorname{diam}(M) \leq D\}$$

is precompact in $\mathcal{C}$. Limits of converging sequences with bounded K are Alexandrov spaces that are not in general manifolds.

§33. The Preissman Theorem

M^n complete, $K < 0 \Rightarrow \tilde{M}^n \cong \mathbb{R}^n \Rightarrow \pi_k(M^n) = 0 \ \ \forall\, k \geq 2$. But how is $\pi_1(M^n)$ when M^n is compact?

Def.: *Free homotopy classes*: $\hat{\pi}_1(M)$.

Def.: *Closed geodesics* and *geodesic loops*.

Theorem 110. *(Cartan) M^n compact $\Rightarrow \exists$ a closed geodesic in each free homotopy class.*

Proof: Fix $w \in \hat{\pi}_1(M)$ nontrivial, and take a sequence of closed piecewise geodesics $\gamma_n : \mathbb{S}^1 \to M$ such that $L(\gamma_n) \to \ell := \inf\{L(c)\colon c \in w\}$. $\{\gamma_n\}$ is equicontinuous $\Rightarrow \gamma_n \to \sigma \in C^0$ uniformly. Define γ as the closed broken geodesic joining $\sigma(t_i)$ to $\sigma(t_{i+1})$, where $\sigma([t_i, t_{i+1}])$ is inside a convex ball $\Rightarrow \gamma \in w \ \Rightarrow L(\gamma) \geq \ell$. But $L(\gamma) \leq \ell \Rightarrow \gamma$ is not broken. ∎

Remark 111. Compactness is necessary. Yet, every compact Riemannian manifold has a closed geodesic (Lyusternik-Fet '51).

Def.: $g \in \mathrm{Iso}(N)$ without fixed points is a *translation along* γ if $g(\gamma) = \gamma$ (the images as sets), for some geodesic γ of N.

Lemma 112. *M compact, $\pi\colon \tilde{M} \to M$ its universal cover with the covering metric. Then, every $f \in \mathrm{Deck}(\pi) \subset \mathrm{Iso}(\tilde{M})$ is a translation.*

Proof: Let j be the isomorphism in Corollary 39 and $\gamma \in j^{-1}(f)$ as in Cartan's Theorem 110 (as a free homotopy class) with lift $\tilde{\gamma}$. Then, $f(\tilde{\gamma}(s)) = \tilde{\gamma}(s + r)$, where r is the period of γ (it is s and not $-s$ since otherwise $\tilde{\gamma}(r/2)$ would be a fixed point of f). ∎

Lemma 113. *If all elements of a subgroup $1 \neq H \subset \mathrm{Deck}(\pi)$ leave invariant the same nonclosed geodesic, then $H \cong \mathbb{Z}$.*

Proof: $h(\gamma(s)) = \gamma(s + \tau(h))$, with $\tau\colon H \to (\mathbb{R}, +)$ an injective group homomorphism. H acts discontinuously $\Rightarrow \tau(H) \cong \mathbb{Z}$. ∎

<u>*Exercise*</u>. Show that the hypothesis of the above Lemma without the nonclosedness of the geodesic implies that H is cyclic. Give an example of this situation for each cyclic group.

Lemma 114. *A, B, C a geodesic triangle in a Hadamard manifold $\Rightarrow$ i) $A^2 + B^2 - 2AB\cos(\gamma) \leq C^2$ ($<$ if $K < 0$), ii) $\alpha + \beta + \gamma \leq \pi$ ($<$ if $K < 0$).*

Proof: Consequence of Proposition 76 ($\exp_p$ is an expansion). ∎

Proposition 115. *Let $\tilde{M}$ be a Hadamard manifold with $K < 0$, and $f \neq Id$ a translation along $\gamma \Rightarrow \gamma$ is unique.*

Proof: Suppose there are two, $\gamma_1, \gamma_2 \Rightarrow \gamma_1 \cap \gamma_2 = \emptyset \Rightarrow$ there is a geodesic quadrilateral which contradicts Lemma 114. ∎

Corollary 116. *If $g \in \mathrm{Iso}(\tilde{M})$ commutes with an f as in Proposition 115 $\Rightarrow g$ is also a translation along γ.*

Theorem 117. *(Preissman) M compact with $K < 0 \Rightarrow$ any nontrivial abelian subgroup of $\pi_1(M)$ is infinite cyclic.*

Proof: Lemma 112 + Corollary 116 + Lemma 113. ∎

Corollary 118. *Many compact manifolds that admit metric with $K \leq 0$ admit no metric with $K < 0$: T^n, $N^2 \times \mathbb{S}^1$ for a compact N^2, $M \times N$ for compact M and N, etc, etc...*

Lemma 119. *If M complete with $K \leq 0$ and* $\mathrm{Deck}(\pi)$ *fixes the same geodesic $\tilde{\gamma}$, then M is not compact (in fact, every geodesic orthogonal to $\pi(\tilde{\gamma})$ is a ray).*

Proof: Take β a unit orthogonal geodesic to γ at $p = \gamma(0)$, α_t a minimizing geodesic joining p to $\beta(t)$, and lift β and α_t to $\tilde{M}$. By Lemma 114 (i), $t \leq L(\tilde{\alpha}_t) = L(\alpha_t) = d(p, \beta(t)) \leq t$. ∎

Corollary 120. *(Preissman) If M is compact with $K < 0$, then $\pi_1(M)$ is not abelian.*

Theorem 121. *(Byers) If M is compact with $K < 0$ and $1 \neq H \subset \pi_1(M)$ is solvable, then $H \cong \mathbb{Z}$. Moreover, any such subgroup has infinite index.*

Proof: $H = H_0 \supset H_1 \supset \cdots \supset H_{k-1} \supset H_k = 1$ with H_i normal in H_{i+1} and abelian quotients $\Rightarrow H_{k-1} = \langle g \rangle \cong \mathbb{Z}$ with g fixing γ. If $h \in H_{k-2}$, $[h, g] = g^m$ for some $m \Rightarrow h$ also leaves γ invariant $\Rightarrow H_{k-2} \cong \mathbb{Z}$, and so on $\Rightarrow H \cong \mathbb{Z}$ (abelian quotients only needed for H_{k-1}).
For the second part, suppose $H =< g >\cong \mathbb{Z} \subset \pi_1(M)$ has finite index, and take $h \in \pi_1(M) \Rightarrow$ for some n, m, $h^n = g^m \Rightarrow h^n$ fixes γ. By Proposition 115 h also fixes $\gamma \Rightarrow \pi_1(M)$ fixes γ. This contradicts Corollary 120 by Lemma 113. ∎

Remark 122. For (much) more about manifolds with non-negative curvature, see [**BGS**].

§34. Busemann functions

These functions are one of the main tools to study the behavior "at infinity" of complete noncompact manifolds.

First, recall: Integration by parts $\Rightarrow$ *weak solutions of PDEs* = good spaces where things converge nicely, as opposed to $C^k(M,\mathbb{R})$. *Regularity theory of elliptic PDEs*: weak solutions are strong. *Max. pple*: $f \in C^2(M,\mathbb{R}), f \geq 0, f(p_0) = 0, \Delta f \leq 0 \Rightarrow f \equiv 0$. *Support functions and the strong maximum principle*: Let $f \in C^0(M,\mathbb{R}),\ f \geq 0,\ f(p_0) = 0$. Suppose that $\forall x \in M$ and $\forall \epsilon > 0$, $\exists\ g_\epsilon^x \in C^2(U_x)$ with $g_\epsilon^x \geq f$, $g_\epsilon^x(x) = f(x)$ and $\Delta g_\epsilon^x(x) \leq \epsilon$. Then, $f \equiv 0$.

Def.: A *ray* $\gamma\colon [0,+\infty) \to M$ is a (normalized) geodesic such that $d(p,\gamma(t)) = t, \forall t > 0$, while a *line* is a (normalized) geodesic $\gamma\colon \mathbb{R} \to M$ with $d(\gamma(t),\gamma(s)) = |t-s|,\ \forall t,s \in \mathbb{R}$.

For a ray γ and $t \geq 0$, set $b_t = b_t^\gamma := d(\gamma(t),\cdot) - t\colon M \to \mathbb{R}$. If $p := \gamma(0)$, triangle inequality $\Rightarrow b_t \leq b_s$ if $t \geq s$, $b_t \geq -d(p,\cdot)$, and $|b_t(x) - b_t(y)| \leq d(x,y)\ \forall x,y \in M \Rightarrow$ the *Busemann function of* γ given by

$$b^\gamma := \lim_{t\to+\infty} b_t^\gamma = \lim_{t\to+\infty} (d(\gamma(t),\cdot) - t)\colon M \to \mathbb{R}$$

is well defined and Lipschitz.
Busemann functions naturally have upper support functions:
Given $x \in M$, take $\mu_+ = \lim_s \mu_s\colon [0,+\infty) \to M$ a future asymptote to γ with $\mu_+(0) = x$. Since μ_+ is a ray starting at x,

$$g_t^x := b_t^{\mu_+} + b^\gamma(x) \tag{4}$$

is smooth at x, with $g_t^x(x) = b^\gamma(x)$. Since $d(\gamma(s),x) - t =$

$d(\gamma(s), \mu_s(t)) \geq d(\gamma(s), \mu_+(t)) - d(\mu_s(t), \mu_+(t))$, we get

$$g_t^x = \lim_{s \to +\infty} (d(\mu_+(t), \cdot) + d(\gamma(s), x) - t - s) \geq b^\gamma.$$

That is, g_t^x is an upper support function for b^γ at x.

Lemma 123. *If $f: M \to \mathbb{R}$ is C^2 with $\|\nabla f\| \equiv 1$, then*

$$-(n-1)Ric(\nabla f) = \nabla f(\Delta f) + \|\mathrm{Hess}_f\|^2 \geq \nabla f(\Delta f) + \frac{(\Delta f)^2}{n-1}.$$

Proof: The first inequality follows taking an o.n.b. diagonalizing Hess_f, while the second one is just Cauchy-Schwarz on $(\nabla f)^\perp$. ∎

Corollary 124. *(Calabi) If $Ric \geq 0$, then for $\rho := d(p, \cdot)$ it holds that $\Delta\rho \leq (n-1)/\rho$ on $M \setminus C_m(p) \cup \{p\}$.*

Proof: If γ is a minimizing geodesic starting at p, and $\lambda := \frac{1}{n-1}\Delta\rho \circ \gamma$, then $\lim_{t\to 0} \frac{1}{\lambda(t)} = \lim_{t \to 0} t = 0$, and $\lambda' + \lambda^2 \leq 0$ by Lemma 123. Hence $(1/\lambda)' \geq 1$, and so $\lambda(t) \leq 1/t = 1/\rho(\gamma(t))$. ∎

Corollary 125. *$Ric \geq 0 \Rightarrow$ a.e. $\Delta b_t^\gamma \leq \frac{n-1}{t-d(p,\cdot)} \to 0$ on compacts as $t \to +\infty$. In particular, b^γ is weakly subharmonic.*

§35. The Cheeger-Gromoll splitting Theorem

While any complete noncompact manifold has a ray, not lines (compute the rays for the paraboloid). In fact, lines only appear in products under nonnegative Ricci curvature:

Theorem 126. *(Cheeger-Gromoll) Let M be complete with $Ric \geq 0$. If M has a line, then M is isometric to $N \times \mathbb{R}$.*

Proof: Let γ be a line, $x \in M$, and let g_t^x and $\tilde{g}_t^x$ be the natural support functions of b^γ and $b^{-\gamma}$ as in (4). The function $b := b^\gamma + b^{-\gamma}$, satisfies $b \geq 0$ and $b = 0$ over γ. Thus $h_t^x := g_t^x + \tilde{g}_t^x$ is an upper support function for b at x and, by Corollary 124, $\Delta h_t^x(x) \leq 2(n-1)/t$. By the strong maximum principle, $b \equiv 0$, and by Corollary 125 both $b^{\pm\gamma}$ are harmonic, hence smooth. By Lemma 123, $\mathrm{Hess}_{b^\gamma} \equiv 0$, ∇b^γ is parallel ($\Rightarrow$ Killing), the level sets $N_t = (b^\gamma)^{-1}(t)$ of b^γ are smooth embedded totally geodesic isometric hypersurfaces, and the (global!) flux of ∇b^γ restricted to $N_0 \times \mathbb{R}$ is a bijective local isometry, hence an isometry. ∎

Exercise. If M is compact with $Ric \geq 0$, then its universal cover splits isometrically as $N \times \mathbb{R}^k$, with N compact and simply connected.

§36. On the differentiable sphere Theorem

Let M^n be a compact manifold with positive sectional curvature. Then, $K_{min} \leq K \leq K_{max}$ (i.e., $K_{min}(p) \leq K(\sigma_p) \leq K_{max}(p)$).

Def.: The function K_{min}/K_{max} is called the *pinching function* of M. We say that M is *δ-pinched*, or that $\delta \in \mathbb{R}$ is a *pinching of M*, if $\delta < K_{min}/K_{max}$, i.e.,

$$\delta K_{max}(p) < K(\sigma_p) \leq K_{max}(p), \quad \forall \sigma_p \subset T_pM, \ \forall p \in M.$$

The old question: $\delta \sim 1 \Rightarrow M^n \cong \mathbb{S}^n/\Gamma$?

The answer was **yes**, but how close δ has to be from 1, and what does "$\cong$" mean? *Lots* of development and people involved.
At least for n even, $\delta \geq 1/4 : \mathbb{CP}^n$.
Extrinsic geometric flows: Curvature flow for closed embedded

curves in compact and complete surfaces. Watch this and this youtube videos to get an intuition.
Very global in nature: smooth a "triangle" at its vertices.
Mean curvature flow (MCF): $f' = -HN$; inverse MCF, etc...: $f' = -\nabla E(f)$ for some *energy functional* E ($E =$ vol for MCF).

Def.: *Hamilton's Ricci flow*: $g_t' = -Ric_{g_t}$.

Def.: *Normalized Ricci flow*: $g_t' = -Ric_{g_t} + \frac{1}{n}(\int_M scal_{g_t})g_t$.

These are diffusion equations that tend to 'distribute' the curvature uniformly over the manifold (preserving the volume for the normalized flow). So they should somehow make the metric more 'symmetric'. In general, although we always have existence of flux for small time (Hamilton), singularities (where $K \to \infty$) appear.

Remark 127. Perelman's proof of Thurston's geometrization (and hence Poincaré's) conjecture is based on the classification of the singularity types of the Ricci flow, and their desingularization using (discrete!) surgeries. The number of surgeries is finite for compact simply connected 3-dimensional manifolds, proving Poincaré's conjecture, since every topological manifold of dim ≤ 3 admits a smooth (even C^ω) structure. Apart from the beautiful and tough math, the story behind this is well known (and quite sad... to say the least: see [**NG**]).

The two important questions for us are:

1. *Which are <u>invariant conditions</u> under the Ricci flow?*
2. *Does the metric converge under an invariant condition?*

Under some invariant conditions the Ricci flow develops no singularities, like it was shown in the seminal work [**BW**]:

Theorem 128. *(Böhm-Wilking) Positive and 2-positive curvature operator are invariant conditions, and the metrics converge to a metric with constant sectional curvature. In particular, M is diffeomorphic to a spherical space form, $\mathbb{S}^n/\Gamma$.*

The key main technique behind this beautiful result is the use of *pinching-families*, that are barriers in the sense of PDEs.

Theorem 129. *(Yau-Zheng) If M is $1/4$-pinched $\Rightarrow K_{\mathbb{C}} > 0$.*

Theorem 130. *(Ni-Wolfson, [**NW**]) Both $K_{\mathbb{C}} \geq 0$ and $K_{\mathbb{C}} > 0$ are invariant conditions under the Ricci flow.*

These three results, together with a pinching-family construction as [**BW**], immediately give the *differentiable sphere theorem*:

Corollary 131. *(Brendle-Schoen) If M is (pointwise) 1/4-pinched, then M is diffeomorphic to a spherical space form.*

Actually, Ni and Wolfson in their beautiful and short work [**NW**] proved a stronger version of the differentiable sphere theorem Corollary 131, where even zero curvatures are allowed:

Theorem 132. *(Ni-Wolfson) Assume there exist continuous functions $k(p), \delta(p) \geq 0$, such that $\mathcal{P} := \{p \in M : k(p) > 0\}$ is dense and $\delta \not\equiv 0$, satisfying that, for all $p \in M$, $\sigma \subset T_pM$,*

$$\frac{1}{4}(1 + \delta(p))k(p) \leq K(\sigma) \leq (1 - \delta(p))k(p).$$

Then, the normalized Ricci flow deforms g into a metric of constant sectional curvature. In particular, $M^n \cong \mathbb{S}^n/\Gamma$.

Remark 133. It is a pity that the paper [**NW**] by Ni and Wolfson was never published in print (as neither were the three papers where Perelman proves Thurston's geometrization conjecture). But the really interesting and sad question is: *WHY?*

For details about the Ricci flow, Böhm-Wilking superb work [**BW**] and the differentiable sphere theorem, see the survey [**Ri**].

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