# Differential Geometry guide 

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## PART I

Prerequisites (part I): Analysis on $\mathbb{R}^{n}$.
Analysis on manifolds, Stokes and de Rham recommended. Bibliography: dC]

## §1. Introduction

Differentiable manifolds: smooth world. Now we're going to measure in them. After all, geometry comes from the Greek: "measurement of the Earth":
Eratosthenes (Cirene, 276 AC - Alexandria, 194 AC)
Posidonius ( $135 \mathrm{AC}-51 \mathrm{AC}$ ) $\Rightarrow$ Colombo
We will study in this first part curves, surfaces and hypersurfaces of Euclidean space $\Rightarrow$ Two aspects: intrinsic and extrinsic.

## §2. Curves

Curve: intrinsically, nothing interesting: $I \subseteq \mathbb{R}$.
Regular curves $\alpha$ in $\mathbb{R}^{2}$ and $\mathbb{R}^{3}$ : arc length $s$, p.b.a.l, curvature $\kappa^{\alpha}$ and torsion $\tau^{\alpha}$. Frenet Trihedron: $\{t, n, b\}$. FTC: explicit vs. ODE.
Curves in $\mathbb{R}^{n}$ : FTC.
Exercise. Prove that $\kappa^{\alpha}=\sqrt{\left\|\alpha^{\prime}\right\|^{2}\left\|\alpha^{\prime \prime}\right\|^{2}-\left\langle\alpha^{\prime}, \alpha^{\prime \prime}\right\rangle^{2}} /\left\|\alpha^{\prime}\right\|^{3}$, independently of the parametrization of $\alpha$.

## §3. Surfaces in $\mathbb{R}^{3}$

Regular (Euclidean) surface: $S=S^{2} \subset \mathbb{R}^{3}$ (embedding!)
Regular (Euclidean) hypersurface: $M^{n} \subset \mathbb{R}^{n+1}$ (embedding!)
Regular (Euclidean) submanifold: $M^{n} \subset \mathbb{R}^{n+p}$ (embedding!)
It is enough to check that: $\forall x \in M^{n}, \exists V \subset \mathbb{R}^{n+p}$ open, $x \in V$, and a smooth map $U \subset \mathbb{R}^{n} \mapsto M^{n} \cap V \subset \mathbb{R}^{n+p}$ that is injective, open and has rank $n$ : coordinates (smooth $=C^{r} / C^{\infty} / \overline{C^{w}}$ ).

Examples:
$\operatorname{graf}(f)$ for $f: U \subset \mathbb{R}^{n} \rightarrow \mathbb{R}$.
$g^{-1}\left(t_{0}\right)$ for a regular value $t_{0}$ (in the image) of $g: W \subset \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ : Sphere $\mathbb{S}^{n} \subset \mathbb{R}^{n+1}$.
Ellipsoid $g^{-1}(r)$ for $g(x, y, z)=x^{2} / a^{2}+y^{2} / b^{2}+z^{2} / c^{2}, r>0$. Hyperboloid $g^{-1}(r)$ for $g=x^{2}+y^{2}-z^{2}$ (Hyperboloid of two sheets for $r<0$, while the Cone $g^{-1}(0)$ is NOT a regular surface). Hyperbolic paraboloid $g^{-1}(0)$ for $g=x^{2} / a^{2}-y^{2} / b^{2}-z$. Circular cylinder and Cylinders over conics.

Def.: Parametrized surface just $\varphi: U \subset \mathbb{R}^{2} \rightarrow \mathbb{R}^{3}$. Singular points of $\varphi$ : $d \varphi_{p}$ singular. We call $\varphi$ regular if it is an immersion.

Proposition 1. Every hypersurface is locally a graph.

Exercise. No need to check that a coordinate system is open (homeomorphism) if we know beforehand that $M^{n} \subset \mathbb{R}^{n+1}$ is a regular hypersurface.

Differentiable functions: now we can use the ambient space.

Examples: $F: U \subset \mathbb{R}^{3} \rightarrow \mathbb{R}$ smooth $\left.\Rightarrow F\right|_{S}$ is smooth $\forall S \subset U$ $\varphi: U \subset \mathbb{R}^{2} \rightarrow \mathbb{R}^{3}$ coordinates $\Rightarrow U$ and $\varphi(U)$ are diffeomorphic $S$ symmetric $\Rightarrow$ the symmetry restricted to $S$ is smooth:

Example: Surfaces of revolution: Meridians, Parallels, Axis, Generatrix (embedded!).

Example: Tangent surfaces to a curve $\alpha: I \rightarrow \mathbb{R}^{3}: \kappa^{\alpha} \neq 0$ $\Rightarrow \varphi(s, t)=\alpha(s)+t \alpha(s)$ is a parametrized surface, regular for $t \neq 0$.

## §4. Tangent space as a subspace

For a regular submanifold $M^{n} \subset \mathbb{R}^{n+q}$ and $p \in M^{n}$, we now have $T_{p} M$ naturally included in $\mathbb{R}^{n+q}$ as an affine subspace: spanned by $\left\{\left(\partial \varphi / \partial u_{i}\right)(p): i=1, \ldots, n\right\}$ for any coordinate $\varphi$ at $p$.

If $\alpha: I \rightarrow S \subset \mathbb{R}^{3}, \alpha(s)=\varphi(u(s), v(s)), \alpha(0)=p=\varphi(0,0) \Rightarrow$

$$
\alpha^{\prime}(0)=u^{\prime}(0) \frac{\partial \varphi}{\partial u}(0,0)+v^{\prime}(0) \frac{\partial \varphi}{\partial v}(0,0) \in T_{p} S \subset \mathbb{R}^{3}
$$

Differential of a function $f: S_{1} \subset \mathbb{R}^{3} \rightarrow S_{2} \subset \mathbb{R}^{3}$ at a point: can be seen as a linear map between subspaces of $\mathbb{R}^{3}$.
Local diffeomorphism, chain rule...
TIP: Use curves to compute differentials!
Example: $L: \mathbb{R}^{3} \rightarrow \mathbb{R}^{3}$ linear, $S \subset \mathbb{R}^{3} \Rightarrow f_{* p}=\left.f\right|_{T_{p} S}$
Since we fixed an orientation in $\mathbb{R}^{3}$, we can talk about the Normal
vector field of our surface:

$$
\begin{equation*}
N=\frac{\frac{\partial \varphi}{\partial u} \wedge \frac{\partial \varphi}{\partial v}}{\left\|\frac{\partial \varphi}{\partial u} \wedge \frac{\partial \varphi}{\partial v}\right\|} \tag{1}
\end{equation*}
$$

Same holds for hypersurfaces.
Angle between surfaces.

## §5. The First Fundamental Form

Curves + inner product $\langle\cdot, \cdot\rangle$ of $\mathbb{R}^{3} \rightarrow$ distance $\rightarrow$ first fundamental form $I$ :

$$
I(p)=\left.\langle\cdot, \cdot\rangle\right|_{T_{p} S \times T_{p} S}: T_{p} S \times T_{p} S \rightarrow \mathbb{R}
$$

is an inner product on $T_{p} S$.
We will denote also by $I$ its associated quadratic form.
Arc length $s$ for $\alpha: I \rightarrow S \subset \mathbb{R}^{3}$ :

$$
s(t)=\int_{t_{0}}^{t} \sqrt{I\left(\alpha^{\prime}(r)\right)} d r
$$

$\varphi: U \rightarrow S \subset \mathbb{R}^{3}$ coordinate system $\Rightarrow$

$$
E=\left\|\frac{\partial \varphi}{\partial u}\right\|^{2}, F=\left\langle\frac{\partial \varphi}{\partial u}, \frac{\partial \varphi}{\partial v}\right\rangle, G=\left\|\frac{\partial \varphi}{\partial v}\right\|^{2} \in C^{\infty}(U)
$$

coefficients of $I$ in the coordinate $\varphi$.
$\operatorname{Im}(\alpha) \subset \operatorname{Im}(\varphi) \Rightarrow s(t)=\int_{t_{0}}^{t} \sqrt{E u^{\prime 2}+2 F u^{\prime} v^{\prime}+G v^{\prime 2}} d r$.
Remark 2. $I$ is a symmetric positive definite $(2,0)$-tensor in $S$, a Riemannian metric on $S: I=i n c^{*}\langle\cdot, \cdot\rangle$.

Same for arbitrary regular Euclidean submanifolds:
Example: Affine plane through $p \in \mathbb{R}^{3}$ or Cylinder over plane curve $\Rightarrow$ everywhere coordinate systems with $E \equiv 1 \equiv G, F \equiv 0$.

Def.: Regular domain $D \subset S$.
If $\varphi: U \rightarrow S$ is a coordinate system, $\left\|\frac{\partial \varphi}{\partial u} \wedge \frac{\partial \varphi}{\partial v}\right\|$ is the area of the parallelogram determined by the coordinate vector fields, and we can define the area of a regular domain $\Omega \subset \varphi(U)$ by

$$
A(\Omega)=\int_{\varphi^{-1}(\Omega)}\left\|\frac{\partial \varphi}{\partial u} \wedge \frac{\partial \varphi}{\partial v}\right\|=\int_{\varphi^{-1}(\Omega)} \sqrt{E G-F^{2}} d u d v
$$

since $\|x \wedge y\|^{2}+\langle x, y\rangle^{2}=\|x\|^{2}\|y\|^{2}$

## §6. Recalling basic concepts of vector bundles

## §7. Orientation

$\operatorname{graf}(f)$ orientable
$M^{n} \subset \mathbb{R}^{n+1}$ orientable $\Longleftrightarrow$ there exist a globally defined smooth unit normal vector field.
$g^{-1}(r)$ orientable $(0 \neq \operatorname{grad}(g) \perp M)$
Theorem 3. $\quad M^{n} \subset \mathbb{R}^{n+1}$ embedded, orientable $\Rightarrow$ there exist $V \subset \mathbb{R}^{n+1}$ open with $M^{n} \subset V$, and $g: V \subset \mathbb{R}$ such that 0 is a regular value of $g$ and $M^{n}=g^{-1}(0)$.
Proof: Existence of tubular neighborhoods.
Remark 4. $M^{n} \subset \mathbb{R}^{n+1}$ embedded and compact $\Rightarrow$ orientable (Jordan Theorem 41 in our last course).

## §8. Gauss map and Second Fundamental Form

For any Euclidean hypersurface $M^{n} \subset \mathbb{R}^{n+1}$, locally we have a unit normal vector field

$$
N: U \subset M^{n} \rightarrow \mathbb{S}^{n} \subset \mathbb{R}^{n+1}
$$

But $T_{p} M$ is parallel to $T_{N(p)} \mathbb{S}^{n}$, and hence $d N_{p} \in \operatorname{End}\left(T_{p} M\right)$. Moreover, $d N_{p}$ is self adjoint (w.r.t. $I$ ), so the quadratic form

$$
I I_{p}(w):=-\left\langle d N_{p} w, w\right\rangle
$$

is called the second fundamental form of $M^{n}$ at $p$. We also give the same name to the associated symmetric tensor,

$$
A_{p}:=-d N_{p} .
$$

Def.: Let $\alpha: I \rightarrow M$ be a regular curve through $p=\alpha(0) \in M$. The normal curvature of $\alpha$ at $p$ is given by

$$
\kappa_{n}:=\kappa\langle N, n\rangle,
$$

where $n$ is the normal vector of $\alpha$ at $p$ and $\kappa$ its curvature.
Remark 5 . (!!!!) It holds that (draw a picture)

$$
I I\left(\alpha^{\prime}(0)\right)=\kappa_{n}(0)
$$

i.e., the normal curvature only depends on the direction of $\alpha^{\prime}(!!)$. In particular, if $v \in T_{p} M$ and $\pi$ is the plane spanned by $v$ and $N(p), \alpha=\pi \cap M \subset \pi$ is a plane curve whose curvature is $I I(v)$ (beware of orientations).

Examples: Second fundamental form of a graph; $y=x^{4} ; \mathbb{S}^{n}$.
$A$ self adjoint $\Rightarrow$ principal curvatures $\left\{k_{i}\right\}$, principal directions $\left\{e_{i}\right\}$, lines of curvature:

$$
v=\sum_{i} v_{i} e_{i} \Rightarrow I I(v)=\sum_{i} k_{i} v_{i}^{2}
$$

For dimension 2, we have the Euler's formula for $\|v\|=1$ :

$$
v=\cos (\theta) e_{1}+\sin (\theta) e_{2} \Rightarrow I I(v)=k_{1} \cos (\theta)^{2}+k_{2} \sin (\theta)^{2}
$$

Remark 6. Ordering the principal curvatures $k_{1} \leq \cdots \leq k_{n}$ we see by Remark 5 that $e_{1}$ is the direction where $M$ "curves" less (w.r.t. $N$ ) in the ambient space, while $e_{n}$ is the one where it curves more. This follows from the usual diagonalization process.

## §9. The two curvatures for surfaces: $K$ and $H$

For a symmetric endomorphism in dimension two, we have two invariants (independent of orthonormal basis): the trace and the determinant.

Def.: For a regular surface $S \subset \mathbb{R}^{3}$ and $p \in S$, the Gaussian curvature of $S$ at $p$ is given by

$$
K(p):=\operatorname{det}\left(A_{p}\right)=k_{1} k_{2} .
$$

Def.: The mean curvature of $S$ at $p$ is $H(p)=-\operatorname{trace}\left(A_{p}\right) / 2$. We will see in a while that the two curvatures have very different nature.

Notice that $K$ does not depend on orientation, while $H$ does. Notice that both are smooth functions that determine $k_{1}$ and $k_{2}$ :

$$
k_{i}=H \pm \sqrt{H^{2}-K}
$$

Def.: A point $p$ in $S$ can be elliptic, hyperbolic, parabolic, planar (or totally geodesic), minimal, umbilical. Accordingly, $S$ itself could be totally geodesic, minimal, umbilical.

Remark 7. The principal curvatures and their eigenspaces are always continuous, and smooth along any open subset where their multiplicities are constant. In particular, they are always smooth along (the connected components of) an open dense subset $W$ of $M$. For surfaces, if $V$ is the set of umbilical points of $S, W$ can be taken as $V^{o} \cup S \backslash V$.

Def.: Asymptotic direction and asymptotic curve of $S \subset \mathbb{R}^{3}$. Notice that there exists an asymptotic direction at $p \Longleftrightarrow$ $K(p) \leq 0$, while there are precisely two asymptotic directions at $p \Longleftrightarrow K(p)<0$.

Proposition 8. Let $M^{n} \subset \mathbb{R}^{n+1}$ regular and connected. Then, $M$ is umbilical $\Longleftrightarrow M^{n}$ is (an open subset of) a round $n$-sphere or a hyperplane.

Def.: Conjugate directions.

## §10. $I I$ and $K$ in coordinates, part 1

For a coordinate system $\varphi=\varphi(u, v)$ in a surface $S \subset \mathbb{R}^{3}$, let as before

$$
N=\frac{\frac{\partial \varphi}{\partial u} \wedge \frac{\partial \varphi}{\partial v}}{\left\|\frac{\partial \varphi}{\partial u} \wedge \frac{\partial \varphi}{\partial v}\right\|} .
$$

Denote by $\left(a_{i j}\right)$ the matrix of $A$ in the coordinate basis,

$$
\begin{aligned}
& N_{u}=a_{11} \varphi_{u}+a_{12} \varphi_{v}, \\
& N_{v}=a_{21} \varphi_{u}+a_{22} \varphi_{v}
\end{aligned}
$$

Define the functions

$$
\begin{gathered}
e:=-\left\langle N_{u}, \varphi_{u}\right\rangle=\left\langle N, \varphi_{u u}\right\rangle, \\
g:=-\left\langle N_{v}, \varphi_{v}\right\rangle=\left\langle N, \varphi_{v v}\right\rangle, \\
f:=-\left\langle N_{u}, \varphi_{v}\right\rangle=-\left\langle N_{v}, \varphi_{u}\right\rangle=\left\langle N, \varphi_{u v}\right\rangle .
\end{gathered}
$$

Hence, if $v=v_{1} \varphi_{u}+v_{2} \varphi_{v}, I I(v)=e v_{1}^{2}+2 f v_{2} v_{2}+g v_{2}^{2}$, and

$$
\left(\begin{array}{ll}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{array}\right)=\frac{-1}{E G-F^{2}}\left(\begin{array}{ll}
e & f \\
f & g
\end{array}\right)\left(\begin{array}{cc}
G & -F \\
-F & E
\end{array}\right)
$$

These are known as the Weingarten equations. In particular, for the Gaussian curvature we obtain

$$
K=\frac{e g-f^{2}}{E G-F^{2}} .
$$

Example: The torus. For $0<r<a$, take the (almost global) chart

$$
\varphi(u, v)=((a+r \cos (u)) \cos (v),(a+r \cos (u)) \sin (v), r \sin (u)) .
$$

Hence, $N=(\cos (u) \cos (v), \cos (u) \sin (v), \sin (u))$ (geometrically!), and hence $E=r^{2}, G=(a+r \cos (u))^{2}, F=0$, and

$$
K=\frac{\cos (u)}{r(a+r \cos (u))}
$$

Make the computation, make a picture, sign $K$ interpretation, elliptic/hyperbolic points, move $a$ and $r$ and see how $K$ varies, independence of $v \ldots$ see everything geometrically!

Remark 9. Observe that $\int K=0$, independently of $a$ and $r$ !
Now, let's compute $K$ for any surface of revolution of a simple closed curve p.b.a.l. as $(a(s), 0, b(s)): K=-a^{\prime \prime} / a, \int K=0!!!!$

Proposition 10. If $p \in S \subset \mathbb{R}^{3}$ is elliptic $\Rightarrow$ a neighborhood of $p$ is in one side of $T_{p} S$. If $p$ hyperbolic, it is not.

Proof: Differentiate $g=\langle\varphi-\varphi(0), N(p)\rangle$ at $p=\varphi(0)$.
Def.: Lines of curvature.
If $\alpha(t)=\varphi(u(t), v(t))$ is a line of curvature, then $\left.d N\left(\alpha^{\prime}\right)\right)=\lambda \alpha^{\prime}$, or equivalently, $(f F-e G) u^{\prime}+(g F-f G) v^{\prime}=\lambda u^{\prime}\left(E G-F^{2}\right)$, $(e F-f E) u^{\prime}+(f F-g E) v^{\prime}=\lambda v^{\prime}\left(E G-F^{2}\right)$, or

$$
(f E-e F)\left(u^{\prime}\right)^{2}+(g E-e G) u^{\prime} v^{\prime}+(g F-f G)\left(v^{\prime}\right)^{2}=0
$$

known as the equation of the lines of curvature. In particular, outside of the umbilical points:
the chart is by lines of curvature $\Leftrightarrow F=f=0$.

Proposition 11. Let $p \in S \subset \mathbb{R}^{3}$, and a sequence of compact regions $B_{i} \subset S$ such that $B_{i} \rightarrow p$. Then,

$$
|K(p)|=\lim _{i \rightarrow \infty} \frac{\operatorname{Area}\left(N\left(B_{i}\right)\right)}{\operatorname{Area}\left(B_{i}\right)} .
$$

Proof: Follows from $\left|N_{u} \wedge N_{v}\right|=|K|\left|\varphi_{u} \wedge \varphi_{v}\right|$, even if $K(p)=0$.
Remark 12. At the non-flat points, $N$ preserves orientation if and only if $K>0$. Hence we can remove the modulus if we define "oriented area".

## §11. Vector fields

Recall: Trajectories (= integral curves), F.T. ODE, local flux. $\mathfrak{X}(M)=\Gamma(T M)$
For $f: N \rightarrow M, \mathfrak{X}_{f}=\Gamma\left(f^{*}(T M)\right)$.
Exercise. If $X, Y \in \mathfrak{X}(M)$ and $\xi$ is the local flux of $X$ around $p$, then $[X, Y](p)=\lim _{t \rightarrow 0} \frac{1}{t}\left(\left(\xi_{-t}\right)_{*} Y\left(\xi_{t}(p)\right)-Y(p)\right)$.

Lemma 13. Let $M$ be any manifold, $p \in M$ and $X \in \mathfrak{X}(M)$ with $X(p) \neq 0$. Then, there is a coordinate system around $p$ such that $\left.X\right|_{U}=\partial / \partial x_{1}$. In particular, if $M$ is a surface, there is a first integral of $X$ in $U$, i.e., a function $f: U \rightarrow \mathbb{R}$ with $d f_{p} \neq 0$ such that $f$ is constant along trajectories of $X$. Proof: Use the flux of $X$ to construct a suitable chart in $p$ for which $\left.X\right|_{U}$ is a coordinate vector field.

Lemma 14. Same hypothesis as in Lemma 13, and $g \in$ $\mathcal{F}(U) \Rightarrow$ there is $\mu \in \mathcal{F}(U), \mu>0$, such that $X(\mu)=g$.

Proof: Use the flux of $X$ to write this as an ODE.
Lemma 15. Two vector fields $X, Y$ satisfy that $[X, Y]=0$ if and only if their fluxes commute: $\phi_{t}^{X} \circ \phi_{s}^{Y}=\phi_{s}^{Y} \circ \phi_{t}^{X} \forall t, s$. Proof: At points where $X=Y=0$ it is obvious, so assume $X \neq$ 0 . By the ' $\varphi$-related' property for Lie brackets of vector fields, we can assume our manifold is $\mathbb{R}^{n}$. Moreover, by Lemma [13, we can assume $X=\partial / \partial x_{1} \cong e_{1}$, and the lemma follows easily.

Lemma 16. Same hypothesis as in Lemma 13 and $Y \in \mathfrak{X}(S)$ linearly independent with $X$ in $p \Rightarrow$ there is a chart $\varphi$ around $p$ whose coordinate vector fields are colinear with $X$ and $Y$.

Proof: Write $[X, Y]=g X+f Y$ and use Lemma 14 to find $\mu, \lambda$ positive functions such that $\left[X^{\prime}, Y^{\prime}\right]=0$, where $X^{\prime}=\mu X$, $Y^{\prime}=\lambda Y$. By Lemma 15 we can use them to build our chart.

Corollary 17. If $p$ is not an umbilical point of $S \subset \mathbb{R}^{3}$, there is a coordinate system by lines of curvature around $p$.

Def.: Isometric and conformal maps, local isometries.
Proposition 18. Any surface ( $S, I$ ) has isothermal charts.
Proof: Several proofs exist, few are elementary, and none easy... ■

## §12. Ruled surfaces

$\varphi(t, s)=\alpha(s)+t v(s), v \in \mathfrak{X}_{\alpha},\|v\|=1$.
$<\{v(s)\}>=$ geratrix line, $\alpha=$ directrix curve.
Remark 19. $S$ ruled $\Rightarrow K \leq 0$.

## Examples:

- $v=\alpha^{\prime}$ : Tangent surface to $\alpha$
- $\alpha(s)=(0,0, a s), v(s)=(\cos (s), \sin (s), 0) \Rightarrow$ helicoid.
- If $v \equiv v_{0}$ constant $\Rightarrow$ cylinder over a plane curve. Hence, we say that $S$ is noncylindrical if $v^{\prime}$ never vanishes.
- $\alpha(s)=(\cos (s), \sin (s), 0), v= \pm \alpha^{\prime}+e_{3} \Rightarrow x^{2}+y^{2}-z^{2}=1$ : the hyperboloid of revolution is doubly ruled.
- $\alpha(s)=(s, 0,0), v(s)=(0,1, s) \Rightarrow x y=z$, the hyperbolic paraboloid, that is also doubly ruled, since $(s, t, s t)=$ $t e_{2}+s(1,0, t)$.

Remark 20. Besides the plane, these are the only 2 doubly ruled surfaces! How would you prove this??

Singularities of a ruled surface are contained in the striction curve:
Def.: For a noncylindrical ruled surface $S$, the striction curve is given by $\sigma(s)=\alpha(s)+t(s) v(s)$ for which $\left\langle\sigma^{\prime}, v^{\prime}\right\rangle=0$ (i.e., $\left.t(s)=-\left\langle\alpha^{\prime}, v^{\prime}\right\rangle /\left\|v^{\prime}\right\|^{2}\right)$.

Notice that the striction curve does not depend on the directrix $\alpha$. In particular, we can assume that $\sigma=\alpha$, that is, $\left\langle\alpha^{\prime}, v^{\prime}\right\rangle=0$.

## §13. Minimal surfaces

The brachistochrone problem was formally posed by Johann Bernoulli as a challenge (he knew the answer using the Fermat Principle), but it appeared first in the Discorsi, of Galileo for lines, and observed that there was a quicker non straight solution (arguing then wrongly that the circle would be the fastest). Leib-
niz persuaded Bernoulli to extend the six month limit to solve the challenge for foreign mathematicians to be able to participate. Five more mathematicians solved the problem: Tschirnhaus, Jacob Bernoulli, Leibniz, de L'Hôpital, and... Isaac Newton, who was teased by Bernoulli and Leibniz, and solved the problem in one night. These solutions eventually lead to a general method by Euler to solve these kind of problems: the calculus of variations. When does a surface minimize the area for "close enough" surfaces?

Proposition 21. Let $S \in \mathbb{R}^{3}$ be a compact surface (with or without boundary). Then, $S$ is a critical point of the area functional $A(S)$ if and only if $H=0$.

Proof: It is enough to consider normal variations $i_{t}(p)=p+$ $t f N(p)$. Now, compute $a^{\prime}(0)$, where $a(t)=\operatorname{Area}\left(i_{t}(S)\right)$.

Exercise. Conclude the same for hypersurfaces adapting the proof using that the volume element is given by $\sqrt{\operatorname{det}\left\langle\varphi_{u_{i}}, \varphi_{u_{j}}\right\rangle} d u_{1} \wedge \cdots \wedge d u_{n}$.
The Plateau problem: Find a minimal surface whose boundary is a given closed curve. Douglas (1931) and Radó (1933) prove general existence for arbitrary simple closed curves, but the surface could have singularities. Osserman (1970) and Gulliver (1973): a minimizing solution cannot have singularities. Regular solutions may not exist.

CMC (hyper)surfaces.

## §14. Intrinsic Geometry

Intrinsic objects of a Riemannian manifold are the ones that only depend on the first fundamental form, i.e., invariant by isometries: distance, angle, area, volume...

Cylinder $\cong$ plane $\cong$ cone (locally): they are intrinsically the same thing, and hence the mean curvature $H$ is not an intrinsic concept. If $\varphi: U \rightarrow M, \varphi^{\prime}: U \rightarrow M^{\prime}$ are charts such that $g_{i j}=g_{i j}^{\prime} \Rightarrow$ $\varphi(U) \subset M$ and $\varphi^{\prime}(U) \subset M^{\prime}$ are isometric.

Example: For a surface of revolution

$$
\varphi(u, v)=(f(v) \cos (u), f(v) \sin (u), g(v))
$$

we have $E=f^{2}, F=0, G=f^{\prime 2}+g^{\prime 2}$. In particular, the catenoid, where $f(v)=a \cosh (v)$ and $g(v)=a v$ for $a>0$, has $E=G=a^{2} \cosh ^{2}(v), F=0$. Now, change variables on the helicoid

$$
\varphi(\bar{u}, \bar{v})=\bar{v}(\cos (\bar{u}), \sin (\bar{u}), 0)+a \bar{u} e_{3},
$$

$\bar{v}=a \sinh (v), \bar{u}=u$ to get

$$
\bar{\varphi}(u, v)=a(\sinh (v) \cos (u), \sinh (v) \sin (u), u),
$$

that also has $E=G=a^{2} \cosh ^{2}(v), F=0$. Therefore, the catenoid and the helicoid are locally isometric (but not globally).

This is a general phenomenon for minimal surfaces in $\mathbb{R}^{3}$ : they have a one parameter family of isometric deformations.
Def.: Isometries, conformal diffeomorphisms.
Existence of isothermal coordinates $\Rightarrow$ any two surfaces are locally conformally equivalent.

## §15. The Gaussian curvature in coordinates, part 2

Given a chart $\varphi=\varphi\left(u_{1}, u_{2}\right): U \rightarrow S$ on a surface $S \subset \mathbb{R}^{3}$, we have seen that $K=\left(e g-f^{2}\right) /\left(E G-F^{2}\right)$. Let's do this computation again in other way, by decomposing the second derivatives on their tangent and normal components:

$$
\varphi_{i j}=\sum_{k} \Gamma_{i j}^{k} \varphi_{k}+r_{i j} N
$$

The functions $\Gamma_{i j}^{k}$ (that of course depend on $\varphi$ ) are called the Christoffel symbols. (In our previous notation, $r_{11}=e, r_{22}=$ $\left.g, r_{12}=f\right)$. Taking inner product with $\varphi_{i}$, we have:

$$
\begin{gathered}
\Gamma_{11}^{1} E+\Gamma_{11}^{2} F=\left\langle\varphi_{11}, \varphi_{1}\right\rangle=\frac{1}{2} E_{u_{1}} \\
\Gamma_{11}^{1} F+\Gamma_{11}^{2} G=\left\langle\varphi_{11}, \varphi_{2}\right\rangle=F_{u_{1}}-\frac{1}{2} E_{u_{2}}
\end{gathered}
$$

that can be written as

$$
\left(\begin{array}{cc}
E & F \\
F & G
\end{array}\right)\binom{\Gamma_{11}^{1}}{\Gamma_{11}^{2}}=\binom{\frac{1}{2} E_{u_{1}}}{F_{u_{1}}-\frac{1}{2} E_{u_{2}}}
$$

and similarly for the other indexes. In other words, we have:
Proposition 22. The Christoffel symbols $\Gamma_{i j}^{k}$ depend only on the first fundamental form and its first derivatives.

Proof: Follows from the Koszul formula:

$$
2\left\langle\varphi_{i j}, \varphi_{k}\right\rangle=\left\langle\varphi_{i}, \varphi_{k}\right\rangle_{j}+\left\langle\varphi_{j}, \varphi_{k}\right\rangle_{i}-\left\langle\varphi_{i}, \varphi_{j}\right\rangle_{k}
$$

Exercise. Show that for a surface of revolution around a curve $\alpha(v)=$ $(f(v), g(v))$, it holds that $\Gamma_{11}^{1}=\Gamma_{12}^{2}=\Gamma_{22}^{1}=0, \Gamma_{11}^{2}=f f^{\prime} /\left(f^{\prime 2}+g^{\prime 2}\right), \Gamma_{12}^{1}=$ $f^{\prime} / f, \Gamma_{22}^{2}=\left(f f^{\prime \prime}+g g^{\prime \prime}\right) /\left(f^{\prime 2}+g^{\prime 2}\right)$.

Now, we get relations that come from taking the tangent and normal components of $\varphi_{r i j}=\varphi_{r j i}$ and $N_{i j}=N_{j i}(3 \times 3$ equations if $n=2$ ) that have the form

$$
\begin{equation*}
\sum_{k=1}^{n} c_{i j r}^{k} \varphi_{k}+d_{i j r}^{k} N=0, \quad \forall 1 \leq i, j, r \leq n=2 \tag{2}
\end{equation*}
$$

In particular, taking the $\varphi_{2}$ component of $\varphi_{112}=\varphi_{121}$ we obtain the Gauss equation
$K=\frac{1}{E}\left(\left(\Gamma_{12}^{2}\right)_{1}-\left(\Gamma_{11}^{2}\right)_{2}+\Gamma_{12}^{1} \Gamma_{11}^{2}+\Gamma_{12}^{2} \Gamma_{12}^{2}-\Gamma_{11}^{2} \Gamma_{22}^{2}-\Gamma_{11}^{1} \Gamma_{12}^{2}\right)$.
We have proved the famous Gauss' Egregium Theorem:
Theorem 23 (Theorema Egregium = "outstanding"). The Gaussian curvature is an intrinsic concept (in fact, it depends only on $I, \partial I$, and $\partial^{2} I$ ). In particular:

$$
\underline{K} \text { is invariant by (local) isometries. }
$$

## Corollary 24. $K_{\text {catenoid }}(p)=K_{\text {helicoid }}(\xi(p))$.

## §16. The Codazzi-Mainardi equations

The other 5 tangential equations are other ways of writting the Gauss equation, or give $0=0$. But the normal components give
two more equations, called the Codazzi-Mainardi equations:

$$
\begin{aligned}
e_{v}-f_{u} & =e \Gamma_{12}^{1}+f\left(\Gamma_{12}^{2}-\Gamma_{11}^{1}\right)-g \Gamma_{11}^{2} . \\
f_{v}-g_{u} & =e \Gamma_{22}^{1}+f\left(\Gamma_{22}^{2}-\Gamma_{12}^{1}\right)-g \Gamma_{12}^{2} .
\end{aligned}
$$

Application: The relative nullity integrate as straight lines. These are the straight lines we 'see' in some ruled surfaces: $K \equiv 0$.

Remark 25. In a coordinate system by lines of curvature of a surface without umbilic points, Codazzi-Mainardi equations have the form

$$
e_{v}=\frac{E_{v}}{2}\left(\frac{e}{E}+\frac{g}{G}\right), \quad g_{u}=\frac{G_{u}}{2}\left(\frac{e}{E}+\frac{g}{G}\right) .
$$

## §17. Global application: The rigidity of $\mathbb{S}^{2} \subset \mathbb{R}^{3}$

Lemma 26. Let $p \in S \subset \mathbb{R}^{3}$ regular such that $K(p)>0$, and $p$ is a local maximum of $k_{2}$ and a local minimum of $k_{1}$ ( $k_{1} \leq k_{2}$ ). Then, $p$ is umbilic.
Proof: Assume not, $k_{1}(p)<k_{2}(p)$, and take a chart at $p$ by lines of curvature, $2 H=k_{1}+k_{2} \Rightarrow k_{1}=e / E, k_{2}=g / G$. Now, by Remark 25 (Codazzi), $e_{v}=E_{v} H, g_{u}=G_{u} H \Rightarrow E\left(k_{1}\right)_{v}=$ $e_{v}-e E_{v} / E=E_{v}\left(k_{2}-k_{1}\right) / 2$, and $G\left(k_{2}\right)_{u}=-G_{u}\left(k_{2}-k_{1}\right) / 2$. In particular, $E_{v}(p)=G_{u}(p)=0$. But the Exercise in page 31 says that $-2 K E G=E_{v v}+G_{u u}+(\cdots) E_{v}+(\cdots) G_{u}$. Then, at $p$, $0>-2 K E G=E_{v v}+G_{u u}=2\left(E\left(k_{1}\right)_{v v}-G\left(k_{2}\right)_{u u}\right) /\left(k_{2}-k_{1}\right) \geq 0$. Theorem 27. (Liebman): If $S \subset \mathbb{R}^{3}$ is a regular connected compact surface with $K=$ constant $\Rightarrow S$ is a round sphere (i.e., $S$ is umbilic).

Proof: Notice that $K>0$ by compactness. Now, consider the minimum of $k_{1}$, and apply Lemma 26.

The same result follows with $H=$ constant (Alexandrov). But a weaker version of it (for $K>0$ ) already follows exactly as above:

Theorem 28. If $S \subset \mathbb{R}^{3}$ is a regular connected compact surface with $K>0$ and $H=$ constant $\Rightarrow S$ is a round sphere (i.e., $S$ is umbilic).

## $\S 18$. The Fundamental Theorem of surfaces in $\mathbb{R}^{3}$

We have seen that $S \subset \mathbb{R}^{3} \Rightarrow$ Gauss eq. (intrinsic) + Codazzi equations (extrinsic). It turns out that there is no more information, or, equivalently, the converse holds locally:

Theorem 29 (FTS: Bonnet). Let E, F, G, e,f,g be differentiable functions on $V \subset \mathbb{R}^{2}$ with $E, G>F^{2}$ that satisfy Gauss and Codazzi-Mainardi equations. Then, each $q \in$ $V$ has a neighborhood $q \in U \subset V$ and a diffeomorphism $\varphi: U \rightarrow \varphi(U) \subset \mathbb{R}^{3}$ such that $E, F, G$ and $e, f, g$ are the coefficients of the first and second fundamental forms of $\varphi(U)$, respectively, in the chart $\varphi$. In addition, if $U$ is connected and $\bar{\varphi}$ is another chart with the same $E, F, G, e, f, g$, then there is a rigid motion $T \in \operatorname{Iso}\left(\mathbb{R}^{3}\right)$ such that $\bar{\varphi}=T \circ \varphi$.

Proof: (Sketch). For a chart $\varphi$, we define $f:=\left(\varphi_{u}, \varphi_{v}, N\right)$ : $V \rightarrow G L(3, \mathbb{R})$ where $N$ is given by ( $\mathbb{I})$. Hence, there are two functions $P, Q: V \rightarrow \mathbb{R}^{3 \times 3}$ such that $f_{u}=f P, f_{v}=f Q$. Gauss and Codazzi equations are precisely the integrability conditions
of this first order system of PDE: $P_{v}-Q_{u}=[P, Q]$ (Frobenius Theorem). Integrating once more, we get $\varphi$, and it is easy to check that it is a surface with the desired first and second fundamental forms (for details, see R. Palais notes here).

Remark 30. Take a long time comparing this with the FTC.

## §19. The covariant derivative: affine connections

We want to differentiate vector fields on our surface (submanifold) $S \subset \mathbb{R}^{3}$. For this, we agree that, if $L \subset \mathbb{R}^{m}$ is a subspace and $v \in L,(v)_{L}$ denotes the orthogonal projection of $v$ onto $L$.

Definition 31. Given $X \in \mathfrak{X}(S)$ and $v \in T_{p} S$, we define the covariant derivative of $X$ in the direction $v$ by

$$
\nabla_{v} X=\left(X_{* p}(v)\right)_{T_{p} S} .
$$

- $\nabla$ is an intrinsic operator: depends only on $\Gamma_{i j}^{k}$;
- $\nabla$ coincides with the usual derivative for $S=\mathbb{R}^{n}$, since $\Gamma_{i j}^{k}=0$;
- $\nabla_{v} X$ is linear in $v$ (tensorial!). So, we define for each $Y \in \mathfrak{X}(S)$ the vector field $\nabla_{Y} X \in \mathfrak{X}(S)$ by

$$
\left(\nabla_{Y} X\right)(p):=\nabla_{Y(p)} X
$$

and this is tensorial in $Y: \nabla_{f Y} X=f \nabla_{Y} X$;

- $\nabla_{v} X$ is a derivation in $X: \forall f \in \mathcal{F}(S), X \in \mathfrak{X}(S), v \in T_{p} S$,

$$
\nabla_{v} f X=v(f) X(p)+f(p) \nabla_{v} X
$$

or, for $Y \in \mathfrak{X}(S)$,

$$
\nabla_{Y} f X=Y(f) X+f \nabla_{Y} X
$$

a) In other words, we have:

$$
\nabla: \mathfrak{X}(S) \times \mathfrak{X}(S) \rightarrow \mathfrak{X}(S)
$$

$(Y, X) \mapsto \nabla_{Y} X$, that is tensorial in $Y$ and a derivation in $X$;
b) $\nabla$ is symmetric:

$$
\nabla_{X} Y-\nabla_{Y} X=[X, Y], \quad \forall X, Y \in \mathfrak{X}(M)
$$

(it is enough to check for $X=\partial_{i}, Y=h \partial_{j}$ for any function $h$ )
c) $\nabla$ is compatible with the metric:

$$
X\langle Y, Z\rangle=\left\langle\nabla_{X} Y, Z\right\rangle+\left\langle Y, \nabla_{X} Z\right\rangle, \quad \forall X, Y, Z \in \mathfrak{X}(M)
$$

- Such an operator satisfying $(a)+(b)+(c)$ always exists and is unique by the Koszul formula:

$$
\begin{aligned}
2\left\langle\nabla_{X} Y, Z\right\rangle= & X\langle Y, Z\rangle+Y\langle X, Z\rangle-Z\langle X, Y\rangle \\
& -\langle X,[Y, Z]\rangle-\langle Y,[X, Z]\rangle+\langle Z,[X, Y]\rangle .
\end{aligned}
$$

In the realm of Riemannian Geometry, $\nabla$ is called the Levi-Civita connection of $(S,\langle\rangle$,$) .$

## §20. Affine connections in vector bundles

Now, observe that, to have an affine connection (i.e., property (a) only), all we need is the vector bundle structure on the second variable, and not necessarily TS. Hence, we have:

Definition 32. Given a vector bundle $\pi: E \rightarrow M$, an affine connection in $E$ is an $\mathbb{R}$-bilinear operator

$$
\nabla: \mathfrak{X}(M) \times \Gamma(E) \rightarrow \Gamma(E)
$$

$(Y, \xi) \mapsto \nabla_{Y} \xi$, that is tensorial in $Y$ and a derivation in $\xi$ :

$$
\begin{gathered}
\nabla_{Y} f \xi=Y(f) \xi+f \nabla_{Y} \xi, \\
\nabla_{f Y} \xi=f \nabla_{Y} \xi
\end{gathered}
$$

$\forall Y \mathfrak{X}(M), f \in \mathcal{F}(M), \xi \in \Gamma(E)$.
Remark 33. Bump functions + local sections $\Rightarrow$ affine connections are first order differential operators. In particular, they are local: computations can be done in coordinates or local sections.

Therefore, if $X, Y \in \mathfrak{X}(M)$ and $\varphi: U \rightarrow M$ is a chart, we write on $V=\varphi(U),\left.X\right|_{V}=\sum_{i} x_{i} \frac{\partial}{\partial u_{i}},\left.\quad Y\right|_{V}=\sum_{i} y_{i} \frac{\partial}{\partial u_{i}}$, and since $\nabla_{\frac{\partial}{\partial u_{i}}} \frac{\partial}{\partial u_{j}}=\sum_{k} \Gamma_{i j}^{k} \frac{\partial}{\partial u_{k}}$, in $V$ we get for the Levi-Civita connection

$$
\begin{equation*}
\left.\left(\nabla_{X} Y\right)\right|_{V}=\sum_{k}\left(\sum_{i} x_{i} \frac{\partial y_{k}}{\partial u_{i}}+\sum_{i j} x_{i} y_{j} \Gamma_{i j}^{k}\right) \frac{\partial}{\partial u_{k}} . \tag{3}
\end{equation*}
$$

Remark 33 also implies:
Proposition 34. Suppose $E$ is a vector bundle with a connection $\hat{\nabla}$. Then, for each smooth map $f: N \rightarrow M$, there is a unique pull-back connection $\hat{\nabla}^{f}$ on $f^{*} E$ satisfying that

$$
\hat{\nabla}_{X}^{f}(\xi \circ f)=\hat{\nabla}_{f_{*} X} \xi, \quad \forall X \in \mathfrak{X}(N), \xi \in \Gamma(E) .
$$

Proof: Since connections are local objects, it is enough to do the computation locally. But if $\left\{\xi_{i}\right\}$ is a local frame of $E$ in $U \subset M$, $\left\{\xi_{i} \circ f\right\}$ is a local frame of $f^{*} E$ in $V=f^{-1}(U)$. So, if $\eta \in \Gamma\left(f^{*} E\right)$,
we write $\eta$ on $V$ as $\left.\eta\right|_{V}=\sum_{i} z_{i} \xi_{i} \circ f$. Just by the definition of a connection and its local nature, on $V$ we get

$$
\begin{equation*}
\hat{\nabla}_{X}^{f} \eta=\sum_{i}\left(X\left(z_{i}\right) \xi_{i} \circ f+z_{i} \hat{\nabla}_{f_{*} X} \xi_{i}\right) . \tag{4}
\end{equation*}
$$

This implies the uniqueness of $\hat{\nabla}^{f}$. But we can define $\hat{\nabla}^{f}$ locally with (4): it is easy to check that $\hat{\nabla}^{f}$ defined this way is well defined, and a connection.

## §21. Affine connections along maps

If $(M,\langle\rangle$,$) is a Riemannian manifold, the ONLY affine connec-$ tion $\nabla$ on $T M$ that we will consider is the Levi-Civita connection of $\langle$,$\rangle . If M$ is an Euclidean submanifold, we know how to construct $\nabla$ from the standard vector field derivative on $\mathbb{R}^{m}$ (that is itself the Levi-Civita connection of $\mathbb{R}^{m}$ with the standard inner product seen as a Riemannian metric).
As a particular case of Proposition 34, we have:
Proposition 35. Given $f: N \rightarrow(M,\langle\rangle$,$) , there is a unique$ affine connection $\nabla^{f}\left(=f^{*} \nabla\right)$ in $f^{*}(T M)$,

$$
\nabla^{f}: \mathfrak{X}(N) \times \mathfrak{X}_{f} \rightarrow \mathfrak{X}_{f}
$$

called the affine connection along $f$, that satisfies

$$
\nabla_{X}^{f}(Y \circ f)=\nabla_{f_{*} X} Y, \quad \forall X \in \mathfrak{X}(N), Y \in \mathfrak{X}(M)
$$

In particular, for a curve $\alpha: I \rightarrow M$, we obtain:

- A notation: if $X \in \mathfrak{X}_{\alpha}$,

$$
X^{\prime}:=\nabla_{d / d t} X
$$

- We have the acceleration of $\alpha$ (intrinsic!):

$$
\alpha^{\prime \prime}:=\nabla_{d / d t} \alpha^{\prime},
$$

- and the geodesic curvature of $\alpha$ (intrinsic!):

$$
\kappa_{g}=\kappa_{g}^{\alpha}:=\left\|\alpha^{\prime \prime}\right\| .
$$

- "Compute derivatives using curves": For any $Z \in \mathfrak{X}(M) \Rightarrow$ $Z \circ \alpha \in \mathfrak{X}_{\alpha}$ and

$$
(Z \circ \alpha)^{\prime}=\nabla_{d / d t}(Z \circ \alpha)=\nabla_{\alpha^{\prime}} Z .
$$

- For curves in a submanifold, $\alpha: I \rightarrow M \subset \mathbb{R}^{m}$, if $X \in \mathfrak{X}_{\alpha}$, $Z \in \mathfrak{X}(M)$, and $v=\alpha^{\prime}(0) \in T_{p} M$, we have:

$$
\begin{gather*}
X^{\prime}:=\nabla_{d / d t} X=\left(\frac{d X}{d t}\right)_{T_{\alpha} M} \in \mathfrak{X}_{\alpha} \\
\alpha^{\prime \prime}=\left(\frac{d^{2} \alpha}{d t^{2}}\right)_{T_{\alpha} M} \in \mathfrak{X}_{\alpha} \\
\nabla_{v} Z=\left(\left.\frac{d}{d t}\right|_{t=0}(Z \circ \alpha)\right)_{T_{p} M} \tag{5}
\end{gather*}
$$

and the three curvatures of $\alpha$ are related by

$$
\kappa^{2}=\kappa_{g}^{2}+\kappa_{n}^{2} .
$$

- Eq. (5) also implies that: if two submanifolds are tangent along a curve $\alpha$, their connections along $\alpha$ coincide.
- If $M$ is a hypersurface, $(v)_{T M}=v-\langle v, N\rangle N$, and (5) becomes

$$
\nabla_{v} Z=\left.\frac{d}{d t}\right|_{t=0}(Z \circ \alpha)-\left\langle A_{p} v, Z(p)\right\rangle N(p) .
$$

Exercise. A connection is compatible with a metric $\langle,\rangle \Leftrightarrow\langle V, W\rangle^{\prime}=$ $\left\langle V^{\prime}, W\right\rangle+\left\langle V, W^{\prime}\right\rangle$, for ever curve $\alpha$ and every $V, W \in \mathfrak{X}_{\alpha}$ (notice that these are different "'").

## §22. Parallel transport

By (3), if we write a curve $\alpha$ locally as $\alpha=\varphi\left(\alpha_{1}, \ldots, \alpha_{n}\right)$, and $Y=\sum_{i} y_{i} \frac{\partial}{\partial u_{i}} \circ \alpha \in \mathfrak{X}_{\alpha}$, then

$$
\begin{equation*}
Y^{\prime}=\sum_{k}\left(y_{k}^{\prime}+\sum_{i j} \alpha_{i}^{\prime} y_{j} \Gamma_{i j}^{k} \circ \alpha\right) \frac{\partial}{\partial u_{k}} \circ \alpha \tag{6}
\end{equation*}
$$

We say that $Y \in \mathfrak{X}_{\alpha}$ is parallel if $Y^{\prime}=0$. Since the last equation is linear, the set of parallel vector fields along $\alpha$, denoted by $\mathfrak{X}_{\alpha}^{\|}$, is a vector space. Also by this equation we easily see:

Proposition 36. Given a curve $\alpha: I \rightarrow M, p=\alpha\left(t_{0}\right)$, for every $v \in T_{p} M$ there exists a unique parallel vector field $\mu_{v} \in \mathfrak{X}_{\alpha}$ such that $\mu_{v}\left(t_{0}\right)=v$.
So, this map $v \mapsto \mu_{v}$ is an isomorphism between $T_{p} M$ and $\mathfrak{X}_{\alpha}^{\|}$. In particular, if $t$ is another point in $I$, we get a linear isomorphism

$$
P_{t_{0}, t}^{\alpha}: T_{\alpha\left(t_{0}\right)} M \rightarrow T_{\alpha(t)} M
$$

given by $P_{t_{0}, t}^{\alpha}(v)=\mu_{v}(t)$. Notice that it depends smoothly on everything.

Def.: This isomorphism $P_{t_{0}, t}^{\alpha}$ is called the parallel transport along $\alpha$ between $\alpha\left(t_{0}\right)$ and $\alpha(t)$.

Beware: along $\alpha$ !! It does depend on $\alpha$, not just on $\alpha\left(t_{0}\right)$ and $\alpha(t)$ (in contrast to $\mathbb{R}^{m}$ ).

Examples: $M=\mathbb{R}^{m}$ : usual. Meridian in $\mathbb{S}^{2}$ : Cylinder. Parallel in $\mathbb{S}^{2}$ : Cone $\Rightarrow$ after a complete turn, the parallel transport does not close $\Rightarrow$ dependency on $\alpha$.

Remark 37. By the previous exercise $P_{t_{0}, t}^{\alpha}$ are linear isometries.
Exercise. Prove that a connection is compatible with the metric $\langle,\rangle \Leftrightarrow$ $\langle V, W\rangle$ is constant, for every curve $\alpha$ and every $V, W \in \mathfrak{X}_{\alpha}^{\|}$.

## §23. Geodesics

Lemma 38. Given $\varphi: U \subset \mathbb{R}^{2} \rightarrow M \Rightarrow \nabla_{\partial_{u}} \varphi_{v}=\nabla_{\partial_{v}} \varphi_{u}$.
Proof: Use coordinates and the symmetry of $\nabla$ (it's equivalent).
Proposition 39. A curve $\alpha$ parametrized by arc-length is a critical point of the arc-length functional if and only if $\alpha^{\prime \prime}=0$.
Proof: Calculus of variations! :-)
Def.: A curve $\alpha$ with $\alpha^{\prime \prime}=0$ is called a geodesic.
Local and intrinsic concept $\Rightarrow$ invariant by local isometries
Remark 40. Let $\alpha$ be a non-constant curve in $(M,\langle\rangle$,$) .$

- $\alpha$ is a geodesic $\Rightarrow\left\|\alpha^{\prime}\right\|=$ constant $\Rightarrow \alpha$ is regular.
- $\alpha$ and $\alpha \circ h$ are geodesics $\Leftrightarrow h(t)=a t+b$.
- $\alpha$ is a geodesic $\Leftrightarrow \kappa_{g}^{\alpha}=0$.
- If $\alpha$ is a straight line segment in $M \subset \mathbb{R}^{m} \Rightarrow \alpha$ is a geodesic (once we parametrize it by arc length).


## Examples:

- Great circles in round spheres are geodesics;
- More generally, meridians in surfaces of revolution are geodesics;
- Any helix inside cylinders;
- Given two points in a cylinder (not in the same parallel), there are infinitely many geodesics joining them. But if we take a line from the cylinder, we recover uniqueness (and existence).
- In $\mathbb{R}^{2} \backslash\{0\},(1,0)$ and $(-1,0)$ have no geodesic joining them.
$\alpha$ is a geodesic $\Leftrightarrow \alpha^{\prime} \in \mathfrak{X}_{\alpha}^{\|}$. Then, by (6), we have that $\alpha=$ $\varphi\left(\alpha_{1}, \ldots, \alpha_{n}\right)$ is a geodesic $\Leftrightarrow$

$$
\begin{equation*}
\alpha_{k}^{\prime \prime}=-\sum_{i, j=1}^{n} \alpha_{i}^{\prime} \alpha_{j}^{\prime} \Gamma_{i j}^{k} \circ \alpha \tag{7}
\end{equation*}
$$

This is the differential equation of geodesics, and implies:
Proposition 41. For every $p \in M$ and every $v \in T_{p} M$, there is $\epsilon>0$ and a unique geodesic $\gamma_{v}:(-\epsilon, \epsilon) \rightarrow M$ such that $\gamma_{v}(0)=p, \gamma_{v}^{\prime}(0)=v$.

In fact, $\gamma_{v}$ also depends smoothly on $v$.

## §24. Geodesics in a surface of revolution

Let $\varphi(u, v)=(f(v) \cos (u), f(v) \sin (u), g(v))$ be a surface of revolution with axis $z$ and geratrix $\alpha(v)=(f(v), g(v))$ parametrized by arc length: $\left\|\alpha^{\prime}\right\|^{2}=f^{\prime 2}+g^{\prime 2}=1, f>0$. Notice that $f$ is the distance from the surface to the axis of revolution. Then, (7) becomes

$$
\begin{equation*}
u^{\prime \prime}=-2 \frac{f^{\prime}}{f} u^{\prime} v^{\prime}, \quad v^{\prime \prime}=f f^{\prime} u^{\prime 2} \tag{8}
\end{equation*}
$$

We get from this:

- Meridians are geodesics (very easy to see without this!)
- Parallels are geodesics $\Leftrightarrow$ the distance function $r=f$ to the axis is critical: we can see this also geometrically with a picture.

Remark 42. The first equation in (8) can also be written as $f^{2} u^{\prime}=c=$ constant. Now, the angle $\theta \in[0, \pi / 2]$ between the geodesic and the parallel that it intersects is given by $\cos (\theta)=\left|\left\langle\varphi_{u} /\left\|\varphi_{u}\right\|, u^{\prime} \varphi_{u}+v^{\prime} \varphi_{v}\right\rangle\right|=\left|u^{\prime} f\right|$. Therefore, we have the Clairaut relation:

$$
r \cos (\theta)=\text { constant }
$$

where $r=$ distance to the axis, $\theta=$ angle with parallel.
Let $\gamma$ be a geodesic (p.b.a.l.) that is neither a parallel nor a meridian. Then, $f^{2} u^{\prime}=c \neq 0$ is constant. But $1=\left\|\gamma^{\prime}\right\|^{2}=f^{2} u^{\prime 2}+v^{\prime 2}$ (by differentiating again, this implies the second equation in (8) ). So, $v^{\prime}=\sqrt{f^{2}-c^{2}} / f, \quad u^{\prime}=c / f^{2}$, and

$$
u=c \int \frac{1}{f \sqrt{f^{2}-c^{2}}} d v+u_{0}
$$

In other words, we have integrated all the geodesic equations. This is extremely rare, and we should thank the Clairaut relation, that resumes the information about the geodesics.

Application: Let $\gamma$ be a geodesic on the paraboloid of revolution $z=x^{2}+y^{2}$ that is not a meridian. Then, $r \cos (\theta)=|c| \neq 0$ $\Rightarrow \theta$ grows with $r$, and $\theta=0$ (i.e. $\gamma$ tangent to a parallel) only at one point, the unique parallel $r=|c|$ (limit of geodesics is a
geodesic, so it cannot accumulate over the parallel). Therefore, $\gamma$ intersects itself infinitely many times since it cannot be asymptotic to a meridian ( $u=$ constant), since otherwise

$$
u-u_{0}=c \int \frac{1}{v} \sqrt{\frac{1+4 v^{2}}{v^{2}-c^{2}}} d v>c \int \frac{d v}{v} \rightarrow+\infty .
$$

## §25. The covariant derivative on oriented surfaces

Let $\left(S^{2},\langle\rangle,\right)$ oriented. Then, the oriented rotation of angle $\pi / 2$ on $T S$ is a skew symmetric orthogonal tensor $J$ with $J^{2}=-I d$. For $w \in T S$ we use the notation $\bar{w}=J w$. Let $c$ be a regular curve in $S^{2}$, and $w \in \mathfrak{X}_{c}$ with $\|w\|=1$. Then, $w^{\prime}=\lambda \bar{w}$ along $c$ for some function $\lambda=:\left[w^{\prime}\right]$, called the algebraic value of $w^{\prime}$. In other words, $\left[w^{\prime}\right]=\left\langle w^{\prime}, \bar{w}\right\rangle$. Accordingly, if $c$ is a curve in $S^{2}$ parametrized by arc-length, we have the (oriented) geodesic curvature of $c$,

$$
\kappa_{g}^{c}=\left[c^{\prime \prime}\right] .
$$

Actually, for any manifold $N$ and any map $f: N \rightarrow S^{2}$, if $w \in \mathfrak{X}_{f}$ is unitary, we have a 1 -form $[\nabla w]$ over $N$ given by

$$
[\nabla w](X)=\left\langle\nabla_{X} w, \bar{w}\right\rangle .
$$

Now, if $w, e \in \mathfrak{X}_{f}$ are unitary $\Rightarrow w=a e+b \bar{e}, a^{2}+b^{2}=1 \Rightarrow$
Lemma 43. With the notations above, assume $N$ is simply connected, and fix $p \in N$. If $\cos \xi_{0}=a(p)$ and $\sin \xi_{0}=b(p)$, then there is a unique differentiable function $\xi=\varangle(w, e)$ : $N \rightarrow \mathbb{R}$ such that $\cos \xi=a, \sin \xi=b$, and $\xi(p)=\xi_{0}$.

Proof: Define $\sigma \in \Omega^{1}(N)$ by $\sigma(X)=a X(b)-b X(a)$. Since $a^{2}+b^{2}=1$ we easily check that $\sigma$ is closed, hence exact. Define now $\xi$ by $d \xi=\sigma, \xi\left(p_{0}\right)=\xi_{0}$. The lemma follows simply by differentiating $(a-\cos \xi)^{2}+(b-\sin \xi)^{2}=2-2(a \cos \xi+b \sin \xi)$.

Def.: Given $w$ and $e \in \mathfrak{X}_{f}$ unitary, the differentiable function $\varangle(w, e)$ is called a determination of the angle between $w$ and $e$. For a non vanishing $X \in \mathfrak{X}(S)$, we set $\varangle(w, X):=\varangle\left(w, \frac{X}{\|X\|} \circ f\right)$. Lemma 44. If $f: N \rightarrow S$ with $N$ simply connected, and $w, e \in \mathfrak{X}_{f}$ are unitary $\Rightarrow[\nabla w]-[\nabla e]=d \xi$, where $\xi=\varangle(w, e)$. Proof: Just compute $[\nabla w]$ using that $w=\cos (\xi) e+\sin (\xi) \bar{e}$. Remark 45. In particular, if $\alpha$ is parametrized by arc-length and $w \in \mathfrak{X}_{\alpha}^{\|} \Rightarrow \kappa_{g}^{\alpha}=\left[\alpha^{\prime \prime}\right]=\xi^{\prime}$, where $\xi=\varangle\left(w, \alpha^{\prime}\right)$. Therefore: the geodesic curvature of a curve is the rate of change of the angle of its tangent and a parallel vector field along it.
From now on, $\varphi: U \rightarrow S$ will be an orthogonal oriented chart of $S$, and $N$ any simply connected manifold.
Lemma 46. Let $f: N \rightarrow \varphi(U) \subset S$, and write $f(x)=$ $\varphi(u(x), v(x))$ for some $u, v: N \rightarrow \mathbb{R}$. If $w \in \mathfrak{X}_{f}$ is unitary,

$$
[\nabla w]=\frac{1}{2 \sqrt{E G}}\left(G_{u} d v-E_{v} d u\right)+d \xi, \quad \text { where } \xi=\varangle\left(w, \varphi_{u}\right) \text {. }
$$

Proof: Define the vector fields $V=\varphi_{u} / \sqrt{E} \in \mathfrak{X}_{\varphi}$ (unitary), and $e(x)=V(u(x), v(x)) \in \mathfrak{X}_{f}$. By Lemma [44, $[\nabla w]-d \xi=[\nabla e]$. But

$$
\begin{aligned}
{[\nabla V]\left(\partial_{u}\right) } & =\left\langle\nabla_{\partial_{u}}\left(\varphi_{u} / \sqrt{E}\right), \varphi_{v} / \sqrt{G}\right\rangle=\left\langle\nabla_{\partial_{u}} \varphi_{u}, \varphi_{v}\right\rangle / \sqrt{E G} \\
& =-\left\langle\nabla_{\partial_{u}} \varphi_{v}, \varphi_{u}\right\rangle / \sqrt{E G}=-E_{v} / 2 \sqrt{E G},
\end{aligned}
$$

by Lemma 38, Analogously, $[\nabla V]\left(\partial_{v}\right)=G_{u} / 2 \sqrt{E G}$. The lemma follows from $\nabla_{X} e=\nabla_{X}(V \circ(u, v))=\nabla_{X(u) \partial_{u}+X(v) \partial_{v}} V$.

Corollary 47. If $\alpha(s)=\varphi(u(s), v(s))$ is a curve p.b.a.l.,

$$
\kappa_{g}^{\alpha}=\frac{1}{2 \sqrt{E G}}\left(G_{u} v^{\prime}-E_{v} u^{\prime}\right)+\xi^{\prime}, \quad \text { where } \xi=\varangle\left(\alpha^{\prime}, \varphi_{u}\right) .
$$

Exercise. In an orthogonal chart, $-2 K \sqrt{E G}=\left(\frac{G_{u}}{\sqrt{E G}}\right)_{u}+\left(\frac{E_{v}}{\sqrt{E G}}\right)_{v}$.

## §26. The local Gauss-Bonnet Theorem

Gauss: Geodesic triangles and excess.
In this section, $S$ will be an oriented surface, charts will be compatible with its orientation, and $\alpha: I \rightarrow S$ will be a simple, closed, piecewise regular curve with vertices $\left\{\alpha\left(t_{1}\right), \ldots, \alpha\left(t_{n}\right)\right\}$ $\Rightarrow$ oriented external angle $\theta_{i} \in[-\pi, \pi]$ at the vertex $\alpha\left(t_{i}\right)$. Let $\varphi: U \cong D^{2}:=\left\{x \in \mathbb{R}^{2}:\|x\|<1\right\} \rightarrow S$ be a chart, $\alpha(I) \subset$ $\varphi(U)$, and $\left.\xi_{i}=\varangle\left(\alpha^{\prime} \mid t_{i}, t_{i+1}\right], \varphi_{u}\right):\left[t_{i}, t_{i+1}\right] \rightarrow \mathbb{R}\left(t_{n+1}:=t_{1}\right)$.

Proposition 48. (Turning tangents) With these notations,

$$
\sum_{i=1}^{n}\left(\xi_{i}\left(t_{i+1}\right)-\xi_{i}\left(t_{i}\right)\right)+\sum_{i=1}^{n} \theta_{i}= \pm 2 \pi
$$

where the RHS sign depends on the orientation of $\alpha$.
Proof: The LHS is the total change of the angle between $\alpha^{\prime}$ and $\varphi_{u}$. Since $\alpha$ is closed, this is $2 k \pi$, for some integer $k$, hence invariant under homotopies of $\alpha$. Since $\alpha$ is simple, $k= \pm 1$.

Def.: We say that a compact region $R \subset S$ is simple if $R \cong \overline{D^{2}}$ and $\partial R$ is the trace of a closed simple piecewise regular curve $\alpha$, that we orient in such a way that $\overline{\alpha^{\prime}}$ points to the interior of $R$.

Theorem 49. (Gauss-Bonnet; local) Let $R \subset S$ be a simple region contained in the image of an orthogonal oriented chart $\varphi: U \cong D^{2} \rightarrow S$, and let $\alpha: I \rightarrow \partial R$ oriented p.b.a.l. with vertices $\alpha\left(s_{i}\right)$ and external oriented angles $\theta_{i}$. Then,

$$
\sum_{i} \int_{s_{i}}^{s_{i+1}} \kappa_{g}^{\alpha}+\sum_{i} \theta_{i}+\int_{R} K=2 \pi
$$

Proof: Write $\alpha(s)=\varphi(u(s), v(s))$. Integrating Corollary 47 and using Green's Theorem and the previous exercise we obtain

$$
\begin{aligned}
\sum_{i} \int_{s_{i}}^{s_{i+1}} \kappa_{g}^{\alpha} d s & -\sum_{i}\left(\xi_{i}\left(s_{i+1}\right)-\xi_{i}\left(s_{i}\right)\right) \\
& =\sum_{i} \int_{s_{i}}^{s_{i+1}}\left(\frac{G_{u}}{2 \sqrt{E G}} v^{\prime}-\frac{E_{v}}{2 \sqrt{E G}} u^{\prime}\right) d s \\
& =\int_{\varphi^{-1}(R)}\left(\left(\frac{G_{u}}{2 \sqrt{E G}}\right)_{u}+\left(\frac{E_{v}}{2 \sqrt{E G}}\right)_{v}\right) d u d v \\
& =-\int_{\varphi^{-1}(R)}(K \circ \varphi) \sqrt{E G} d u d v=-\int_{R} K
\end{aligned}
$$

Now the result follows from Proposition 48.
Remark 50. Let $\alpha: I \rightarrow S$ a regular simple closed curve parametrized by arc-length, and $w \in \mathfrak{X}_{\alpha}^{\|}$unitary. Then,

$$
0=\int_{0}^{\ell}\left[w^{\prime}\right]=\int_{0}^{\ell}\left(\frac{G_{u}}{2 \sqrt{E G}} v^{\prime}-\frac{E_{v}}{2 \sqrt{E G}} u^{\prime}\right) d s+\int_{0}^{\ell} \xi^{\prime}
$$

$$
=-\int_{R} K+\xi(\ell)-\xi(0)=\Delta \xi-\int_{R} K,
$$

where $\xi=\varangle\left(w, \varphi_{u}\right)$. So, $\lim _{R \rightarrow p} \frac{\Delta \xi}{A(R)}=K(p)$, and we conclude:
The parallel transport is (locally) independent of the path $\alpha \Leftrightarrow K \equiv 0$.

## §27. Gauss-Bonnet: What's happening?

By Lemma 46 and the previous exercise, if $w$ is any unit vector field on an open set $V \subset S$, then on $V$ it holds that

$$
\begin{equation*}
-K d A=d[\nabla w] . \tag{9}
\end{equation*}
$$

(Notice that Lemma 44 explains why the LHS does not depend on $w$ ). Thus, if $R$ is a small simple region with smooth boundary parametrized by an oriented $\alpha$, and $\xi=\varangle\left(\alpha^{\prime}, w\right)$, we get from Lemma 44 and Stokes' theorem applied to (9) that

$$
\int_{R} K=-\int_{\partial R}[\nabla w]=-\int_{0}^{\ell}\left[(w \circ \alpha)^{\prime}\right]=\int_{0}^{\ell}\left(\xi^{\prime}-\kappa_{g}^{\alpha}\right)=2 \pi-\int_{0}^{\ell} \kappa_{g}^{\alpha} .
$$

Now, we get Theorem 49 from this by approximating $R$ with domains with regular boundaries.

## §28. The global Gauss-Bonnet Theorem

Recall: Triangulations and the Euler characteristic of manifolds $\chi(M)$ (see Theorem 33 in our Analysis on Manifolds notes here).

Example: If $n$-torus $:=\mathbb{S}^{2}+n$ handles $\Rightarrow \chi(n$-torus $)=2-2 n$.

For compact connected surfaces it holds that:

$$
\chi(S)=\chi\left(S^{\prime}\right) \Leftrightarrow S \text { is (diffeo)homeomorphic to } S^{\prime}(!!)
$$

If $S$ is orientable $\Rightarrow 4-\chi(S) \in 2 \mathbb{N}$. Therefore, by Jordan's Theorem, the only compact regular surfaces in $\mathbb{R}^{3}$ are the $n$-tori (up to diffeomorphism). Thus, the number $g$ of "handles" of $S$, $g:=\frac{2-\chi(S)}{2} \in \mathbb{N}_{0}$, is called the genus of $S$.
Def.: A region $R \subset S$ is regular if it is compact and $\partial R$ is a disjoint union of simple closed piecewise differential curves.

Theorem 51. (Gauss-Bonnet; global) Let $R$ be a regular region of an oriented surface $S$, and let $\partial R=\cup_{i=1}^{k} C_{i}$ positively oriented with external angles $\theta_{1}, \ldots, \theta_{m}$. Then,

$$
\sum_{i=1}^{k} \int_{C_{i}} \kappa_{g}^{\alpha}+\sum_{j=1}^{m} \theta_{j}+\int_{R} K=2 \pi \chi(R)
$$

Proof: Take a fine triangulation of $R, \mathcal{T}=\left\{T_{i}, \ldots, T_{F}\right\}$, such that each triangle lies in a simple orthogonal coordinate system, and orient each triangle $T_{i}$ according to the orientation of $S$. Let $E$ be the number of edges and $V$ the number of vertices of the triangulation, $E_{i}$ and $E_{e}$ the number of internal and external edges, respectively, and $V_{i}$ and $V_{e}$ the number of internal and external vertices, respectively. In addition, we have $V_{e}=V_{e c}+V_{e t}$, where $V_{e c}$ is the number of external vertices that are vertices of $C_{i}$, while $V_{e t}$ is the number of vertices on regular points of $\partial R$.

Adding the local Gauss-Bonnet Theorem 49 for each $T_{i}$ gives

$$
\sum_{r=1}^{k} \int_{C_{r}} \kappa_{g}^{\alpha}+\sum_{r=1}^{F} \sum_{j=1}^{3} \theta_{r j}+\int_{R} K=2 \pi F,
$$

where $\theta_{r j}$ are the 3 oriented external angles of $T_{r}$, since the internal edges of each triangle get opposite orientations. Call $\beta_{r j}:=\pi-\theta_{r j}$ the internal angles of the triangle $T_{r}$. Thus,

$$
3 \pi F-\sum_{r j} \theta_{r j}=\sum_{r j} \beta_{r j}=2 \pi V_{i}+\pi V_{e t}+\sum_{l=1}^{m}\left(\pi-\theta_{l}\right)
$$

Now, since each $C_{r}$ is closed, $E_{e}=V_{e}$. Moreover, by counting each triangle edges we get $3 F=2 E_{i}+E_{e}$. And, since $m=V_{e c}$,

$$
\sum_{r j} \theta_{r j}-\sum_{l=1}^{m} \theta_{l}=\pi\left(2 E_{i}+E_{e}-2 V_{i}-V_{e t}-m\right)=\pi(2 E-2 V)
$$

Corollary 52. The total curvature of a compact oriented surface is a purely topological invariant:

$$
\int_{S} K=2 \pi \chi(S)
$$

Corollary 53. Local Gauss-Bonnet for simple regions.
Corollary 54. $S$ compact orientable surface with $K \geq 0 \Rightarrow$ $S \cong \mathbb{S}^{2}$ or $S \equiv S^{1} \times S^{1}$. If, in addition, $S \subset \mathbb{R}^{3} \Rightarrow S \cong \mathbb{S}^{2}$.

Corollary 55. $S$ orientable with $K \leq 0 \Rightarrow 2$ geodesics do not enclose a simple region. In particular, a closed geodesic or a geodesic loop do not enclose a simple region.

Corollary 56. $S \cong$ cylinder with $K<0 \Rightarrow S$ has at most one closed geodesic (compare with the flat cylinder).

Corollary 57. $S$ compact, $K>0 \Rightarrow$ two closed geodesics intersect (compare with $K \geq 0$ : a flat cylinder with two caps).

Corollary 58. (Gauss) The excess in the internal angles of a geodesic triangle is equal to its total curvature. $\Rightarrow$

Corollary 59. $\mathbb{H}^{2}$ : the fifth Euclid axiom is independent.

## §29. Application: Total Index of a vector field

Def.: The index $I(p)$ of isolated singularity $p$ of a vector field $X \in \mathfrak{X}(S)$ is the integer given by

$$
2 \pi I(p)=\int_{0}^{\ell} \tau^{\prime}=\tau(\ell)-\tau(0)=\Delta \tau
$$

$\tau=\varangle\left(X, \varphi_{u}\right) \circ \alpha$, for a small curve $\alpha$ around $p$. By Remark 50,

$$
\int_{R} K-2 \pi I(p)=\Delta(\xi-\tau)=\Delta(\varangle(w, X)),
$$

that does not depend on $\varphi$ (it is also independent of $\alpha$ ).
Therefore, if $X$ is a vector field in a compact oriented surface with isolated singularities $\left\{p_{1}, \ldots, p_{n}\right\}$ (a generic property), by choosing a smart triangulation we get $\int_{S} K-2 \pi \sum_{i} I\left(p_{i}\right)=0$, since the boundaries of the triangles appear twice with opposite orientations. By Corollary 52 we thus have for the total index $\sum_{i} I\left(p_{i}\right)$ of $X$ :

Theorem 60. (Poincaré-Hopf) The total index of a vector field in a compact oriented surface $S$ with isolated singularities is equal to the Euler characteristic of $S: \sum_{i} I\left(p_{i}\right)=\chi(S)$.

## PART II

Prerequisites (part II): Analysis on manifolds, Stokes and de Rham cohomology.
Bibliography: [KN] Vol.II, Ch.12; [S] Vol.V Ch.13; [MS]

## §30. Fiber and principal bundles ([KN], vol. I, Ch. 1.4, 1.5)

Lie group $G$; left-invariant vector fields $\cong$ Lie algebra $\mathfrak{g}=T_{e} G$ : $V_{v}(g):=\left(L_{g}\right)_{* e} v$.

Proposition 61. $v \in \mathfrak{g} \leftrightarrow$ one parameter subgroup $\beta_{t}^{v}$ of $G$.
Proof: If $\xi_{t}(g)$ the flux of $V_{v}, \beta_{s}^{v}=\xi_{s}(e) \Rightarrow \beta_{0}^{v \prime}=v$, and $V_{v}\left(\beta_{s}^{v} \beta_{t}^{v}\right)=\left(L_{\beta_{s}^{v}}\right)_{* \beta_{t}^{v}} V_{v}\left(\beta_{t}^{v}\right)=\left(L_{\beta_{s}^{v}}\right)_{* \beta_{t}^{v}} \beta_{t}^{v \prime} \Rightarrow \beta_{s}^{v} \beta_{t}^{v}$ is an integral curve of $V_{v}$ passing at $\beta_{s}^{v}$ for $t=0 \Rightarrow \beta_{s}^{v} \beta_{t}^{v}=\beta_{s+t}^{v}$.

Exercise. Show that the flux of $V_{v}$ is $(g, s) \mapsto g \beta_{s}^{v}=R_{\beta_{s}^{v}}(g)$.
Representations; Adjoint representation: $a d_{G}: G \rightarrow \operatorname{End}(\mathfrak{g})$
Lemma 62. If $v \in \mathfrak{g},[v, \cdot]=\left(a d_{G}\right)_{* e}(v)=\frac{d}{\left.d s\right|_{s=0}} a d_{\beta_{s}^{v}}$.
Proof: By the exercise in $\S 11$, since $R_{\beta_{s}^{v}}(g)$ is the flux of $V_{v}$,

$$
\begin{aligned}
{[v, w] } & =\left[V_{v}, V_{w}\right](e)=\lim _{s \rightarrow 0} \frac{1}{s}\left(\left(R_{\beta_{-s}^{v}}\right)_{*} V_{w}\left(\beta_{s}^{v}\right)-V_{w}(e)\right) \\
& =\lim _{s \rightarrow 0} \frac{1}{s}\left(\left(R_{\beta_{-s}^{v}}\right) *\left(L_{\beta_{s}^{v}}\right)_{* e} w-w\right)=\left.\frac{d}{d s}\right|_{s=0} a d_{\beta_{s}^{v}}(w) .
\end{aligned}
$$

Group actions on manifolds: $R: E \times G \rightarrow E$; free actions Fiber bundles $F \rightarrow E \xrightarrow{\pi} B$ with typical fiber $E_{p} \cong F$, total space $E$ and base $B$
Transition functions: $\hat{\xi}_{U V}: U \cap V \rightarrow \operatorname{Diff}(F)$
Structure group of a fiber bundle: $G$-bundles:

Def.: A $G$-bundle is a fiber bundle $F \rightarrow E \xrightarrow{\pi} B$ together with a left action on $F$ by $G, \rho: G \times F \rightarrow F$, such that the transition functions are given through $\rho$, that is, there are $\xi_{U V}: U \cap V \rightarrow G$ and $\hat{\xi}_{U V}(x)(f)=\rho\left(\xi_{U V}(x), f\right)$.
(transition functions act on the left)
Exercise. Show that a rank $k$ vector bundle is a $G l(k, \mathbb{R})$-bundle.
Exercise. Show that the pull-back and Whitney sum of vector bundles is a vector bundle.

Exercise. Show that the tangent bundle of $\mathbb{S}^{3}$ is trivial.
Example: Sphere bundles
Bundle maps, bundle isomorphism
Def.: A principal $G$-bundle is as a $G$-bundle $\pi: E \rightarrow B$ with fiber $G$ where the structure group acts on the fibers by left multiplication.
Remark 63. By the associativity of the group, the right multiplication by $G$ on the fiber commutes with the action of the structure group (left multiplication). So we get an invariant right action by $G$ on $E$. This action preserves the fibers of $E$ and acts freely and transitively on them. Actually, principal bundles can be defined with the use of a free right action transitive on the fibers.

Examples: Product $B \times G$. If $(S,\langle\rangle$,$) is an oriented surface,$ the unit tangent bundle $T_{1} S$ is an $\mathbb{S}^{1}$-principal bundle. The (orthonormal) frame bundle of a rank $k$ vector bundle is a principal ( $O(k)$-bundle) $G L(k, \mathbb{R})$-bundle. Projective spaces.

Exercise. Show that all these are principal bundles.
Exercise. Show that the $\mathbb{S}^{1}$ and $\mathbb{S}^{3}$ Hopf bundles are principal bundles.
Exercise. Show that the lens spaces $\mathbb{S}^{3} / \mathbb{Z}_{k}$ for $\mathbb{Z}_{k} \subset \mathbb{S}^{1}$ are circle bundles over $\mathbb{S}^{2}$.

Fact: Principal $G$-bundles "generate" all $G$-bundles, via the associated bundles (we will see this in §34).
(Cross-)sections
Local sections $\Leftrightarrow$ (equivariant) local trivializations
Obs.: A principal bundle $E$ has a global section $\Leftrightarrow E$ is trivial. Compare to vector bundles, that always have 0 as a global section.

## §31. Connections ([KN], Vol. I, Ch. 2.1, 2.5)

Def.: The vertical subbundle $\mathcal{V} \subset T E$ is the vector bundle over $B$ given by $\mathcal{V}=\operatorname{Ker} \pi_{*}$.
Fundamental vec. fields: If $\xi_{p}(g)=R_{g}(p)=p g, v \in \mathfrak{g} \mapsto v^{*} \in \mathfrak{X}(\mathcal{V})$,

$$
v^{*}(p):=\left.\frac{d}{d t}\right|_{t=0}\left(p \beta_{t}^{v}\right)=\xi_{p * e}(v) .
$$

Notice that $(t, p) \mapsto p \beta_{t}^{v}=R_{\beta_{t}^{u}}(p)$ is the flux of $v^{*}$.
Exercise. Show that $v^{*} \circ \xi_{p}=\xi_{p_{*}} \circ V_{v}$, i.e., $v^{*} \stackrel{\xi_{p}}{\sim} V_{v}$ for all $p \in E$. In particular, $\left[v^{*}, w^{*}\right]=[v, w]^{*}$, i.e., $v \mapsto v^{*}$ is an algebra homomorphism.

Def.: A connection on a principal bundle $E$ is a differentiable map that assigns to each $x \in E$ a subspace $\mathcal{H}_{x} \subset T_{x} E$ such that:

- $T E=\mathcal{V} \oplus \mathcal{H}$;
- $\mathcal{H}$ is $G$-invariant: $\mathcal{H}_{p g}=\left(R_{g}\right)_{* p}\left(\mathcal{H}_{p}\right), \quad \forall p \in E, g \in G$.

Obs.: $T E=\left.\mathcal{V} \oplus \mathcal{H} \Rightarrow \pi_{* p}\right|_{\mathcal{H}_{p}}: \mathcal{H}_{p} \rightarrow T_{\pi(p)} B$ is isomorphism
Def.: A type $a d_{G} k$-form on $E$ is a $\mathfrak{g}$-valued $k$-form $\sigma: T E \times$ $\cdots \times T E \rightarrow \mathfrak{g}$ that satisfies

$$
R_{g}^{*} \sigma=a d_{g^{-1}} \circ \sigma, \quad \forall g \in G
$$

Def.: A principal connection on $E$ is a type $a d_{G} 1$-form $w$ : $T E \rightarrow \mathfrak{g}$ such that $\mathcal{H}=\operatorname{Ker} w$, and $w\left(v^{*}\right)=v$ for all $v \in \mathfrak{g}$.

Exercise. There is a 1-1 correspondence between the two type of connections ( $w$ is of type $a d_{G} \Leftrightarrow \mathcal{H}$ is $G$-invariant).

Exercise. Principal connections always exist (use partitions of unitiy).
$\mathcal{H}$ and $\mathcal{V}$ components: $X=X^{h}+X^{v}$. Define $h(X):=X^{h}$.
Obs.: Given $X \in \mathfrak{X}(B)$, there is a unique $X^{*} \in \mathfrak{X}(E)$, called the lift of $X$, such that $X \in \mathcal{H}$ and $\pi_{*}\left(X^{*}\right)=X \circ \pi$ (i.e., $X^{*} \stackrel{\pi}{\sim} X$ ). Moreover, the lift is $G$-invariant (i.e., $X^{*} \stackrel{R_{g}}{\sim} X^{*}$ ), and, conversely, every $G$-invariant horizontal vector field on $E$ is a lift.

Exercise. $\left[X^{*}, Y^{*}\right]^{h}=[X, Y]^{*}$.
Lemma 64. If $v^{*}$ is a fundamental vector field and $Y \in \mathcal{H}$, $\left[v^{*}, Y\right] \in \mathcal{H}$. If, in addition, $Y$ is a lift, then $\left[v^{*}, Y\right]=0$.

Proof: By exercise in $\S 11,\left[v^{*}, Y\right]=\lim _{t \rightarrow 0} \frac{1}{t}\left(\left(\xi_{-t}\right)_{*} Y \circ \xi_{t}-Y\right)$, where $\xi_{t}=R_{\beta_{t}^{v}}$ is the flux of $v^{*}$. Thus, $\left(\left(\xi_{-t}\right)_{*} Y\right)\left(\xi_{t}(p)\right) \in \mathcal{H}_{p}$.

Def.: A $k$-form $\sigma$ on $E$ is horizontal if $\mathcal{V} \subseteq \operatorname{Ker} \sigma$, i.e., if it vanishes if one of the entries is vertical.

Lemma 65. Let $\sigma$ be a horizontal $k$-form in $E$ that is $G$ invariant, i.e., $R_{g}^{*} \sigma=\sigma$. Then, $\sigma \underline{\text { projects }}$ to $\bar{\sigma}$, i.e., there exist a $k$-form $\bar{\sigma}$ in $B$ such that $\sigma=\pi^{*} \bar{\sigma}$.

Proof: Define $\bar{\sigma}\left(Z_{1}, \ldots, Z_{k}\right)(x)=\sigma\left(Z_{1}^{*}, \ldots, Z_{k}^{*}\right)(p)$, where $p \in$ $\pi^{-1}(x)$. This is independent of $p$ since the lift is $G$-invariant.

If $\sigma$ is a $k$-form on $E$, we define the $k$-form $\sigma^{h}$ by

$$
\sigma^{h}\left(X_{1}, \ldots, X_{k}\right):=\sigma\left(X_{1}^{h}, \ldots, X_{k}^{h}\right)
$$

If $\sigma$ is of type $a d_{G}$, then so $d \sigma$ and $\sigma^{h}$ are (because $\left(R_{g}\right)_{*} \circ h=$ $\left.h \circ\left(R_{g}\right)_{*}\right)$. In addition, $\sigma^{h}$ is always horizontal. So:
Def.: The horizontal $(k+1)$-form $D \sigma=(d \sigma)^{h}$ is called the exterior covariant derivative of $\sigma$ (it is of type $a d_{G}$ if $\sigma$ is).

Lemma 66. If $\sigma$ projects to $\bar{\sigma}$, then $D \sigma=d \sigma$.
Proof: The obvious: $d \sigma\left(Y_{0}, \ldots, Y_{k}\right)=d\left(\pi^{*} \bar{\sigma}\right)\left(Y_{0}, \ldots, Y_{k}\right)=$ $d \bar{\sigma}\left(\pi_{*} Y_{0}, \ldots, \pi_{*} Y_{k}\right)=d \bar{\sigma}\left(\pi_{*} Y_{0}^{h}, \ldots, \pi_{*} Y_{k}^{h}\right)=d \sigma\left(Y_{0}^{h}, \ldots, Y_{k}^{h}\right)$.

## §32. The curvature of a principal connection

Def.: If $w$ is a connection form on $E$, the horizontal 2-form $D w$ is of type $a d_{G}$ and is called the curvature form of $E$.

From now on, $\Omega:=D w$ will be the curvature 2-form of $(E, w)$.
Proposition 67. (Structure equation) $\Omega=d w+[w, w]$, i.e.,

$$
\Omega(X, Y)=d w(X, Y)+[w(X), w(Y)], \quad \forall X, Y \in T E .
$$

Proof: If $X, Y \in \mathcal{H}$, follows from the definition of $D$. If $X=A^{*} \in \mathcal{V}$ is fundamental, $\Omega\left(A^{*}, \cdot\right)=0$ because it is vertical. Now, if $B^{*} \in \mathcal{V}$ is fundamental, $d w\left(A^{*}, B^{*}\right)=A^{*}\left(w\left(B^{*}\right)\right)-$ $B^{*}\left(w\left(A^{*}\right)\right)-w\left(\left[A^{*}, B^{*}\right]\right)=A^{*}(B)-B^{*}(A)-[A, B]=-[A, B]=$ $-\left[w\left(A^{*}\right), w\left(B^{*}\right)\right]$. Now, if $Y \in \mathcal{H},\left[w\left(A^{*}\right), w(Y)\right]=0$ and $d w\left(A^{*}, Y\right)=-w\left(\left[A^{*}, Y\right]\right)=0$ by Lemma 64.

Proposition 68. (Bianchi's identity) $D \Omega=0$.
Proof: We need to check that $d \Omega=d([w, w])=0$ for 3 horizontal vector fields, but this is immediate from $\mathcal{H} \subset \operatorname{Ker} w$.

## Lemma 69. If $\sigma$ is a horizontal 1-form of type $\operatorname{ad}_{G}$,

$$
D \sigma=d \sigma+[\sigma, w]+[\omega, \sigma] .
$$

Proof: The only nontrivial case is for $v^{*} \in \mathcal{V}$ fundamental and $Y^{*} \in \mathcal{H}$ a lift. But $D \sigma\left(v^{*}, Y^{*}\right)=0$, and $d \sigma\left(v^{*}, Y^{*}\right)=v^{*}\left(\sigma\left(Y^{*}\right)\right)$ since $\left[v^{*}, Y^{*}\right]=0$ by Lemma 64. Since $Y^{*}$ is $G$-invariant, by Lemma 62,

$$
\begin{aligned}
v^{*}\left(\sigma\left(Y^{*}\right)\right)(p) & =\sigma\left(Y^{*}\left(p \beta_{s}^{v}\right)\right)^{\prime}=\sigma\left(\left(R_{\beta_{s}^{v}}\right)_{* p} Y^{*}(p)\right)^{\prime} \\
& =\left(\left(R_{\beta_{s}^{v}}^{*} \sigma\right)\left(Y^{*}(p)\right)\right)^{\prime}=\left(a d_{\beta_{s}^{-v}}\left(\sigma\left(Y^{*}(p)\right)\right)\right)^{\prime} \\
& =\left[-v, \sigma\left(Y^{*}(p)\right)\right]=-\left[w\left(v^{*}\right), \sigma\left(Y^{*}\right)\right](p) .
\end{aligned}
$$

## §33. Weil homomorphism (KN), Vol. II, Ch. 12.1)

Let $I^{k}(G)$ be the set of symmetric $k$-multilinear maps over $\mathfrak{g}, f$ : $\mathfrak{g} \times \cdots \times \mathfrak{g} \rightarrow \mathbb{R}$ and $a d_{G}$-invariant, i.e. $f\left(a d_{g} X_{1}, \ldots, a d_{g} X_{k}\right)=$ $f\left(X_{1}, \ldots, X_{k}\right)$. This is a vector space, and $I(G)=\oplus_{k=0}^{\infty} I^{k}(G)$ is a graded algebra with the natural product $(f g)\left(t_{1}, \ldots, t_{k+s}\right)=$ $\frac{1}{(k+s)!} \sum_{\sigma} f\left(t_{\sigma_{1}}, \ldots, t_{\sigma_{k}}\right) g\left(t_{\sigma_{k+1}}, \ldots, t_{\sigma_{k+s}}\right)$.

Now, let $E$ be a principal $G$-bundle with principal connection 1-form $w$ and curvature 2-form $\Omega$. For $f \in I^{k}(G)$, we define the $2 k$-form $f(\Omega)=f(\Omega, \ldots, \Omega)$ on $E$ by

$$
\begin{aligned}
& f(\Omega)\left(X_{1}, \ldots, X_{2 k}\right) \\
& \quad=\frac{1}{(2 k)!} \sum_{\sigma} \operatorname{sign}(\sigma) f\left(\Omega\left(X_{\sigma_{1}}, X_{\sigma_{2}}\right), \ldots, \Omega\left(X_{\sigma_{2 k-1}}, X_{\sigma_{2 k}}\right)\right) .
\end{aligned}
$$

Theorem 70. (A.Weil) For each $f \in I^{k}(G)$, the ( $2 k$ )-form $f(\Omega) \in \Omega^{2 k}(E)$ projects to a unique closed $(2 k)$-form $\overline{f(\Omega)} \in$ $\Omega^{2 k}(B)$. Moreover, its cohomology class

$$
\omega_{f}=[\overline{f(\Omega)}] \in H^{2 k}(B)
$$

is independent of the choice of the connection, and $\omega: I(G) \rightarrow$ $H^{*}(B)$ is an algebra homomorphism, called Weil homomorp.. Proof: Since $\Omega$ is horizontal by definition, so is $f(\Omega)$. Since $\Omega$ is of type $a d_{G}$ and $f$ is $a d_{G}$-invariant, $f(\Omega)$ is $G$-invariant: $R_{g}^{*}(f(\Omega))=f(\Omega)$. By Lemma $65 f(\Omega)$ projects: $f(\Omega)=\pi^{*} \overline{f(\Omega)}$. Proposition 68 says that $D \Omega=0$, and hence $D(f(\Omega))=0$. By Lemma 66, $f(\Omega)$ is closed, and so is $\overline{f(\Omega)}$ since $\pi_{*}$ is onto.
For the second part, take $w_{1}$ and $w_{2}$ two principal connections on $E$, and define $w_{t}:=w_{0}+t\left(w_{1}-w_{0}\right)$. Obviously, $w_{t}$ and $\alpha=w_{1}-w_{0}$ are also of type $a d_{G}$, and $\alpha(\mathcal{V})=0$. Let $D_{t}$ and $\Omega_{t}$ be the exterior covariant differentiation and curvature form of $w_{t}$, respectively. By Proposition 67, $\Omega_{t}=D_{t} w_{t}=d w_{t}+\left[w_{t}, w_{t}\right]$, so, by Lemma [69, $\frac{d}{d t} \Omega_{t}=D_{t} \alpha$. Therefore, by Proposition 68,

$$
\begin{aligned}
\frac{d}{d t} f\left(\Omega_{t}\right) & =k f\left(D_{t} \alpha, \Omega_{t}, \ldots, \Omega_{t}\right)=k D_{t}\left(f\left(\alpha, \Omega_{t}, \ldots, \Omega_{t}\right)\right) \\
& =k d\left(f\left(\alpha, \Omega_{t}, \ldots, \Omega_{t}\right)\right)
\end{aligned}
$$

where the last equality follows from Lemma 65 and Lemma 66: since both $\alpha$ and $\Omega_{t}$ are horizontal and of type $a d_{G}$, and $f$ is $a d_{G^{-}}$ invariant, so $\beta_{t}:=f\left(\alpha, \Omega_{t}, \ldots, \Omega_{t}\right)$ is horizontal and $G$-invariant. But then $\beta_{t}$ also projects to a $(k-1)$-form on $B$, and so does $\Phi=$ $k \int_{0}^{1} \beta_{t} d t$. We conclude from the above that $d \Phi=f\left(\Omega_{1}\right)-f\left(\Omega_{0}\right)$ also projects, and thus $\overline{f\left(\Omega_{1}\right)}-\overline{f\left(\Omega_{0}\right)}=d \bar{\Phi}$ is exact.
It's easy to check that $\omega$ is an algebra homomorphism (exercise).
Remark 71. Notice that the homology class is in the base $B$, not in the total space $E$ !!

Def.: The class $\omega_{f}$ is called the characteristic class of $E$ associated to $f$, that, by Theorem [70, depends only on the isomorphism class of the bundle, and not on the choice of the connection.

## §34. Associated bundles

Take a $G$-bundle $F \rightarrow E \rightarrow B$ with its $G$-action $\rho: G \times F \rightarrow F$ and transition functions $\xi_{U V}: U \cap V \rightarrow G$. If $F^{\prime}$ is another manifold where $G$ acts via $\rho^{\prime}: G \times F^{\prime} \rightarrow F^{\prime}$, we can construct another $G$-bundle $F^{\prime} \rightarrow E^{\prime} \rightarrow B$ associated to the original one by using the same transition functions $\xi_{U V}$ but simply changing $F$ by $F^{\prime}$ and $\rho$ by $\rho^{\prime}$ (see here for details).
In particular, we can take $F^{\prime}=G$ and $\rho^{\prime}=$ left multiplication to get the $G$-principal bundle associated to the original one. This allows us to define the characteristic classes of any $G$-bundle as the characteristic classes of its associated principal bundle. In particular, for a real vector bundle of rank $k$ its associated $G l(k, \mathbb{R})$-principal bundle is nothing but its frame bundle.

## §35. The shortcut for vector bundles ([MS], Appendix C)

Let's show a much more direct approach for vector bundles. Notice that, since $G=G l(n, \mathbb{R}) \subset \mathbb{R}^{n \times n}$ is open, $\mathfrak{g}=\mathbb{R}^{n \times n}$.
Let $\mathbb{R}^{n} \rightarrow P \xrightarrow{\pi} M$ be a real rank $n$ vector bundle over a manifold $M$, and $\nabla$ an affine connection on $P$. Given a local frame $e=$ $\left\{\xi_{1}, \ldots \xi_{n}\right\}$ of $\pi^{-1}(U) \cong U \times \mathbb{R}^{n}, U \subset M$, write

$$
\nabla_{X} \xi_{j}=\sum_{i} \Gamma_{j}^{i}(e)(X) \xi_{i} .
$$

So, $\omega(e)=\left(\Gamma_{j}^{i}(e)\right)$ are $\mathfrak{g}$-valued 1 -forms on $U$ that determine $\nabla$.
Exercise. For $g: U \rightarrow G, \omega(e g)=g^{-1} d g+a d_{g^{-1}}(\omega(e))$.
Define the $\mathfrak{g}$-valued curvature 2-form $\Omega(e)$ of $\nabla$ by

$$
\nabla_{X} \nabla_{Y} \xi_{j}-\nabla_{Y} \nabla_{X} \xi_{j}-\nabla_{[X, Y]} \xi_{j}=\sum_{i} \Omega_{j}^{i}(e)(X, Y) \xi_{i}
$$

It is easy to check that $\Omega$ satisfies the structure equation (compare with Proposition 677):

$$
\Omega(e)=d \omega(e)+[\omega(e), \omega(e)],
$$

the Bianchi identity (compare with Proposition 68):

$$
d \Omega(e)=[\Omega(e), \omega(e)]
$$

(i.e., $[\Omega, \omega]_{j}^{i}=\sum_{k}\left(\Omega \Omega_{k}^{i} \wedge w_{j}^{k}-w_{k}^{i} \wedge \Omega_{j}^{k}\right)$, or $[\Omega, \omega]\left(X_{1}, X_{2}, X_{3}\right)=\frac{1}{2} \sum_{\sigma \in S_{3}}\left[\Omega\left(X_{\sigma_{1}}, X_{\sigma_{2}}\right), \omega\left(X_{\sigma_{3}}\right]\right)$ and changes as $\Omega(e g)=g^{-1} \Omega(e) g=a d_{g^{-1}}(\Omega(e))$ (exercises). Thus, if $f$ is an $a d_{G}$-invariant homogeneous polynomial of degree $k$ as before, $f(\Omega)$ is a well defined (i.e. independent of the local frames $e$ ) and thus global ( $2 k$ )-form on $M^{n}$. In addition, $f(\Omega)$ is closed (easy exercise using Bianchi), so $[f(\Omega)] \in H^{2 k}(M)$.

Now, $[f(\Omega)]$ does not depend on the affine connection. If $\nabla^{1}, \nabla^{2}$ are two affine connections on $P$, then $\nabla^{t}=(1-t) \nabla^{0}+t \nabla^{1}$ is also an affine connection on $P$. Consider the projection $\pi_{1}$ : $M \times \mathbb{R} \rightarrow M$ and $i_{t}: M \rightarrow M \times \mathbb{R}, i_{t}(x)=(x, t)$. The connection $\hat{\nabla}=\pi_{1}^{*} \nabla^{t}$ is an affine connection on the vector bundle $\pi_{1}^{*}(P) \rightarrow M \times \mathbb{R}$, so the corresponding $f(\hat{\Omega})$ is closed on $M \times \mathbb{R}$. But $i_{\epsilon}^{*}(f(\hat{\Omega}))=f\left(\Omega_{\epsilon}\right)$, for $\epsilon=0,1$ and, since $i_{0}$ and $i_{1}$ are homotopic, $\left[f\left(\Omega_{0}\right)\right]=\left[f\left(\Omega_{1}\right)\right]$.

### 35.1 Affine connections $\Leftrightarrow$ principal connections

Let's see that the two constructions agree. Given $P$ the vector bundle above, its frame bundle of $G \rightarrow \mathcal{F}(P) \xrightarrow{\hat{\pi}} M$ is a principal $G$-bundle, a trivializing neighborhood of which is $\mathcal{F}\left(\hat{\pi}^{-1}(U)\right) \cong$ $U \times G$. If $w$ is a principal connection on $\mathcal{F}(P)$, and $e: U \subset$ $M \rightarrow \hat{\pi}^{-1}(U) \subset \mathcal{F}(P)$ is a local section, the equation

$$
\begin{equation*}
\omega(e)=e^{*} w \tag{10}
\end{equation*}
$$

relates $w$ with the affine connection form $\omega$. Then, check that $\Omega_{\omega}(e)=e^{*} \Omega_{w}$, and so the forms $f\left(\Omega_{w}\right)$ project precisely to $f\left(\Omega_{\omega}\right)$ :

$$
\begin{equation*}
\hat{\pi}^{*}\left(f\left(\Omega_{w}\right)\right)=\hat{\pi}^{*}\left(f\left(e^{*} \Omega_{w}\right)\right)=\hat{\pi}^{*} e^{*}\left(f\left(\Omega_{w}\right)\right)=f\left(\Omega_{w}\right), \tag{11}
\end{equation*}
$$

where for the last equality we used that $f\left(\Omega_{w}\right)$ projects.
Exercise. If $\omega$ and $w$ are forms related by (10), then $\omega$ is well defined, and it is a principal connection $\Leftrightarrow w$ is an affine connection (form).

Now, put a Riemannian metric on $P$ and work with the orthonormal frame bundle. If $e$ is an orthonormal frame and $\nabla$ is compatible we get $\Gamma_{j}^{i}(e)=-\Gamma_{i}^{j}(e)$, so $\omega(e)$ is still a $\mathfrak{g}$-valued 1-form on $U$ but now for $\mathfrak{g}=\mathfrak{o}(n)$, and we play as before.

In particular, all this holds for $P=T M$ when $M$ is Riemannian.

### 35.2 Gauss-Bonnet: What's REALLY happening??

We can now understand more deeply the Gauss-Bonnet theorem:

1. TS as an oriented vector bundle. If $S$ is a oriented Riemannian surface, its Levi-Civita connection (form) of an orthonormal oriented frame $e=\left\{e_{1}, e_{2}\right\}$ is a standard 1-form since $\mathfrak{s o}(2)=\mathbb{R}$, and is given by $\omega(e)=-\left[\nabla e_{1}\right]$. Its curvature form is $\Omega(e)=$ $d \omega(e)=K d A$ by (9). Taking $f(t)=t, \Omega=f(\Omega(e))=K d A$ is a well-defined and global closed 2 -form, whose cohomology class is independent of the compatible affine connection, in particular, independent of the metric, and so is $\int K$.
Now, if $e_{1}$ is globally defined but in a finite set $\left\{p_{i}\right\}$ the curvature form is then exact almost everywhere. We remove $\epsilon$-small disks $D_{i}^{\epsilon}$ around each $p_{i}$ and we use Stokes and Theorem 60 to get

$$
\int K=\lim _{\epsilon \rightarrow 0} \int_{M \backslash \cup_{i} D_{i}} \Omega=\sum_{i} \lim _{\epsilon \rightarrow 0} \int_{\partial D_{i}^{\epsilon}}-\left[\nabla e_{1}\right]=2 \pi \sum_{i} I\left(p_{i}\right)=2 \pi \chi(S) .
$$

2. $T_{1} S$ as a $S O(2)=\mathbb{S}^{1}$-principal bundle. The $\mathbb{S}^{1}$ action on $T_{1} S$ is given by $u \theta=\cos (\theta) u+\sin (\theta) \bar{u}$. We can choose as a principal connection 1-form $w\left(u_{*}(X)\right)=-[\nabla u](X)$, where $u$ is a section of $T_{1} S$ and $X \in T S$. Since $\mathbb{S}^{1}$ is abelian, the curvature 2-form of $w$ is $\hat{\Omega}=d w$ and projects to a closed two form on $S$ whose cohomology class does not depend on the metric. Indeed,

$$
u^{*} \hat{\Omega}=u^{*} d w=d u^{*} w=d \omega(\{u, \bar{u}\})=\Omega(\{u, \bar{u}\})=\Omega
$$

does not depend on $u$ and therefore $\pi^{*} \Omega=\hat{\Omega}$.

## §36. Invariant polynomials ([KN], Vol. II, Ch. XII.2)

Let $P^{k}(\mathbb{V})$ be the homogeneous polynomial functions on the (finite dimensional) vector space $\mathbb{V}$ of degree $k$ (polynomial by taking a basis), and $P(\mathbb{V})=\oplus_{k=0}^{\infty} P^{k}(\mathbb{V})$ the natural algebra of polynomial functions. Let $S^{k}(\mathbb{V})$ be the set of symmetric $k$-multilinear functions on $\mathbb{V}$, with $S(\mathbb{V})=\oplus_{k=0}^{\infty} S^{k}(\mathbb{V})$ its natural commutative algebra.

Proposition 72. (Polarization). The map $\tau: S(\mathbb{V}) \mapsto P(\mathbb{V})$ given by $\tau(h)(t)=h(t, \ldots, t)$ is an algebra isomorphism.
Proof: If $\left\{\xi^{1}, \ldots, \xi^{n}\right\}$ is a basis of $\mathbb{V}^{*}, f \in P^{k}(V)$ can be written as $\sum f_{i_{1} \ldots i_{k}} \xi^{i_{1}} \ldots \xi^{i_{k}}$, for some $f_{i_{1} \ldots i_{k}} \in \mathbb{R}$ symmetric in the indexes. The function $\Phi(f)\left(t_{1}, \ldots, t_{k}\right)=\sum f_{i_{1} \ldots i_{k}} \xi^{i_{1}}\left(t_{1}\right) \cdots \xi^{i_{k}}\left(t_{k}\right)$ is the inverse of $\tau$ (exercise).

Exercise. If $G \subset L(\mathbb{V})$ is a subgroup, the isomorphism above induces an isomorphism between the $G$-invariant subalgebras $S_{G}(\mathbb{V})$ and $P_{G}(\mathbb{V})$.

Corollary 73. $I(G) \cong P(G)$, where $P(G)$ are the ad $G_{G^{-}}$ invariant polynomial functions in $\mathfrak{g}$.
36.1 The unitary group: $U(n)=\left\{X \in \mathbb{C}^{n \times n}: X \bar{X}^{t}=I\right\}$. Its Lie algebra is $\mathfrak{u}(n)=\left\{A \in \mathbb{C}^{n \times n}: \bar{A}^{t}=-A\right\}$. If $A \in \mathfrak{u}(n)$, $\operatorname{det}\left(\lambda I+\frac{i}{2 \pi} A\right)=\lambda^{n}-\sigma_{1}(A) \lambda^{n-1}+\sigma_{2}(A) \lambda^{n-2}-\cdots+(-1)^{n} \sigma_{n}(A)$. Then, the polynomial functions $\sigma_{i}$ are $a d_{U(n)}$-invariant. In fact, if $i t_{1}, \ldots, i t_{n}$ are the eigenvalues of $A$, then $\sigma_{i}(A)$ is the $i$-th symmetric function on $t_{1}, \ldots, t_{n}$. And these are all:

Proposition 74. The polynomial functions $\sigma_{1}, \ldots, \sigma_{n}$ are $a d_{U(n)}$-invariant, algebraically independent, and generate (as algebra) $P_{U(n)}(\mathfrak{u}(n))$.

Proof: See Theorem 2.5 in Kobayashi-Nomizu, Vol 2, Cap. XII. I
Corollary 75. The characteristic classes $c_{k}(E):=\omega_{\sigma_{k}} \in$ $H^{2 k}(B), 1 \leq k \leq n$, generate all the characteristic classes of an $U(n)$-principal bundle $U(n) \rightarrow E \rightarrow B$ as an algebra. They are called the Chern classes of the bundle.
36.2 The orthogonal group: $O(n)=\left\{X \in \mathbb{R}^{n \times n}: X X^{t}=I\right\}$.

Its Lie algebra is $\mathfrak{o}(n)=\left\{A \in \mathbb{R}^{n \times n}: A^{t}=-A\right\}$. Define

$$
\operatorname{det}\left(\lambda I-\frac{1}{2 \pi} A\right)=\lambda^{n}+p_{1}(A) \lambda^{n-2}+p_{2}(A) \lambda^{n-4}+\cdots+\cdots
$$

Then, the polynomial functions $p_{1}, \ldots, p_{[n / 2]}$ are $a d_{O(n) \text {-invariant. }}$ In fact, if $\pm i t_{1}, \ldots, \pm i t_{[n / 2]}$ are the eigenvalues of $A$ (besides the 0 if $n$ is odd), then $p_{i}(A)$ is the $i$-th symmetric function on $t_{1}^{2}, \ldots, t_{[n / 2]}^{2}$. And these are all:
Proposition 76. The polynomial functions $p_{1}, \ldots, p_{[n / 2]}$ are $a_{O(n)}$-invariant, algebraically independent, and generate (as algebra) $P_{O(n)}(\mathfrak{o}(n))$.

Proof: See Theorem 2.6 in Kobayashi-Nomizu, Vol 2, Cap. XII. I
Corollary 77. The characteristic classes $p_{k}(E):=\omega_{p_{k}} \in$ $H^{4 k}(B), 1 \leq k \leq[n / 2]$, generate all the characteristic classes of an $O(n)$-principal bundle as an algebra. They are called the Pontrjagin classes of the bundle.
36.3 Special orthogonal group: $S O(n)=\{X \in O(n): \operatorname{det}(X)=1\}$. Being $S O(n)$ the connected component of $O(n)$ containing the identity, their Lie algebras coincide $\mathfrak{s o}(n)=\mathfrak{o}(n)$, and the situation is very similar to that of $O(n)$. However, for $n=2 m$ even, there is a (unique up to sign) $S O(n)$ invariant homogeneous polynomial function $p f$ such that $p f^{2}=p_{m}$, called the pfaffian. In terms of matrixes, $p f(A)^{2}=\operatorname{det}(A)$, and is given by

$$
\begin{equation*}
p f(A)=\frac{1}{2^{m} m!} \sum_{\sigma \in S_{2 m}} \operatorname{sign}(\sigma) \prod_{i=1}^{m} a_{\sigma(2 i-1) \sigma(2 i)} \tag{12}
\end{equation*}
$$

Hence, we have:
Proposition 78. For $n=2 m-1$ (resp. $n=2 m$ ) the polynomial functions $p_{1}, \ldots, p_{m-1}$ (resp. $p_{1}, \ldots, p_{m-1}, p f$ ) are $a d_{S O(n)}$-invariant, algebraically independent, and generate (as algebra) $P_{S O(n)}(\mathfrak{s o}(n))$.

Proof: See Theorem 2.7 in Kobayashi-Nomizu, Vol 2, Cap. XII.
Corollary 79. The characteristic classes $p_{k}(E):=\omega_{p_{k}} \in$ $H^{4 k}(B), 1 \leq k \leq\left[\frac{n-1}{2}\right]$, together with $e(E):=(2 \pi)^{-n / 2} \omega_{p f} \in$ $H^{n}(B)$ if $n$ is even, generate all the characteristic classes of an $S O(n)$-principal bundle as an algebra. The classes $p_{i}(E)$ are called the Pontrjagin classes of the bundle, while, for $n$ even, $e(E)$ is called the Euler class of the bundle.

Remark 80. In particular, the three subsections apply for complex, real, and oriented real vector bundles, where the terminology Chern, Pontrjagin and Euler classes are usually applied (resp.),
by means of Section 35. By definition, the classes of a vector bundle are the classes of its frame principal bundle.

Total Chern and Pontrjagin classes:

$$
\begin{aligned}
& c(E)=1+c_{1}(E)+c_{2}(E)+\cdots \in H^{*}(B), \\
& p(E)=1+p_{1}(E)+p_{2}(E)+\cdots \in H^{*}(B) .
\end{aligned}
$$

## §37. The axiomatic approach

Suppose $E_{1} \oplus E_{2}$ is a Whitney sum of two (real or complex) vector bundles. By the previous section, since the classes come from determinants, the total class for $E$ is the product of the total classes of $E_{1}$ and $E_{2}: c\left(E_{1} \oplus E_{2}\right)=c\left(E_{1}\right) \wedge c\left(E_{2}\right)$ (for complex) $p\left(E_{1} \oplus E_{2}\right)=p\left(E_{1}\right) \wedge p\left(E_{2}\right)$ (for real). Moreover, extending the definition of the Euler class to odd dimensional real vector bundles as 0 , for the Euler class it also holds that $e\left(E_{1} \oplus E_{2}\right)=e\left(E_{1}\right) \wedge e\left(E_{2}\right)$; see [ $\left.\mathbf{S}\right]$, Vol.5, Ch.13, Theorem 22. In particular: if the orientable vector bundle $E$ has a nowhere vanishing section, then $e(E)=0$.

In fact, there is a way of defining characteristic classes for vector bundles in an axiomatic way: we proved that they exist, and it is not hard to see that they are unique. For example, for Chern classes for complex vector bundles (CVB) we have:

- Axiom 1: For each CVB $E$ over $M$, and each integer $k \geq 0$, there exist a class $c_{k}(E) \in H^{2 k}(M)$, with $c_{0}(E)=1$ (so we can define $c(E)=\sum_{i=0}^{\infty} c_{k}(E) \in H^{*}(M)$, the total Chern class of $E$.) - Axiom 2 (Naturality): If $E$ is a CVB over $M$ and $f: M^{\prime} \rightarrow M$ is smooth, then $c\left(f^{*} E\right)=f^{*} c(E)$.
- Axiom 3 (Whitney sum formula): If $E, E^{\prime}$ are CVBs over $M$ and $E_{1} \oplus E^{\prime}$ their Whitney sum, then $c\left(E \oplus E^{\prime}\right)=c(E) \wedge c\left(E^{\prime}\right)$. - Axiom 4 (Normalization): If $\mathbb{C P}^{1}$ is the complex projective line and $\mathbb{P}$ its canonical complex line bundle, then $\int_{\mathbb{C P}^{1}} c_{1}(\mathbb{P})=-1$.

Exercise. Show that the Chern classes for CVBs as we defined satisfy the four axioms above, thus proving existence; see $\$ 35$ and $[\mathbf{K N}]$ II c. 13 .

Axiomatically, the Pontrjagin classes of a real vector bundle $E$ are defined simply by $p_{k}(E)=c_{2 k}(E \otimes \mathbb{C})$.
For oriented real vector bundles of rank $k$, the Euler class is defined with the same axioms as the Chern classes, except that, in Axiom 1, we require $e(E) \in H^{k}(M)$, and $e(M)=0$ if $k$ is odd.

## §38. The Poincaré-Hopf Theorem in all dimensions

Let $X \in \mathfrak{X}\left(M^{n}\right)$ be a vector field on an oriented $M^{n}$ with an isolated singularity at $p \in M^{n}$. If we restrict $X$ to the boundary of a small ball $B_{\epsilon}$ around $p$, we have $\mathbb{S}_{p}^{n-1}:=T_{1} M \cap T_{p} M$ and

$$
V=X /\|X\|: \partial B_{\epsilon} \cong \mathbb{S}^{n-1} \rightarrow T_{1} B_{\epsilon} \cong B_{\epsilon} \times \mathbb{S}_{p}^{n-1}
$$

for some trivializing chart $\varphi: B_{\epsilon} \times \mathbb{S}_{p}^{n-1} \rightarrow T_{1} B_{\epsilon}$ with $\varphi \circ i_{p}$ being the inclusion $\mathbb{S}_{p}^{n-1} \subset T_{1} B_{\epsilon}$, where $i_{p}: \mathbb{S}_{p}^{n-1} \rightarrow B_{\epsilon} \times \mathbb{S}_{p}^{n-1}$, $i_{p}(v)=(p, v)$. We define the index of $X$ at $p$ as the integer

$$
I(p)=\operatorname{deg}(\hat{V}),
$$

where $\hat{V}=\pi_{2} \circ \varphi^{-1} \circ V: \partial B_{\epsilon} \cong \mathbb{S}^{n-1} \rightarrow \mathbb{S}_{p}^{n-1}$. Notice that, for $\epsilon$ small, $V \cong \varphi \circ i_{p} \circ \hat{V}$ as smooth functions.

With these definitions, the Poincaré-Hopf Theorem 60 holds for any compact oriented manifold and any vector field with isolated singularities, and not just for surfaces. Indeed, by the proof of the Gauss-Bonnet-Chern Theorem 833 below it follows that the total index of a vector field is a topological invariant, i.e., does not depend on the vector field. But it is easy to construct a vector field whose total index is the Euler characteristic: for a triangulation $\mathcal{T}$, define $V_{\mathcal{T}}$ as having precisely one singularity on the 'center' of each simplex of $\mathcal{T}$ in a way so that the flow lines of the vector field point from the centers of higher dimensional simplexes towards the lower dimensional simplexes. Such a vector field has total index equal to the Euler characteristic.

## §39. The Gauss-Bonnet-Chern Theorem (园)

Consider a compact oriented even dimensional Riemannian manifold $M^{2 m}$. Its tangent bundle is an $S O(2 m)$-bundle, and so it has its Euler Class, $e(T M) \in H^{2 m}(M) \cong \mathbb{R}$. So its integral

$$
\int e(T M) \in \mathbb{R}
$$

is a topological invariant that does not depend on the Riemannian metric. In terms of the curvature $\Omega$ of the Levi-Civita connection, (12) $\Rightarrow(2 \pi)^{m} e(T M)$ is represented by the $2 m$-form

$$
p f(\Omega)=\frac{1}{2^{m} m!} \sum_{\sigma \in S_{2 m}} \operatorname{sign}(\sigma) \Omega_{\sigma_{2}}^{\sigma_{1}} \wedge \cdots \wedge \Omega_{\sigma_{2 m}}^{\sigma_{2 m-1}}
$$

Now, consider the sphere bundle $\pi: T_{1} M \rightarrow M$. Then,

$$
\pi^{*}(p f(\Omega)) \in \Omega^{2 m}\left(T_{1} M\right)
$$

Lemma 81. (S.S.Chern; Transgression Lemma) There is $\lambda \in \Omega^{2 m-1}\left(T_{1} M\right)$ such that $\pi^{*}(p f(\Omega))=d \lambda$. In addition, $\left.\int_{\mathbb{S}_{p}^{2 m-1}} \lambda\right|_{\mathbb{S}_{p}^{2 m-1}}=(2 \pi)^{m}$, for all $p \in M$.
Proof: A long algebraic construction... See [L], Lemma 3.2.3.
Remark 82. For $m=1$, since $G=S O(2) \cong \mathbb{S}^{1}$ is abelian, by (9), (10), (11) and Proposition 67 we can take $\lambda=w$ in Lemma 81 .

Theorem 83. (Generalized Gauss-Bonnet-Chern Theorem) If $M^{2 m}$ is compact and orientable, then

$$
\int e(T M)=\chi\left(M^{2 m}\right)
$$

Proof: Let $X$ be a vector field with isolated singularities only $\left\{p_{1}, \ldots, p_{r}\right\}$. Remove small balls $B_{\epsilon}\left(p_{i}\right)$ from $M$, and define $M_{\epsilon}=M \backslash \cup_{i} B_{\epsilon}\left(p_{i}\right)$ and $V=X /\|X\|: M_{\epsilon} \rightarrow T_{1} M_{\epsilon}$. Then, by the Transgression Lemma 81 and Stoke's Theorem,

$$
\begin{aligned}
\int w_{p f} & =\lim _{\epsilon \rightarrow 0} \int_{M_{\epsilon}} p f(\Omega)=\lim _{\epsilon \rightarrow 0} \int_{M_{\epsilon}} V^{*}\left(\pi^{*}(p f(\Omega))\right) \\
& =\lim _{\epsilon \rightarrow 0} \int_{M_{\epsilon}} V^{*}(d \lambda)=\lim _{\epsilon \rightarrow 0} \int_{M_{\epsilon}} d\left(V^{*} \lambda\right) \\
& =\sum_{i} \lim _{\epsilon \rightarrow 0} \int_{\partial B_{\epsilon}\left(p_{i}\right)} V^{*} \lambda=\sum_{i} \lim _{\epsilon \rightarrow 0} \int_{\partial B_{\epsilon}\left(p_{i}\right)} \hat{V}^{*}\left(\left(\varphi \circ i_{p_{i}}\right)^{*} \lambda\right) \\
& =\left.\sum_{i} I\left(p_{i}\right) \int_{\mathbb{S}_{p_{i}^{2 m-1}}} \lambda\right|_{\mathbb{S}_{p_{i}}^{2 m-1}}=(2 \pi)^{m} \sum_{i} I\left(p_{i}\right) .
\end{aligned}
$$

Therefore, the total index is a topological invariant, independent of the vector field $X$. But we saw in $\oint 38$ that there exists a vector field whose total index is equal to $\chi\left(M^{2 m}\right)$.

Remark 84. The above is essentially Chern's original proof in [C]. For an alternative proof using characteristic classes more deeply, see [S], Vol 5, Ch. 13, Theorem 26.

Remark 85. Again, notice that we not only proved the Gauss-Bonnet-Chern Theorem, but also Poincaré-Hopf Theorem 60] for any dimensions.

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