Differential Geometry guide

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PART I

Prerequisites (part I): Analysis on \mathbb{R}^n . Analysis on manifolds, Stokes and de Rham recommended. Bibliography: [dC]

§1. Introduction

Differentiable manifolds: smooth world. Now we're going to <u>measure</u> in them. After all, *geometry* comes from the Greek: "measurement of the Earth":

Eratosthenes (Cirene, 276 AC - Alexandria, 194 AC)

Posidonius (135 AC - 51 AC) \Rightarrow Colombo

We will study in this first part curves, surfaces and hypersurfaces of Euclidean space \Rightarrow Two aspects: *intrinsic* and *extrinsic*.

§2. Curves

Curve: intrinsically, nothing interesting: $I \subseteq \mathbb{R}$. Regular curves α in \mathbb{R}^2 and \mathbb{R}^3 : arc length s, p.b.a.l, curvature κ^{α} and torsion τ^{α} . Frenet Trihedron: $\{t, n, b\}$. FTC: explicit vs. ODE. Curves in \mathbb{R}^n : FTC.

Exercise. Prove that $\kappa^{\alpha} = \sqrt{\|\alpha'\|^2 \|\alpha''\|^2 - \langle \alpha', \alpha'' \rangle^2} / \|\alpha'\|^3$, independently of the parametrization of α .

§3. Surfaces in \mathbb{R}^3

Regular (Euclidean) surface: $S = S^2 \subset \mathbb{R}^3$ (embedding!) Regular (Euclidean) hypersurface: $M^n \subset \mathbb{R}^{n+1}$ (embedding!) Regular (Euclidean) submanifold: $M^n \subset \mathbb{R}^{n+p}$ (embedding!)

It is enough to check that: $\forall x \in M^n, \exists V \subset \mathbb{R}^{n+p}$ open, $x \in V$, and a <u>smooth</u> map $U \subset \mathbb{R}^n \mapsto M^n \cap V \subset \mathbb{R}^{n+p}$ that is <u>injective</u>, open and has <u>rank n</u>: coordinates (smooth = $C^r/C^{\infty}/\overline{C^w}$).

Examples:

 $\operatorname{graf}(f)$ for $f: U \subset \mathbb{R}^n \to \mathbb{R}$.

 $g^{-1}(t_0)$ for a regular value t_0 (in the image) of $g: W \subset \mathbb{R}^{n+1} \to \mathbb{R}$: Sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$.

Ellipsoid $g^{-1}(r)$ for $g(x, y, z) = x^2/a^2 + y^2/b^2 + z^2/c^2$, r > 0. Hyperboloid $g^{-1}(r)$ for $g = x^2 + y^2 - z^2$ (Hyperboloid of two sheets for r < 0, while the Cone $g^{-1}(0)$ is NOT a regular surface). Hyperbolic paraboloid $g^{-1}(0)$ for $g = x^2/a^2 - y^2/b^2 - z$. Circular cylinder and Cylinders over conics.

Def.: Parametrized surface just $\varphi : U \subset \mathbb{R}^2 \to \mathbb{R}^3$. Singular points of φ : $d\varphi_p$ singular. We call φ regular if it is an immersion.

Proposition 1. Every hypersurface is locally a graph.

Exercise. No need to check that a coordinate system is open (homeomorphism) if we know beforehand that $M^n \subset \mathbb{R}^{n+1}$ is a regular hypersurface.

Differentiable functions: now we can use the ambient space.

Examples: $F: U \subset \mathbb{R}^3 \to \mathbb{R}$ smooth $\Rightarrow F|_S$ is smooth $\forall S \subset U$ $\varphi: U \subset \mathbb{R}^2 \to \mathbb{R}^3$ coordinates $\Rightarrow U$ and $\varphi(U)$ are diffeomorphic S symmetric \Rightarrow the symmetry restricted to S is smooth:

Example: Surfaces of revolution: Meridians, Parallels, Axis, Generatrix (embedded!).

Example: Tangent surfaces to a curve $\alpha : I \to \mathbb{R}^3$: $\kappa^{\alpha} \neq 0$ $\Rightarrow \varphi(s,t) = \alpha(s) + t\alpha(s)$ is a parametrized surface, regular for $t \neq 0$.

§4. Tangent space as a subspace

For a regular submanifold $M^n \subset \mathbb{R}^{n+q}$ and $p \in M^n$, we now have $T_p M$ naturally included in \mathbb{R}^{n+q} as an affine subspace: spanned by $\{(\partial \varphi/\partial u_i)(p) : i = 1, \ldots, n\}$ for any coordinate φ at p.

If
$$\alpha: I \to S \subset \mathbb{R}^3$$
, $\alpha(s) = \varphi(u(s), v(s))$, $\alpha(0) = p = \varphi(0, 0) \Rightarrow$
 $\alpha'(0) = u'(0) \frac{\partial \varphi}{\partial u}(0, 0) + v'(0) \frac{\partial \varphi}{\partial v}(0, 0) \in T_p S \subset \mathbb{R}^3$

Differential of a function $f: S_1 \subset \mathbb{R}^3 \to S_2 \subset \mathbb{R}^3$ at a point: can be seen as a linear map between subspaces of \mathbb{R}^3 .

Local diffeomorphism, chain rule...

TIP: Use curves to compute differentials!

Example: $L : \mathbb{R}^3 \to \mathbb{R}^3$ linear, $S \subset \mathbb{R}^3 \Rightarrow f_{*p} = f|_{T_pS}$

Since we fixed an orientation in \mathbb{R}^3 , we can talk about the *Normal*

vector field of our surface:

$$N = \frac{\frac{\partial \varphi}{\partial u} \wedge \frac{\partial \varphi}{\partial v}}{\left\| \frac{\partial \varphi}{\partial u} \wedge \frac{\partial \varphi}{\partial v} \right\|}$$
(1)

Same holds for hypersurfaces. Angle between surfaces.

§5. The First Fundamental Form

Curves + inner product $\langle \cdot, \cdot \rangle$ of $\mathbb{R}^3 \to \text{distance} \to \text{first funda-mental form } I$:

 $I(p) = \langle \cdot, \cdot \rangle|_{T_p S \times T_p S} : T_p S \times T_p S \to \mathbb{R}$

is an inner product on T_pS .

We will denote also by I its associated quadratic form. Arc length s for $\alpha : I \to S \subset \mathbb{R}^3$:

$$s(t) = \int_{t_0}^t \sqrt{I(\alpha'(r))} dr.$$

 $\varphi: U \to S \subset \mathbb{R}^3 \text{ coordinate system } \Rightarrow$

$$E = \|\frac{\partial \varphi}{\partial u}\|^2, \ F = \langle \frac{\partial \varphi}{\partial u}, \frac{\partial \varphi}{\partial v} \rangle, \ G = \|\frac{\partial \varphi}{\partial v}\|^2 \in C^{\infty}(U)$$

coefficients of I in the coordinate φ . $\operatorname{Im}(\alpha) \subset \operatorname{Im}(\varphi) \implies s(t) = \int_{t_0}^t \sqrt{Eu'^2 + 2Fu'v' + Gv'^2} dr.$

Remark 2. *I* is a symmetric positive definite (2,0)-tensor in *S*, a *Riemannian metric* on *S*: $I = inc^* \langle \cdot, \cdot \rangle$.

Same for arbitrary regular Euclidean submanifolds:

Example: Affine plane through $p \in \mathbb{R}^3$ or Cylinder over plane curve \Rightarrow everywhere coordinate systems with $E \equiv 1 \equiv G, F \equiv 0$.

Def.: Regular domain $D \subset S$.

If $\varphi: U \to S$ is a coordinate system, $\|\frac{\partial \varphi}{\partial u} \wedge \frac{\partial \varphi}{\partial v}\|$ is the area of the parallelogram determined by the coordinate vector fields, and we can define the *area* of a regular domain $\Omega \subset \varphi(U)$ by

$$\begin{split} A(\Omega) &= \int_{\varphi^{-1}(\Omega)} \|\frac{\partial \varphi}{\partial u} \wedge \frac{\partial \varphi}{\partial v}\| = \int_{\varphi^{-1}(\Omega)} \sqrt{EG - F^2} \; du dv \\ \text{since } \|x \wedge y\|^2 + \langle x, y \rangle^2 = \|x\|^2 \|y\|^2 \end{split}$$

§6. Recalling basic concepts of vector bundles

§7. Orientation

 $\operatorname{graf}(f)$ orientable

 $M^n \subset \mathbb{R}^{n+1}$ orientable \iff there exist a globally defined smooth unit normal vector field.

 $g^{-1}(r)$ orientable $(0 \neq \text{grad } (g) \perp M)$

Theorem 3. $M^n \subset \mathbb{R}^{n+1}$ embedded, orientable \Rightarrow there exist $V \subset \mathbb{R}^{n+1}$ open with $M^n \subset V$, and $g: V \subset \mathbb{R}$ such that 0 is a regular value of g and $M^n = g^{-1}(0)$.

Proof: Existence of tubular neighborhoods. \blacksquare

Remark 4. $M^n \subset \mathbb{R}^{n+1}$ embedded and compact \Rightarrow orientable (Jordan Theorem 41 in our last course).

§8. Gauss map and Second Fundamental Form

For any Euclidean hypersurface $M^n \subset \mathbb{R}^{n+1}$, <u>locally</u> we have a unit normal vector field

 $N: U \subset M^n \to \mathbb{S}^n \subset \mathbb{R}^{n+1}.$

But T_pM is parallel to $T_{N(p)}\mathbb{S}^n$, and hence $dN_p \in End(T_pM)$. Moreover, dN_p is self adjoint (w.r.t. I), so the quadratic form

$$II_p(w) := -\langle dN_p w, w \rangle$$

is called the *second fundamental form* of M^n at p. We also give the same name to the associated symmetric tensor,

$$A_p := -dN_p.$$

Def.: Let $\alpha : I \to M$ be a regular curve through $p = \alpha(0) \in M$. The normal curvature of α at p is given by

$$\kappa_n := \kappa \langle N, n \rangle,$$

where n is the normal vector of α at p and κ its curvature.

Remark 5. (!!!!) It holds that (draw a picture)

$$II(\alpha'(0)) = \kappa_n(0)$$

i.e., the normal curvature only depends on the direction of α' (!!). In particular, if $v \in T_p M$ and π is the plane spanned by v and $N(p), \alpha = \pi \cap M \subset \pi$ is a plane curve whose curvature is II(v) (beware of orientations). Examples: Second fundamental form of a graph; $y = x^4$; \mathbb{S}^n . A self adjoint \Rightarrow principal curvatures $\{k_i\}$, principal directions $\{e_i\}$, lines of curvature:

$$v = \sum_{i} v_i e_i \quad \Rightarrow \quad II(v) = \sum_{i} k_i v_i^2.$$

For dimension 2, we have the *Euler's formula* for ||v|| = 1:

$$v = \cos(\theta)e_1 + \sin(\theta)e_2 \Rightarrow II(v) = k_1\cos(\theta)^2 + k_2\sin(\theta)^2.$$

Remark 6. Ordering the principal curvatures $k_1 \leq \cdots \leq k_n$ we see by Remark 5 that e_1 is the direction where M "curves" less (w.r.t. N) in the ambient space, while e_n is the one where it curves more. This follows from the usual diagonalization process.

§9. The two curvatures for surfaces: K and H

For a symmetric endomorphism in dimension two, we have two invariants (independent of orthonormal basis): the trace and the determinant.

Def.: For a regular surface $S \subset \mathbb{R}^3$ and $p \in S$, the *Gaussian* curvature of S at p is given by

$$K(p) := \det(A_p) = k_1 k_2.$$

Def.: The mean curvature of S at p is $H(p) = -\text{trace}(A_p)/2$. We will see in a while that the two curvatures have very different nature. Notice that K does not depend on orientation, while H does. Notice that both are <u>smooth functions</u> that determine k_1 and k_2 :

$$k_i = H \pm \sqrt{H^2 - K}.$$

Def.: A point p in S can be *elliptic*, *hyperbolic*, *parabolic*, *planar* (or *totally geodesic*), *minimal*, *umbilical*. Accordingly, S itself could be *totally geodesic*, *minimal*, *umbilical*.

Remark 7. The principal curvatures and their eigenspaces are always continuous, and smooth along any open subset where their multiplicities are constant. In particular, they are always smooth along (the connected components of) an open dense subset W of M. For surfaces, if V is the set of umbilical points of S, W can be taken as $V^o \cup S \setminus V$.

Def.: Asymptotic direction and asymptotic curve of $S \subset \mathbb{R}^3$.

Notice that there exists an asymptotic direction at $p \iff K(p) \le 0$, while there are precisely two asymptotic directions at $p \iff K(p) < 0$.

Proposition 8. Let $M^n \subset \mathbb{R}^{n+1}$ regular and connected. Then, M is umbilical $\iff M^n$ is (an open subset of) a round n-sphere or a hyperplane.

Def.: Conjugate directions.

§10. II and K in coordinates, part 1

For a coordinate system $\varphi = \varphi(u, v)$ in a surface $S \subset \mathbb{R}^3$, let as before

$$N = \frac{\frac{\partial \varphi}{\partial u} \wedge \frac{\partial \varphi}{\partial v}}{\|\frac{\partial \varphi}{\partial u} \wedge \frac{\partial \varphi}{\partial v}\|}.$$

Denote by (a_{ij}) the matrix of A in the coordinate basis,

$$N_u = a_{11}\varphi_u + a_{12}\varphi_v,$$
$$N_v = a_{21}\varphi_u + a_{22}\varphi_v.$$

Define the functions

$$e := -\langle N_u, \varphi_u \rangle = \langle N, \varphi_{uu} \rangle,$$
$$g := -\langle N_v, \varphi_v \rangle = \langle N, \varphi_{vv} \rangle,$$
$$f := -\langle N_u, \varphi_v \rangle = -\langle N_v, \varphi_u \rangle = \langle N, \varphi_{uv} \rangle.$$

Hence, if $v = v_1 \varphi_u + v_2 \varphi_v$, $II(v) = ev_1^2 + 2fv_2v_2 + gv_2^2$, and

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \frac{-1}{EG - F^2} \begin{pmatrix} e & f \\ f & g \end{pmatrix} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix}$$

These are known as the *Weingarten equations*. In particular, for the Gaussian curvature we obtain

$$K = \frac{eg - f^2}{EG - F^2}.$$

Example: The torus. For 0 < r < a, take the (almost global) chart

$$\varphi(u,v) = \left((a + r\cos(u))\cos(v), (a + r\cos(u))\sin(v), r\sin(u) \right).$$

Hence, $N = (\cos(u)\cos(v), \cos(u)\sin(v), \sin(u))$ (geometrically!), and hence $E = r^2$, $G = (a + r\cos(u))^2$, F = 0, and

$$K = \frac{\cos(u)}{r(a + r\cos(u))}.$$

Make the computation, make a picture, sign K interpretation, elliptic/hyperbolic points, move a and r and see how K varies, independence of v... see everything geometrically!

Remark 9. Observe that $\int K = 0$, *independently* of a and r!

Now, let's compute K for any surface of revolution of a simple closed curve p.b.a.l. as (a(s), 0, b(s)): K = -a''/a, $\int K = 0!!!!$

Proposition 10. If $p \in S \subset \mathbb{R}^3$ is elliptic \Rightarrow a neighborhood of p is in one side of T_pS . If p hyperbolic, it is not.

Proof: Differentiate $g = \langle \varphi - \varphi(0), N(p) \rangle$ at $p = \varphi(0)$.

Def.: *Lines of curvature.*

If $\alpha(t) = \varphi(u(t), v(t))$ is a line of curvature, then $dN(\alpha') = \lambda \alpha'$, or equivalently, $(fF - eG)u' + (gF - fG)v' = \lambda u'(EG - F^2)$, $(eF - fE)u' + (fF - gE)v' = \lambda v'(EG - F^2)$, or

$$(fE - eF)(u')^{2} + (gE - eG)u'v' + (gF - fG)(v')^{2} = 0,$$

known as the *equation of the lines of curvature*. In particular, outside of the umbilical points:

the chart is by lines of curvature $\Leftrightarrow F = f = 0$.

Proposition 11. Let $p \in S \subset \mathbb{R}^3$, and a sequence of compact regions $B_i \subset S$ such that $B_i \to p$. Then,

$$|K(p)| = \lim_{i \to \infty} \frac{Area(N(B_i))}{Area(B_i)}.$$

Proof: Follows from $|N_u \wedge N_v| = |K| |\varphi_u \wedge \varphi_v|$, even if K(p) = 0.

Remark 12. At the non-flat points, N preserves orientation if and only if K > 0. Hence we can remove the modulus if we define "oriented area".

§11. Vector fields

Recall: Trajectories (= integral curves), F.T. ODE, local flux. $\mathfrak{X}(M) = \Gamma(TM)$ For $f: N \to M, \ \mathfrak{X}_f = \Gamma(f^*(TM)).$

Exercise. If $X, Y \in \mathfrak{X}(M)$ and ξ is the local flux of X around p, then $[X, Y](p) = \lim_{t \to 0} \frac{1}{t} \left((\xi_{-t})_* Y(\xi_t(p)) - Y(p) \right).$

Lemma 13. Let M be any manifold, $p \in M$ and $X \in \mathfrak{X}(M)$ with $X(p) \neq 0$. Then, there is a coordinate system around p such that $X|_U = \partial/\partial x_1$. In particular, if M is a surface, there is a first integral of X in U, i.e., a function $f: U \to \mathbb{R}$ with $df_p \neq 0$ such that f is constant along trajectories of X.

Proof: Use the flux of X to construct a suitable chart in p for which $X|_U$ is a coordinate vector field.

Lemma 14. Same hypothesis as in Lemma 13, and $g \in \mathcal{F}(U) \Rightarrow$ there is $\mu \in \mathcal{F}(U), \mu > 0$, such that $X(\mu) = g$.

Proof: Use the flux of X to write this as an ODE. \blacksquare

Lemma 15. Two vector fields X, Y satisfy that [X, Y] = 0if and only if their fluxes commute: $\phi_t^X \circ \phi_s^Y = \phi_s^Y \circ \phi_t^X \ \forall t, s.$

Proof: At points where X = Y = 0 it is obvious, so assume $X \neq 0$. By the ' φ -related' property for Lie brackets of vector fields, we can assume our manifold is \mathbb{R}^n . Moreover, by Lemma 13, we can assume $X = \partial/\partial x_1 \cong e_1$, and the lemma follows easily.

Lemma 16. Same hypothesis as in Lemma 13 and $Y \in \mathfrak{X}(S)$ linearly independent with X in $p \Rightarrow$ there is a chart φ around p whose coordinate vector fields are collinear with X and Y.

Proof: Write [X, Y] = gX + fY and use Lemma 14 to find μ, λ positive functions such that [X', Y'] = 0, where $X' = \mu X$, $Y' = \lambda Y$. By Lemma 15 we can use them to build our chart.

Corollary 17. If p is not an umbilical point of $S \subset \mathbb{R}^3$, there is a coordinate system by lines of curvature around p.

Def.: Isometric and conformal maps, local isometries.

Proposition 18. Any surface (S, I) has isothermal charts.

Proof: Several proofs exist, few are elementary, and none easy... \blacksquare

§12. Ruled surfaces

 $\varphi(t,s) = \alpha(s) + tv(s), v \in \mathfrak{X}_{\alpha}, ||v|| = 1.$ < {v(s)} > = geratrix line, α = directrix curve.

Remark 19. S ruled $\Rightarrow K \leq 0$.

Examples:

• $v = \alpha'$: Tangent surface to α

• $\alpha(s) = (0, 0, as), v(s) = (\cos(s), \sin(s), 0) \implies helicoid.$

• If $v \equiv v_0$ constant \Rightarrow cylinder over a plane curve. Hence, we say that S is noncylindrical if v' never vanishes.

• $\alpha(s) = (\cos(s), \sin(s), 0), v = \pm \alpha' + e_3 \implies x^2 + y^2 - z^2 = 1$: the hyperboloid of revolution is doubly ruled.

• $\alpha(s) = (s, 0, 0), v(s) = (0, 1, s) \Rightarrow xy = z$, the hyperbolic paraboloid, that is also doubly ruled, since $(s, t, st) = te_2 + s(1, 0, t)$.

Remark 20. Besides the plane, these are the only 2 doubly ruled surfaces! How would you prove this??

Singularities of a ruled surface are contained in the striction curve:

Def.: For a noncylindrical ruled surface S, the *striction curve* is given by $\sigma(s) = \alpha(s) + t(s)v(s)$ for which $\langle \sigma', v' \rangle = 0$ (i.e., $t(s) = -\langle \alpha', v' \rangle / ||v'||^2$).

Notice that the striction curve does not depend on the directrix α . In particular, we can assume that $\sigma = \alpha$, that is, $\langle \alpha', v' \rangle = 0$.

§13. Minimal surfaces

The *brachistochrone problem* was formally posed by Johann Bernoulli as a challenge (he knew the answer using the *Fermat Principle*), but it appeared first in the *Discorsi*, of Galileo for lines, and observed that there was a quicker non straight solution (arguing then wrongly that the circle would be the fastest). Leibniz persuaded Bernoulli to extend the six month limit to solve the challenge for foreign mathematicians to be able to participate. Five more mathematicians solved the problem: Tschirnhaus, Jacob Bernoulli, Leibniz, de L'Hôpital, and... Isaac Newton, who was teased by Bernoulli and Leibniz, and solved the problem in one night. These solutions eventually lead to a general method by Euler to solve these kind of problems: the *calculus of variations*.

When does a surface minimize the area for "close enough" surfaces?

Proposition 21. Let $S \in \mathbb{R}^3$ be a compact surface (with or without boundary). Then, S is a critical point of the area functional A(S) if and only if H = 0.

Proof: It is enough to consider normal variations $i_t(p) = p + tfN(p)$. Now, compute a'(0), where $a(t) = Area(i_t(S))$.

Exercise. Conclude the same for hypersurfaces adapting the proof using that the volume element is given by $\sqrt{\det\langle\varphi_{u_i},\varphi_{u_j}\rangle} du_1 \wedge \cdots \wedge du_n$.

The Plateau problem: Find a minimal surface whose boundary is a given closed curve. Douglas (1931) and Radó (1933) prove general existence for arbitrary simple closed curves, but the surface could have singularities. Osserman (1970) and Gulliver (1973): a minimizing solution cannot have singularities. Regular solutions may not exist.

CMC (hyper)surfaces.

§14. Intrinsic Geometry

Intrinsic objects of a Riemannian manifold are the ones that only depend on the first fundamental form, i.e., *invariant by isometries*: distance, angle, area, volume...

Cylinder \cong plane \cong cone (locally): they are intrinsically the same thing, and hence the mean curvature H is <u>not</u> an intrinsic concept.

If $\varphi : U \to M$, $\varphi' : U \to M'$ are charts such that $g_{ij} = g'_{ij} \Rightarrow \varphi(U) \subset M$ and $\varphi'(U) \subset M'$ are isometric.

Example: For a surface of revolution

 $\varphi(u,v) = (f(v)\cos(u),f(v)\sin(u),g(v))$

we have $E = f^2$, F = 0, $G = f'^2 + g'^2$. In particular, the *catenoid*, where $f(v) = a \cosh(v)$ and g(v) = av for a > 0, has $E = G = a^2 \cosh^2(v)$, F = 0. Now, change variables on the helicoid

$$\varphi(\overline{u},\overline{v}) = \overline{v}(\cos(\overline{u}),\sin(\overline{u}),0) + a\overline{u}e_3,$$

 $\overline{v} = a \sinh(v), \, \overline{u} = u \text{ to get}$

$$\overline{\varphi}(u,v) = a(\sinh(v)\cos(u),\sinh(v)\sin(u),u),$$

that also has $E = G = a^2 \cosh^2(v)$, F = 0. Therefore, the catenoid and the helicoid are locally isometric (but not globally).

This is a general phenomenon for minimal surfaces in \mathbb{R}^3 : they have a one parameter family of isometric deformations.

Def.: Isometries, conformal diffeomorphisms.

Existence of isothermal coordinates \Rightarrow any two surfaces are locally conformally equivalent.

§15. The Gaussian curvature in coordinates, part 2

Given a chart $\varphi = \varphi(u_1, u_2) : U \to S$ on a surface $S \subset \mathbb{R}^3$, we have seen that $K = (eg - f^2)/(EG - F^2)$. Let's do this computation again in other way, by decomposing the second derivatives on their tangent and normal components:

$$\varphi_{ij} = \sum_{k} \Gamma^k_{ij} \varphi_k + r_{ij} N$$

The functions Γ_{ij}^k (that of course depend on φ) are called the *Christoffel symbols*. (In our previous notation, $r_{11} = e, r_{22} = g, r_{12} = f$). Taking inner product with φ_i , we have:

$$\Gamma_{11}^{1}E + \Gamma_{11}^{2}F = \langle \varphi_{11}, \varphi_{1} \rangle = \frac{1}{2}E_{u_{1}},$$

$$\Gamma_{11}^{1}F + \Gamma_{11}^{2}G = \langle \varphi_{11}, \varphi_{2} \rangle = F_{u_{1}} - \frac{1}{2}E_{u_{2}},$$

that can be written as

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} \Gamma_{11}^1 \\ \Gamma_{21}^2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}E_{u_1} \\ F_{u_1} - \frac{1}{2}E_{u_2} \end{pmatrix}$$

and similarly for the other indexes. In other words, we have:

Proposition 22. The Christoffel symbols Γ_{ij}^k depend only on the first fundamental form and its first derivatives.

Proof: Follows from the *Koszul formula*:

$$2\langle \varphi_{ij}, \varphi_k \rangle = \langle \varphi_i, \varphi_k \rangle_j + \langle \varphi_j, \varphi_k \rangle_i - \langle \varphi_i, \varphi_j \rangle_k. \quad \blacksquare$$

Exercise. Show that for a surface of revolution around a curve $\alpha(v) = (f(v), g(v))$, it holds that $\Gamma_{11}^1 = \Gamma_{12}^2 = \Gamma_{22}^1 = 0$, $\Gamma_{11}^2 = ff'/(f'^2 + g'^2)$, $\Gamma_{12}^1 = f'/f$, $\Gamma_{22}^2 = (ff'' + gg'')/(f'^2 + g'^2)$.

Now, we get relations that come from taking the tangent and normal components of $\varphi_{rij} = \varphi_{rji}$ and $N_{ij} = N_{ji}$ (3×3 equations if n = 2) that have the form

$$\sum_{k=1}^{n} c_{ijr}^{k} \varphi_{k} + d_{ijr}^{k} N = 0, \quad \forall 1 \le i, j, r \le n = 2.$$
(2)

In particular, taking the φ_2 component of $\varphi_{112} = \varphi_{121}$ we obtain the *Gauss equation*

$$K = \frac{1}{E} \left((\Gamma_{12}^2)_1 - (\Gamma_{11}^2)_2 + \Gamma_{12}^1 \Gamma_{11}^2 + \Gamma_{12}^2 \Gamma_{12}^2 - \Gamma_{11}^2 \Gamma_{22}^2 - \Gamma_{11}^1 \Gamma_{12}^2 \right)$$

We have proved the famous *Gauss' Egregium Theorem*:

Theorem 23 (Theorema Egregium = "outstanding"). The Gaussian curvature is an intrinsic concept (in fact, it depends only on I, ∂I , and $\partial^2 I$). In particular:

K is invariant by (local) isometries.

Corollary 24. $K_{catenoid}(p) = K_{helicoid}(\xi(p)).$

§16. The Codazzi-Mainardi equations

The other 5 tangential equations are other ways of writting the Gauss equation, or give 0 = 0. But the normal components give

two more equations, called the <u>Codazzi-Mainardi</u> equations:

$$e_v - f_u = e\Gamma_{12}^1 + f(\Gamma_{12}^2 - \Gamma_{11}^1) - g\Gamma_{11}^2.$$

$$f_v - g_u = e\Gamma_{22}^1 + f(\Gamma_{22}^2 - \Gamma_{12}^1) - g\Gamma_{12}^2.$$

Application: The relative nullity integrate as straight lines. These are the straight lines we 'see' in some ruled surfaces: $K \equiv 0$.

Remark 25. In a coordinate system by lines of curvature of a surface without umbilic points, Codazzi-Mainardi equations have the form

$$e_v = \frac{E_v}{2} \left(\frac{e}{E} + \frac{g}{G}\right), \quad g_u = \frac{G_u}{2} \left(\frac{e}{E} + \frac{g}{G}\right).$$

§17. Global application: The rigidity of $\mathbb{S}^2 \subset \mathbb{R}^3$

Lemma 26. Let $p \in S \subset \mathbb{R}^3$ regular such that K(p) > 0, and p is a local maximum of k_2 and a local minimum of k_1 $(k_1 \leq k_2)$. Then, p is umbilic.

Proof: Assume not, $k_1(p) < k_2(p)$, and take a chart at p by lines of curvature, $2H = k_1 + k_2 \Rightarrow k_1 = e/E$, $k_2 = g/G$. Now, by Remark 25 (Codazzi), $e_v = E_v H$, $g_u = G_u H \Rightarrow E(k_1)_v =$ $e_v - eE_v/E = E_v(k_2 - k_1)/2$, and $G(k_2)_u = -G_u(k_2 - k_1)/2$. In particular, $E_v(p) = G_u(p) = 0$. But the Exercise in page 31 says that $-2KEG = E_{vv} + G_{uu} + (\cdots)E_v + (\cdots)G_u$. Then, at p,

$$0 > -2KEG = E_{vv} + G_{uu} = 2(E(k_1)_{vv} - G(k_2)_{uu})/(k_2 - k_1) \ge 0.$$

Theorem 27. (Liebman): If $S \subset \mathbb{R}^3$ is a regular connected compact surface with $K = constant \Rightarrow S$ is a round sphere (i.e., S is umbilic). *Proof:* Notice that K > 0 by compactness. Now, consider the minimum of k_1 , and apply Lemma 26.

The same result follows with H = constant (Alexandrov). But a weaker version of it (for K > 0) already follows exactly as above:

Theorem 28. If $S \subset \mathbb{R}^3$ is a regular connected compact surface with K > 0 and $H = constant \Rightarrow S$ is a round sphere (i.e., S is umbilic).

§18. The Fundamental Theorem of surfaces in \mathbb{R}^3

We have seen that $S \subset \mathbb{R}^3 \Rightarrow$ Gauss eq. (intrinsic) + Codazzi equations (extrinsic). It turns out that there is no more information, or, equivalently, the converse holds locally:

Theorem 29 (FTS: Bonnet). Let E, F, G, e, f, g be differentiable functions on $V \subset \mathbb{R}^2$ with $E, G > F^2$ that satisfy Gauss and Codazzi-Mainardi equations. Then, each $q \in$ V has a neighborhood $q \in U \subset V$ and a diffeomorphism $\varphi: U \to \varphi(U) \subset \mathbb{R}^3$ such that E, F, G and e, f, g are the coefficients of the first and second fundamental forms of $\varphi(U)$, respectively, in the chart φ . In addition, if U is connected and $\overline{\varphi}$ is another chart with the same E, F, G, e, f, g, then there is a rigid motion $T \in \operatorname{Iso}(\mathbb{R}^3)$ such that $\overline{\varphi} = T \circ \varphi$.

Proof: (Sketch). For a chart φ , we define $f := (\varphi_u, \varphi_v, N) : V \to GL(3, \mathbb{R})$ where N is given by (1). Hence, there are two functions $P, Q : V \to \mathbb{R}^{3\times 3}$ such that $f_u = fP$, $f_v = fQ$. Gauss and Codazzi equations are precisely the integrability conditions

of this first order system of PDE: $P_v - Q_u = [P, Q]$ (Frobenius Theorem). Integrating once more, we get φ , and it is easy to check that it is a surface with the desired first and second fundamental forms (for details, see R. Palais notes <u>here</u>).

Remark 30. Take a long time comparing this with the FTC.

§19. The covariant derivative: affine connections

We want to <u>differentiate</u> vector fields on our surface (submanifold) $S \subset \mathbb{R}^3$. For this, we agree that, if $L \subset \mathbb{R}^m$ is a subspace and $v \in L$, $(v)_L$ denotes the orthogonal projection of v onto L.

Definition 31. Given $X \in \mathfrak{X}(S)$ and $v \in T_pS$, we define the covariant derivative of X in the direction v by

$$\nabla_v X = (X_{*p}(v))_{T_p S} \,.$$

- ∇ is an <u>intrinsic</u> operator: depends only on Γ_{ij}^k ;
- ∇ coincides with the usual derivative for $S = \mathbb{R}^n$, since $\Gamma_{ij}^k = 0$;

• $\nabla_v X$ is linear in v (tensorial!). So, we define for each $Y \in \mathfrak{X}(S)$ the vector field $\nabla_Y X \in \mathfrak{X}(S)$ by

$$(\nabla_Y X)(p) := \nabla_{Y(p)} X,$$

and this is tensorial in Y: $\nabla_{fY}X = f\nabla_Y X$;

• $\nabla_v X$ is a derivation in X: $\forall f \in \mathcal{F}(S), X \in \mathfrak{X}(S), v \in T_p S$,

$$\nabla_v f X = v(f) X(p) + f(p) \nabla_v X,$$

or, for $Y \in \mathfrak{X}(S)$,

$$\nabla_Y f X = Y(f) X + f \nabla_Y X.$$

a) In other words, we have:

$$\nabla:\mathfrak{X}(S)\times\mathfrak{X}(S)\to\mathfrak{X}(S)$$

 $(Y, X) \mapsto \nabla_Y X$, that is <u>tensorial in Y</u> and <u>a derivation in X</u>; b) ∇ is symmetric:

$$\nabla_X Y - \nabla_Y X = [X, Y], \ \forall X, Y \in \mathfrak{X}(M)$$

(it is enough to check for $X = \partial_i, Y = h\partial_j$ for any function h) c) ∇ is compatible with the metric:

$$X\langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle, \quad \forall \ X, Y, Z \in \mathfrak{X}(M)$$

• Such an operator satisfying (a) + (b) + (c) always exists and is unique by the *Koszul formula*:

$$2\langle \nabla_X Y, Z \rangle = X \langle Y, Z \rangle + Y \langle X, Z \rangle - Z \langle X, Y \rangle$$
$$- \langle X, [Y, Z] \rangle - \langle Y, [X, Z] \rangle + \langle Z, [X, Y] \rangle.$$

In the realm of Riemannian Geometry, ∇ is called the *Levi-Civita* connection of (S, \langle , \rangle) .

§20. Affine connections in vector bundles

Now, observe that, to have an affine connection (i.e., property (a) only), all we need is the vector bundle structure on the second variable, and not necessarily TS. Hence, we have:

Definition 32. Given a vector bundle $\pi : E \to M$, an *affine* connection in E is an <u> \mathbb{R} -bilinear</u> operator

$$\nabla: \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E),$$

 $(Y,\xi) \mapsto \nabla_Y \xi$, that is <u>tensorial in Y</u> and a derivation in ξ :

$$\nabla_Y f\xi = Y(f)\xi + f\nabla_Y \xi$$
$$\nabla_{fY}\xi = f\nabla_Y \xi,$$
$$\forall Y \mathfrak{X}(M), f \in \mathcal{F}(M), \xi \in \Gamma(E).$$

Remark 33. Bump functions + local sections \Rightarrow affine connections are first order differential operators. In particular, they are local: computations can be done in coordinates or local sections.

,

Therefore, if $X, Y \in \mathfrak{X}(M)$ and $\varphi : U \to M$ is a chart, we write on $V = \varphi(U), X|_V = \sum_i x_i \frac{\partial}{\partial u_i}, \quad Y|_V = \sum_i y_i \frac{\partial}{\partial u_i}$, and since $\nabla_{\frac{\partial}{\partial u_i}} \frac{\partial}{\partial u_j} = \sum_k \Gamma_{ij}^k \frac{\partial}{\partial u_k}$, in V we get for the Levi-Civita connection

$$(\nabla_X Y)|_V = \sum_k \left(\sum_i x_i \frac{\partial y_k}{\partial u_i} + \sum_{ij} x_i y_j \Gamma_{ij}^k \right) \frac{\partial}{\partial u_k}.$$
 (3)

Remark 33 also implies:

Proposition 34. Suppose E is a vector bundle with a connection $\hat{\nabla}$. Then, for each smooth map $f: N \to M$, there is a unique pull-back connection $\hat{\nabla}^f$ on f^*E satisfying that

$$\hat{\nabla}^f_X(\xi \circ f) = \hat{\nabla}_{f_*X}\xi, \quad \forall \ X \in \mathfrak{X}(N), \xi \in \Gamma(E).$$

Proof: Since connections are local objects, it is enough to do the computation locally. But if $\{\xi_i\}$ is a local frame of E in $U \subset M$, $\{\xi_i \circ f\}$ is a local frame of f^*E in $V = f^{-1}(U)$. So, if $\eta \in \Gamma(f^*E)$,

we write η on V as $\eta|_V = \sum_i z_i \ \xi_i \circ f$. Just by the definition of a connection and its local nature, on V we get

$$\hat{\nabla}_X^f \eta = \sum_i \left(X(z_i)\xi_i \circ f + z_i \hat{\nabla}_{f_*X}\xi_i \right). \tag{4}$$

This implies the uniqueness of $\hat{\nabla}^f$. But we can define $\hat{\nabla}^f$ locally with (4): it is easy to check that $\hat{\nabla}^f$ defined this way is well defined, and a connection.

§21. Affine connections along maps

If (M, \langle , \rangle) is a Riemannian manifold, the ONLY affine connection ∇ on TM that we will consider is the Levi-Civita connection of \langle , \rangle . If M is an Euclidean submanifold, we know how to construct ∇ from the standard vector field derivative on \mathbb{R}^m (that is itself the Levi-Civita connection of \mathbb{R}^m with the standard inner product seen as a Riemannian metric).

As a particular case of Proposition 34, we have:

Proposition 35. Given $f: N \to (M, \langle , \rangle)$, there is a unique affine connection $\nabla^f (= f^* \nabla)$ in $f^*(TM)$,

 $\nabla^f:\mathfrak{X}(N)\times\mathfrak{X}_f\to\mathfrak{X}_f,$

called the <u>affine connection along f</u>, that satisfies

 $\nabla^f_X(Y \circ f) = \nabla_{f_*X}Y, \quad \forall X \in \mathfrak{X}(N), Y \in \mathfrak{X}(M).$

In particular, for a curve $\alpha : I \to M$, we obtain:

• A notation: if $X \in \mathfrak{X}_{\alpha}$,

$$X' := \nabla_{d/dt} X.$$

• We have the *acceleration* of α (intrinsic!):

$$\alpha'' := \nabla_{d/dt} \ \alpha',$$

• and the *geodesic curvature* of α (intrinsic!):

$$\kappa_g = \kappa_g^\alpha := \|\alpha''\|.$$

• "Compute derivatives using curves": For any $Z \in \mathfrak{X}(M) \Rightarrow Z \circ \alpha \in \mathfrak{X}_{\alpha}$ and

$$(Z \circ \alpha)' = \nabla_{d/dt}(Z \circ \alpha) = \nabla_{\alpha'} Z.$$

• For curves in a submanifold, $\alpha : I \to M \subset \mathbb{R}^m$, if $X \in \mathfrak{X}_{\alpha}$, $Z \in \mathfrak{X}(M)$, and $v = \alpha'(0) \in T_pM$, we have:

$$X' := \nabla_{d/dt} X = \left(\frac{dX}{dt}\right)_{T_{\alpha}M} \in \mathfrak{X}_{\alpha}$$
$$\alpha'' = \left(\frac{d^{2}\alpha}{dt^{2}}\right)_{T_{\alpha}M} \in \mathfrak{X}_{\alpha}$$
$$\nabla_{v} Z = \left(\frac{d}{dt}|_{t=0}(Z \circ \alpha)\right)_{T_{p}M}$$
(5)

and the three curvatures of α are related by

$$\kappa^2 = \kappa_g^2 + \kappa_n^2.$$

• Eq. (5) also implies that: if two submanifolds are tangent along a curve α , their connections along α coincide.

• If M is a hypersurface, $(v)_{TM} = v - \langle v, N \rangle N$, and (5) becomes

$$\nabla_{v} Z = \frac{d}{dt} |_{t=0} (Z \circ \alpha) - \langle A_{p} v, Z(p) \rangle N(p).$$

Exercise. A connection is compatible with a metric $\langle , \rangle \Leftrightarrow \langle V, W \rangle' = \langle V', W \rangle + \langle V, W' \rangle$, for ever curve α and every $V, W \in \mathfrak{X}_{\alpha}$ (notice that these are different "'").

§22. Parallel transport

By (3), if we write a curve α locally as $\alpha = \varphi(\alpha_1, \ldots, \alpha_n)$, and $Y = \sum_i y_i \frac{\partial}{\partial u_i} \circ \alpha \in \mathfrak{X}_{\alpha}$, then

$$Y' = \sum_{k} \left(y'_{k} + \sum_{ij} \alpha'_{i} y_{j} \Gamma^{k}_{ij} \circ \alpha \right) \frac{\partial}{\partial u_{k}} \circ \alpha \tag{6}$$

We say that $Y \in \mathfrak{X}_{\alpha}$ is *parallel* if Y' = 0. Since the last equation is linear, the set of parallel vector fields along α , denoted by $\mathfrak{X}_{\alpha}^{\parallel}$, is a vector space. Also by this equation we easily see:

Proposition 36. Given a curve $\alpha : I \to M$, $p = \alpha(t_0)$, for every $v \in T_pM$ there exists a unique parallel vector field $\mu_v \in \mathfrak{X}_{\alpha}$ such that $\mu_v(t_0) = v$.

So, this map $v \mapsto \mu_v$ is an isomorphism between $T_p M$ and $\mathfrak{X}_{\alpha}^{\parallel}$. In particular, if t is another point in I, we get a linear isomorphism

$$P^{\alpha}_{t_0,t}: T_{\alpha(t_0)}M \to T_{\alpha(t)}M,$$

given by $P_{t_0,t}^{\alpha}(v) = \mu_v(t)$. Notice that it depends smoothly on everything.

Def.: This isomorphism $P_{t_0,t}^{\alpha}$ is called the *parallel transport* <u>along α </u> between $\alpha(t_0)$ and $\alpha(t)$. <u>Beware</u>: along α !! It does depend on α , not just on $\alpha(t_0)$ and $\alpha(t)$ (in contrast to \mathbb{R}^m).

Examples: $M = \mathbb{R}^m$: usual. Meridian in \mathbb{S}^2 : Cylinder. Parallel in \mathbb{S}^2 : Cone \Rightarrow after a complete turn, the parallel transport does not close \Rightarrow dependency on α .

Remark 37. By the previous exercise $P_{t_0,t}^{\alpha}$ are linear isometries.

Exercise. Prove that a connection is compatible with the metric $\langle , \rangle \Leftrightarrow \langle V, W \rangle$ is constant, for every curve α and every $V, W \in \mathfrak{X}_{\alpha}^{\parallel}$.

§23. Geodesics

Lemma 38. Given $\varphi: U \subset \mathbb{R}^2 \to M \Rightarrow \nabla_{\partial_u} \varphi_v = \nabla_{\partial_v} \varphi_u$.

Proof: Use coordinates and the symmetry of ∇ (it's equivalent).

Proposition 39. A curve α parametrized by arc-length is a critical point of the arc-length functional if and only if $\alpha'' = 0$.

Proof: Calculus of variations! :-) \blacksquare

Def.: A curve α with $\alpha'' = 0$ is called a *geodesic*.

Local and intrinsic concept \Rightarrow invariant by local isometries

Remark 40. Let α be a non-constant curve in (M, \langle , \rangle) .

- α is a geodesic $\Rightarrow ||\alpha'|| = \text{constant} \Rightarrow \alpha$ is regular.
- α and $\alpha \circ h$ are geodesics $\Leftrightarrow h(t) = at + b$.
- α is a geodesic $\Leftrightarrow \kappa_q^{\alpha} = 0.$
- If α is a straight line segment in $M \subset \mathbb{R}^m \Rightarrow \alpha$ is a geodesic (once we parametrize it by arc length).

Examples:

- Great circles in round spheres are geodesics;
- More generally, meridians in surfaces of revolution are geodesics;
- Any helix inside cylinders;

• Given two points in a cylinder (not in the same parallel), there are infinitely many geodesics joining them. But if we take a line from the cylinder, we recover uniqueness (and existence).

• In $\mathbb{R}^2 \setminus \{0\}$, (1,0) and (-1,0) have no geodesic joining them.

 α is a geodesic $\Leftrightarrow \alpha' \in \mathfrak{X}_{\alpha}^{\parallel}$. Then, by (6), we have that $\alpha = \varphi(\alpha_1, \ldots, \alpha_n)$ is a geodesic \Leftrightarrow

$$\alpha_k'' = -\sum_{i,j=1}^n \alpha_i' \alpha_j' \Gamma_{ij}^k \circ \alpha.$$
⁽⁷⁾

This is the *differential equation of geodesics*, and implies:

Proposition 41. For every $p \in M$ and every $v \in T_pM$, there is $\epsilon > 0$ and a unique geodesic $\gamma_v : (-\epsilon, \epsilon) \to M$ such that $\gamma_v(0) = p, \ \gamma'_v(0) = v.$

In fact, γ_v also depends smoothly on v.

§24. Geodesics in a surface of revolution

Let $\varphi(u, v) = (f(v) \cos(u), f(v) \sin(u), g(v))$ be a surface of revolution with axis z and geratrix $\alpha(v) = (f(v), g(v))$ parametrized by arc length: $\|\alpha'\|^2 = f'^2 + g'^2 = 1$, f > 0. Notice that f is the distance from the surface to the axis of revolution. Then, (7) becomes

$$u'' = -2\frac{f'}{f}u'v', \quad v'' = ff'u'^2.$$
 (8)

We get from this:

• Meridians are geodesics (very easy to see without this!)

• Parallels are geodesics \Leftrightarrow the distance function r = f to the axis is critical: we can see this also geometrically with a picture.

Remark 42. The first equation in (8) can also be written as $f^2u' = c$ = constant. Now, the angle $\theta \in [0, \pi/2]$ between the geodesic and the parallel that it intersects is given by $\cos(\theta) = |\langle \varphi_u / || \varphi_u ||, u' \varphi_u + v' \varphi_v \rangle| = |u' f|$. Therefore, we have the *Clairaut relation:*

$$r\cos(\theta) = \text{constant},$$

where r = distance to the axis, $\theta = \text{angle}$ with parallel.

Let γ be a geodesic (p.b.a.l.) that is neither a parallel nor a meridian. Then, $f^2u' = c \neq 0$ is constant. But $1 = ||\gamma'||^2 = f^2u'^2 + v'^2$ (by differentiating again, this implies the second equation in (8)). So, $v' = \sqrt{f^2 - c^2}/f$, $u' = c/f^2$, and

$$u = c \int \frac{1}{f\sqrt{f^2 - c^2}} \, dv + u_0.$$

In other words, we have *integrated* all the geodesic equations. This is extremely rare, and we should thank the Clairaut relation, that resumes the information about the geodesics.

Application: Let γ be a geodesic on the paraboloid of revolution $z = x^2 + y^2$ that is not a meridian. Then, $r \cos(\theta) = |c| \neq 0$ $\Rightarrow \theta$ grows with r, and $\theta = 0$ (i.e. γ tangent to a parallel) only at one point, the unique parallel r = |c| (limit of geodesics is a geodesic, so it cannot accumulate over the parallel). Therefore, γ intersects itself infinitely many times since it cannot be asymptotic to a meridian (u = constant), since otherwise

$$u - u_0 = c \int \frac{1}{v} \sqrt{\frac{1 + 4v^2}{v^2 - c^2}} \, dv > c \int \frac{dv}{v} \to +\infty.$$

§25. The covariant derivative on oriented surfaces

Let (S^2, \langle , \rangle) oriented. Then, the oriented rotation of angle $\pi/2$ on TS is a skew symmetric orthogonal tensor J with $J^2 = -Id$. For $w \in TS$ we use the notation $\overline{w} = Jw$.

Let c be a regular curve in S^2 , and $w \in \mathfrak{X}_c$ with ||w|| = 1. Then, $w' = \lambda \overline{w}$ along c for some function $\lambda =: [w']$, called the *algebraic* value of w'. In other words, $[w'] = \langle w', \overline{w} \rangle$. Accordingly, if c is a curve in S^2 parametrized by arc-length, we have the (oriented) geodesic curvature of c,

$$\kappa_g^c = [c''].$$

Actually, for any manifold N and any map $f : N \to S^2$, if $w \in \mathfrak{X}_f$ is unitary, we have a 1-form $[\nabla w]$ over N given by

$$[\nabla w](X) = \langle \nabla_X w, \overline{w} \rangle.$$

Now, if $w, e \in \mathfrak{X}_f$ are unitary $\Rightarrow w = ae + b\overline{e}, a^2 + b^2 = 1 \Rightarrow$

Lemma 43. With the notations above, assume N is simply connected, and fix $p \in N$. If $\cos \xi_0 = a(p)$ and $\sin \xi_0 = b(p)$, then there is a unique <u>differentiable</u> function $\xi = \sphericalangle(w, e)$: $N \to \mathbb{R}$ such that $\cos \xi = a$, $\sin \xi = b$, and $\xi(p) = \xi_0$. **Proof:** Define $\sigma \in \Omega^1(N)$ by $\sigma(X) = aX(b) - bX(a)$. Since $a^2 + b^2 = 1$ we easily check that σ is closed, hence exact. Define now ξ by $d\xi = \sigma$, $\xi(p_0) = \xi_0$. The lemma follows simply by differentiating $(a - \cos \xi)^2 + (b - \sin \xi)^2 = 2 - 2(a \cos \xi + b \sin \xi)$.

Def.: Given w and $e \in \mathfrak{X}_f$ unitary, the differentiable function $\triangleleft(w, e)$ is called a *determination of the angle between* w and e. For a non vanishing $X \in \mathfrak{X}(S)$, we set $\triangleleft(w, X) := \triangleleft(w, \frac{X}{\|X\|} \circ f)$. **Lemma 44.** If $f : N \to S$ with N simply connected, and $w, e \in \mathfrak{X}_f$ are unitary $\Rightarrow [\nabla w] - [\nabla e] = d\xi$, where $\xi = \triangleleft(w, e)$. *Proof:* Just compute $[\nabla w]$ using that $w = \cos(\xi) e + \sin(\xi) \overline{e}$.

Remark 45. In particular, if α is parametrized by arc-length and $w \in \mathfrak{X}_{\alpha}^{\parallel} \Rightarrow \kappa_{g}^{\alpha} = [\alpha''] = \xi'$, where $\xi = \sphericalangle(w, \alpha')$. Therefore: the geodesic curvature of a curve is the rate of change of the angle of its tangent and a parallel vector field along it.

From now on, $\varphi: U \to S$ will be an orthogonal oriented chart of S, and N any simply connected manifold.

Lemma 46. Let $f : N \to \varphi(U) \subset S$, and write $f(x) = \varphi(u(x), v(x))$ for some $u, v : N \to \mathbb{R}$. If $w \in \mathfrak{X}_f$ is unitary,

$$[\nabla w] = \frac{1}{2\sqrt{EG}}(G_u dv - E_v du) + d\xi, \quad \text{where } \xi = \sphericalangle(w, \varphi_u).$$

Proof: Define the vector fields $V = \varphi_u / \sqrt{E} \in \mathfrak{X}_{\varphi}$ (unitary), and $e(x) = V(u(x), v(x)) \in \mathfrak{X}_f$. By Lemma 44, $[\nabla w] - d\xi = [\nabla e]$. But

$$\begin{aligned} [\nabla V](\partial_u) &= \langle \nabla_{\partial_u}(\varphi_u/\sqrt{E}), \varphi_v/\sqrt{G} \rangle = \langle \nabla_{\partial_u}\varphi_u, \varphi_v \rangle/\sqrt{EG} \\ &= -\langle \nabla_{\partial_u}\varphi_v, \varphi_u \rangle/\sqrt{EG} = -E_v/2\sqrt{EG}, \end{aligned}$$

by Lemma 38. Analogously, $[\nabla V](\partial_v) = G_u/2\sqrt{EG}$. The lemma follows from $\nabla_X e = \nabla_X (V \circ (u, v)) = \nabla_{X(u)\partial_u + X(v)\partial_v} V$.

Corollary 47. If $\alpha(s) = \varphi(u(s), v(s))$ is a curve p.b.a.l.,

$$\kappa_g^{\alpha} = \frac{1}{2\sqrt{EG}}(G_u v' - E_v u') + \xi', \quad \text{where } \xi = \sphericalangle(\alpha', \varphi_u).$$

Exercise. In an orthogonal chart, $-2K\sqrt{EG} = \left(\frac{G_u}{\sqrt{EG}}\right)_u + \left(\frac{E_v}{\sqrt{EG}}\right)_v$.

§26. The local Gauss-Bonnet Theorem

Gauss: Geodesic triangles and excess.

In this section, S will be an oriented surface, charts will be compatible with its orientation, and $\alpha : I \to S$ will be a simple, closed, piecewise regular curve with vertices $\{\alpha(t_1), \ldots, \alpha(t_n)\}$ \Rightarrow oriented external angle $\theta_i \in [-\pi, \pi]$ at the vertex $\alpha(t_i)$. Let $\varphi : U \cong D^2 := \{x \in \mathbb{R}^2 : ||x|| < 1\} \to S$ be a chart, $\alpha(I) \subset \varphi(U)$, and $\xi_i = \sphericalangle(\alpha'|_{[t_i, t_{i+1}]}, \varphi_u) : [t_i, t_{i+1}] \to \mathbb{R}$ $(t_{n+1} := t_1)$.

Proposition 48. (Turning tangents) With these notations,

$$\sum_{i=1}^{n} (\xi_i(t_{i+1}) - \xi_i(t_i)) + \sum_{i=1}^{n} \theta_i = \pm 2\pi,$$

where the RHS sign depends on the orientation of α .

Proof: The LHS is the total change of the angle between α' and φ_u . Since α is closed, this is $2k\pi$, for some integer k, hence invariant under homotopies of α . Since α is simple, $k = \pm 1$.

Def.: We say that a compact region $R \subset S$ is *simple* if $R \cong \overline{D^2}$ and ∂R is the trace of a closed simple piecewise regular curve α , that we orient in such a way that $\overline{\alpha'}$ points to the interior of R.

Theorem 49. (Gauss-Bonnet; local) Let $R \subset S$ be a simple region contained in the image of an orthogonal oriented chart $\varphi: U \cong D^2 \to S$, and let $\alpha: I \to \partial R$ oriented p.b.a.l. with vertices $\alpha(s_i)$ and external oriented angles θ_i . Then,

$$\sum_{i} \int_{s_i}^{s_{i+1}} \kappa_g^{\alpha} + \sum_{i} \theta_i + \int_R K = 2\pi.$$

Proof: Write $\alpha(s) = \varphi(u(s), v(s))$. Integrating Corollary 47 and using Green's Theorem and the previous exercise we obtain

$$\begin{split} \sum_{i} \int_{s_{i}}^{s_{i+1}} \kappa_{g}^{\alpha} \, ds &- \sum_{i} \left(\xi_{i}(s_{i+1}) - \xi_{i}(s_{i}) \right) \\ &= \sum_{i} \int_{s_{i}}^{s_{i+1}} \left(\frac{G_{u}}{2\sqrt{EG}} v' - \frac{E_{v}}{2\sqrt{EG}} u' \right) ds \\ &= \int_{\varphi^{-1}(R)} \left(\left(\frac{G_{u}}{2\sqrt{EG}} \right)_{u} + \left(\frac{E_{v}}{2\sqrt{EG}} \right)_{v} \right) du dv \\ &= -\int_{\varphi^{-1}(R)} (K \circ \varphi) \sqrt{EG} \, du dv = -\int_{R} K. \end{split}$$

Now the result follows from Proposition 48. \blacksquare

Remark 50. Let $\alpha : I \to S$ a regular simple closed curve parametrized by arc-length, and $w \in \mathfrak{X}_{\alpha}^{\parallel}$ unitary. Then,

$$0 = \int_0^{\ell} [w'] = \int_0^{\ell} \left(\frac{G_u}{2\sqrt{EG}} v' - \frac{E_v}{2\sqrt{EG}} u' \right) ds + \int_0^{\ell} \xi'$$

$$= -\int_R K + \xi(\ell) - \xi(0) = \Delta \xi - \int_R K,$$

where $\xi = \sphericalangle(w, \varphi_u)$. So, $\lim_{R \to p} \frac{\Delta \xi}{A(R)} = K(p)$, and we conclude:

The parallel transport is (locally) independent of the path $\alpha \iff K \equiv 0$.

§27. Gauss-Bonnet: What's happening?

By Lemma 46 and the previous exercise, if w is any unit vector field on an open set $V \subset S$, then on V it holds that

$$-KdA = d\left[\nabla w\right].\tag{9}$$

(Notice that Lemma 44 explains why the LHS does not depend on w). Thus, if R is a small simple region with smooth boundary parametrized by an oriented α , and $\xi = \sphericalangle(\alpha', w)$, we get from Lemma 44 and Stokes' theorem applied to (9) that

$$\int_R K = -\int_{\partial R} [\nabla w] = -\int_0^\ell [(w \circ \alpha)'] = \int_0^\ell (\xi' - \kappa_g^\alpha) = 2\pi - \int_0^\ell \kappa_g^\alpha.$$

Now, we get Theorem 49 from this by approximating R with domains with regular boundaries.

§28. The global Gauss-Bonnet Theorem

Recall: Triangulations and the Euler characteristic of manifolds $\chi(M)$ (see Theorem 33 in our Analysis on Manifolds notes <u>here</u>).

 $Example: \text{ If } n\text{-torus} := \mathbb{S}^2 + n \text{ handles } \Rightarrow \chi(n\text{-torus}) = 2 - 2n.$

For compact connected surfaces it holds that:

 $\chi(S) = \chi(S') \Leftrightarrow S$ is (diffeo) homeomorphic to S' (!!)

If S is orientable $\Rightarrow 4 - \chi(S) \in 2\mathbb{N}$. Therefore, by Jordan's Theorem, the only compact regular surfaces in \mathbb{R}^3 are the *n*-tori (up to diffeomorphism). Thus, the number g of "handles" of S, $g := \frac{2-\chi(S)}{2} \in \mathbb{N}_0$, is called the *genus* of S.

Def.: A region $R \subset S$ is *regular* if it is compact and ∂R is a disjoint union of simple closed piecewise differential curves.

Theorem 51. (Gauss-Bonnet; global) Let R be a regular region of an oriented surface S, and let $\partial R = \bigcup_{i=1}^{k} C_i$ positively oriented with external angles $\theta_1, \ldots, \theta_m$. Then,

$$\sum_{i=1}^{k} \int_{C_i} \kappa_g^{\alpha} + \sum_{j=1}^{m} \theta_j + \int_R K = 2\pi \chi(R).$$

Proof: Take a fine triangulation of R, $\mathcal{T} = \{T_i, \ldots, T_F\}$, such that each triangle lies in a simple orthogonal coordinate system, and orient each triangle T_i according to the orientation of S. Let E be the number of edges and V the number of vertices of the triangulation, E_i and E_e the number of internal and external edges, respectively, and V_i and V_e the number of internal and external external vertices, respectively. In addition, we have $V_e = V_{ec} + V_{et}$, where V_{ec} is the number of external vertices on regular points of ∂R .

Adding the local Gauss-Bonnet Theorem 49 for each T_i gives

$$\sum_{r=1}^{k} \int_{C_r} \kappa_g^{\alpha} + \sum_{r=1}^{F} \sum_{j=1}^{3} \theta_{rj} + \int_R K = 2\pi F,$$

where θ_{rj} are the 3 oriented external angles of T_r , since the internal edges of each triangle get opposite orientations. Call $\beta_{rj} := \pi - \theta_{rj}$ the internal angles of the triangle T_r . Thus,

$$3\pi F - \sum_{rj} \theta_{rj} = \sum_{rj} \beta_{rj} = 2\pi V_i + \pi V_{et} + \sum_{l=1}^m (\pi - \theta_l).$$

Now, since each C_r is closed, $E_e = V_e$. Moreover, by counting each triangle edges we get $3F = 2E_i + E_e$. And, since $m = V_{ec}$,

$$\sum_{rj} \theta_{rj} - \sum_{l=1}^{m} \theta_l = \pi (2E_i + E_e - 2V_i - V_{et} - m) = \pi (2E - 2V). \quad \blacksquare$$

Corollary 52. The total curvature of a compact oriented surface is a purely topological invariant:

$$\int_{S} K = 2\pi \chi(S)$$

Corollary 53. Local Gauss-Bonnet for simple regions.

Corollary 54. S compact orientable surface with $K \ge 0 \Rightarrow$ $S \cong \mathbb{S}^2$ or $S \equiv S^1 \times S^1$. If, in addition, $S \subset \mathbb{R}^3 \Rightarrow S \cong \mathbb{S}^2$.

Corollary 55. S orientable with $K \leq 0 \Rightarrow 2$ geodesics do not enclose a simple region. In particular, a closed geodesic or a geodesic loop do not enclose a simple region.

Corollary 56. $S \cong$ cylinder with $K < 0 \implies S$ has at most one closed geodesic (compare with the flat cylinder).

Corollary 57. S compact, $K > 0 \Rightarrow$ two closed geodesics intersect (compare with $K \ge 0$: a flat cylinder with two caps).

Corollary 58. (Gauss) The excess in the internal angles of a geodesic triangle is equal to its total curvature. \Rightarrow

Corollary 59. \mathbb{H}^2 : the fifth Euclid axiom is independent.

§29. Application: Total Index of a vector field

Def.: The *index* I(p) of isolated singularity p of a vector field $X \in \mathfrak{X}(S)$ is the integer given by

$$2\pi I(p) = \int_0^\ell \tau' = \tau(\ell) - \tau(0) = \Delta \tau,$$

 $au = \sphericalangle(X, \varphi_u) \circ \alpha$, for a small curve α around p. By Remark 50, $\int_R K - 2\pi I(p) = \Delta(\xi - \tau) = \Delta(\sphericalangle(w, X)),$

that does not depend on φ (it is also independent of α).

Therefore, if X is a vector field in a compact oriented surface with isolated singularities $\{p_1, \ldots, p_n\}$ (a generic property), by choosing a smart triangulation we get $\int_S K - 2\pi \sum_i I(p_i) = 0$, since the boundaries of the triangles appear twice with opposite orientations. By Corollary 52 we thus have for the *total index* $\sum_i I(p_i)$ of X:

Theorem 60. (Poincaré-Hopf) The total index of a vector field in a compact oriented surface S with isolated singularities is equal to the Euler characteristic of $S: \sum_i I(p_i) = \chi(S)$.

PART II

Prerequisites (part II): Analysis on manifolds, Stokes and de Rham cohomology. Bibliography: [KN] Vol.II, Ch.12; [S] Vol.V Ch.13; [MS]

§30. Fiber and principal bundles ([KN], Vol. I, Ch. 1.4, 1.5)

Lie group G; left-invariant vector fields \cong Lie algebra $\mathfrak{g} = T_eG$: $V_v(g) := (L_g)_{*e}v.$

Proposition 61. $v \in \mathfrak{g} \leftrightarrow$ one parameter subgroup β_t^v of G.

Proof: If $\xi_t(g)$ the flux of V_v , $\beta_s^v = \xi_s(e) \Rightarrow \beta_0^{v'} = v$, and $V_v(\beta_s^v\beta_t^v) = (L_{\beta_s^v})_{*\beta_t^v}V_v(\beta_t^v) = (L_{\beta_s^v})_{*\beta_t^v}\beta_t^{v'} \Rightarrow \beta_s^v\beta_t^v$ is an integral curve of V_v passing at β_s^v for $t = 0 \Rightarrow \beta_s^v\beta_t^v = \beta_{s+t}^v$.

Exercise. Show that the flux of V_v is $(g, s) \mapsto g\beta_s^v = R_{\beta_s^v}(g)$.

Representations; Adjoint representation: $ad_G: G \to \operatorname{End}(\mathfrak{g})$

Lemma 62. If $v \in \mathfrak{g}$, $[v, \cdot] = (ad_G)_{*e}(v) = \frac{d}{ds}_{|_{s=0}} ad_{\beta_s^v}$.

Proof: By the exercise in §11, since $R_{\beta_s^v}(g)$ is the flux of V_v ,

$$[v,w] = [V_v, V_w](e) = \lim_{s \to 0} \frac{1}{s} ((R_{\beta_{-s}^v})_* V_w(\beta_s^v) - V_w(e))$$

= $\lim_{s \to 0} \frac{1}{s} ((R_{\beta_{-s}^v})_* (L_{\beta_s^v})_{*e} w - w) = \frac{d}{ds} ad_{\beta_s^v}(w).$

Group actions on manifolds: $R : E \times G \to E$; free actions Fiber bundles $F \to E \xrightarrow{\pi} B$ with typical fiber $E_p \cong F$, total space E and base BTransition functions: $\hat{\xi}_{UV} : U \cap V \to Diff(F)$ Structure group of a fiber bundle: G-bundles: **Def.:** A *G*-bundle is a fiber bundle $F \to E \xrightarrow{\pi} B$ together with a left action on F by $G, \rho : G \times F \to F$, such that the transition functions are given through ρ , that is, there are $\xi_{UV} : U \cap V \to G$ and $\hat{\xi}_{UV}(x)(f) = \rho(\xi_{UV}(x), f)$.

(transition functions act on the left)

Exercise. Show that a rank k vector bundle is a $Gl(k, \mathbb{R})$ -bundle.

Exercise. Show that the pull-back and Whitney sum of vector bundles is a vector bundle.

Exercise. Show that the tangent bundle of \mathbb{S}^3 is trivial.

Example: Sphere bundles

Bundle maps, bundle isomorphism

Def.: A principal G-bundle is as a G-bundle $\pi : E \to B$ with fiber G where the structure group acts on the fibers by left multiplication.

Remark 63. By the associativity of the group, the right multiplication by G on the fiber commutes with the action of the structure group (left multiplication). So we get an invariant right action by G on E. This action preserves the fibers of E and acts freely and transitively on them. Actually, principal bundles can be defined with the use of a free right action transitive on the fibers.

Examples: Product $B \times G$. If (S, \langle , \rangle) is an oriented surface, the unit tangent bundle T_1S is an \mathbb{S}^1 -principal bundle. The (orthonormal) frame bundle of a rank k vector bundle is a principal (O(k)-bundle) $GL(k, \mathbb{R})$ -bundle. Projective spaces.

Exercise. Show that all these are principal bundles.

Exercise. Show that the \mathbb{S}^1 and \mathbb{S}^3 Hopf bundles are principal bundles.

Exercise. Show that the lens spaces $\mathbb{S}^3/\mathbb{Z}_k$ for $\mathbb{Z}_k \subset \mathbb{S}^1$ are circle bundles over \mathbb{S}^2 .

Fact: Principal *G*-bundles "generate" all *G*-bundles, via the *as*sociated bundles (we will see this in $\S34$).

(Cross-)sections

Local sections \Leftrightarrow (equivariant) local trivializations

Obs.: A principal bundle E has a global section $\Leftrightarrow E$ is trivial. Compare to vector bundles, that always have 0 as a global section.

§31. Connections ([KN], Vol. I, Ch. 2.1, 2.5)

Def.: The *vertical* subbundle $\mathcal{V} \subset TE$ is the vector bundle over B given by $\mathcal{V} = \operatorname{Ker} \pi_*$.

Fundamental vec. fields: If $\xi_p(g) = R_g(p) = pg, v \in \mathfrak{g} \mapsto v^* \in \mathfrak{X}(\mathcal{V}),$

$$v^*(p) := \frac{d}{dt}_{|_{t=0}}(p\beta_t^v) = \xi_{p*e}(v).$$

Notice that $(t, p) \mapsto p\beta_t^v = R_{\beta_t^v}(p)$ is the flux of v^* .

Exercise. Show that $v^* \circ \xi_p = \xi_{p_*} \circ V_v$, i.e., $v^* \stackrel{\xi_p}{\sim} V_v$ for all $p \in E$. In particular, $[v^*, w^*] = [v, w]^*$, i.e., $v \mapsto v^*$ is an algebra homomorphism.

Def.: A connection on a principal bundle E is a differentiable map that assigns to each $x \in E$ a subspace $\mathcal{H}_x \subset T_x E$ such that:

• $TE = \mathcal{V} \oplus \mathcal{H};$

• \mathcal{H} is *G*-invariant: $\mathcal{H}_{pg} = (R_g)_{*p}(\mathcal{H}_p), \quad \forall \ p \in E, g \in G.$ **Obs.:** $TE = \mathcal{V} \oplus \mathcal{H} \Rightarrow \pi_{*p}|_{\mathcal{H}_p} : \mathcal{H}_p \to T_{\pi(p)}B$ is isomorphism **Def.:** A type ad_G k-form on E is a g-valued k-form $\sigma : TE \times \cdots \times TE \to \mathfrak{g}$ that satisfies

$$R_q^*\sigma = ad_{q^{-1}} \circ \sigma, \quad \forall \ g \in G.$$

Def.: A principal connection on E is a type ad_G 1-form w: $TE \to \mathfrak{g}$ such that $\mathcal{H} = \operatorname{Ker} w$, and $w(v^*) = v$ for all $v \in \mathfrak{g}$.

Exercise. There is a 1-1 correspondence between the two type of connections (w is of type $ad_G \Leftrightarrow \mathcal{H}$ is G-invariant).

Exercise. Principal connections always exist (use partitions of unity).

 \mathcal{H} and \mathcal{V} components: $X = X^h + X^v$. Define $h(X) := X^h$.

Obs.: Given $X \in \mathfrak{X}(B)$, there is a unique $X^* \in \mathfrak{X}(E)$, called the *lift of* X, such that $X \in \mathcal{H}$ and $\pi_*(X^*) = X \circ \pi$ (i.e., $X^* \stackrel{\pi}{\sim} X$). Moreover, the lift is *G*-invariant (i.e., $X^* \stackrel{R_g}{\sim} X^*$), and, conversely, every *G*-invariant horizontal vector field on E is a lift.

Exercise. $[X^*, Y^*]^h = [X, Y]^*$.

Lemma 64. If v^* is a fundamental vector field and $Y \in \mathcal{H}$, $[v^*, Y] \in \mathcal{H}$. If, in addition, Y is a lift, then $[v^*, Y] = 0$.

Proof: By exercise in §11, $[v^*, Y] = \lim_{t\to 0} \frac{1}{t} ((\xi_{-t})_* Y \circ \xi_t - Y),$ where $\xi_t = R_{\beta_t^v}$ is the flux of v^* . Thus, $((\xi_{-t})_* Y)(\xi_t(p)) \in \mathcal{H}_p$.

Def.: A k-form σ on E is *horizontal* if $\mathcal{V} \subseteq \text{Ker } \sigma$, i.e., if it vanishes if one of the entries is vertical.

Lemma 65. Let σ be a horizontal k-form in E that is Ginvariant, i.e., $R_g^*\sigma = \sigma$. Then, σ projects to $\overline{\sigma}$, i.e., there exist a k-form $\overline{\sigma}$ in B such that $\sigma = \pi^*\overline{\sigma}$.

Proof: Define $\overline{\sigma}(Z_1, \ldots, Z_k)(x) = \sigma(Z_1^*, \ldots, Z_k^*)(p)$, where $p \in \pi^{-1}(x)$. This is independent of p since the lift is G-invariant.

If σ is a k-form on E, we define the k-form σ^h by

$$\sigma^h(X_1,\ldots,X_k) := \sigma(X_1^h,\ldots,X_k^h).$$

If σ is of type ad_G , then so $d\sigma$ and σ^h are (because $(R_g)_* \circ h = h \circ (R_g)_*$). In addition, σ^h is always horizontal. So:

Def.: The horizontal (k + 1)-form $D\sigma = (d\sigma)^h$ is called the *exterior covariant derivative* of σ (it is of type ad_G if σ is).

Lemma 66. If σ projects to $\overline{\sigma}$, then $D\sigma = d\sigma$.

Proof: The obvious: $d\sigma(Y_0, \ldots, Y_k) = d(\pi^*\overline{\sigma})(Y_0, \ldots, Y_k) = d\overline{\sigma}(\pi_*Y_0, \ldots, \pi_*Y_k) = d\overline{\sigma}(\pi_*Y_0^h, \ldots, \pi_*Y_k^h) = d\sigma(Y_0^h, \ldots, Y_k^h).$

§32. The curvature of a principal connection

Def.: If w is a connection form on E, the horizontal 2-form Dw is of type ad_G and is called the *curvature form of* E.

From now on, $\Omega := Dw$ will be the curvature 2-form of (E, w).

Proposition 67. (Structure equation) $\Omega = dw + [w, w]$, *i.e.*,

 $\Omega(X,Y) = dw(X,Y) + [w(X),w(Y)], \quad \forall X,Y \in TE.$

Proof: If $X, Y \in \mathcal{H}$, follows from the definition of D. If $X = A^* \in \mathcal{V}$ is fundamental, $\Omega(A^*, \cdot) = 0$ because it is vertical. Now, if $B^* \in \mathcal{V}$ is fundamental, $dw(A^*, B^*) = A^*(w(B^*)) - B^*(w(A^*)) - w([A^*, B^*]) = A^*(B) - B^*(A) - [A, B] = -[A, B] = -[w(A^*), w(B^*)]$. Now, if $Y \in \mathcal{H}$, $[w(A^*), w(Y)] = 0$ and $dw(A^*, Y) = -w([A^*, Y]) = 0$ by Lemma 64. ∎

Proposition 68. (Bianchi's identity) $D\Omega = 0$.

Proof: We need to check that $d\Omega = d([w, w]) = 0$ for 3 horizontal vector fields, but this is immediate from $\mathcal{H} \subset \operatorname{Ker} w$.

Lemma 69. If σ is a horizontal 1-form of type ad_G ,

$$D\sigma = d\sigma + [\sigma, w] + [\omega, \sigma].$$

Proof: The only nontrivial case is for $v^* \in \mathcal{V}$ fundamental and $Y^* \in \mathcal{H}$ a lift. But $D\sigma(v^*, Y^*) = 0$, and $d\sigma(v^*, Y^*) = v^*(\sigma(Y^*))$ since $[v^*, Y^*] = 0$ by Lemma 64. Since Y^* is *G*-invariant, by Lemma 62,

$$\begin{split} v^*(\sigma(Y^*))(p) &= \sigma(Y^*(p\beta_s^v))' = \sigma((R_{\beta_s^v})_{*p}Y^*(p))' \\ &= ((R_{\beta_s^v}^*\sigma)(Y^*(p)))' = (ad_{\beta_s^{-v}}(\sigma(Y^*(p))))' \\ &= [-v, \sigma(Y^*(p))] = -[w(v^*), \sigma(Y^*)](p). \blacksquare \end{split}$$

§33. Weil homomorphism ([KN], Vol. II, Ch. 12.1)

Let $I^k(G)$ be the set of symmetric k-multilinear maps over \mathfrak{g} , f: $\mathfrak{g} \times \cdots \times \mathfrak{g} \to \mathbb{R}$ and ad_G -invariant, i.e. $f(ad_gX_1, \ldots, ad_gX_k) = f(X_1, \ldots, X_k)$. This is a vector space, and $I(G) = \bigoplus_{k=0}^{\infty} I^k(G)$ is a graded algebra with the natural product $(fg)(t_1, \ldots, t_{k+s}) = \frac{1}{(k+s)!} \sum_{\sigma} f(t_{\sigma_1}, \ldots, t_{\sigma_k}) g(t_{\sigma_{k+1}}, \ldots, t_{\sigma_{k+s}}).$ Now, let E be a principal G-bundle with principal connection 1-form w and curvature 2-form Ω . For $f \in I^k(G)$, we define the 2k-form $f(\Omega) = f(\Omega, \ldots, \Omega)$ on E by

$$f(\Omega)(X_1,\ldots,X_{2k}) = \frac{1}{(2k)!} \sum_{\sigma} \operatorname{sign}(\sigma) f(\Omega(X_{\sigma_1},X_{\sigma_2}),\ldots,\Omega(X_{\sigma_{2k-1}},X_{\sigma_{2k}})).$$

Theorem 70. (A. Weil) For each $f \in I^k(G)$, the (2k)-form $f(\Omega) \in \Omega^{2k}(E)$ projects to a unique <u>closed</u> (2k)-form $\overline{f(\Omega)} \in \Omega^{2k}(B)$. Moreover, its cohomology class

$$\omega_f = [\overline{f(\Omega)}] \in H^{2k}(B)$$

is independent of the choice of the connection, and $\omega: I(G) \to H^{*}(B)$ is an algebra homomorphism, called Weil homomorph.

Proof: Since Ω is horizontal by definition, so is $f(\Omega)$. Since Ω is of type ad_G and f is ad_G -invariant, $f(\Omega)$ is G-invariant: $R_g^*(f(\Omega)) = f(\Omega)$. By Lemma 65 $f(\Omega)$ projects: $f(\Omega) = \pi^* \overline{f(\Omega)}$. Proposition 68 says that $D\Omega = 0$, and hence $D(f(\Omega)) = 0$. By Lemma 66, $f(\Omega)$ is closed, and so is $\overline{f(\Omega)}$ since π_* is onto.

For the second part, take w_1 and w_2 two principal connections on E, and define $w_t := w_0 + t(w_1 - w_0)$. Obviously, w_t and $\alpha = w_1 - w_0$ are also of type ad_G , and $\alpha(\mathcal{V}) = 0$. Let D_t and Ω_t be the exterior covariant differentiation and curvature form of w_t , respectively. By Proposition 67, $\Omega_t = D_t w_t = dw_t + [w_t, w_t]$, so, by Lemma 69, $\frac{d}{dt}\Omega_t = D_t\alpha$. Therefore, by Proposition 68,

$$\frac{d}{dt}f(\Omega_t) = kf(D_t\alpha, \Omega_t, \dots, \Omega_t) = kD_t(f(\alpha, \Omega_t, \dots, \Omega_t))$$
$$= kd(f(\alpha, \Omega_t, \dots, \Omega_t)),$$

where the last equality follows from Lemma 65 and Lemma 66: since both α and Ω_t are horizontal and of type ad_G , and f is ad_G invariant, so $\beta_t := f(\alpha, \Omega_t, \ldots, \Omega_t)$ is horizontal and G-invariant. But then β_t also projects to a (k-1)-form on B, and so does $\Phi = k \int_0^1 \beta_t dt$. We conclude from the above that $d\Phi = f(\Omega_1) - f(\Omega_0)$ also projects, and thus $\overline{f(\Omega_1)} - \overline{f(\Omega_0)} = d\overline{\Phi}$ is exact.

It's easy to check that ω is an algebra homomorphism (exercise).

Remark 71. Notice that the homology class is in the base B, not in the total space E!!

Def.: The class ω_f is called the *characteristic class of* E associated to f, that, by Theorem 70, depends <u>only</u> on the isomorphism class of the bundle, and <u>not</u> on the choice of the connection.

§34. Associated bundles

Take a *G*-bundle $F \to E \to B$ with its *G*-action $\rho: G \times F \to F$ and transition functions $\xi_{UV}: U \cap V \to G$. If F' is another manifold where *G* acts via $\rho': G \times F' \to F'$, we can construct another *G*-bundle $F' \to E' \to B$ associated to the original one by using the same transition functions ξ_{UV} but simply changing *F* by *F'* and ρ by ρ' (see here for details).

In particular, we can take F' = G and $\rho' =$ left multiplication to get the *G*-principal bundle associated to the original one.

This allows us to define the characteristic classes of any G-bundle as the characteristic classes of its associated principal bundle. In particular, for a real vector bundle of rank k its associated $Gl(k, \mathbb{R})$ -principal bundle is nothing but its frame bundle.

§35. The shortcut for vector bundles ([MS], Appendix C)

Let's show a much more direct approach for vector bundles. Notice that, since $G = Gl(n, \mathbb{R}) \subset \mathbb{R}^{n \times n}$ is open, $\mathfrak{g} = \mathbb{R}^{n \times n}$. Let $\mathbb{R}^n \to P \xrightarrow{\pi} M$ be a real rank *n* vector bundle over a manifold M, and ∇ an affine connection on P. Given a local frame $e = \{\xi_1, \ldots, \xi_n\}$ of $\pi^{-1}(U) \cong U \times \mathbb{R}^n$, $U \subset M$, write

$$\nabla_X \xi_j = \sum_i \Gamma_j^i(e)(X)\xi_i.$$

So, $\omega(e) = (\Gamma_j^i(e))$ are \mathfrak{g} -valued 1-forms on U that determine ∇ .

Exercise. For $g: U \to G$, $\omega(eg) = g^{-1}dg + ad_{g^{-1}}(\omega(e))$.

Define the \mathfrak{g} -valued *curvature 2-form* $\Omega(e)$ of ∇ by

$$\nabla_X \nabla_Y \xi_j - \nabla_Y \nabla_X \xi_j - \nabla_{[X,Y]} \xi_j = \sum_i \Omega_j^i(e)(X,Y) \xi_i.$$

It is easy to check that Ω satisfies the *structure equation* (compare with Proposition 67):

$$\Omega(e) = d\omega(e) + [\omega(e), \omega(e)],$$

the *Bianchi identity* (compare with Proposition 68):

$$d\Omega(e) = [\Omega(e), \omega(e)]$$

(i.e., $[\Omega, \omega]_j^i = \sum_k (\Omega_k^i \wedge w_j^k - w_k^i \wedge \Omega_j^k)$, or $[\Omega, \omega](X_1, X_2, X_3) = \frac{1}{2} \sum_{\sigma \in S_3} [\Omega(X_{\sigma_1}, X_{\sigma_2}), \omega(X_{\sigma_3})])$ and changes as $\Omega(eg) = g^{-1}\Omega(e)g = ad_{g^{-1}}(\Omega(e))$ (exercises). Thus, if f is an ad_G -invariant homogeneous polynomial of degree k as before, $f(\Omega)$ is a <u>well defined</u> (i.e. independent of the local frames e) and thus <u>global</u> (2k)-form on M^n . In addition, $f(\Omega)$ is closed (easy exercise using Bianchi), so $[f(\Omega)] \in H^{2k}(M)$. Now, $[f(\Omega)]$ does not depend on the affine connection. If ∇^1, ∇^2 are two affine connections on P, then $\nabla^t = (1-t)\nabla^0 + t\nabla^1$ is also an affine connection on P. Consider the projection π_1 : $M \times \mathbb{R} \to M$ and $i_t : M \to M \times \mathbb{R}, i_t(x) = (x, t)$. The connection $\hat{\nabla} = \pi_1^* \nabla^t$ is an affine connection on the vector bundle $\pi_1^*(P) \to M \times \mathbb{R}$, so the corresponding $f(\hat{\Omega})$ is closed on $M \times \mathbb{R}$. But $i_{\epsilon}^*(f(\hat{\Omega})) = f(\Omega_{\epsilon})$, for $\epsilon = 0, 1$ and, since i_0 and i_1 are homotopic, $[f(\Omega_0)] = [f(\Omega_1)]$.

35.1 Affine connections \Leftrightarrow principal connections

Let's see that the two constructions agree. Given P the vector bundle above, its frame bundle of $G \to \mathcal{F}(P) \xrightarrow{\hat{\pi}} M$ is a principal G-bundle, a trivializing neighborhood of which is $\mathcal{F}(\hat{\pi}^{-1}(U)) \cong$ $U \times G$. If w is a principal connection on $\mathcal{F}(P)$, and $e: U \subset$ $M \to \hat{\pi}^{-1}(U) \subset \mathcal{F}(P)$ is a local section, the equation

$$\omega(e) = e^* w \tag{10}$$

relates w with the affine connection form ω . Then, check that $\Omega_{\omega}(e) = e^* \Omega_w$, and so the forms $f(\Omega_w)$ project precisely to $f(\Omega_{\omega})$:

$$\hat{\pi}^*(f(\Omega_\omega)) = \hat{\pi}^*(f(e^*\Omega_w)) = \hat{\pi}^*e^*(f(\Omega_w)) = f(\Omega_w),$$
 (11)

where for the last equality we used that $f(\Omega_w)$ projects.

Exercise. If ω and w are forms related by (10), then ω is well defined, and it is a principal connection $\Leftrightarrow w$ is an affine connection (form).

Now, put a Riemannian metric on P and work with the orthonormal frame bundle. If e is an orthonormal frame and ∇ is compatible we get $\Gamma_j^i(e) = -\Gamma_i^j(e)$, so $\omega(e)$ is still a \mathfrak{g} -valued 1-form on U but now for $\mathfrak{g} = \mathfrak{o}(n)$, and we play as before. In particular, all this holds for P = TM when M is Riemannian.

35.2 Gauss-Bonnet: What's REALLY happening??

We can now understand more deeply the Gauss-Bonnet theorem: **1.** TS as an oriented vector bundle. If S is a oriented Rieman-

1. If S as an oriented vector bundle. If S is a oriented Riemannian surface, its Levi-Civita connection (form) of an orthonormal oriented frame $e = \{e_1, e_2\}$ is a standard 1-form since $\mathfrak{so}(2) = \mathbb{R}$, and is given by $\omega(e) = -[\nabla e_1]$. Its curvature form is $\Omega(e) =$ $d\omega(e) = KdA$ by (9). Taking f(t) = t, $\Omega = f(\Omega(e)) = KdA$ is a well-defined and global closed 2-form, whose cohomology class is independent of the compatible affine connection, in particular, independent of the metric, and so is $\int K$.

Now, if e_1 is globally defined but in a finite set $\{p_i\}$ the curvature form is then exact almost everywhere. We remove ϵ -small disks D_i^{ϵ} around each p_i and we use Stokes and Theorem 60 to get

2. T_1S as a $SO(2) = \mathbb{S}^1$ -principal bundle. The \mathbb{S}^1 action on T_1S is given by $u\theta = \cos(\theta)u + \sin(\theta)\overline{u}$. We can choose as a principal connection 1-form $w(u_*(X)) = -[\nabla u](X)$, where u is a section of T_1S and $X \in TS$. Since \mathbb{S}^1 is abelian, the curvature 2-form of w is $\hat{\Omega} = dw$ and projects to a closed two form on Swhose cohomology class does not depend on the metric. Indeed,

$$u^*\hat{\Omega} = u^*dw = du^*w = d\omega(\{u,\overline{u}\}) = \Omega(\{u,\overline{u}\}) = \Omega$$

does not depend on u and therefore $\pi^*\Omega = \hat{\Omega}$.

§36. Invariant polynomials ([KN], Vol. II, Ch. XII.2)

Let $P^k(\mathbb{V})$ be the homogeneous polynomial functions on the (finite dimensional) vector space \mathbb{V} of degree k (polynomial by taking a basis), and $P(\mathbb{V}) = \bigoplus_{k=0}^{\infty} P^k(\mathbb{V})$ the natural algebra of polynomial functions. Let $S^k(\mathbb{V})$ be the set of symmetric k-multilinear functions on \mathbb{V} , with $S(\mathbb{V}) = \bigoplus_{k=0}^{\infty} S^k(\mathbb{V})$ its natural commutative algebra.

Proposition 72. (Polarization). The map $\tau: S(\mathbb{V}) \mapsto P(\mathbb{V})$ given by $\tau(h)(t) = h(t, \ldots, t)$ is an algebra isomorphism.

Proof: If $\{\xi^1, \ldots, \xi^n\}$ is a basis of \mathbb{V}^* , $f \in P^k(V)$ can be written as $\sum f_{i_1 \ldots i_k} \xi^{i_1} \cdots \xi^{i_k}$, for some $f_{i_1 \ldots i_k} \in \mathbb{R}$ symmetric in the indexes. The function $\Phi(f)(t_1, \ldots, t_k) = \sum f_{i_1 \ldots i_k} \xi^{i_1}(t_1) \cdots \xi^{i_k}(t_k)$ is the inverse of τ (exercise).

Exercise. If $G \subset L(\mathbb{V})$ is a subgroup, the isomorphism above induces an isomorphism between the G-invariant subalgebras $S_G(\mathbb{V})$ and $P_G(\mathbb{V})$.

Corollary 73. $I(G) \cong P(G)$, where P(G) are the ad_G -invariant polynomial functions in \mathfrak{g} .

36.1 The unitary group: $U(n) = \{X \in \mathbb{C}^{n \times n} : X\overline{X}^t = I\}.$

Its Lie algebra is $\mathfrak{u}(n) = \{A \in \mathbb{C}^{n \times n} : \overline{A}^t = -A\}$. If $A \in \mathfrak{u}(n)$,

$$\det(\lambda I + \frac{i}{2\pi}A) = \lambda^n - \sigma_1(A)\lambda^{n-1} + \sigma_2(A)\lambda^{n-2} - \dots + (-1)^n \sigma_n(A).$$

Then, the polynomial functions σ_i are $ad_{U(n)}$ -invariant. In fact, if it_1, \ldots, it_n are the eigenvalues of A, then $\sigma_i(A)$ is the *i*-th symmetric function on t_1, \ldots, t_n . And these are all:

Proposition 74. The polynomial functions $\sigma_1, \ldots, \sigma_n$ are $ad_{U(n)}$ -invariant, algebraically independent, and generate (as algebra) $P_{U(n)}(\mathfrak{u}(n))$.

Proof: See Theorem 2.5 in Kobayashi-Nomizu, Vol 2, Cap. XII. ∎

Corollary 75. The characteristic classes $c_k(E) := \omega_{\sigma_k} \in H^{2k}(B), 1 \leq k \leq n$, generate all the characteristic classes of an U(n)-principal bundle $U(n) \to E \to B$ as an algebra. They are called the <u>Chern classes</u> of the bundle.

36.2 The orthogonal group: $O(n) = \{X \in \mathbb{R}^{n \times n} : XX^t = I\}.$

Its Lie algebra is $\mathfrak{o}(n) = \{A \in \mathbb{R}^{n \times n} : A^t = -A\}$. Define

$$\det(\lambda I - \frac{1}{2\pi}A) = \lambda^n + p_1(A)\lambda^{n-2} + p_2(A)\lambda^{n-4} + \dots + \dots$$

Then, the polynomial functions $p_1, \ldots, p_{[n/2]}$ are $ad_{O(n)}$ -invariant. In fact, if $\pm it_1, \ldots, \pm it_{[n/2]}$ are the eigenvalues of A (besides the 0 if n is odd), then $p_i(A)$ is the *i*-th symmetric function on $t_1^2, \ldots, t_{[n/2]}^2$. And these are all:

Proposition 76. The polynomial functions $p_1, \ldots, p_{[n/2]}$ are $ad_{O(n)}$ -invariant, algebraically independent, and generate (as algebra) $P_{O(n)}(\mathfrak{o}(n))$.

Proof: See Theorem 2.6 in Kobayashi-Nomizu, Vol 2, Cap. XII. ∎

Corollary 77. The characteristic classes $p_k(E) := \omega_{p_k} \in H^{4k}(B), 1 \le k \le \lfloor n/2 \rfloor$, generate all the characteristic classes of an O(n)-principal bundle as an algebra. They are called the Pontrjagin classes of the bundle.

36.3 Special orthogonal group: $SO(n) = \{X \in O(n) : \det(X) = 1\}$.

Being SO(n) the connected component of O(n) containing the identity, their Lie algebras coincide $\mathfrak{so}(n) = \mathfrak{o}(n)$, and the situation is very similar to that of O(n). However, for n = 2m even, there is a (unique up to sign) SO(n) invariant homogeneous polynomial function pf such that $pf^2 = p_m$, called the *pfaffian*. In terms of matrixes, $pf(A)^2 = \det(A)$, and is given by

$$pf(A) = \frac{1}{2^m m!} \sum_{\sigma \in S_{2m}} \operatorname{sign}(\sigma) \prod_{i=1}^m a_{\sigma(2i-1)\sigma(2i)}.$$
 (12)

Hence, we have:

Proposition 78. For n = 2m - 1 (resp. n = 2m) the polynomial functions p_1, \ldots, p_{m-1} (resp. p_1, \ldots, p_{m-1}, pf) are $ad_{SO(n)}$ -invariant, algebraically independent, and generate (as algebra) $P_{SO(n)}(\mathfrak{so}(n))$.

Proof: See Theorem 2.7 in Kobayashi-Nomizu, Vol 2, Cap. XII. ∎

Corollary 79. The characteristic classes $p_k(E) := \omega_{p_k} \in H^{4k}(B), 1 \leq k \leq [\frac{n-1}{2}]$, together with $e(E) := (2\pi)^{-n/2}\omega_{pf} \in H^n(B)$ if n is even, generate all the characteristic classes of an SO(n)-principal bundle as an algebra. The classes $p_i(E)$ are called the <u>Pontrjagin classes</u> of the bundle, while, for n even, e(E) is called the <u>Euler class</u> of the bundle.

Remark 80. In particular, the three subsections apply for complex, real, and oriented real vector bundles, where the terminology Chern, Pontrjagin and Euler classes are usually applied (resp.), by means of Section 35. By definition, the classes of a vector bundle are the classes of its frame principal bundle.

Total Chern and Pontrjagin classes:

$$c(E) = 1 + c_1(E) + c_2(E) + \dots \in H^*(B),$$

$$p(E) = 1 + p_1(E) + p_2(E) + \dots \in H^*(B).$$

§37. The axiomatic approach

Suppose $E_1 \oplus E_2$ is a Whitney sum of two (real or complex) vector bundles. By the previous section, since the classes come from determinants, the total class for E is the product of the total classes of E_1 and E_2 : $c(E_1 \oplus E_2) = c(E_1) \wedge c(E_2)$ (for complex) $p(E_1 \oplus E_2) = p(E_1) \wedge p(E_2)$ (for real). Moreover, extending the definition of the Euler class to odd dimensional real vector bundles as 0, for the Euler class it also holds that $e(E_1 \oplus E_2) = e(E_1) \wedge e(E_2)$; see [**S**], Vol.5, Ch.13, Theorem 22. In particular: *if the orientable vector bundle E has a nowhere vanishing section, then* e(E) = 0.

In fact, there is a way of defining characteristic classes for vector bundles in an axiomatic way: we proved that they exist, and it is not hard to see that they are unique. For example, for Chern classes for complex vector bundles (CVB) we have:

• Axiom 1: For each CVB E over M, and each integer $k \ge 0$, there exist a class $c_k(E) \in H^{2k}(M)$, with $c_0(E) = 1$ (so we can define $c(E) = \sum_{i=0}^{\infty} c_k(E) \in H^*(M)$, the total Chern class of E.) • Axiom 2 (Naturality): If E is a CVB over M and $f: M' \to M$ is smooth, then $c(f^*E) = f^*c(E)$. Axiom 3 (Whitney sum formula): If E, E' are CVBs over M and E₁ ⊕ E' their Whitney sum, then c(E ⊕ E') = c(E) ∧ c(E').
Axiom 4 (Normalization): If CP¹ is the complex projective line and P its canonical complex line bundle, then ∫_{CP¹}c₁(P) = -1.

Exercise. Show that the Chern classes for CVBs as we defined satisfy the four axioms above, thus proving existence; see §35 and $[\mathbf{KN}]$ II c.13.

Axiomatically, the Pontrjagin classes of a real vector bundle E are defined simply by $p_k(E) = c_{2k}(E \otimes \mathbb{C})$.

For oriented real vector bundles of rank k, the Euler class is defined with the same axioms as the Chern classes, except that, in Axiom 1, we require $e(E) \in H^k(M)$, and e(M) = 0 if k is odd.

§38. The Poincaré-Hopf Theorem in all dimensions

Let $X \in \mathfrak{X}(M^n)$ be a vector field on an oriented M^n with an isolated singularity at $p \in M^n$. If we restrict X to the boundary of a small ball B_{ϵ} around p, we have $\mathbb{S}_p^{n-1} := T_1 M \cap T_p M$ and

$$V = X/||X|| : \partial B_{\epsilon} \cong \mathbb{S}^{n-1} \to T_1 B_{\epsilon} \cong B_{\epsilon} \times \mathbb{S}_p^{n-1}$$

for some trivializing chart $\varphi : B_{\epsilon} \times \mathbb{S}_{p}^{n-1} \to T_{1}B_{\epsilon}$ with $\varphi \circ i_{p}$ being the inclusion $\mathbb{S}_{p}^{n-1} \subset T_{1}B_{\epsilon}$, where $i_{p} : \mathbb{S}_{p}^{n-1} \to B_{\epsilon} \times \mathbb{S}_{p}^{n-1}$, $i_{p}(v) = (p, v)$. We define the *index of* X *at* p as the integer

$$I(p) = \deg(V),$$

where $\hat{V} = \pi_2 \circ \varphi^{-1} \circ V : \partial B_{\epsilon} \cong \mathbb{S}^{n-1} \to \mathbb{S}_p^{n-1}$. Notice that, for ϵ small, $V \cong \varphi \circ i_p \circ \hat{V}$ as smooth functions.

With these definitions, the Poincaré-Hopf Theorem 60 holds for any compact oriented manifold and any vector field with isolated singularities, and not just for surfaces. Indeed, by the proof of the Gauss-Bonnet-Chern Theorem 83 below it follows that the total index of a vector field is a topological invariant, i.e., does not depend on the vector field. But it is easy to construct a vector field whose total index is the Euler characteristic: for a triangulation \mathcal{T} , define $V_{\mathcal{T}}$ as having precisely one singularity on the 'center' of each simplex of \mathcal{T} in a way so that the flow lines of the vector field point from the centers of higher dimensional simplexes towards the lower dimensional simplexes. Such a vector field has total index equal to the Euler characteristic.

§39. The Gauss-Bonnet-Chern Theorem ([L])

Consider a compact oriented even dimensional Riemannian manifold M^{2m} . Its tangent bundle is an SO(2m)-bundle, and so it has its Euler Class, $e(TM) \in H^{2m}(M) \cong \mathbb{R}$. So its integral

$$\int e(TM) \in \mathbb{R}$$

is a topological invariant that does not depend on the Riemannian metric. In terms of the curvature Ω of the Levi-Civita connection, $(12) \Rightarrow (2\pi)^m e(TM)$ is represented by the 2*m*-form

$$pf(\Omega) = \frac{1}{2^m m!} \sum_{\sigma \in S_{2m}} \operatorname{sign}(\sigma) \ \Omega_{\sigma_2}^{\sigma_1} \wedge \dots \wedge \Omega_{\sigma_{2m}}^{\sigma_{2m-1}}$$

Now, consider the sphere bundle $\pi: T_1M \to M$. Then,

$$\pi^*(pf(\Omega)) \in \Omega^{2m}(T_1M).$$

Lemma 81. (S.S. Chern; Transgression Lemma) There is $\lambda \in \Omega^{2m-1}(T_1M)$ such that $\pi^*(pf(\Omega)) = d\lambda$. In addition, $\int_{\mathbb{S}_p^{2m-1}} \lambda|_{\mathbb{S}_p^{2m-1}} = (2\pi)^m$, for all $p \in M$.

Proof: A long algebraic construction... See [L], Lemma 3.2.3.

Remark 82. For m = 1, since $G = SO(2) \cong \mathbb{S}^1$ is abelian, by (9), (10), (11) and Proposition 67 we can take $\lambda = w$ in Lemma 81.

Theorem 83. (Generalized Gauss-Bonnet-Chern Theorem) If M^{2m} is compact and orientable, then

$$\int e(TM) = \chi(M^{2m}).$$

Proof: Let X be a vector field with isolated singularities only $\{p_1, \ldots, p_r\}$. Remove small balls $B_{\epsilon}(p_i)$ from M, and define $M_{\epsilon} = M \setminus \bigcup_i B_{\epsilon}(p_i)$ and $V = X/||X|| : M_{\epsilon} \to T_1 M_{\epsilon}$. Then, by the Transgression Lemma 81 and Stoke's Theorem,

$$\int w_{pf} = \lim_{\epsilon \to 0} \int_{M_{\epsilon}} pf(\Omega) = \lim_{\epsilon \to 0} \int_{M_{\epsilon}} V^*(\pi^*(pf(\Omega)))$$
$$= \lim_{\epsilon \to 0} \int_{M_{\epsilon}} V^*(d\lambda) = \lim_{\epsilon \to 0} \int_{M_{\epsilon}} d(V^*\lambda)$$
$$= \sum_{i} \lim_{\epsilon \to 0} \int_{\partial B_{\epsilon}(p_i)} V^*\lambda = \sum_{i} \lim_{\epsilon \to 0} \int_{\partial B_{\epsilon}(p_i)} \hat{V}^*((\varphi \circ i_{p_i})^*\lambda)$$
$$= \sum_{i} I(p_i) \int_{\mathbb{S}_{p_i}^{2m-1}} \lambda|_{\mathbb{S}_{p_i}^{2m-1}} = (2\pi)^m \sum_{i} I(p_i).$$

Therefore, the total index is a topological invariant, independent of the vector field X. But we saw in §38 that there exists a vector field whose total index is equal to $\chi(M^{2m})$. **Remark 84.** The above is essentially Chern's original proof in **[C]**. For an alternative proof using characteristic classes more deeply, see **[S]**, Vol 5, Ch. 13, Theorem 26.

Remark 85. Again, notice that we not only proved the Gauss-Bonnet-Chern Theorem, but also Poincaré-Hopf Theorem 60 for any dimensions.

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