Analysis on Manifolds

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Bibliography: [Tu], [Sp], [Le], [Ha], [Hi], [Hr]...

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§1. Manifolds

We want to extend calculus: object needs to be *locally* a vector space. *Example:* $\mathbb{S}^n$.
Topological space, neighborhood, covering.
Countable basis.
Hausdorff (T_2).
REM: Countable basis and Hausdorff are inherited by subspaces.
Locally Euclidean Topological space: charts and coordinates.
Dimension, notation: $\dim\, M^n = n$.

Topological manifold = Topological space + Locally Euclidean + Countable basis + Hausdorff.
Examples: $\mathbb{R}^n$, graph, cusp. Not a manifold: '$\times$' ($\subset \mathbb{R}^2$).

Compatible C^∞–charts, transition functions, atlas (always C^∞).
Example: $\mathbb{S}^n$: $\pi_N : \mathbb{S}^n \setminus \{-N\} \to N^\perp$ *stereographic projection*:

$$\pi_N(x) = \frac{x_{N^\perp}}{1-\langle x, N\rangle}, \quad \pi_N^{-1}(y) = \frac{2y-(1-\|y\|^2)N}{1+\|y\|^2}, \quad \pi_{-N} \circ \pi_N^{-1}(y) = \frac{y}{\|y\|^2}.$$

Differentiable structure = maximal (C^∞) atlas.

REM: By a theorem due to Whitney, every maximal C^k–atlas for $k > 0$ contains a "unique" C^∞–atlas. Not true for $k = 0$: there exist topological manifolds which admit no C^1–structure.

From now on, for us: Manifold = differentiable manifold = smooth manifold = Topological manifold + maximal atlas.
Examples: $\mathbb{R}^n$, $\mathrm{End}(\mathbb{V}^n)$, $\mathbb{S}^n$, $U \subset M^n$ open, $GL(n, \mathbb{R})$, graphs, products.

§2. Differentiable functions between manifolds

Definition, composition, diffeomorphism, local diffeomorphism.
Examples: Every chart is a diffeo with its image; function from/to a product. Ex.: $f : \mathbb{R}^2 \to \mathbb{R}$ defined as $f(s,t) = st/(s^2+t^2)$ outside the origin and $f(0,0) = 0$ satisfies that, for all $x \in \mathbb{R}$, $f(x,\cdot) \in C^\infty$, $f(\cdot,x) \in C^\infty$, yet f is not even continuous at the origin. So video 2 at 42:00 is awfully wrong...
Partial derivatives, Jacobian matrix, Jacobian.
Lie Groups, examples: $Gl(n,\mathbb{R})$, $\mathbb{S}^1$, $\mathbb{S}^3$.
Right and left translations: L_g, R_g for $g \in G$.

§3. The moduli space

As you know, $\mathbb{R}^{n^2}$ and the set of square matrices $\mathbb{R}^{n\times n}$ are isomorphic as vector spaces. This means that, although they are different *as sets*, they are indistinguishable *as vector spaces*: every inherent property *of vector spaces* is satisfied by both. In fact, the dimension is the only vector space property that distinguishes between vector spaces (of finite dimension over the same field). Now, regard $M := \mathbb{R}$ as a topological manifold, and $N := \mathbb{R}$ as a smooth manifold. Consider the map $\tau : M \to N$ given by $\tau(t) = t^3$. Since τ is a homeomorphism, the topologies and therefore the sets of continuous functions on M and N agree: $C^0(M) = C^0(N)$. On the other hand, since τ is a bijection, there is a unique differentiable structure on M such that τ is a diffeomorphism, that is, the one induced by $\{\tau\}$ as an atlas. Let $\hat{M}$ be M with this differentiable structure. Now, although $\hat{M} = N$ as sets (and as topological manifolds), $\hat{M} \neq N$ as smooth manifolds, since τ is not even an immersion when we regard on $M = \mathbb{R}$ the

standard differentiable structure of $\mathbb{R}$. In fact, $\mathcal{F}(\hat{M}) \neq \mathcal{F}(N)$. However, $\tau : \hat{M} \to N$ is a diffeomorphism by definition (hence $\mathcal{F}(\hat{M}) = \{g \circ \tau : g \in \mathcal{F}(N)\}$), and thus, by the above discussion, *as smooth manifolds* they should be indistinguishable! Huh????
Answer: As a general fact in math, when studying a mathematical structure as such, we should distinguish them only *up to the isomorphism of the category.* That is, we should not really study the set $\mathcal{M}_n$ of differentiable n-manifolds, but its *moduli space* $\mathcal{M}_n/\sim$, where two manifolds are identified if they are diffeomorphic. So we finally obtain $\hat{M} \sim N$, as we got $\mathbb{R}^{n^2} \sim \mathbb{R}^{n \times n}$. In fact, every topological manifold of dimension $n \leq 3$ has a differentiable structure, which is also unique (in the above sense). Yet, in 1956 John Milnor showed that the topological 7-sphere $\mathbb{S}^7$ has more than one differentiable structure! We now know exactly how many smooth structures exist on each $\mathbb{S}^n$... except for $n = 4$ for which almost nothing is known. See here. (Don't worry, you will understand more of this Wiki article by the end of the course).

§4. Quotients

Exercise: Show that on any topological space quotient there is a unique minimal topological structure, called *quotient topology*, such that the projection π is continuous (i.e., the *final topology of* π). But the quotient of a manifold is not necessarily a manifold...

Examples: Möbius strip, $\mathbb{R}^2/\mathbb{Z}^2$, $[0,1]/\{0,1\} = \mathbb{S}^1$.
Open equivalence relations: X has countable basis $\Rightarrow X/\sim$ has, and $\{(x,y) \in X \times X : x \sim y\}$ is closed $\Rightarrow X/\sim$ is Hausdorff.
Example: $\mathbb{RP}^n$.
A *properly discontinuous action* $\varphi : G \times M \to M$ satisfies:

1) $\forall p \in M, \exists\, U_p \subset M$ *such that* $(g \cdot U_p) \cap U_p = \emptyset,\ \forall g \in G \backslash \{e\}$,

2) $\forall p, q \in M$ *in different orbits,* $\exists\, U_p, U_q \subset M$ *such that* $(G \cdot U_p) \cap U_q = \emptyset$ (this is necessary to ensure Hausdorff!).

In this situation, $M/\sim\ (= M/\varphi)$ is a manifold.

Exercise: Consider $\mathbb{S}^3$ as the unit quaternions, and define the map $P : \mathbb{S}^3 \to SO(3)$ by $P_u x = u x u^{-1}$, where $x \in \mathbb{R}^3$ is identified with the imaginary quaternions. Prove that this map is well defined, a homomorphism and a 2-1 surjective local diffeomorphism. Conclude that $SO(3) \cong \mathbb{S}^3/\{\pm I\}$.

§5. The tangent space

Germs of functions: $\mathcal{F}_p(M) = \{f : U \subset M \to \mathbb{R} : p \in U\}/\sim$
T_pM, $x : U_p \subset M^n \to \mathbb{R}^n$ chart $\Rightarrow \frac{\partial}{\partial x_i}|_p \in T_pM$, $1 \le i \le n$.
Differential of functions $\Rightarrow$ chain rule.
f local diffeomorphism $\Rightarrow f_{*p}$ isomorphism $\Rightarrow$ dimension is preserved by local diffeomorphisms.
Converse: Inverse function Theorem (it *has* to hold!).
Since every chart x is a diffeomorphism with its image and since

$$x_{*p}(\partial/\partial x_i|_p) = \partial/\partial u_i|_{x(p)} \ \ \forall 1 \le i \le n,$$

then $\{\frac{\partial}{\partial x_1}|_p, \dots, \frac{\partial}{\partial x_n}|_p\}$ is a basis of $T_pM \Rightarrow \dim T_pM = \dim M$.
Local expression of the differential.
Curves: speed, local expression.
Differential using curves: every vector is the derivative of a curve.
REM: $T_p\mathbb{R}^n = \mathbb{R}^n$. Therefore, if $f \in \mathcal{F}_p(U), v \in T_pM$, then $f_{*p}(v) = v(f)$.
Differential of curves, and computation of differentials using curves.
Immersion, submersion, embedding. Rank.

Exercise: Every injective immersion from a compact manifold is an embedding.

Examples: projections and injections in product manifolds. Identification of the tangent space of a product manifold:

$$T_pM \times T_{p'}M' \cong T_{(p,p')}(M \times M').$$

Definition 1. The point $p \in M$ is a *regular point* of $f : M \to N$ if f_{*p} is surjective. Otherwise, p is a *critical point*. The point $q \in N$ is a *critical value* of f if it the image of *some* critical point. Otherwise, q is a *regular value* of f (in particular, $q \in N, q \notin \mathrm{Im}\,(f) \Rightarrow q$ is a regular value of f).

§6. Submanifolds

Regular submanifolds $S \subset M$. Codimension. Topology.
Adapted charts $x_S \Rightarrow$ the inclusion $i_S : S \to M$ is an embedding.
Examples: $\sin(1/t) \cup I$; points and open sets.
The φ_S give an atlas of S.
Differentiable functions from and to regular submanifolds.
Level sets: $f^{-1}(q)$. Regular level sets.
Examples: $\mathbb{S}^n$, $SL(n, \mathbb{R})$: use the curve $t \mapsto \det(tA)$!!

Exercise: $S \subset M$ is a submanifold $\iff$ $\exists$ covering C of S such that $S \cap U$ is a submanifold of U, for all $U \in C$.

Theorem 2. *If $q \in \mathrm{Im}\,(f) \subset N^n$ is a regular value of $f : M^m \to N^n$, then $f^{-1}(q) \subset M^m$ is a regular submanifold of M^m of dimension $m - n$.*

Proof: Let $p \in M^m$ with $f(p) = q$ and local charts (x, U) and (y, V) in p and q. We can assume that $y(q) = 0$, $f(U) \subset V$ and that $\mathrm{span}\{f_{*p}(\frac{\partial}{\partial x_i}|_p) : i = 1, \ldots, n\} = T_qN$. Define $\varphi : U \to \mathbb{R}^m$

by $\varphi = (y \circ f, x_{n+1}, \dots, x_m)$. Then, since φ_{*p} is a isomorphism, $\exists U' \subset U$ such that $x' = \varphi|_{U'} : U' \to \mathbb{R}^m$ is a chart of M^m in p. Moreover, since $y \circ f \circ x'^{-1} = \pi_n$, we have that $f^{-1}(q) \cap U' = \{r \in U' : x'_1(r) = \cdots = x'_n(r) = 0\}$. Therefore, x' is an adapted chart to $f^{-1}(q)$. ∎

Exercise: If $p \in L := f^{-1}(q) \subset M^m$ in Theorem 2, then $T_pL = \operatorname{Ker} f_{*p}$.

Exercise: Adapting the proof of Theorem 2, prove the following: Let $f : M^m \to N^n$ a function whose rank is a constant k in a neighborhood of $p \in M$. Then, there are charts in p and $f(p)$ such that the expression of f in those coordinates is given by

$$\pi_k := (x_1, \dots, x_m) \mapsto (x_1, \dots, x_k, 0, \dots, 0) \in \mathbb{R}^n.$$

Conclude from this the normal form of immersions and submersions as particular cases.

Exercise: Conclude for the previous exercise that, if f has constant rank $= k$ in a neighborhood U of $f^{-1}(q) \neq \emptyset$, then $U \cap f^{-1}(q)$ is a regular submanifold of M^m with dimension $m - k$.

Example: $f : GL(n, \mathbb{R}) \to GL(n, \mathbb{R})$, $f(A) = AA^t$ has constant rank $n(n+1)/2$ (since $f \circ L_C = L_C \circ R_{C^t} \circ f \ \forall C$) $\Rightarrow O(n)$ is a submanifold of dimension $n(n-1)/2$ (no needed for constant rank: enough to see that I is a regular value of f thought the $\operatorname{Im}(f) \subset Sim(n, \mathbb{R})$).

REM: Since *"having maximal rank"* is an open condition, if a function f is an immersion (or a submersion) at point p, then it is an immersion (or a submersion) at a neighborhood of p.

$SL(n, \mathbb{R})$, $SO(n)$, $O(n)$, $\mathbb{S}^3$, $U(n)$,... are all Lie groups.
Immersed and embedded submanifolds. Figure 8.
Identify: $p \in S \subset M \Rightarrow T_pS \subset T_pM$; $S \subset \mathbb{R}^n \Rightarrow T_pS \subset \mathbb{R}^n$.

Exercise: Show that 0 is a regular value of $F : \mathbb{S}^{n-1} \times \mathbb{R} \times \mathbb{R}^{n \times n} \to \mathbb{R}^n$, $F(v, s, A) = (A - sI)v$. Conclude the smooth dependence of eigenvectors and eigenvalues near simple real eigenvalues (what happens at non-simple eigenvalues?). Complexify the exercise.

§7. Tangent and vector bundles (see [Zi])

Topological and differentiable structure of TM.
$\pi : TM \to M$. Vector fields over M:

$$\mathcal{X}(M) = \{X : M \to TM : \pi \circ X = \mathrm{Id}_M\}.$$

Differentiability, module structure of $\mathcal{X}(M)$.
Vector fields on $M \cong$ Derivations on M:

$$\mathcal{D}(M) = \{X \in \mathrm{End}(\mathcal{F}(M)) : X(fg) = X(f)g + fX(g)\}.$$

Lie bracket: $\mathcal{X}(M)$ is a *Lie algebra*: $[\,\cdot\,,\,\cdot\,]$ is bilinear, skewsymmetric and satisfy Jacobi identity.
Given $f : M \to N \Rightarrow f$-related vector fields: $\mathcal{X}_f$. *Ex.:* $X|_U$.
$X_i \sim_f Y_i \Rightarrow [X_1, X_2] \sim_f [Y_1, Y_2] \Rightarrow [X|_U, X'|_U] = [X, X']|_U$.
Fields along f: local expression.
Integral curves, local flux and Fundamental Theorem ODE.
Vector bundles, local trivializations, transition functions. TM.
Trivial vector bundle, product vector bundle.
Whitney sum of of vector bundles.
Pull-back of vector bundles: $f^*(E)$.
Bundle maps, isomorphism. *Example:* $f_* : TM \to TN$.
Sections. Frames. Differentiability.

Exercise: A vector bundle is trivial if and only if exists a *global* frame.

Cotangent bundle: T^*M, $\{dx_i, i = 1, \ldots, n\}$.
Vector bundles $\Rightarrow$ local basis of sections (as $\mathcal{F}(U)$-module) $\Rightarrow$ *All linear algebra constructions apply to vector bundles !!*
General bundles and G-bundles. Reduction.

§8. Partitions of unity

Exercise: Show that any differentiable manifold M has a countable basis of pre-compact sets. Conclude that M is a countable union of nested compact sets $K_1 \subset K_2 \subset ...$

Support of functions. Bump functions.
Global extensions of locally defined objects: functions, C^∞ fields, sections of vector bundles, etc.
Locally finite partitions of unity subordinated to coverings.

Theorem 3. *For any open cover $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ of a smooth manifold there is a locally finite partition of unit subordinated to $\mathcal{U}$.*

Proof for compact manifolds.

Exercise: Read (and understand!) the proof of the existence of partitions of unity in general (better than in Tu, see this simple proof by Thurston).

Application: Existence of Riemannian metrics.

Exercise: Give a well defined meaning for a subset $K \subset M^n$ to have (Lebesgue) *measure zero*. Show that, if $f : M \to N$ is smooth with $\dim M = \dim N$ and $K \subset M$ has measure 0, then $f(K) \subset N$ has measure zero.

Exercise (mini Sard's Theorem): If $m < n$ and $f : M^m \to N^n$, then $f(M) \subset N$ has measure zero.

Application: Whitney's embedding theorem (for compact manifolds): use mini Sard exercise. (You can see the general proof here).

§9. Orientation

$\mathbb{V}^n$ a real vector space $\Rightarrow \mathcal{O}(\mathbb{V}^n)$ =Bases/ $\sim$ two orientations.
Moebius strip: paper trick, knot: intrinsic vs extrinsic topology.
Orientability: bundle!

For every real vector bundle $E \to M$, the *orientation bundle of* E is the 2:1 cover $\mathcal{O}(E) \to M$. We say that E is *orientable* (as a vector bundle) $\Leftrightarrow \Gamma(\mathcal{O}(E)) \neq \emptyset$. Each element in $\Gamma(\mathcal{O}(E))$ is called an *orientation* of E.
We have a natural homomorphism $\tau_E : \pi_1(M) \to \mathbb{Z}_2$.

Exercise: Show that $\tau_{E\oplus E'} = \tau_E + \tau_{E'}$.

E is orientable $\Leftrightarrow \mathcal{O}(E) \cong M \cup M$.
We say that a manifold is orientable when its tangent bundle is.
Example: TM is orientable *as a manifold.*

Exercise: If M is orientable, then, a vector bundle $E \to M$ is orientable as a vector bundle if and only if E is orientable *as a manifold.*

§10. Differential 1–forms

$\Omega^1(M) = \Gamma(T^*M) = \{w : \mathcal{X}(M) \to \mathcal{F}(M)/w$ is $\mathcal{F}(M)-$linear$\}$:
Local operator $\Rightarrow$ point-wise operator $\Rightarrow \mathcal{F}(M)$-linear.
$f \in \mathcal{F}(M) \Rightarrow df \in \Omega^1(M)$, and $df \cong f_*$.
(x, U) chart $\Rightarrow \{\frac{\partial}{\partial x_1}|_p, \ldots, \frac{\partial}{\partial x_n}|_p\}$ is basis of T_pM whose dual basis is $\{dx_1|_p, \ldots, dx_n|_p\}$ (i.e., basis of T_p^*M).
$\{dx_1, \ldots, dx_n\}$ are then a frame of T^*U: local expression.

Example: Liouville form on T^*M: $\lambda(w) := w \circ \pi_{*w}$.
Pull-back: $\varphi \in End(\mathbb{V}, \mathbb{W}) \Rightarrow \varphi^* \in End(\mathbb{W}^*, \mathbb{V}^*)$;
$f : M \to N \Rightarrow f^* : \mathcal{F}(N) \to \mathcal{F}(M)$; $f^* : \Omega^1(N) \to \Omega^1(M)$.
Importance of pull-back!
Restriction of 1-forms to a submanifold $i : S \to M$: $w|_S = i^*w$.

§11. Multilinear algebra

Let $\mathbb{V}$ and $\mathbb{V}'$ $\mathbb{R}$–vector spaces. $\mathbb{V}^* = \mathrm{Hom}(\mathbb{V}, \mathbb{R})$.

Bi/tri/multi linear functions on vector spaces: $\mathbb{V} \otimes \mathbb{V}$.
Tensors and k–forms on $\mathbb{V}$: $\mathrm{Bil}(\mathbb{V}) = (\mathbb{V} \otimes \mathbb{V})^* = \mathbb{V}^* \otimes \mathbb{V}^*$.
$\mathbb{V} \otimes \mathbb{V}'$, $\mathbb{V} \wedge \mathbb{V}$, $\wedge^0 \mathbb{V} = \mathbb{V}^{\otimes 0} := \mathbb{R}$,

$$\mathbb{V}^{\otimes k} := \mathbb{V} \otimes \cdots \otimes \mathbb{V}, \quad \dim \mathbb{V}^{\otimes k} = (\dim \mathbb{V})^k$$

$$\wedge^k \mathbb{V} := \mathbb{V} \wedge \cdots \wedge \mathbb{V} \subset \mathbb{V}^{\otimes k}, \quad \dim \wedge^k \mathbb{V} = \binom{\dim \mathbb{V}}{k}$$

Operators $\otimes$ and $\wedge$ (bil. and assoc.) over multilinear maps:

$$\sigma \in \wedge^k \mathbb{V}, \ \omega \in \wedge^s \mathbb{V} \Rightarrow \omega \wedge \sigma := \frac{1}{k!s!} A(\omega \otimes \sigma) \in \wedge^{(k+s)} \mathbb{V}$$

REM: $\omega \wedge \sigma = (-1)^{ks} \sigma \wedge \omega$.

§12. Differential k – forms and tensor fields

ALL the multilinear algebra extends to vector bundles: $\mathrm{Hom}(E, E')$
Examples: T^*M; Riemannian metric: $\langle \, , \rangle|_U = \sum g_{ij} dx_i \otimes dx_j$
Tensor (field) and (differential) k-form:

$$\mathcal{X}^k(M^n), \quad \Omega^k(M^n)$$

are simply the sections of the bundles $(T^*M)^{\otimes k}$, $\Lambda^k(T^*M)$.
Tensors = $\mathcal{F}(M)$-multilinear maps (bump-functions).

REM: $\Omega^0(M) = \mathcal{X}^0(M) = \mathcal{F}(M)$, $\Omega^1(M) = \mathcal{X}^1(M)$.

Notation: $\mathcal{J}_{k,n} := \{(i_1, \ldots, i_k) : 1 \le i_1 < \cdots < i_k \le n\}$, and for $I = (i_1, \ldots, i_k) \in \mathcal{J}_{k,n}$, we set $dx_I := dx_{i_1} \wedge \cdots \wedge dx_{i_k}$.

Local expression:

$$df_1 \wedge \cdots \wedge df_n = \det([\partial f_i / \partial x_j]_{1 \le i,j \le n}) \, dx_1 \wedge \cdots \wedge dx_n \ , \quad (1)$$

and, for $J = (j_1, \ldots, j_k) \in \mathcal{J}_{k,n}$ and $y_1, \ldots, y_n \in \mathcal{F}(M)$,

$$dy_J = \sum_{I \in \mathcal{J}_{k,n}} \det(A_{JI})\, dx_I, \quad \text{onde} \ \ A_{JI} = \left[\frac{\partial y_{j_r}}{\partial x_{i_s}}\right]_{1 \le r,s \le k}.$$

Wedge operator $\wedge : \Omega^k(M) \times \Omega^s(M) \to \Omega^{k+s}(M)$ bilinear, tensorial

$$\Omega^\bullet(M) := \bigoplus_{k=0}^{n} \Omega^k(M)$$

is a *graded algebra* with $\wedge$.

Pull-back of tensors and forms: linear, tensorial, respects $\wedge$:

$$F^* f := f \circ F, \ \ \forall f \in \mathcal{F}(M),$$

$$F^*(\omega \wedge \sigma) = F^*\omega \wedge F^*\sigma,$$

$$(F \circ G)^* = G^* \circ F^*.$$

§13. Orientation and n − forms

Recall: if $B = \{v_1, \ldots, v_n\}$, $B' = \{v'_1, \ldots, v'_n\}$ are bases of $\mathbb{V}^n \Rightarrow \beta(v_1, \ldots, v_n) = \det C(B, B')\beta(v'_1, \ldots, v'_n), \forall\, \beta \in \Lambda^n(\mathbb{V}^n)$. We say that β *determines an orientation* $[B]$ if $\beta(v_1, ..., v_n) > 0$.

REM: M^n orientable $\Leftrightarrow$ exists $\beta \in \mathcal{V}$, where

$$\mathcal{V} = \{\sigma \in \Omega^n(M^n) : \sigma(p) \neq 0, \ \forall\, p \in M^n\}.$$

Orientations of $M \cong \mathcal{V}/\mathcal{F}_+(M)$.

Diffeomorphisms that preserve/revert orientation.

Exercise: Do this exercise again, but now use forms: If M is orientable, then, a vector bundle $E \to M$ is orientable as a vector bundle if and only if E is orientable *as a manifold.*

§14. Exterior derivative: VIP!!

Definition 4. The *exterior derivative* on $\Omega^\bullet(M)$ is the linear map $d : \Omega^\bullet(M) \to \Omega^\bullet(M)$ that satisfies the following properties:

1. $d(\Omega^k(M)) \subset \Omega^{k+1}(M)$;
2. $f \in \mathcal{F}(M) = \Omega^0(M) \;\Rightarrow\; df(X) = X(f),\ \forall\, X \in \mathcal{X}(M)$;
3. $\forall \omega \in \Omega^k(M), \sigma \in \Omega^\bullet(M) \Rightarrow d(\omega \wedge \sigma) = d\omega \wedge \sigma + (-1)^k \omega \wedge d\sigma$;
4. $d^2 = 0$.

- Props (2) + (3) + bump functions: $\omega|_U = 0 \Rightarrow d\omega|_U = 0$.
- Then, $d\omega|_U = d(\omega|_U)$, and we can carry local computations.
- Props (3) + (4) + induction $\Rightarrow d(df_1 \wedge \cdots \wedge df_k) = 0$.
- d exists and is unique: coordinate local expression.

For every $F : M \to N$ we have that (see first for Ω^0):

$$\boxed{F^* \circ d = d \circ F^*}$$

i.e., $F^* : \Omega^\bullet(N) \to \Omega^\bullet(M)$ *is a morphism of differential graded algebras* (i.e., preserves degree and commutes with d).

REM: This also explains why $d\omega|_U = d(\omega|_U)$ via inc^*.

Exercise: $\forall\, k$, $\forall\, \omega \in \Omega^k(M)$, $\forall\, Y_0, \ldots, Y_k \in \mathcal{X}(M)$, it holds that $dw(Y_0, \ldots, Y_k) =$

$$\sum_{i=0}^{k} (-1)^i Y_i \omega(Y_0, \ldots, \hat{Y}_i, \ldots, Y_k) + \sum_{0 \le i < j \le k}^{k} (-1)^{i+j} \omega([Y_i, Y_j], Y_0, \ldots, \hat{Y}_i, \ldots, \hat{Y}_j, \ldots, Y_k).$$

Given $X \in \mathcal{X}(M)$ we define the *interior multiplication*

$$i_X : \Omega^{k+1}(M) \to \Omega^k(M)$$

by $(i_X\omega)(Y_1, \ldots, Y_k) = \omega(X, Y_1, \ldots, Y_k)$.

- $i_X\omega$ is tensorial ($= \mathcal{F}(M)$-bilinear) on X and on ω.
- $\forall\ \omega \in \Omega^k(M), \sigma \in \Omega^r(M)$,

$$i_X(\omega \wedge \sigma) = (i_X\omega) \wedge \sigma + (-1)^k \omega \wedge (i_X\sigma).$$

- $i_X \circ i_X = 0$.

§15. Manifolds with boundary

C^∞ functions and diffeos over arbitrary subsets $S \subset M^n$.

Proposition 5. *Let $U \subset M^n$ open, $S \subset \hat{M}^n$ arbitrary, and $f : U \to S$ a diffeomorphism. Then, S is open on $\hat{M}^n$.*

Corolary 6. *Let U and V open of $\mathcal{H}^n := \mathbb{R}^n_+ = \{x_n \geq 0\}$ and $f : U \to V$ a diffeomorphism. Then f takes interior (resp. boundary) points to interior (resp. of boundary) points.*

Manifold with boundary: definition. (Rough idea of *orbifold*).
Interior points.
The boundary of $M^n = \partial M^n$ is a manifold of dimension $n - 1$.
∂M versus topological boundary.
If $p \in \partial M$: $\mathcal{F}_p(M)$, T_p^*M, $v \in T_pM$ (yet, it could be no curve with $\alpha'(0) = v$), TM, orientation, submanifolds (with boundary!): SAME as before. In particular, ∂M is an embedded hypersurface of M.

If $p \in \partial M$: $v \in T_pM$ *interior* and *exterior*.
REM: In any manifold with boundary M there exists an *exterior* vector field X along ∂M (i.e., considering the inclusion $inc : \partial M \to M$ we have that $X \in \mathcal{X}_{inc}$). Then, ∂M is orientable if M is, with the induced orientation $inc^* i_X \omega$. In fact, X is defined in a neighborhood U of ∂M, which in turnin turn defines a <u>*collar*</u> $\partial M \subset U_\epsilon \subset M$ by means of the flux of X.
Examples: $\mathcal{H}^n, [a,b]$; B^n, $\overline{B^n}$.
Example: If $j = inc : \mathbb{S}^{n-1} = \partial \overline{B^n} \to \overline{B^n}$, $Z(p) = p \in \mathcal{X}_{inc}$ is exterior $\Rightarrow$ orientation σ in $\mathbb{S}^{n-1} \subset \overline{B^n}$ via $\overline{B^n} \subset \mathbb{R}^n$ and $dv_{\mathbb{R}^n}$:

$$\sigma = j^*(i_Z dv_{\mathbb{R}^n}) = \sum_i (-1)^{i-1}\, u_i\, du_1 \wedge \cdots \wedge \widehat{du_i} \wedge \cdots \wedge du_n. \quad (2)$$

§16. Integration (Riemann)

Forms with compact support $= \Omega_c^\bullet(M)$: preserved by pull-backs of diffeomorphisms (and, more generally, proper maps).
• If $\omega \in \Omega_c^n(U)$, $U \subset \mathcal{H}^n$ write $\omega = f dx_1 \wedge \cdots \wedge dx_n$. Given a diffeo $\xi : V \subset \mathcal{H}^n \to U \subset \mathcal{H}^n$ with $\epsilon_\xi = 1$ (resp. $\epsilon_\xi = -1$) if ξ preserves (resp. reverses) orientation, we get from (1) and the Change of Variables Theorem (CVT: VIP!!!) that

$$\begin{aligned}
\int_V \xi^*\omega &= \int_V \xi^*(f dx_1 \wedge \cdots \wedge dx_n) \\
&= \int_V f \circ \xi \ (\xi^* dx_1 \wedge \cdots \wedge \xi^* dx_n) \\
&= \int_V f \circ \xi \ (d\xi_1 \wedge \cdots \wedge d\xi_n) \\
&= \int_V f \circ \xi \ \det(J_\xi)\, dx_1 \wedge \cdots \wedge dx_n = \epsilon_\xi \int_U \omega.
\end{aligned}$$

• If $U \subset \mathcal{H}^n$, we define the *linear operator*

$$\omega \in \Omega_c^n(U) \mapsto \int_U \omega = \int_{\mathcal{H}^n} \omega := \int_{\mathcal{H}^n} f dx = \text{Riemann integral.}$$

REM: Same for ω n-form continuous on U, $A \subset U$ bounded with measure zero boundary (e.g., $A =$ cube) $\Rightarrow \int_A \omega$.
• If M^n is oriented, $\varphi : U \subset M^n \to \mathcal{H}^n$ oriented chart, we define the *linear operator*

$$\omega \in \Omega_c^n(U) \mapsto \int_U \omega = \int_M \omega := \int_{\varphi(U)} (\varphi^{-1})^* \omega.$$

• If M^n oriented, we define the *linear operator*

$$\omega \in \Omega_c^n(M^n) \mapsto \int_M \omega := \sum_\alpha \int_M \rho_\alpha \omega.$$

CVT: $\int_N \varphi^* \omega = \int_M \omega,\ \forall\, \varphi \in Dif_+(N, M), \forall\, \omega \in \Omega_c^n(M^n)$.
Dim $M = 0$ case: $\int_M f := \sum_i f(p_i) - \sum_j f(q_j)$.
$M = (M, o)$ oriented, $-M := (M, -o) \Rightarrow \int_{-M} \omega = - \int_M \omega$.

§17. Stokes Theorem 1.0

...which was not proved by Stokes, but by Klein (dim 2) and E.Cartan in general... :o/

Theorem 7 (Stokes v.1.0). *M^n oriented, $\omega \in \Omega_c^{n-1}(M^n) \Rightarrow$*

$$\int_M d\omega = \int_{\partial M} \omega.$$

Underlying idea: Sum integrals over small cubes, since the interior faces cancel down due to orientation (dim 1 and 2 pictures).
Cor.: M^n compact oriented $\partial M = \emptyset \Rightarrow \int_M d\omega = 0, \forall \omega \in \Omega^{n-1}(M)$.

Exercise 1: The classical calculus theorems all follow from Stokes.

Exercise 2. M compact orientable $\Rightarrow \nexists\ f : M \to \partial M$ with $f|_{\partial M} = Id$ (retraction).

Theorem 8 (Brouwer's fixed point Theorem)**.** *If $B \subset \mathbb{R}^n$ is a closed ball (or a compact convex subset), then every continuous function $f\colon B \to B$ has fixed points.*

OBS (!!): $i : N^k \subset M$, N^k compact oriented regular submanifold, and $\omega \in \Omega^k(M)$ (or N^k oriented and $\omega \in \Omega^k_c(M)$) $\Rightarrow \int_N \omega\ (= \int_N i^*\omega)$.
If $\rho \in Diff_+(N^k) \Rightarrow \int_N \rho^*\omega = \int_N \omega \Rightarrow$ we only care about the image $i(N)$, nor really on the map i (!) $\Rightarrow$ Notation:

$$\int_i w := \int_N i^*\omega, \quad \forall i : N^k \to M^n.$$

It makes sense for <u>*any*</u> differentiable function i: $\int_i w$ (even if M is not orientable!), and $\int_{i\circ\rho} w = \int_i w$ (we only care about $i(N)$...).

Curiosity: Palais' Theorem. Let $D : \Omega^k \to \Omega^r$ such that $Df^* = f^*D$, $\forall f : M \to N$. Then, either $k = l$ and $D = cId$, or $r = k + 1$ and $D = c\,d$, or $k = dimM$, $r = 0$, and $D = c\int_M$.

§18. Stokes Theorem 2.0 (Spivak vol.1 chap.8)

k-cube: $I^k\colon [0,1]^k \hookrightarrow \mathbb{R}^k$. *Singular k-cube*: $c\colon [0,1]^k \to M$.
c singular k-cube, $\omega \in \Omega^k(M) \Rightarrow \int_c \omega := \int_{[0,1]^k} c^*\omega$.
$C_k(M) = C_k(M; G) :=$ *k-chains of M* $=$ free G-module over singular cubes, for $G = \mathbb{R}$ (or $\mathbb{Z}$ or $\mathbb{Q}$ or $\mathbb{Z}_2$ or...).
$\int : C_k(M) \times\ \Omega^k(M) \to \mathbb{R}$ is defined $\forall\, M$ and is bilinear!
$I^k_{i,\alpha}(x_1, \ldots, x_{k-1}) := I^k(x_1, \ldots, x_{i-1}, \alpha, x_i, \ldots, x_{k-1})), \alpha = 0{,}1$.
$c_{i,\alpha} := c \circ I^k_{i,\alpha}$, $\partial c = \sum_{i=1}^k \sum_{\alpha=0}^1 (-1)^{i+\alpha} c_{i,\alpha}$ (dim 2 picture).
Extend linearly $\partial\colon C_k(M) \to C_{k-1}(M)$: $\partial c =$ *boundary* of c.

Defs: $c \in C_k(M)$ is *closed* if $\partial c = 0$; c is um *boundary* if $c = \partial\tilde{c}$.
Examples: c_1, c_2 1-cubes. c_1 closed $\Leftrightarrow c_1(0)=c_1(1)$; $c=c_1-c_2$ is closed $\Leftrightarrow c_1(0)=c_2(0)$ and $c_1(1)=c_2(1)$, or c_1 and c_2 closed.
Since $(I^k_{i,\alpha})_{j,\beta} = (I^k_{j+1,\beta})_{i,\alpha}\ \forall\, 1 \leq i \leq j \leq k-1 \Rightarrow \boxed{\partial^2 = 0}$.
What we proved in Theorem 7 is, in fact, the following:

Theorem 9 (Stokes v.2.0). *For every differentiable manifold M, $w \in \Omega^{k-1}(M)$, and $c \in C_k(M)$, we have that*

$$\int_c d\omega = \int_{\partial c} \omega.$$

In other words, ∂ (over $\mathbb{R}$) is the dual (with respect to $\int$) of d.
Everything works the same with k-*simplex* instead of k-cubes.

DO ALL EXERCISES
IN CHAPS. 8 AND 11 OF SPIVAK!!

§19. De Rham cohomology (Spivak, vol.1 chap.8)

If $w \in \Omega^1(\mathbb{R}^n)$, when $w = df$ for certain $f \in \mathcal{F}(\mathbb{R}^n)$? Necessary condition: $dw = 0$. Is it enough?? YES: taking singular 1-cube c, $c(0) = 0, c(1) = p$, define $f(p) = \int_c w$. It is well defined by Stokes(!), since every closed curve on $\mathbb{R}^n$ is a boundary. In fact, $c_s(t) = sc_1(t) + (1-s)c_0(t)$. That is: *solutions of certain PDEs are related to the topology of the space.*
Poincaré's Lemma (seen later): $Z^k(\mathbb{R}^n) = B^k(\mathbb{R}^n)$.
That is, locally we can always solve the problem, but globally...
depends on the topology! $\Rightarrow$ likewise orientability!

System of linear PDEs: integrability condition.
Obstructions to solve PDEs, or globalize certain local objects.

$Z^k(M) := \operatorname{Ker} d_k =$ *closed* forms (local condition)

$B^k(M) := \operatorname{Im} d_{k-1} =$ *exact* forms (global condition!)

Definition: The k-th *de Rham cohomology* of the manifold M (with or without boundary) is given by

$$H^k(M) := Z^k(M)/B^k(M).$$

$H^0(M) = \mathbb{R}^r$, where r is the number of connected comp. of M.
$H^n(M^n) \neq 0$ if M^n is a compact orientable manifold (Stokes).
$H^{n+k}(M^n) = 0, \ \forall\ k \geq 1$.
Ex: $\dim H^k(T^n) \geq \binom{n}{k}$: if $\omega_I := [d\theta_{i_1} \wedge \cdots \wedge d\theta_{i_k}] \Rightarrow \int_{T_J} w_I = \delta^I_J$.
Pull-back: $F : M \to N \Rightarrow F^*(= F^\#) : H^k(N) \to H^k(M)$.
$(F \circ G)^* = G^* \circ F^* \Rightarrow H^k(M)$ is an <u>invariant</u> of the differentiable structure (!), and invariant under diffeomorphisms.
$\wedge : H^k(M) \times H^r(M) \to H^{k+r}(M)$, $[\omega] \wedge [\sigma] := [\omega \wedge \sigma]$ (well!).
$H^\bullet(M) := \oplus_{k\in\mathbb{Z}} H^k(M)$ is the *de Rham cohomology ring of* M.
In fact, $H^\bullet(M)$ is a *anticommutative graded algebra*, and F^* is a homomorphism of graded algebras.

§20. Homotopy invariance (Spivak, vol.1 chap.8)

Definition 10. Given two manifolds (with or without boundary) M and N, we say that $f, g : M \to N$ are (differentiably) *homotopic* if there is a smooth function $T : M \times [0,1] \to N$ such that $T_0 := T \circ i_0 = f$, $T_1 := T \circ i_1 = g$, where $i_s(p) = (p, s)$.

This is an equivalence relation on $\mathcal{F}(M, N)$: $f \sim g$.
Example: M is contractible $\Leftrightarrow Id_M \sim cte$.

REM: Continuously homotopic $\Rightarrow$ differentiably homotopic ([Le])

Proposition 11. *If M is a manifold with or without boundary, for all k there is a linear map $\tau : \Omega^k(M \times [0,1]) \to \Omega^{k-1}(M)$ (called cochain homotopy) such that*

$$i_1^*\omega - i_0^*\omega = d\tau\omega + \tau d\omega, \quad \forall\, \omega \in \Omega^k(M \times [0,1]).$$

Proof: Define $\tau(\omega) = \int_0^1 i_s^*(i_{\partial/\partial t}(\omega))ds$. It is enough to check two cases (identify via π_1^* and π_2^*). If $\omega = f dx_I$, $d\omega = \cdots + (\partial f/\partial t)dt \wedge dx_I$, and therefore it is just the Fundamental Theorem of Calculus. If $\omega = f dt \wedge dx_I$, then $i_1^*\omega = i_0^*\omega = 0$, and an easy computation gives $\Rightarrow d\tau\omega + \tau d\omega = 0$. ∎

More than a differential invariant $H^\bullet(M)$ is a <u>homotopic invariant</u>:

Theorem 12 (!!!!!!). $f \sim g \Rightarrow f^* = g^*$ (in $H^\bullet(M)$).

Proof: Immediate from Proposition 11. (The same holds true for the singular homology: see Theorem 2.10 on [Ha] and its proof). ∎

Corolary 13. *M contractible $\Rightarrow H^k(M) = 0,\ \forall\ k \geq 1$.*

Corolary 14. *(Poincaré's Lemma) $Z^k(\mathbb{R}^n) = B^k(\mathbb{R}^n)\ \forall k \geq 1$.*

Corolary 15. *M^n compact orient. $\Rightarrow M^n$ not contractible.*

Definition 16. $f : M \to N$ is a *homotopic equivalence* if there exists $g : N \to M$ such that $g \circ f \sim Id_M$ and $f \circ g \sim Id_N$. In this case, we say that M and N are *homotopically equivalent*, or that M and N *have the same homotopy type: $M \sim N$.*

Example: M contractible $\iff M \sim$ point.

Exercise. The "letters" X and Y as subsets of $\mathbb{R}^2$ are homotopically equivalent but not homeomorphic.

REM: *Whitehead's Theorem* states that, if a continuous function (between CW complexes) induces isomorphisms between all homotopy groups, then f is a homotopy equivalence. Yet, it is not enough to assume that all homotopy groups are isomorphic: $\mathbb{RP}^2 \times \mathbb{S}^3 \not\sim \mathbb{S}^2 \times \mathbb{RP}^3$ since they are covered by $\mathbb{S}^2 \times \mathbb{S}^3$ and $\pi_1 = \mathbb{Z}_2$. By Hurewicz Theorem, this implies that a continuous function f between simply connected CW complexes that induces isomorphisms between the singular homologies with integer coefficients is also a homotopy equivalence.

Corolary 17 (!!!!!). *Let $f : M \to N$ be a homotopy equivalence between manifolds with or without boundary. Then $f^* : H^\bullet(N) \to H^\bullet(M)$ is an isomorphism.*

Corolary 18. *If M has boundary, then $H^\bullet(M) = H^\bullet(M^\circ)$.*

Definition 19. A *retract* of M to a submanifold $S \subset M$ is a function $f : M \to S$ such that $f|_S$ $(= f \circ inc_S) = Id_S$. S is called a *retract* of M ($\Rightarrow f^*$ is injective and inc_S^* is surjective).

Exercise 1. Using deRham cohomology prove again Brouwer fixed point Theorem 8.

Exercise 2. Using deRham cohomology prove again Exercise 2 on page 15: if M is compact and orientable then there is no retraction $f : M \to \partial M$.

Exercise 3. Prove the previous exercise without orientability. (Suggestion: pick a regular value p of f and notice that $\#\partial(f^{-1}(p))$ is even by Sard, which is a contradiction).

Definition 20. A *deformation retract* from M to $S \subset M$ is a function $T : M \times [0,1] \to M$ such that $T_0 = Id_M$, $\mathrm{Im}\,(T_1) \subseteq S$, and $T_1|_S = Id_S$ (i.e., retract $T_1 \sim T_0 = Id_M \Rightarrow T_1^*$ and inc_S^* are isomorphisms).

In other words, a deformation retract is a homotopy between a retract from M to S and the identity of M. In particular, if S is a deformation retract of M, then $M \sim S$.

Corolary 21. *If E is a vector bundle over M, then $H^\bullet(E) = H^\bullet(M)$.*

Application: tubular neighborhoods. Given an embedded compact submanifold $N \subset M$, for each $0 < \epsilon < \epsilon_0$ there exists an open subset $N \subset V_\epsilon \subset M$, such that N is a deformation retract of V_ϵ, $V_\epsilon \subset V_{\epsilon'}$ if $\epsilon < \epsilon'$, and $\cap_\epsilon V_\epsilon = N$. (Proof: use Whitney's Theorem for M, or Riemannian metrics; see Theorem 5.2 on [Hr]). In particular, $H^\bullet(V_\epsilon) = H^\bullet(N)$.

Definition 22. A *strong deformation retract* is a deformation retract T as in Definition 20 such that $T_t|_S = Id_S,\ \forall\ t \in [0,1]$ (e.g, H below).

Example: Möbius strip $F \sim \mathbb{S}^1$ ($\Rightarrow H^2(F) = 0$).

Example: $\mathbb{R}^n \setminus \{0\} \sim \mathbb{S}^{n-1} \not\sim \mathbb{R}^n$: $H(x,t) = ((1-t) + t/\|x\|)x$.

§21. Integrating cohomology: degree (Spivak, vol.1 chap.8)

For noncompact M (without boundary) we also work with

$$H^k_c(M) := Z^k_c(M)/B^k_c(M), \quad k \in \mathbb{Z}.$$

REM: If M^n is orientable, then $\int : H^n_c(M^n) \to \mathbb{R}$ is of course a well defined linear map. And more:

Theorem 23. *If M^n is a connected orientable manifold, then $\int : H^n_c(M^n) \to \mathbb{R}$ is a isomorphism* ($\Rightarrow \dim H^n_c(M^n) = 1$).

Proof: We only need to check that, if $\int_M \omega = 0$, then $\omega = d\beta$ with β with compact support.

(a) *It is true for $M = \mathbb{R}$.* Se $g(t) = \int_{-\infty}^t \omega \Rightarrow \omega = dg$.

(b) *If it holds for $\mathbb{S}^{n-1}$, then it holds for $\mathbb{R}^n$.* If $\omega \in \Omega^n_c(\mathbb{R}^n) \subset \Omega^n(\mathbb{R}^n)$, since $\mathbb{R}^n$ is contractible we know that $\omega = d\eta$ for some $\eta \in \Omega^{n-1}(\mathbb{R}^n)$ (but η not necessarily with compact support!). Now, since ω has compact support (say, inside the ball B^n_1) and $\int_{\mathbb{R}^n} \omega = 0$, we have $\int_{\mathbb{S}^{n-1}} j^*\eta' = \int_{\mathbb{S}^{n-1}} i^*\eta = \int_{\mathbb{R}^n} \omega = 0$ by Stokes, where $i : \mathbb{S}^{n-1} \to \mathbb{R}^n$ and $j : \mathbb{S}^{n-1} \to \mathbb{R}^n \setminus \{0\}$ are the inclusions, and $\eta' = \eta|_{\mathbb{R}^n\setminus\{0\}}$. Then, by hypothesis, $j^*[\eta'] = 0$. But j^* is a isomorphism since $\mathbb{S}^{n-1}$ is deformation retract of $\mathbb{R}^n \setminus \{0\}$. We conclude that $\eta' = d\lambda$ for some $\lambda \in \Omega^{n-2}(\mathbb{R}^n\setminus\{0\})$. In particular, if $h : \mathbb{R}^n \to \mathbb{R}$ satisfies $h \equiv 1$ outside of B^n_1 and $h \equiv 0$ inside B^n_ϵ, then $\beta = \eta - d(h\lambda) \in \Omega^{n-1}(\mathbb{R}^n)$ has compact support on B^n_1, and $\omega = d\beta$.

Another, more explicit proof of **(b)**: If $\omega = f dv_{\mathbb{R}^n} \in \Omega^n(\mathbb{R}^n)$ has compact sup. on B^n_1, then define $g : \mathbb{R}^n \to \mathbb{R}$ by $g(p) = \int_0^1 t^{n-1} f(tp)dt$, $r : \mathbb{R}^n \setminus \{0\} \to \mathbb{S}^{n-1}$, $r(x) = x/\|x\|$ (*retract*), $i : \mathbb{S}^{n-1} \to \mathbb{R}^n$ the inclusion and $\sigma = i_X dv_{\mathbb{R}^n} \in \Omega^{n-1}(\mathbb{R}^n)$ as in (2).

- Computation $\Rightarrow w = d(g\sigma)$ (yet $g\sigma$ not necessarily with compact support!)
- $\int_{\mathbb{S}^{n-1}} (g \circ i) i^*\sigma = \int_{B^n} f dv_{\mathbb{R}^n} = \int_{\mathbb{R}^n} \omega = 0 \Rightarrow i^*(g\sigma) = d\lambda$, by hypothesis.
- $g\sigma = r^*(i^*(g\sigma)) = d(r^*\lambda)$ outside B^n_1, since $(i \circ r)_{*p} = \|p\|^{-1}\Pi_{p^\perp}$, $(i \circ r)^*\sigma(p) = \|p\|^{-n}\sigma(p)$, and $g(p) = \|p\|^{-n}(g \circ i \circ r)(p)$, if $\|p\| \geq 1$.
- If $\beta := g\sigma - d(hr^*\lambda) \Rightarrow w = d(g\sigma) = d\beta$, with $sup(\beta) \subseteq B^n_1$.

(c) (!!!) *If it holds for $\mathbb{R}^n$ it holds for every M^n.* Fix any $w_0 \in \Omega^n_c(U_0)$ with $U_0 \subset M^n$ diffeo to $\mathbb{R}^n$, with $\int_M w_0 \neq 0$. Let $w \in \Omega^n_c(M^n)$. It is enough to see that there is $a \in \mathbb{R}$ and $\eta \in \Omega^{n-1}_c(M^n)$ such that $w = aw_0 + d\eta$. Taking partitions of unity we can assume that $\sup(w) \subset U$, U diffeo a $\mathbb{R}^n$. Since M^n is connected, there exists a sequence $\{U_i, 1 \leq i \leq m\}$, U_i diffeo a $\mathbb{R}^n$, with $U_m = U$ and $U_i \cap U_{i+1} \neq \emptyset$. Let w_i with compact support, $\sup(w_i) \subset U_i \cap U_{i+1}$, and such that $\int_M w_i \neq 0$. Since it holds for $\mathbb{R}^n \cong U_{i+1}$, $w_{i+1} - c_{i+1}w_i = d\eta_{i+1}$. Done! :) ∎

Theorem 24. *M^n connected not orientable $\Rightarrow H^n_c(M^n) = 0$.*

Proof: Use the idea in **(c)** above. ∎

Exercise. Prove Theorem 24 using the orientable double cover.

Theorem 25. *M^n connected non compact, with or without boundary $\Rightarrow H^n(M^n) = 0$.*

Proof: Use the idea in **(c)**. Suppose first M^n orientable and use exhaustion by compact sets (or by Theorem 52). For non orientable M^n, prove that $\pi^* : H^n(M^n) \to H^n(\tilde{M}^n)$ is injective. ∎

By Theorem 23, for any <u>proper</u> differentiable function between connected orientable manifolds, $f : M^n \to N^n$ (same dimension!), there exists $deg(f) \in \mathbb{R}$, the *degree of* f, such that

$$\int_M f^*\omega = deg(f) \int_N \omega, \quad \forall\, \omega \in \Omega^n_c(N^n).$$

Theorem 26. *Under the above hypothesis, if $q \in N^n$ is a regular value of f and $f(p) = q$, set $sgn_f(p) = \pm 1$, according to f_{*p} preserving or reversing orientation. Then,*

$$deg(f) = \sum_{p \in f^{-1}(q)} sgn_f(p).$$

In particular, $deg(f) \in \mathbb{Z}$, and $deg(f) = 0$ for f not surjective.

Proof: If $\{p_1, \ldots, p_k\} = f^{-1}(q)$, choose small disjoint neighborhoods U_i of p_i and V of q such that $f : U_i \to V$ is diffeo. Let ω with compact support on V such that $\int_N \omega \neq 0$. Then, $\int_{U_i} f^*\omega = sgn_f(p_i) \int_V \omega$. So, the result is immediate... if it only holds that $sup(f^*\omega) \subset U_1 \cup \cdots \cup U_k$. But we fix it like this:

Let $K \subset V$ compact such that $q \in K^o$. Then, $K' = f^{-1}(K) \setminus (U_1 \cup \cdots \cup U_k)$ is compact, and thus $f(K')$ is closed not containing q. Now just change V by any $V' \subset K^o \setminus f(K') \subset K$, with $q \in V'$, that automatically satisfies $f^{-1}(V') \subset U_1 \cup \cdots \cup U_k$. ∎

REM: The set of regular values is open and dense, and the sum in Theorem 26 is finite.

REM: $H^n_c(M^n) \not\subset H^n(M^n)$ in general: $H^n_c(\mathbb{R}^n) = \mathbb{R}$, yet $H^n(\mathbb{R}^n)=0$, $n \geq 1$. In fact, $f \sim g \not\Rightarrow f^* = g^*$ on $H^\bullet_c$. But:

Corolary 27. *$f, g : M^n \to N^n$ as above, $f \sim g$ (properly homotopic) $\Rightarrow deg(f) = deg(g)$.*

Example: $deg(-Id_{\mathbb{S}^n}) = (-1)^{n+1}$.

Corolary 28. *Hairy even dimensional dog Theorem.*

REM: We can always comb odd dimensional dogs!

Corolary 29. *Fundamental Theorem of Algebra.*

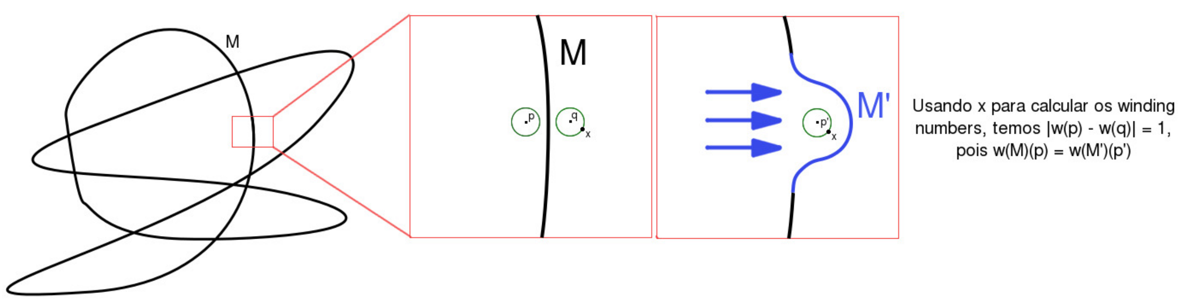
Proof: Extend $g(z) = z^k + a_1 z^{k-1} + \cdots + a_k$ to $\mathbb{C} \cup \infty = \mathbb{S}^2$ via $g(\infty) = \infty$. It is smooth since $1/g(1/z) = \frac{z^k}{1+a_1z+\cdots a_k z^k}$, and it is homotopic to $h(z) = z^k$ via $g_t(z) = z^k + t(a_1 z^{k-1} + \cdots + a_k)$.
Let $w = f(r)dx \wedge dy = f(r)rdr \wedge d\theta$ with f with compact support. Then, $\int_{\mathbb{R}^2} h^* w = k \int_{\mathbb{R}^2} w \Rightarrow deg(g) = deg(h) = k > 0 \Rightarrow$ g is surjective.
Another proof: h is a proper orientation preserving local diffeo of $\mathbb{C} \setminus \{0\}$, and $\forall u \in \mathbb{C} \setminus \{0\}$, $\#h^{-1}(u) = k \Rightarrow deg(h) = k$. ∎

Exercise. Using degree prove (again!) Brouwer fixed point Theorem 8. (Suggestion: for $0 \leq t \leq 1$ consider $f_t(x) = (x - tf(x))/s_t$, where $s_t = \sup\{y - tf(y) : y \in B\}$).

§22. Application: winding number (video 25)

$f : M^n \to \mathbb{R}^{n+1}$ an immersion of a compact connected orientable manifold, $p \in \mathbb{R}^{n+1} \setminus M^n$, $r > 0$ such that $\overline{B_r(p)} \cap M^n = \emptyset \Rightarrow$ $\pi \circ f : M^n \to \partial B_r(p) \cong \mathbb{S}^n \Rightarrow w(p) := deg(\pi \circ f) \in \mathbb{Z}$ is the *winding number of M^n around p* (independent on r) $\Rightarrow$ w is constant on each connected component of $\mathbb{R}^{n+1} \setminus M^n$.
See for curves, in particular, the effect of the orientation.
M^n is not orientable? Theorem 26 $\Rightarrow$ winding number mod 2: exercises 23 to 26 Spivak chap.8: $f : M^n \times I \to N^n$ homotopy, $y \in N^n$ regular value de $f, f_0, f_1 \Rightarrow \#f_0^{-1}(y) = \#f_1^{-1}(y)$ mod 2. Picture $\Rightarrow$ w is never constant and jumps at $M^n \Rightarrow$

Corolary 30. *M^n orientable or not, $b_0(\mathbb{R}^{n+1} \setminus M^n) \geq 2$.*



§23. The birth of exact sequences

Let $U, V \subset M$ open such that $M = U \cup V$, $k \in \mathbb{Z} \Rightarrow i_U : U \hookrightarrow M$, $j_U : U \cap V \hookrightarrow U \Rightarrow i_U^* : \Omega^k(M) \to \Omega^k(U)$, $j_U^* : \Omega^k(U) \to \Omega^k(U \cap V)$. Idem for i_V, j_V. We then have:

$$i = i_U^* \oplus i_V^* : \Omega^k(M) \to \Omega^k(U) \oplus \Omega^k(V),$$

$$j = j_V^* \circ \pi_2 - j_U^* \circ \pi_1 : \Omega^k(U) \oplus \Omega^k(V) \to \Omega^k(U \cap V),$$

i.e., $i(\omega) = (\omega|_U, \omega|_V)$, $j(\sigma,\omega) = j_V^*\omega - j_U^*\sigma = \omega|_{U\cap V} - \sigma|_{U\cap V}$. Joining, we get

$$0 \to \Omega^k(M) \xrightarrow{i} \Omega^k(U) \oplus \Omega^k(V) \xrightarrow{j} \Omega^k(U\cap V) \to 0, \qquad (3)$$

with each image contained in the kernel of the next. Now, the fundamental point is that, in fact, they equal! (the only not obvious is that j is surjective, but, if $\{\rho_U, \rho_V\}$ is a partition of unity subordinated to $\{U, V\}$ and $\omega \in \Omega^k(U\cap V)$, then $\omega_U := \rho_V\omega \in \Omega^k(U)$, $\omega_V := \rho_U\omega \in \Omega^k(V)$, and $j(-\omega_U, \omega_V) = \omega$).

§24. Complexes (Spivak, Vol. I, Chap. 11)

Exact sequences of abelian groups: short, long.

Exercise. The dual of an exact sequence is an exact sequence.

$A \xrightarrow{f} B \to 0 \Leftrightarrow f$ epimorphism
$0 \to A \xrightarrow{f} B \Leftrightarrow f$ monomorphism
$0 \to A \xrightarrow{f} B \to 0 \Leftrightarrow f$ isomorphism
$A \xrightarrow{f} B \to C \to 0 \Rightarrow C \cong B/\mathrm{Im}\, f$
$0 \to A \to B \to C \to 0 \Rightarrow C \cong B/A$

Proposition 31. (General linear algebra dimension Theorem) If $0 \xrightarrow{\alpha} \mathbb{V}_1 \xrightarrow{\beta} \mathbb{V}_2 \to \cdots \to \mathbb{V}_k \to 0$ is exact $\Rightarrow \sum_i (-1)^i \dim \mathbb{V}_i = 0$.

Proof: Induction on k, changing to $0 \to \mathbb{V}_2/\mathrm{Im}\,\alpha \xrightarrow{\beta[\,]} \mathbb{V}_3 \to \cdots$ ∎

Cochain complex: $\mathcal{C} = \{C^k\}_{k\in\mathbb{Z}}$ + *'differentials'* $\{d_k\}_{k\in\mathbb{Z}}$:

$$\cdots C^{-1} \xrightarrow{d_{-1}} C^0 \xrightarrow{d_0} C^1 \xrightarrow{d_1} C^2 \cdots, \quad d_k \circ d_{k-1} = 0.$$

Direct sum of cochain complexes.
$a \in C^k$ is a $k-$*cochain* of $\mathcal{C}$.
$a \in Z^k(\mathcal{C}) := \operatorname{Ker} d_k \subset C^k$ is a $k-$*cocycle* of $\mathcal{C}$.
$a \in B^k(\mathcal{C}) := \operatorname{Im} d_{k-1} \subset C^k$ is a $k-$*coboundary* of $\mathcal{C}$.
The *k-th cohomology of $\mathcal{C}$* is given by

$$H^k(\mathcal{C}) := Z^k(\mathcal{C})/B^k(\mathcal{C}).$$

If $a \in Z^k(\mathcal{C}) \Rightarrow [a] \in H^k(\mathcal{C})$ is the *cohomology class of a*.

Um *cochain map* $\varphi : \mathcal{A} \to \mathcal{B}$ is a sequence $\{\varphi_k : A^k \to B^k\}_{k\in\mathbb{Z}}$ such that $d \circ \varphi_k = \varphi_{k+1} \circ d$. This gives maps $\varphi^* : H^\bullet(\mathcal{A}) \to H^\bullet(\mathcal{B})$. The sequence $0 \to \mathcal{A} \xrightarrow{i} \mathcal{B} \xrightarrow{j} \mathcal{C} \to 0$ is said to be *short exact* if at each level k is exact. In this situation,

$$H^k(\mathcal{A}) \xrightarrow{i^*} H^k(\mathcal{B}) \xrightarrow{j^*} H^k(\mathcal{C})$$

is exact for all k. Yet, it is NOT exact with 0 at the left or at the right... BUT:

Theorem 32 (!!!!!!!). *If $0 \to \mathcal{A} \xrightarrow{i} \mathcal{B} \xrightarrow{j} \mathcal{C} \to 0$ is short exact, then there exist (explicit and natural) homomorphisms*

$$\delta^* : H^k(\mathcal{C}) \to H^{k+1}(\mathcal{A}),$$

called <u>*connection homomorphisms*</u>*, that induce the following long exact sequence in cohomology:*

$$\cdots \xrightarrow{j^*} H^{k-1}(\mathcal{C}) \xrightarrow{\delta^*} H^k(\mathcal{A}) \xrightarrow{i^*} H^k(\mathcal{B}) \xrightarrow{j^*} H^k(\mathcal{C}) \xrightarrow{\delta^*} H^{k+1}(\mathcal{A}) \xrightarrow{i^*} \cdots ,$$

Proof: ("Diagram chasing": make with students) Given $c \in Z^k(\mathcal{C})$, there exists $b \in B^k$ such that $jb = c$. But then $db \in \operatorname{Ker} j$ ($jdb = djb = dc = 0$), and, since $\operatorname{Ker} j = \operatorname{Im} i$, there is $a \in A^{k+1}$ such that $db = ia$ (given b, a is unique since i is injective). Now, $ida = dia = d^2b = 0 \Rightarrow da = 0$. Define then $\delta^*[c] := [a]$ (independent of the choice of b and c).

Let's check, e.g., that the long sequence is exact on $H^k(\mathcal{C})$.

• $\operatorname{Im} j^* \subset \operatorname{Ker} \delta^*$: for $[b] \in H^k(\mathcal{B})$, we have $\delta^* j^*[b] = \delta^*[jb]$. By definition of δ^*, we can choose as b itself the element that goes to $c = jb$. But b is a cocycle: $db = 0$. Therefore, in the definition of δ^*, $ia = db = 0 \Rightarrow a = 0 \Rightarrow \delta^*[jb] = [0] = 0$. (Idem $i^*\delta^* = 0$).

• $\operatorname{Ker} \delta^* \subset \operatorname{Im} j^*$: if $\delta^*[c] = 0$, the a in the definition of δ^* is a coboundary and the b is a cocycle: $a = da'$. Thus $db = ida' = dia'$, i.e., $d(b - ia') = 0$. So $j^*[b - ia'] = [jb - jia'] = [jb] = [c]$. ∎

§25. The Mayer–Vietoris sequence

As we saw, (3) is exact for all k, hence we conclude:

Theorem 33 (!!!!). *The following long sequence of cohomology, called the sequence of Mayer–Vietoris, is* *<u>exact</u>:*

$$0 \to H^0(M) \overset{i^*}{\to} H^0(U) \oplus H^0(V) \overset{j^*}{\to} H^0(U \cap V) \overset{\delta^*}{\to} \cdots$$

$$\cdots$$

$$\cdots \overset{\delta^*}{\to} H^k(M) \overset{i^*}{\to} H^k(U) \oplus H^k(V) \overset{j^*}{\to} H^k(U \cap V) \overset{\delta^*}{\to}$$

$$\overset{\delta^*}{\to} H^{k+1}(M) \overset{i^*}{\to} H^{k+1}(U) \oplus H^{k+1}(V) \overset{j^*}{\to} H^{k+1}(U \cap V) \overset{\delta^*}{\to} \cdots$$

And, for the same price we got the recipe to construct δ^*:

• If $\omega \in \Omega^k(U \cap V)$, with part. of unity we get forms ω_U and ω_V on U and V such that $j(-\omega_U, \omega_V) = \omega_V|_{U \cap V} + \omega_U|_{U \cap V} = \omega$;
• Now, if ω is closed, $-d\omega_U$ and $d\omega_V$ agree on $U \cap V$ (!!!), since $j(-d\omega_U, d\omega_V) = dj(-\omega_U, \omega_V) = d\omega = 0$;
• Therefore, $-d\omega_U$ and $d\omega_V$ define a form $\sigma \in \Omega^{k+1}(M)$, that is clearly closed (yet not necessarily exact!). We conclude that $\delta^*[\omega] = [\sigma] \in H^{k+1}(M)$.

REM: If U, V and $U \cap V$ are connected we begin at $k = 1$, i.e.,

$$0 \to H^0(M) \overset{i^*}{\to} H^0(U) \oplus H^0(V) \overset{j^*}{\to} H^0(U \cap V) \to 0,$$

$$0 \to H^1(M) \overset{i^*}{\to} H^1(U) \oplus H^1(V) \overset{j^*}{\to} \cdots$$

are exact (since M is connected, and $H^0(U \cap V) \overset{\delta^*}{\to} H^1(M)$ is the zero function, since $j^* : H^0(U) \oplus H^0(V) \to H^0(U \cap V)$ is surjective).

Examples: $M = \bigcup_i M_i$ disjoint $\Rightarrow H^k(M) = \oplus_i H^k(M_i)$. $H^\bullet(\mathbb{S}^n)$. $H^\bullet(T^2)$.

§26. The Euler characteristic

In this section we assume that all cohomologies of M have finite dimension (we will see that this is always the case for M compact).

Definition 34. The *Euler characteristic of* M is the homotopic invariant

$$\chi(M) := \sum_i (-1)^i b_i(M) \in \mathbb{Z},$$

where $b_k(M) := \dim H^k(M)$ is the *k-th Betti number of* M*.*

Mayer–Vietoris + Proposition 31 $\Rightarrow$

$$\chi(M) = \chi(U) + \chi(V) - \chi(U \cap V). \tag{4}$$

Simplex $\Rightarrow$ triangulations: always exist (by countable basis).

Theorem 35. *For* <u>*any*</u> *triangulation of* M^n *it holds that*

$$\chi(M^n) = \sum_{i=0}^{n} (-1)^i \alpha_k,$$

where $\alpha_k = \alpha_k(\mathcal{T})$ *is the number of* k*-simplexes in* $\mathcal{T}$*.*

Proof: For each n-simplex σ_i of $\mathcal{T}$, choose $p_i \in \sigma_i^o$ and $p_i \in B_{p_i} \subset \sigma_i^o$ (think about p_i as a small ball too). Let U_1 be the disjoint union of these α_n balls, and $V_{n-1} = M \setminus \{p_1, \ldots, p_{\alpha_n}\}$. Then, (4) $\Rightarrow \chi(M^n) = \chi(V_{n-1}) + (-1)^n \alpha_n$.
For each $(n-1)$-face τ_j of $\mathcal{T}$, pick the "long" ball B_{τ_j} joining the two B_{p_i}'s of each n-simplex touching τ_j. Call U_2 the union of these disjoint α_{n-1} balls. Pick an arc (inside B_{τ_j}) joining the

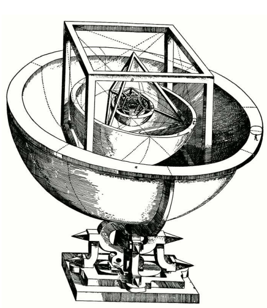
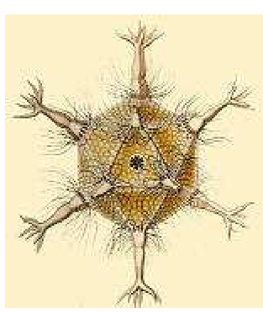
boundaries of the two B_{p_i}'s , and let V_{n-2} be the complement of these α_{n-1} arcs. Again, (4) $\Rightarrow \chi(V_{n-1}) = \chi(V_{n-2}) + (-1)^{n-1}\alpha_{n-1}$. Inductively we obtain $V_{n-3}, \cdots, V_0$, the last one being the union of α_0 contractible sets (each one a neighborhood of a vertex of $\mathcal{T}$), so that $\chi(V_0) = \alpha_0$ and $\chi(V_k) = \chi(V_{k-1}) + (-1)^k\alpha_k$. ∎

Corolary 36. *(Descartes-Euler) If a convex polyhedron has V vertices, F faces, and E edges, then $V - E + F = 2$.*

Corolary 37. *There are only 5 Platonic solids.*

Proof: If $r \geq 3$ is the number of edges (= vertices) on each face, and $s \geq 3$ is the number of edges (= faces) that arrive at each vertex, we have that $rF = 2E = sV$. But $V - E + F = 2 \Rightarrow 1/s + 1/r = 1/E + 1/2 > 1/2$, or $(r-2)(s-2) < 4$. Since $F = 4s/(2s + 2r - sr)$ we get (r, s) = (3,3) = tetrahedron = Fire, (4,3) = cube = Earth, (3,4) = octahedron = Air, (3,5) = icosahedron = Water, and (5,3) = dodecahedron... which, according to Plato, was *"...used by God to distribute the (12!) Constellations in the Universe"* (I was unable to prove this last assertion). ∎

Exercise: Show that if M is compact and $\hat{M} \to M$ is a p-fold cover, then $\chi(\hat{M}) = p\,\chi(M)$.


Platonic model of the solar system by Kepler; Circogonia icosahedra; Stones from 2000 AC

STRONG advice: Watch this video about Kepler's life, from the spectacular **Cosmos** TV series (the one from the 80s!).

REM: On dimension $n = 4$ there are 6 regular solids (there is one with 24 faces), and for $n \geq 5$ there are only 3: the simplex (tetrahedron), the hypercube (of course), and the hyperoctahedron, that is the convex hull of $\{\pm e_i\}$.

§27. Mayer–Vietoris: compact support

We cannot simply switch H^k by H^k_c in Mayer–Vietoris, since $\omega \in \Omega^k_c(M) \not\Rightarrow i^*_U(\omega) \in \Omega^k_c(U)$. However, if $\omega \in \Omega^k_c(U)$, the *extension as 0* of ω, $\hat{i}_U(\omega)$, satisfies $\hat{i}_U(\omega) \in \Omega^k_c(M)$. And this works! ($j := \hat{j}_U \oplus \hat{j}_V$, $i := \hat{i}_U - \hat{i}_V$):

Lemma 38. *The following sequence is exact $\forall k$ (exercise):*

$$0 \to \Omega^k_c(U \cap V) \xrightarrow{j} \Omega^k_c(U) \oplus \Omega^k_c(V) \xrightarrow{i} \Omega^k_c(U \cup V) \to 0.$$

Then, Theorem 32 + Lemma 38 $\Rightarrow$

Theorem 39. *The following long sequence is* <u>*exact*</u>*:*

$$\cdots \xrightarrow{\delta^*} H^k_c(U \cap V) \xrightarrow{j^*} H^k_c(U) \oplus H^k_c(V) \xrightarrow{i^*} H^k_c(M) \xrightarrow{\delta^*}$$

$$\xrightarrow{\delta^*} H^{k+1}_c(U \cap V) \xrightarrow{j^*} H^{k+1}_c(U) \oplus H^{k+1}_c(V) \xrightarrow{i^*} H^{k+1}_c(M) \xrightarrow{\delta^*} \cdots$$

REM: Compare both Mayer–Vietoris.
REM: <u>BEWARE not to mix them!!!</u>
REM: Theorem 32 is a factory of theorems!

§28. Mayer–Vietoris for pairs

Let $i\colon N \hookrightarrow M$ be a compact embedded submanifold, and $k \in \mathbb{Z}$. Then, $W = M \setminus N$ is a manifold and thus

$$\Omega_c^k(M \setminus N) \overset{\hat{j}_W}{\to} \Omega_c^k(M) \overset{i^*}{\to} \Omega^k(N).$$

But this is not exact on $\Omega_c^k(M)$: the kernel of i^* are the forms that vanish on N, while the image of $\hat{j}_W$ are the ones that vanish on a neighborhood of N. But we fix this with a standard trick: Let V be a tubular neighborhood with compact closure of N, $j : N \hookrightarrow V$ the inclusion, and $\pi : V \to N$ a deformation retract, i.e., $\pi \circ j = id_N$, $j \circ \pi \sim id_V$. We construct now a sequence of such V, $V = V_1 \supset V_2 \supset \cdots$, such that $\cap_i V_i = N$. Then, we say that ω and ω' on $\Omega^k(U)$ for some open $U \subset M$ containing N are *equivalent* if there is $r > i, j$ such that $\omega|_{V_r} = \omega'|_{V_r}$. The set of these classes is a vector space $\mathcal{G}^k(N)$, that of "*germs of k-forms defined in a neighborhood of N*", which has an obvious differential induced by d, and is therefore a cochain complex $\mathcal{G} = (\mathcal{G}^\bullet(N), d)$. This gives a cochain map $\Omega_c^k(M) \overset{\hat{i}^*}{\to} \mathcal{G}^k(N)$, where $\hat{i}^*(\omega) =$ class of $\omega|_{V_1}$.

Lemma 40. *The following sequence is exact (exercise):*

$$0 \to \Omega_c^k(M \setminus N) \overset{\hat{j}_W}{\to} \Omega_c^k(M) \overset{\hat{i}^*}{\to} \mathcal{G}^k(N) \to 0.$$

Now, since $j^* : H^k(V_i) \to H^k(N)$ is an isomorphism for all i and for all k, $H^k(N)$ is isomorphic to $H^k(\mathcal{G})$ (exercise). Then, Theorem 32 + Lemma 40 $\Rightarrow$

Theorem 41. *There is a long* <u>*exact*</u> *sequence:*

$$\cdots \to H_c^k(M\backslash N) \to H_c^k(M) \to H^k(N) \overset{\delta^*}{\to} H_c^{k+1}(M\backslash N) \to \cdots$$

In a completely analogous way to Theorem 41 we conclude:

Theorem 42. *Let M be a compact manifold with boundary. Then there exists a long* <u>*exact*</u> *sequence:*

$$\cdots \to H_c^k(M\backslash\partial M) \to H^k(M) \to H^k(\partial M) \overset{\delta^*}{\to} H_c^{k+1}(M\backslash\partial M) \to \cdots$$

Corolary 43. $H_c^k(\mathbb{R}^n) \cong H^{n-k}(\mathbb{R}^n) \cong (H^{n-k}(\mathbb{R}^n))^*$, $\forall\, k$.

Proof: By Corolary 18, if $B \subset \mathbb{R}^n$ is an open ball, $H_c^k(\mathbb{R}^n) = H_c^k(B) \cong H_c^k(\overline{B}) = H^k(\overline{B}) = H^k(B) = 0$, $\forall\ k \neq n$. ∎

Exercise: Compute $H^\bullet(\mathbb{S}^n \times \mathbb{S}^m)$. Suggestion: $\mathbb{S}^n \times \mathbb{S}^m = \partial(\overline{B} \times \mathbb{S}^m)$.

§29. Application: Jordan's theorem

Theorem 44 *(Jordan generalized). Let $M^n \subset \mathbb{R}^{n+1}$ be a connected embedded compact hypersurface. Then, M^n is orientable, $\mathbb{R}^{n+1} \setminus M^n$ has exactly 2 connected components, one bounded and one not, and M^n is the boundary of each one.*

Proof: By Theorem 41 and Corolary 43 we have that

$$0 \cong H_c^n(\mathbb{R}^{n+1}) \to H^n(M^n) \to H_c^{n+1}(\mathbb{R}^{n+1} \setminus M) \to H_c^{n+1}(\mathbb{R}^{n+1}) \cong \mathbb{R} \to 0.$$

That is, $\dim H^n(M^n) + 1 = b_0(\mathbb{R}^{n+1} \setminus M^n) \geq 2$ (Corolary 30). Hence, by Theorem 23 and Theorem 24, $H^n(M^n) \cong \mathbb{R}$, M^n is orientable, and $\#\{$connected components of $\mathbb{R}^{n+1} \setminus M^n\} = 2$. By the same argument for winding numbers, each point of M^n is arbitrarily close to points in both connected components. ∎

Corolary 45. *Neither the Klein bottle nor the projective plane can be embedded in* $\mathbb{R}^3$.

§30. Poincaré duality

Let $U \subset \mathbb{R}^n$ open bounded and star shaped with respect to 0, i.e.,

$$U = U_\rho = \{tx : 0 \le t < \rho(x),\ x \in \mathbb{S}^{n-1}\}$$

for some bounded function $\rho : \mathbb{S}^{n-1} \to \mathbb{R}_{>0}$.

Lemma 46. *If* $\rho \in C^\infty$, *U is diffeomorphic to* $\mathbb{R}^n$.

Proof: Clearly we can assume $\rho \ge 1$, so just choose the diffeomorphism $h\colon B_1 \to U$ as $h(tx) = (t + (\rho(x) - 1)f(t))x$, for any smooth function f with $f = 0$ on $[0, \epsilon)$, $f' \ge 0, f(1) = 1$. ∎

But ρ does not even need to be continuous... yet, it is semicontinuous:

Lemma 47. *Given* $x \in \mathbb{S}^{n-1}$ *and* $\epsilon > 0$, *there exist a neighborhood* $V_x = V(x, \epsilon)$ *of* x *such that* $\rho|_{V_x} > \rho(x) - \epsilon$.

Proof: U is open. ∎

Lemma 48. $H^\bullet(U) \cong H^\bullet(\mathbb{R}^n)$ *and* $H^\bullet_c(U) \cong H^\bullet_c(\mathbb{R}^n)$. *(In fact, U is diffeomorphic to* $\mathbb{R}^n$ *even if* ρ *is not* C^∞, *but this is a difficult result).*

Proof: The first is obvious since U is contractible. By Corolary 43 we thus only need to verify that $H^k_c(U) = 0$ for $k < n$. But if $[\omega] \in H^k_c(U)$, suppose that there is $\overline{\rho} \in C^\infty(\mathbb{R})$ such that $K = sup(\omega) \subset U_{\overline{\rho}} \subset U$ (i.e., $\overline{\rho} < \rho$). Then $U_{\overline{\rho}} \cong \mathbb{R}^n$ and

$[\omega] \in H^k_c(U_{\overline{\rho}}) = 0$. So there is $\eta \in \Omega^{k-1}_c(U_{\overline{\rho}}) \subset \Omega^{k-1}_c(U)$ with $\omega = d\eta$.
To show that there exists such a $\overline{\rho}$, let $2\epsilon = d(K, \mathbb{R}^n \setminus U) > 0$ and, for $x \in \mathbb{S}^{n-1}$, $t(x) := \max\{t : tx \in K\} \leq \rho(x) - 2\epsilon$. At a neighborhood V_x of x we have that $t|_{V_x} < \rho(x) - \epsilon < \rho|_{V_x}$ by Lemma 47 and the definition of ϵ. Pick a finite subcover $\{V_{x_i}\}$ of $\mathbb{S}^{n-1}$ and a partition of unity $\{\varphi_i\}$ subordinated to it, and define $\overline{\rho} = \sum_i (\rho(x_i) - \epsilon)\varphi_i$. Then, $t < \overline{\rho} < \rho$, and $K \subset U_{\overline{\rho}} \subset U$. ∎

Definition 49. We say that M^n is of *finite type* if there is a finite covering $\mathcal{U}$ of M^n such that every nonempty intersection V of elements of $\mathcal{U}$ satisfies that $H^\bullet(V) = H^\bullet(\mathbb{R}^n)$ and $H^\bullet_c(V) = H^\bullet_c(\mathbb{R}^n)$. Such a covering $\mathcal{U}$ is called *good*.

Lemma 50. *Every compact manifold has a good covering.*

Proof: Totally convex neighborhoods (Riemannian geometry). ∎

Proposition 51. *If M is of finite type (e.g. M compact), then $H^\bullet(M)$ and $H^\bullet_c(M)$ have finite dimension.*

Proof: Induction on $\#\,\mathcal{U}$ using Mayer–Vietoris. ∎

Now, observing that $H^k(M) \wedge H^r_c(M) \subset H^{k+r}_c(M)$ we obtain:

Theorem 52 (Poincaré duality). *If M^n is connected and orientable, the linear function $PD\colon H^k(M) \to (H^{n-k}_c(M))^*$,*

$$PD([\omega])([\sigma]) := \int_M \omega \wedge \sigma$$

is an isomorphism, for all k.

Proof: The proof for manifolds of finite type follows by induction in the number of elements of a good covering by the next lemma. ∎

Lemma 53. *If U and V are open such that PD is an isomorphism for all k in U, V and $U \cap V$, then PD is an isomorphism for all k in $U \cup V$.*

Proof: Let $M = U \cup V$ and $l = n - k$. Mayer–Vietoris gives

$$\begin{array}{ccccccccc} H^{k-1}(U) \oplus H^{k-1}(V) & \to & H^{k-1}(U \cap V) & \to & H^k(M) & \to & H^k(U) \oplus H^k(V) & \to & H^k(U \cap V) \\ \downarrow PD \oplus PD & & \downarrow PD & & \downarrow PD & & \downarrow PD \oplus PD & & \downarrow PD \\ (H_c^{l+1}(U) \oplus H_c^{l+1}(V))^* & \to & H_c^{l+1}(U \cap V)^* & \to & H_c^l(M)^* & \to & (H_c^l(U) \oplus H_c^l(V))^* & \to & H_c^l(U \cap V)^* \end{array}$$

where all vertical maps are isomorphisms, except maybe the middle one. Moreover, all squares commute up to signs (exercise), and hence up to some signs in the PD's everything commutes. The lemma follows now from *the five Lemma* (prove!), which says precisely that the middle one must also be an isomorphism. ∎

Corolary 54. *If M^n is compact, connected and orientable, then $b_k(M^n) = b_{n-k}(M^n)$. In particular $\chi(M^n) = 0$ if n is odd.*

Corolary 55. *Theorem 25 follows from Poincaré duality.*

Exercise. Show that the signature $s(M)$ of a compact oriented $4n$-manifold is an oriented cobordism invariant. Namely, from Theorem 52 we know that $\varphi_M = PD : H^{2n}(M^{4n}) \times H^{2n}(M^{4n}) \to \mathbb{R}$ is symmetric and non-degenerate, so we define

$$s(M) := \dim H^{2n}(M^{4n}) - 2\,\mathrm{index}(\varphi_M) = \mathrm{coindex}(\varphi_M) - \mathrm{index}(\varphi_M).$$

Show that, if there is a compact oriented manifold W^{4n+1} such that $\partial W = M^{4n} \cup -N^{4n}$, then $s(M) = s(N)$. (SUG: First notice that M does not need to be connected, so you can assume $N = \emptyset$. Then, if $i : M \to W$ is the inclussion, show that $Im\ i^*$ is isotropic for φ_M by Stokes. Finally, show that $Im\ i^*$ has half of the dimension of $H^{2n}(M^{4n})$ using the connection homomorphism δ^* and Stokes again.)

Exercise. Using the Five Lemma prove the Künneth Formula, which is true in general, when one of the factors is of finite type:

$$H^k(M \times N) = \oplus_{i+j=k} H^i(M) \otimes H^j(N), \quad \forall k.$$

30.1 The Poincaré sphere

Henri Poincaré conjectured that a 3-manifold with the homology of a sphere must be homeomorphic to the 3-sphere $\mathbb{S}^3$. Poincaré himself found a counterexample, essentially creating the concept of fundamental group. Indeed, by Hurewicz theorem, it would be enough to take $\mathbb{S}^3/G$, with $G \subset SO(4)$ a nontrivial perfect group (i.e., $G = [G, G]$) acting freely. The simplest such example that we can think of is $G = A_5 \subset SO(3)$ as the order 60 icosahedral group since A_5 is simple. This almost works, except that G has to be extended to the binary icosahedral group $G = 2A_5$ of order 120, which is still perfect, though not simple (or work with A_5 but on $SO(3) \cong \mathbb{S}^3/\{\pm I\}$ instead) . Then, $H_1(\mathbb{S}^3/G, \mathbb{Z}) = G/[G, G] = 0$, and $H_2(\mathbb{S}^3/G) = H_1(\mathbb{S}^3/G) = 0$ by e.g. Poincaré duality. Thus, $H_*(\mathbb{S}^3/G) = H_*(\mathbb{S}^3)$, yet $\mathbb{S}^3/G$ is not simply connected. It is remarkable that *this is the only example with finite fundamental group* (there are plenty with infinite fundamental group). After Poincaré found this counterexample to his own conjecture, he made another one: *the 3-sphere is the only simply connected homology 3-sphere*. This is of course the very famous *Poincaré conjecture*, proved (among other things!) by G.Perelman in 2002. Notice that, by Perelman's result, any homology 3-sphere with finite fundamental group **must** be $\mathbb{S}^3/G$, with $G \subset SO(4)$ perfect, reducing the original problem to a group one: find the finite perfect subgroups of $SO(4)$ that act freely. It turns out that $2A_5$ is the only one!

§31. Singular homology and de Rham Theorem

As seen in Section 18, we have the boundary operator between chains (of simplex) with any abelian group G as coefficients, $\partial_k : C_k(M) \to C_{k-1}(M)$, that satisfies $\partial^2 = 0$. That is, chains form a complex (for any topological space). The homology of this complex is called the *singular homology* of M:

$$H_k(M) = H_k(M; G) := \mathrm{Ker}\,\partial_k / \mathrm{Im}\,\partial_{k+1}.$$

Now, if $M = U \cup V$, the composition of chains with the inclusions gives the next (obviously exact) Mayer–Vietoris sequence:

$$0 \to C_k(U \cap V) \to C_k(U) \oplus C_k(V) \to C_k(U + V) \to 0,$$

where $C_k(U + V)$ are the k-chains of M that decompose as sum of k-chains on U and V. By Theorem 32 we get then the corresponding long exact sequence on homology. But, with an idea conceptually similar to the one used to construct $\mathcal{G}$ ("barycentric decomposition") we prove (with a bit of work) that

$$H_\bullet(U \cup V) \cong H_\bullet(U + V).$$

Therefore we have the long exact sequence of singular homology:

$$\cdots H_{k+1}(M) \to H_k(U \cap V) \to H_k(U) \oplus H_k(V) \to H_k(M) \to H_{k-1}(U \cap V) \to \cdots \quad (5)$$

Compare with Theorem 39 and use Theorem 9!

Exercise 2. Prove Exercise 3 on page 19 using $\mathbb{Z}_2$ relative homology: if M is compact then there is no retraction $f : M \to \partial M$. (Suggestion: $0 \to H_{n-1}(M, \partial M; \mathbb{Z}_2) \cong \mathbb{Z}_2 \to H_{n-1}(\partial M; \mathbb{Z}_2) \cong \mathbb{Z}_2 \to H_{n-1}(M; \mathbb{Z}_2)$).

For the singular (differentiable) homology $H_\bullet(M; \mathbb{R})$, by Stokes and in an analogous way to Poincaré duality (Lemma 53 in the proof of Theorem 52), we prove the following (recall Section 18):

Theorem 56 (deRham). *For every manifold M, the linear* *functional* $DR\colon H^k(M)\to(H_k(M;\mathbb{R}))^*$ *given by*

$$DR([\omega])([c]) = \int_c \omega$$

is an isomorphism, for all k*.*

Proof: See here for a general argument, even for manifolds that are not of finite type. ∎

The End. :o)

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