Localization and diffusivity for directed polymers in random environment

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2018 update

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Acknowledgements: The author of these notes is much indebted to Francis Comets, who put at his disposition material from directed polymer lectures given at Université Paris Diderot that has been a plentiful source of inspiration. He would also like to thank the organizer of this Summer School for their invitation to give this course and the Institudo de Matematica Pura e Applicada of Rio de Janeiro, for the hospitality during the preparation of this lecture support.

Chapter 1

Introduction and generalities

1 Physical context and modelization

1.1 Polymer in a random environment

We are interested in the following physical problem: consider a polymer chain, that is composed of a finite but large number of hydrophilic monomers in a watery solution. The solution is inhomogeneous in the sense that it contains randomly placed impurities, e.g. hydrophobic molecules, that repel the hydrophilic monomers. These interaction modify the shape of the polymer chain on a microscopic scale. The spatial configuration of the chain changes rapidly due to thermal agitation and thus it becomes reasonable to consider it as a random object. The question we want to address is the following:

"On a macroscopic scale, how do these impurities affect the global shape of the polymer chain."

We want to give a mathematical answer to this question, using the framework of statistical mechanics. To have a chance to do so, we must consider a rather simplified picture, our aim being to have a model that we can mathematically handle and that keeps the essential features of the original problem.

The first simplification is quite common in mathematical physics, is to make our polymer chain and impurities live on a discrete lattice: the spatial configuration of a polymer chain with N monomer is represented an N-steps path on the lattice and impurities, with deterministic or random hydrophilic strength are randomly placed on the sites of the lattice (see Figure 1.1). The distribution of the polymer shape is then given by a standard Gibbs-Boltzmann formalism.

The second simplification is to consider that our polymer chain tends to stretch in a given direction (this is the reason why the model is referred to as **directed** polymer). Whereas it might be physically questionable, this simplification allows us to disregard a lot of possible complication in the shape of the polymer such as entanglements and self-intersections. This of great help for the mathematical treatment of the model.

In this short course our aim is to give a mathematical overview of the known answers and conjectures about the above question. We can schematically distinguish two cases:

(i) When the space in which the polymer lives is high dimensional and the temperature is high-enough, the impurities do not affect the global shape of the polymer, this is due to averaging over space of the effect of the impurities.

(ii) When the dimension is small or when the temperature is low, impurities drastically change the behavior of the polymer.

1.2 The main model

Let us present now the main model that we are to study: the directed polymer on $\mathbb{N}\times\mathbb{Z}^d$. Our polymer configurations live on the lattice $\mathbb{N}\times\mathbb{Z}^d$; more precisely a possible spatial configuration of a chain with N monomer is described by $[(n,S_n)]_{n\in[0,N]}$ where S belongs to the set of nearest neighbor paths:

$$\mathcal{S}_N := \{(S_n)_{n\in[0,N]} \in (\mathbb{Z}^d)^{N+1} \mid S_0 = 0,\ \forall n \in [1,N], |S_n - S_{n-1}| = 1\}. \tag{1.1}$$

The set of impurities is modeled by $\omega := (\omega_{n,x})_{n\geq 0, x\in\mathbb{Z}^d}$ a field of IID centered variables of unit variance,

$$\mathbb{E}[\omega_{n,x}] = 0 \text{ and } \mathbb{E}[\omega_{n,x}^2] = 1.$$

and having finite exponential moment (let $\mathbb{P}$ and $\mathbb{E}$ denote the associated probability law and expectation),

$$\lambda(\beta) := \log \mathbb{E}[e^{\beta\omega_{1,0}}] < \infty, \forall \beta > 0. \tag{1.2}$$

The variable $\omega_{n,x}$ is the opposite of the energy of interaction between the impurity at site (n,x) and a monomer: it quantifies how much the site (n,x) attracts or repels a monomer.

The polymer measure $\bar{\mu}_N^{\beta,\omega}$ that describes the behavior of the polymer chain is a probability measure on $\mathcal{S}_N$ that is defined via a Gibbs-Boltzmann formalism for a fixed realization of the environment ω: to each path in $\mathcal{S}_N$ one associates an energy given by the Hamiltonian

$$H_N^\omega(S) := -\sum_{n=1}^N \omega_{n,S_n}. \tag{1.3}$$

Then at an inverse temperature $\beta > 0$, one associates to each path a probability that is proportional to its Boltzmann weight, i.e.

$$\bar{\mu}_N^{\beta,\omega}(S) = \frac{1}{\bar{Z}_N^{\beta,\omega}} e^{-\beta H_N^\omega(S)}, \tag{1.4}$$

where $\bar{Z}_N^{\beta,\omega}$, the sum of the Boltzmann weights, is the necessary renormalization factor that makes $\bar{\mu}_N^{\beta,\omega}(S)$ a probability measure on $\mathcal{S}_N$:

$$\bar{Z}_N^{\beta,\omega} = \sum_{S\in\mathcal{S}_N} e^{-\beta H_N^\omega(S)}. \tag{1.5}$$

We call it the partition function.

For technical convenience, in what follows we will rather consider the polymer measure as a probability law on the set of semi-infinite nearest neighbor trajectories $(S_n)_{n\geq 0}$. We remind the reader that for μ_N^β constructed below, only the restriction of S to its N first step has some physical relevance. Let P denote the law of the symmetric nearest neighbor random walk on $\mathbb{Z}^d$ (and E the associated expectation), the polymer measure μ_N^β, is a probability measure absolutely continuous w.r.t. P (expectation denoted by E), and whose Radon-Nikodym derivative is equal to

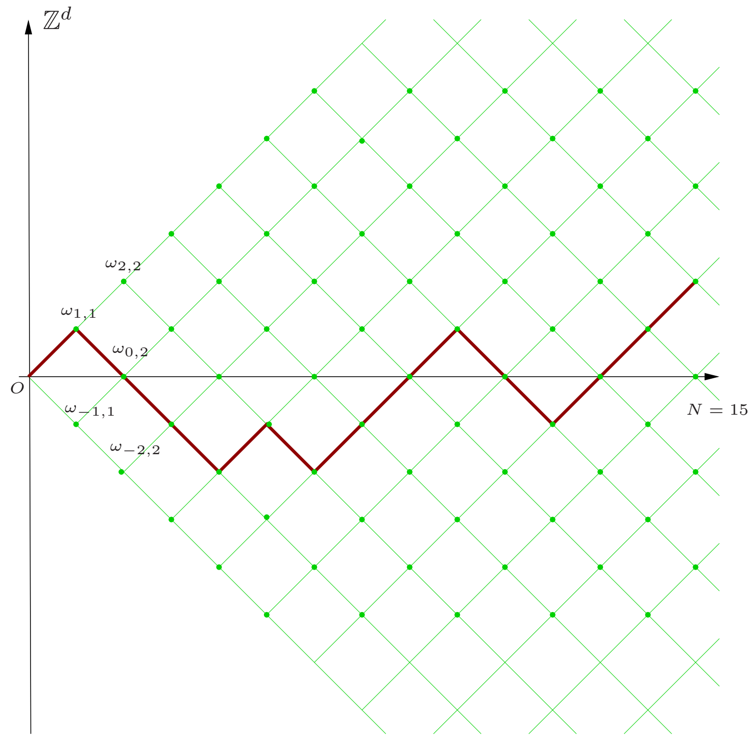

Figure 1.1: Representation of the polymer trajectory and of the random environment in the case where the transversal dimension is equal to one. The energy of a trajectory corresponds to the sum of the energies collected on the different sites visited.

$$\frac{\mathrm{d}\mu_N^{\beta,\omega}}{\mathrm{d}P}(S) := \frac{1}{Z_N^{\beta,\omega}} e^{-\beta H_N^{\omega}(S)}, \tag{1.6}$$

where

$$Z_N^{\beta,\omega} := E[e^{-\beta H_N^{\omega}(S)}] = (2d)^{-N} \bar{Z}_N^{\beta,\omega}. \tag{1.7}$$

Note that the law of $S_{|[0,N]}$ under $\mu_N^{\beta,\omega}$ is given by $\bar{\mu}_N^{\beta,\omega}$ so that the two definitions are equivalent.

We are interested in knowing the asymptotic behavior of $S_{|[0,N]}$ under the law μ_N and specifically whether and how the introduction of random inhomogeneity makes it different from the asymptotic behavior of the simple random walk. The answer to this question may depend on the temperature β and the dimension d.

According to folklore predictions by physicists, there are two alternatives:

(a) Disorder only modifies the local properties of the trajectories: under μ_N the macroscopic feature of the simple random walk are preserved, *e.g.* $(S_{\lceil tN \rceil}/\sqrt{N})_{t\in[0,1]}$ converges to the law of a d-dimensional Brownian motion.

(b) Disorder modifies the structure of the walk on a macroscopic levels, there exists corridors where the values of the potential $-\omega$ are particularly favorable (i.e. low) and thus under μ_N, trajectories are pinned on those corridors. Moreover, in this regime, the trajectories are superdiffusive in the sense that $|S_N|$ is typically of order N^ξ where $\xi > 1/2$ (i.e. the trajectories are ready to pay some entropic cost to visit energetically favorable region).

In the rest of this introductory chapter, we will make these conjectures more precise. As for many systems in statistical mechanics, many information about the behavior of the model can be retrieved by analyzing the behavior of the partition function $Z_N^{\beta,\omega}$. In fact, the right-way to decide (at least heuristically) whether (a) or (b) holds is to compare $Z_N^{\beta,\omega}$ with its expectation (or more especially their respective growth rate). We detail this in the next section.

Remark 1.1.1. Originally the model was introduced for $d = 1$ by Henley and Huse in [14] as a model of effective interface for the two-dimensional disordered Ising model at low temperature. Generalization of the model to arbitrary dimension and the polymer interpretation came soon afterwards [15]. Note that the model can also be used to described wave diffusion in a fast-changing random media, as the partition function is a time-inverted version of the total mass of the solution of the Parabolic Anderson Model (see e.g. [5]).

2 Properties of the partition function and its relation to trajectory properties

The analysis of the behavior of the partition function plays a central role in the study of directed polymer, and in statistical mechanics in general. We explain in this section how different asymptotic behaviors of the sequence Z_N are linked to different behavior for the polymer trajectories.

First let us compute the expectation of $Z_N^{\beta,\omega}$. This is done easily by using Fubini-Tonelli identity (recall definition (1.2))

$$\mathbb{E}\left[Z_N^{\beta,\omega}\right] = E\mathbb{E}\left[e^{\beta\sum_{n=1}^N \omega_{n,S_n}}\right] = e^{N\lambda(\beta)}. \tag{2.1}$$

We define $W_N := Z_N^{\beta,\omega}/\mathbb{E}\left[Z_N^{\beta,\omega}\right]$ to be the renormalized version of the partition function. One has

$$W_N = E[e^{\sum_{n=1}^N (\beta\omega_{n,S_n} - \lambda(\beta))}]. \tag{2.2}$$

Whether the ratio W_N tends to zero or not gives a valuable information, that is, whether the quantity $e^{\sum_{n=1}^N (\beta\omega_{n,S_n} - \lambda(\beta))}$ (which mean is equal to 1, but with typical value $e^{-N\lambda(\beta)+O(\sqrt{N})}$ for a given path) averages when one sums among all paths. If averaging occurs, it indicates that a typical polymer paths sees around him an averaged environment so that disorder does not really count on large scales. On the contrary if there is no averaging, it announces a different behavior, where disorder plays a role also on large scale. One has a nice dichotomy concerning the asymptotic behavior of W_N that allows to present clearly this alternative.

Proposition 1.2.1. *The sequence $(W_N)_{N\geq 0}$ is a martingale with respect to the filtration $(\mathcal{F}_N)_{N\geq 0}$ where*

$$\mathcal{F}_N = \sigma((\omega_{n,x})_{n\leq N, x\in\mathbb{Z}^d}).$$

Consequently the limit $\lim_{N\to\infty} W_N = W_\infty$ exists, and we have the following $0-1$ law

$$\mathbb{P}[W_\infty > 0] \in \{0, 1\}. \tag{2.3}$$

Proof. Using Fubini-Tonelli and the fact that $\mathbb{E}[\exp(\beta\omega - \lambda(\beta))] = 1$, one has

$$\mathbb{E}[W_N \mid \mathcal{F}_{N-1}] = E[\mathbb{E}[e^{\sum_{n=1}^{N}(\beta\omega_{n,S_n} - \lambda(\beta))} \mid \mathcal{F}_{N-1}]] = E[e^{\sum_{n=1}^{N-1}(\beta\omega_{n,S_n} - \lambda(\beta))}] = W_{N-1}, \quad (2.4)$$

so that W_N is a martingale sequence. A positive martingale always converges so that the limit W_∞ exists.

Moreover it is easy to remark that the event $\{W_\infty = 0\}$ is independent of $(\omega_{n,x})_{x\in\mathbb{Z}^d, n\leq N}$ for all values of N so that it belongs to the tail sigma-algebra of the field ω. It is then a classic result that the tail sigma-algebra of an IID field is trivial, i.e. that all events in it have probability zero or one. □

When $W_\infty > 0$ a.s. , we say that the influence of the disorder is weak, or that **weak disorder** holds. This spatial averaging implies that trajectories have diffusive behavior. However this is far from being an easy statement. We will be prove a weak version of the following theorem in Chapter 2.

Theorem 1.2.2 ([9, Theorem 1.2])**.** *When $W_\infty > 0$, $\mathbb{P}$-a.s. one has that the law of the process*

$$\left(\frac{S_{\lceil Nt\rceil}}{\sqrt{N}}\right)_{t\in[0,1]}, \quad (2.5)$$

under $\mu_N^{\beta,\omega}$ converges weakly in law to the law $\mathbf{P}$ of a d-dimensional Brownian motion $B_{t\in[0,1]}$ starting from the origin and with covariance matrix $\frac{1}{d}I$ where I is the identity matrix in $\mathbb{Z}^d$.

On the contrary when $W_\infty = 0$, it is believed that the polymer trajectories have a singular behavior with respect to the law of the simple random walk: more precisely it is predicted that in this case, trajectories tend to pin on a special corridor of where the environment is peculiarly favorable. so that diffusivity do not occurs and one observes instead a phenomenon of localization of the trajectories. For this reason, this case is often referred to as **strong disorder** in the litterature. There are several characterization of the strong disorder regime, which are often not fully satisfactory, but that give at least a heuristic evidence of the phenomenon. One of those is overlap of the terminal point.

Define

$$I_n := \sum_{x\in\mathbb{Z}^d} (\mu_{n-1}(S_n = x))^2 = \mu_{n-1}^{\otimes 2}(S_n^{(1)} = S_n^{(2)}). \quad (2.6)$$

It is almost equal to the probability that two independent polymer have their end-points coincide.

Theorem 1.2.3 ([6, Theorem 1.1])**.** *When weak disorder holds one as*

$$\sum_{n=1}^{\infty} I_n < \infty, \ \mathbb{P} \ a.s. \quad (2.7)$$

On the contrary when $W_\infty = 0$, one has

$$\sum_{n=1}^{\infty} I_n = \infty, \ \mathbb{P} \ a.s.$$

and there exist a constant c (depending on β and the distribution of the ω) such that a.s.

$$-c^{-1}\log W_N \leq \sum_{n=1}^{N} I_n \leq -c\log W_N. \tag{2.8}$$

Of course this result just indicates a kind of overlap for the terminal point of the trajectory, but from the definition of the model there is no reason that the terminal point should play a special role. Hence we might think that the above result indicates that the mean overlap between two polymer trajectories is infinite. I.e. that we can replace $\sum_{n=1}^{N} I_n$ in (2.8) by

$$J_N := \sum_{n=1}^{N} \mu_N^{\otimes 2}(S_n^{(1)} = S_n^{(2)}). \tag{2.9}$$

The statement (2.8) bears a much stronger signification when W_N decays exponentially fast to zero: it indicates that J_N should be proportional to N, so that two independent polymer configuration tend to have a positive overlap fraction. This gives an incentive to study the rate of decay of the partition function (or its radius of convergence). According to the following proposition this rate of decay is perfectly well defined.

Proposition 1.2.4. *The limit*

$$p(\beta) := \lim_{N\to\infty} \frac{1}{N} \log Z_N^{\beta,\omega}, \tag{2.10}$$

exists $\mathbb{P}$-a.s. and is non-random. It is equal to

$$\lim_{N\to\infty} \frac{1}{N}\mathbb{E}\left[\log Z_N^{\beta,\omega}\right]. \tag{2.11}$$

We call $p(\beta)$ the free-energy of the directed polymer. One has

$$p(\beta) \leq \frac{1}{N}\log \mathbb{E} Z_N^{\beta,\omega} = \lambda(\beta). \tag{2.12}$$

When the above inequality is strict we say that **very strong disorder holds***.*

Note that trivially, very strong disorder implies strong disorder. An important issue is thus to know when weak, strong and very strong disorder holds. A first answer to the question is given by the following proposition that says that there is at most one phase transition for each notion of strong disorder

Proposition 1.2.5 ([9, Theorem 1.1])**.** *There exists $\beta_c \leq \bar{\beta}_c$ in $[0,\infty]$ such that the following holds*

(i) If $\beta < \beta_c$ then weak-disorder holds, and if $\beta > \beta_c$ strong disorder holds,

(ii) If $\beta \leq \bar{\beta}_c$ the $p(\beta) = \lambda(\beta)$ and if $\beta > \bar{\beta}_c$ the $p(\beta) < \lambda(\beta)$.

the second point is a consequence that $\beta \mapsto \lambda(\beta) - p(\beta)$ is a non-decreasing function. Furthermore, if ω has an unbounded distribution $\beta_c \leq \bar{\beta}_c < \infty$.

The above result (that we will not prove in these notes) leaves two questions unanswered (see Figure 1.2), the first one concerns the existence of the weak disorder phase: is $\beta_c > 0$? The other is whether there is really two phase transition: are $\beta_c = \bar{\beta}_c$ different?

The answer to the first question is not too difficult in dimension $d \geq 3$ and is yes as we will see in the next chapter. In low dimension however, when $d = 1$ or $d = 2$ the space is not large enough for self-averaging at any temperature and $\bar{\beta}_c = 0$. Concerning the second question, physicists believe that $\beta_c = \bar{\beta}_c$ but it remains a very challenging mathematical issue to prove it when $d \geq 3$ and it is a long standing open problem of the field.

Theorem 1.2.6 ([7, Theorem 1.1], [17, Theorem 1.6]). *When $d = 1$ or $d = 2$, one has very strong disorder at all temperature (i.e. for all value of β).*

Theorems 1.2.3 and 1.2.6 as well as Proposition 1.2.4 are partially proved in Chapter 3.

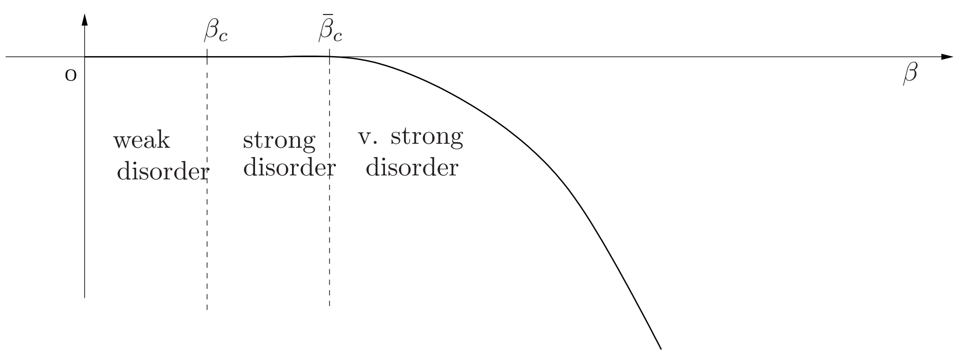

Figure 1.2: Representation of the difference between quenched and annealed free-energy $p(\beta)$ and $\lambda(\beta)$ as a function of β, critical values for β and the different possible regime. In dimension 1 and 2, $\bar{\beta}_c = 0$ and only the strong disorder regime exists. When $d \geq 3$, it is a challenging question to prove that there is no intermediate regime, i.e. that $\beta_c = \bar{\beta}_c$.

Some bibliographical comment

The first work establishing the existence of a diffusive phase for directed polymer is from Imbrie and Spencer [15]. In [3] comes the idea of studying the renormalized partition function W_N to extract information on the trajectorial behavior, and idea that has then been developed in various paper, including [9] where the most precise result up to now is proved. Rigorous localization results are more recent: Carmona and Hu proved some link between the overlap fraction and the asymptotic behavior of W_N in [4] for Gaussian environment, an approach that was generalized and complemented in [6]. In those paper, additionally, strong disorder at all temperature when $d = 1$ and $d = 2$ was established, while very strong disorder was established in [7], [17] respectively for $d = 1$ and $d = 2$. Note that when the environment is either Poissonian or Gaussian, localization results concerning the overlap J_N of equation (2.9) have also been established (see [4, Section 7], or more recently [5, 10]).

3 Superdiffusivity

Apart from localization of the trajectories, there is another interesting phenomenon that should occur in the strong disorder phase. For the simple random walk S on $\mathbb{Z}^d$, it is known that after N step, S_N stands typically at a distance of order $\sqrt{N}$ from the origin. When strong disorder occurs, it is believed that polymer trajectories want to wander away farther from the origin to reach zones where the environment is more favorable (i.e. where ω takes large values).

Physicists predicts that in this case the typical distance of the end-point to the origin is equal to N^ξ where $\xi > 1/2$ depends on the dimension but not on the temperature (as long as $\beta > \beta_c$).

In dimension 1 this conjecture is even more precise, as the value of ξ is predicted to be equal to 2/3. The fact that ξ is a rational number is believed to be a special setup of dimension 1 and determining the value of ξ in higher dimension seems an hopeless task.

On the mathematical side, the case of dimension $1+1$ is the one for which substantial results have been obtained, and they can be divided in two kinds:

- Results obtained using on an entropy versus energy competition approach, giving mainly bounds like $3/5 \leq \xi \leq 3/4$ see for instance [20] for the lower bound and [19] for the upper bound. Similar results have been obtained for Brownian Motion among random obstacles [22, 23].

- Results obtained using exact computations: for very special version of the model, like oriented last passage percolation with exponential environment [16] or polymer with log-gamma environment [21]. For some of these models the formal existence of ξ is established as the equality $\xi = 2/3$.

A formal introduction of these results would be too long in this introduction. Note that for the model presented above, which is also the most studied in the literature, superdiffusivity remains an open and challenging question.

Chapter 2

The diffusive phase

In this chapter our aim is to establish a weak version of Theorem 1.2.2. The approach we present is the one from [9]. We complement the result by showing exhibiting an explicit criterion for having weak disorder (known as the $\mathbb{L}_2$ criterion), and a remark concerning the singularity of the polymer measure with respect to P.

1 Preliminaries

1.1 The $\mathbb{L}_2$ criterion for weak disorder

Before starting a proof that weak disorder implies diffusivity, we might check that the weak disorder phase truly exists, i.e. that $\beta_c > 0$ when $d \geq 3$. This is not too difficult to check using the so-called second moment method: we extract information on the sequence $(W_N)_{N\geq 0}$ by computing $(\mathbb{E}\left[W_N^2\right])_{N\geq 0}$.

Set p_d to be the probability that independent two walks on $\mathbb{Z}^d$ starting from the origin meet at positive time:

$$p_d := P^{\otimes 2}[\exists n \geq 1, S_n^{(1)} = S_n^{(2)}]. \tag{1.1}$$

Note that $p_1 = p_2 = 1$ and that $p_d < 1$ for $d \geq 3$.

Proposition 2.1.1. *If β is such that*

$$e^{\lambda(2\beta)-2\lambda(\beta)} p_d < 1, \tag{1.2}$$

then $W_\infty > 0$, $\mathbb{P}$-a.s. so that in particular

$$\beta_c \geq \beta_2 := \inf\{\beta \mid \exp(\lambda(2\beta) - 2\lambda(\beta))p_d \geq 1\}. \tag{1.3}$$

Proof. One will check that our condition on β is equivalent to

$$\sup_{N\geq 0} \mathbb{E}\left[W_N^2\right] < \infty. \tag{1.4}$$

When (1.4) holds, our martingale is uniformly integrable, which implies that $\mathbb{E}[W_\infty] = \mathbb{E}[W_0] = 1$. Thus $\mathbb{P}[W_\infty = 0] > 0$. The $0 - 1$ law of Proposition 1.2.1 allows to conclude that $W_\infty > 0$ $\mathbb{P}$-a.s.

Let us now prove (1.4). Let $P^{\otimes 2}$ denote the law of two independent random walks $S^{(1)}$, $S^{(2)}$. By the very definition of a product measure, one has

$$W_N^2 = E^{\otimes 2}\left[\exp\left(\sum_{n=1}^{N}\beta(\omega_{n,S_n^{(1)}}+\omega_{n,S_n^{(2)}})-2\lambda(\beta)\right)\right]. \tag{1.5}$$

We have thus by Fubini-Tonelli formula

$$\mathbb{E}[W_N^2] = E^{\otimes 2}\left[\mathbb{E}\left[\exp\left(\sum_{n=1}^{N}\beta(\omega_{n,S_n^{(1)}}+\omega_{n,S_n^{(2)}})-2\lambda(\beta)\right)\right]\right]. \tag{1.6}$$

Then, we note that

$$\mathbb{E}\left[\exp\left(\sum_{n=1}^{N}\beta(\omega_{n,S_n^{(1)}}+\omega_{n,S_n^{(2)}})-2\lambda(\beta)\right)\right] = \begin{cases} 1 \text{ if } S_n^{(1)} \neq S_n^{(2)}, \\ e^{\lambda(2\beta)-2\lambda(\beta)} \text{ if } S_n^{(1)} = S_n^{(2)}. \end{cases} \tag{1.7}$$

so that

$$W_N^2 = E^{\otimes 2}\left[e^{(\lambda(2\beta)-2\lambda(\beta))\sum_{n=1}^{\infty}\mathbf{1}_{S_n^{(1)}=S_n^{(2)}}}\right], \tag{1.8}$$

and the monotone limit Theorem gives

$$\lim_{N\to\infty}\mathbb{E}[W_N^2] := C\left[e^{(\lambda(2\beta)-2\lambda(\beta))\sum_{n=1}^{\infty}\mathbf{1}_{S_n^{(1)}=S_n^{(2)}}}\right]. \tag{1.9}$$

By the strong Markov property for the process $(S^{(2)}-S^{(1)})$, the total overlap between two independent random walk

$$L(S^{(1)},S^{(2)}) := \sum_{n=1}^{\infty}\mathbf{1}_{S_n^{(1)}=S_n^{(2)}},$$

is a geometric variable of parameter p_d:

$$P^{\otimes}[L(S^{(1)},S^{(2)})=k] = (1-p_d)p_d^k.$$

Thus one has

$$\lim_{N\to\infty}\mathbb{E}[W_N^2] = \infty \quad \Leftrightarrow \quad e^{\lambda(2\beta)-2\lambda(\beta)}p_d < 1. \tag{1.10}$$

□

Remark 2.1.2. This criterion does not allow to determine entirely what the weak disorder phase is, as in [2] it was shown that the inequality (1.3) is a strict one. The fact that the martingale is never bounded in $\mathbb{L}_2$ when $d=1,2$ is not surprising in view of Theorem 1.2.3.

1.2 A remark: the polymer measure is very singular with respect to P in the weak disorder phase

To underline the fact that Theorem 1.2.2 is not a trivial result, we want to show here that even in the weak disorder phase, the polymer measure becomes very singular with respect to the simple random walk one when N grows. The result we present here is not optimal: a version of it holds also in the strong disorder case.

Consider $\beta < \beta_c$ and define $\mathcal{H}(\beta, N, \omega)$ to be the set of paths whose potential energy is atypically low:

$$\mathcal{H}(\beta, N, \omega) := \left\{ S \mid \sum_{n=1}^{N} \omega_{n,S_n} \geq N\lambda(\beta)/2\beta \right\}. \tag{1.11}$$

Indeed for most paths S, according to the central limit theorem $\sum_{n=1}^{N} \omega_{n,S_n} = O(\sqrt{N})$ (recall that $\omega_{n,x}$ has mean zero and unit variance). We show that $\mathcal{H}(\beta, N, \omega)$ is an event of almost full probability under the polymer measure whereas its probability is going to zero under P.

Proposition 2.1.3. *There exists $c(\beta)$ such that with probability one, one has for all large enough N*

$$\begin{aligned} \mu_N^{\beta,\omega}(\mathcal{H}(\beta, N, \omega)) &\geq 1 - e^{c(\beta)N}, \\ P(\mathcal{H}(\beta, N, \omega)) &\leq e^{-c(\beta)N}. \end{aligned} \tag{1.12}$$

As a consequence the total variation distance between the two measures tends to one exponentially fast, i.e. for N large enough

$$\|\mu_N - P\|_{TV} \geq 1 - 2\exp(-c(\beta)N). \tag{1.13}$$

Recall that the total variation distance between two measure is defined by

$$\|\mu - \nu\|_{TV} = \max_{A} |\mu(A) - \nu(A)| \tag{1.14}$$

where the max is taken over all measurable events.

Proof. By the very definition of $\mu_N^{\beta,\omega}$ one has

$$\mu_N^{\beta,\omega}((\mathcal{H}(\beta, N, \omega))^c) \leq \frac{1}{W_N} E[e^{\beta(\sum_{n=1}^{N} \omega_{n,S_n}) - N\lambda(\beta)} \mathbf{1}_{(\mathcal{H}(\beta,N,\omega))^c}]. \tag{1.15}$$

Then for N large enough $W_N \geq W_\infty/2 > 0$, so that using the definition of $\mathcal{H}(\beta, N, \omega)$

$$\mu_N^{\beta,\omega}((\mathcal{H}(\beta, N, \omega))^c) \leq \frac{2}{W_\infty} e^{-N\lambda(\beta)/2}. \tag{1.16}$$

wich gives the first part of the result. For the second one note that by Cramer's Theorem for sum of IID variables (see [11, Theorem 2.2.1]), there exists $c(\beta)$ such that

$$\mathbb{E}[P(\mathcal{H}(\beta, N, \omega))] = E\left[\mathbb{P}\left[\sum_{n=1}^{N} \omega_{n,S_n} \geq N\lambda(\beta)/2\beta\right]\right] \leq e^{-2c(\beta)N}. \tag{1.17}$$

Then one has by Markov inequality

$$\mathbb{P}[P(\mathcal{H}(\beta, N, \omega)) \geq e^{-c(\beta)N}] \leq e^{-c(\beta)N}, \tag{1.18}$$

and the Borel-Cantelli Lemma allows us to conclude.

□

1.3 The infinite volume limit

In this section, we define an infinite volume version of the polymer measure that we call μ. The general idea is that it is easier to compare the behavior of $(S_n)_{n\in N}$ under a measure μ at different scales N than to have a different measure for each scale. Then we can use that μ_N is somehow close to μ to get results for μ_N (see Proposition 2.2.6) .

A naive way to define μ so would be for each measurable event A with $P(A) > 0$ to consider

$$\mu_\infty(A) := \lim_{N\to\infty} \mu_N(A) = \frac{1}{W_\infty} \lim_{N\to\infty} E[e^{\beta(\sum_{n=1}^N \omega_{n,S_n})-N\lambda(\beta)} \mathbf{1}_{S\in A}]. \tag{1.19}$$

For fixed A, one can show that the sequence

$$W_N(A) := E[e^{\beta(\sum_{n=1}^N \omega_{n,S_n})-N\lambda(\beta)} \mathbf{1}_{S\in A}], \tag{1.20}$$

is a martingale, as in Proposition 1.2.1. Hence convergence holds.

The problem is that if we can to show the existence of such a limit, we cannot show that it is a probability measure : one has trouble with σ-additivity when passing to the limit. What we can show is that μ_∞ restricted to the sigma algebra

$$\mathcal{G}_N := \sigma(S_n, n \in [0,N]), \tag{1.21}$$

is a measure for every N.

A way to avoid trouble is to consider the probability measure μ which is the projective limit of $\mu_{\infty|\mathcal{G}_N}$. This is necessary as it can be shown that μ_∞ is **not** a probability measure. We can in fact construct μ directly without calling for Kolmogorov's Extension Theorem and we describe this construction now.

We given $X \in \mathbb{Z}^d$ and $N \geq 0$, we define $\theta_{X,N}$ to be the shift operator on the environment

$$(\theta_{X,N}\omega_{x,n})_{x\in\mathbb{Z}^d, n\geq 0} := (\omega_{X+x,N+n})_{x\in\mathbb{Z}^d, n\geq 0} \tag{1.22}$$

We define then for $N \in \mathbb{N} \cup \{\infty\}$

$$W_N(n,x) := W_N \circ \theta_{n,x} = E[e^{\beta(\sum_{k=1}^N \omega_{n+k,S_k+x})-N\lambda(\beta)}]. \tag{1.23}$$

to be the renormalized partition function constructed from $\theta_{x,n}\omega$ (or their limit in the case $N = \infty$). Then we can define μ as the probability measure under which S is a simple random walk whose transitions are random and given by

$$\mu(S_{n+1} = y \mid S_n = x) := \frac{\mathbf{1}_{x\sim y} \exp(\beta\omega_{n+1,y} - \lambda(\beta)) W(n+1,y)}{2dW(n,x)}. \tag{1.24}$$

The reader can check that the transition rates sum to one and that

$$\forall N \geq 0,\ \forall A \in \mathcal{G}_N, \mu(A) = \mu_\infty(A). \tag{1.25}$$

This construction of an infinite volume measure is a specific feature of the weak disorder case and there is no natural way of doing something similar in the strong disorder setup. Note also that μ is singular with respect to P (this can be proved as we proved Proposition 2.1.3).

2 Proof of diffusivity in the weak disorder phase

For the sake of simplicity we do not prove diffusivity directly for the polymer measure but only for the averaged version of it $\mathbb{P}\mu_N$ obtained after integrating over all possible environment.

Our plan for the proof is the following:

- First, one shows that, contrary to μ, the infinite volume measure averaged over the environment $\mathbb{P}\mu$ defined by
$$\mathbb{P}\mu(A) := \mathbb{E}\left[\mu(A)\right], \tag{2.1}$$
is absolutely continuous with respect to P the simple random walk measure, and that in fact it is equal to
$$\mathbb{P}\mu_\infty(A) := \mathbb{E}\left[\mu_\infty(A)\right] \tag{2.2}$$
- Second, one shows that for any measure that is absolutely continuous with respect to P, Donsker's Theorem (i.e. convergence to Brownian motion) holds. The idea there is that if a measure is absolutely continuous with respect to P the difference with P has to lie mainly in the first few steps of the walk (see Proposition 2.2.3).
- Finally, we show that restricted to a certain set of events, μ_N and μ are close to each other (Lemma 2.2.6) and use this remark to show that convergence to Brownian Motion for $\mathbb{P}\mu_N$ is implied by the one for $\mathbb{P}\mu$ (Proposition 2.2.8).

To prove the convergence to Brownian Motion for the non-averaged measure, one needs to go one step further and to show the absolute-continuity with respect to $P^{\otimes 2}$ of the averaged two-replica infinite-volume polymer measure $\mathbb{P}\mu^{\otimes 2}$, which is a bit too long to develop here.

Proposition 2.2.1. *The measure $\mathbb{P}\mu$ is absolutely continuous with respect to P.*

We will in fact prove the following

Lemma 2.2.2. *For any value of N and any event A of probability $P(A)$ one has*
$$\mathbb{P}\mu_N(A) \le \kappa(P(A)), \tag{2.3}$$
where $\kappa : \mathbb{R}_+ \to \mathbb{R}$ is an increasing function such that $\lim_{x\to 0}\kappa(x) = 0$. As a consequence $\mathbb{P}\mu_\infty$ is σ-additive and is thus a probability measure. Furthermore $\mathbb{P}\mu_\infty = \mathbb{P}\mu$.

Proposition 2.2.1 follows directly from the Lemma. Indeed if $P(A) = 0$, $\mu_N(A) = 0$ for all N so that $\mu_\infty(A) = 0$, $\mathbb{P}$ a.s. and thus from the above Lemma $\mathbb{P}\mu(A) = \mathbb{E}\mu_\infty(A) = 0$.

Proof. Set $\delta := \sqrt{P(A)}$ Then one remarks that
$$\mathbb{E}[\mu_N(A)] \le \mathbb{E}[\mu_N(A)\mathbf{1}_{W_N\ge\delta}] + \mathbb{P}[W_N \le \delta]. \tag{2.4}$$
As W is a positive martingale
$$\mathbb{P}[W_N \le \delta] \le 2\mathbb{P}[W_\infty \le 2\delta]. \tag{2.5}$$

On the other hand

$$\begin{aligned}\mathbb{E}[\mu_N(A)\mathbf{1}_{W_N\geq\delta}] &\leq \delta^{-1}\mathbb{E}[\mu_N(A)W_N] \\ &= \delta^{-1}E\left[\mathbb{E}[e^{\beta(\sum_{n=1}^N \omega_{n,S_n})-N\lambda(\beta)}\mathbf{1}_{S\in A}]\right] = \delta^{-1}P(A) = \delta, \end{aligned} \quad (2.6)$$

that converges to zero, so that (2.3) holds for $\kappa(x) = \sqrt{x} + 2\mathbb{P}[W_\infty \leq 2\sqrt{x}]$. Note that by dominated convergence Theorem (2.3) holds also for $N = \infty$.

The function $\mathbb{P}\mu_\infty$ is finitely additive as μ_∞ is. To show that it is sigma additive is not too complicated: if $(A_m)_{m\geq 0}$ is a sequence of distinct event, for any ε there exists an N_ε such that

$$P(\bigcup_{m\geq N} A_m) \leq \varepsilon. \quad (2.7)$$

so that

$$\mathbb{P}\mu_\infty(\bigcup_{m\geq N} A_m) \leq \sum_{k=1}^{N-1}\mathbb{P}\mu_N(A_k) + \mathbb{P}\mu_N(\bigcup_{m\geq N} A_m)) \leq \sum_{k=1}^{\infty}\mathbb{P}\mu_N(A_k) + \kappa(\varepsilon). \quad (2.8)$$

As ε is arbitrary it completes the proof (the other inequality comes from finite additivity).

The measures $\mathbb{P}\mu_\infty$ and $\mathbb{P}\mu$ coincides on $\cup_{N\geq 0}\mathcal{G}_N$ and thus are equal. □

Now let us prove a general statement for product measure that we want to apply to the sequence of random walks increments.

Proposition 2.2.3. *Let P be equal to $Q^{\otimes\mathbb{N}}$ a probability measure on $(\mathbb{R}^d)^{\mathbb{N}}$ under which $(X_n)_{n\in\mathbb{N}}$ is an IID sequence of variable. Let P' be a measure which is absolutely continuous with respect to Q, and P_k and P'_k denote the law of $(X_n)_{n\geq k}$. Then the total variation distance between P_k and P'_k goes to zero when k tends to infinity.*

Proof. By density in $\mathbb{L}_1$ of simple functions (i.e. of function taking only finitely many values), one can restrict our proof to the case where $\mathrm{d}P'/\mathrm{d}P$ is a simple function. Then by linearity, one can in fact restrict ourselves to the case where $P' = P(\cdot \mid A)$ where A is an event of positive P probability. Then by density again one can restrict ourselves to the case where $A \in \bigcup_{n\geq 0}\mathcal{G}_n$ and $\mathcal{G}_n$ is the sigma algebra generated by $X_1, \dots, X_n$ (i.e. A is a cylinder event). Then if $A \in \mathcal{G}_n$, and $P' = P(\cdot \mid A)$, $P_k = P'_k$ for $k \geq n$, which ends the proof. □

Now let us recall Donsker's Theorem concerning the simple random walk in $\mathbb{Z}^d$. Let $\mathcal{C}([0,1])$ denote the space of continuous function $[0,1]$ to $\mathbb{R}^d$ equipped with the $\mathbb{L}_\infty$ norm. And set

$$\begin{aligned} S^{(n)} &: [0,1] \to \mathbb{R}^d, \\ t &\mapsto \frac{\bar{S}_{nt}}{\sqrt{n}}, \end{aligned} \quad (2.9)$$

where $\bar{S}: \mathbb{R}_+ \to \mathbb{R}$ is the linear interpolation of S between integer coordinate.

Theorem 2.2.4 ([12]). *Under P, the sequence of processes $S^{(n)}$ converges in law to $\mathbf{P}$ the law of a standard Brownian motion $(B_t)_{t\in[0,1]}$ of covariance matrix $d^{-1}I$. More precisely for any continuous bounded function F, $\mathcal{C}([0,1]) \mapsto \mathbb{R}$ one has*

$$\lim_{n\to\infty} E[F(S^{(n)})] = \mathbf{E}[F(B)]. \quad (2.10)$$

An easy consequence of Proposition 2.2.3 is the following

Proposition 2.2.5. *Donsker's Theorem still holds if P is replaced by P' absolutely continuous with respect to P. In particular, for every for any continuous bounded function F, $\mathcal{C}([0,1]) \mapsto \mathbb{R}$ one has*

$$\lim_{n\to\infty} \mathbb{E}[\mu(F(S^{(n)}))] = \mathbf{E}[F(B)]. \tag{2.11}$$

Proof. First remark that given ε if k is large enough, from Proposition 2.2.3 one can couple P and P' such that $(S_n - S_k)_{n\geq k}$ and $(S'_n - S'_k)_{n\geq k}$ coincide with probability $(1-\varepsilon)$.

Thus with this coupling the probability that $\|S^{(n)} - (S')^{(n)}\|_\infty \geq k/\sqrt{n}$ is smaller than ε. The convergence of $S^{(n)}$ and the fact that ε is arbitrary allow to conclude the proof. □

The last step to prove diffusivity of the trajectories is to show that somehow μ and μ_N are somehow close to each other.

Given two measures on the set of infinite nearest-neighbor path μ and ν,let $\|\mu-\nu\|_{\mathcal{G}_n}$ denote the total variation distance between $\mu_{|\mathcal{G}_n}$ and $\nu_{|\mathcal{G}_n}$, the measure restricted to $\mathcal{G}_n$ (recall (1.7))

Lemma 2.2.6. *One has*

$$\lim_{k\to\infty} \sup_n \mathbb{E}\left[\|\mu_{n+k} - \mu\|_{\mathcal{G}_n}\right] = 0. \tag{2.12}$$

In particular

$$\lim_{k\to\infty} \sup_n \|\mathbb{P}\mu_{n+k} - \mathbb{P}\mu\|_{\mathcal{G}_n} = 0. \tag{2.13}$$

To this purpose one has to admit that

Proposition 2.2.7. *When weak disorder holds the sequence W_N is uniformly integrable so that the convergence $W_n \to W_\infty$ holds in $\mathbb{L}_1(\mathbb{P})$.*

Proof. From the definition of total variation measure one has (recall notation 1.23)

$$\begin{aligned}\|\mu_{n+k} - \mu\|_{\mathcal{G}_n} &= E\left[\left|\frac{\mathrm{d}(\mu_{n+k})_{|\mathcal{G}_n}}{\mathrm{d}P_{|\mathcal{G}_n}} - \frac{\mathrm{d}\mu_{|\mathcal{G}_n}}{\mathrm{d}P_{|\mathcal{G}_n}}\right|\right] \\ &= E\left[e^{\sum_{l=1}^n \beta\omega_{l,S_l} - n\lambda(\beta)}\left|\frac{W_k(n,S_n)}{W_{n+k}} - \frac{W_\infty(n,S_n)}{W_\infty}\right|\right]. \end{aligned} \tag{2.14}$$

Note that the above quantity is always smaller or equal to one. Moreover when $W_{n+k} \geq \delta$, $W_\infty \geq \delta$ and $|W_k - W_\infty| \leq \varepsilon$ one has

$$\begin{aligned}\left|\frac{W_k(n,S_n)}{W_{n+k}} - \frac{W_\infty(n,S_n)}{W_\infty}\right| &\leq W_k(n,S_n)\left|\frac{1}{W_{n+k}} - \frac{1}{W_\infty}\right| + \frac{1}{W_\infty}\left|W_k(n,S_n) - W_\infty(n,S_n)\right| \\ &\leq \frac{\varepsilon}{\delta^2} W_k(n,S_n) + \delta^{-1}\left|W_k(n,S_n) - W_\infty(n,S_n)\right|. \end{aligned} \tag{2.15}$$

Combining (2.15) with (2.14) one gets that

$$\begin{aligned}\mathbb{E}[\|\mu_{n+k} - \mu\|_{\mathcal{G}_n}] &\leq \mathbb{P}(\min(W_\infty, W_k) < \delta) + \mathbb{P}(|W_k - W_\infty| > \varepsilon) \\ &+ \frac{\varepsilon}{\delta^2}\mathbb{E}\left[e^{\sum_{l=1}^n \beta\omega_{l,S_l} - n\lambda(\beta)} W_k(n,S_n)\right] + \delta^{-1}\mathbb{E}\left[e^{\sum_{l=1}^n \omega_{l,S_l}) - n\lambda(\beta)}\left|W_k(n,S_n) - W_\infty(n,S_n)\right|\right]. \end{aligned} \tag{2.16}$$

The first terms in the r.h.s. can be made small by taking δ small, because $W_\infty > 0$, $\mathbb{P}$-a.s. the second one tends to zero when k tends to infinity, the third one is equal to $\frac{\varepsilon}{\delta^2}$ which can be chosen arbitrarily small by choosing ε accordingly. By translation invariance the fourth term is equal to (δ^{-1} times) the $\mathbb{L}_1$ distance between W_k and W_∞ which tends to zero according to Proposition (2.2.7). This is enough to conclude.

□

Now we can conclude this section that by proving that

Proposition 2.2.8. *The law of $S^{(n)}$ under $\mathbb{P}\mu_n$ converges weakly to the one of Brownian Motion.*

Proof. Indeed by the previous Lemma, one can choose k large enough so that for all $N \geq k$ one has $\|\mathbb{P}\mu_n - \mathbb{P}\mu\|_{\mathcal{G}_{n-k}} \leq \varepsilon$.

Thus for each value of n one can couple S of law $\mathbb{P}\mu_n$ and S' of law $\mathbb{P}\mu$ such that $S_{|[0,n-k]} = S'_{[0,n-k]}$ with probability $(1-\varepsilon)$. This implies in particular that

$$\|S^{(n)} - (S')^{(n)}\| \leq k/\sqrt{n} \text{ with probability } 1-\varepsilon.$$

Thus, as the law of $(S')^{(n)}$ converges to the law of Brownian motion (cf. Proposition 2.2.5), so does the law of $S^{(n)}$.

□

Chapter 3

The localized phase

In this chapter, our aim is to prove some kind of localization result about the polymer measure when strong disorder holds. This chapter is mainly based on [4], [6], [7] and [17].

1 Existence of the free-energy

The first step is to prove the existence of the quenched free-energy. We do this in two stages: first we show that the mean of $\frac{1}{N}\mathbb{E}\log Z_N$ converges and then couple it with a concentration result (Azuma's inequality) to get the existence of the non averaged limit.

For technical convenience we make in this chapter the assumption that the distribution of ω is bounded, i.e. that there exists a constant K such that $|\omega_{n,x}| \leq K$, $\mathbb{P}$-a.s.

Theorem 3.1.1. *The sequence* $(\mathbb{E}\log W_N)_{N\geq 0}$ *is superadditive so that the limit*

$$p(\beta) - \lambda(\beta) := \frac{1}{N}\mathbb{E}\log W_N, \tag{1.1}$$

exists. Moreover the quantity $\log W_N$ *concentrates around its mean: there exists a constant* c *such that*

$$\mathbb{P}\left[\log W_N - \mathbb{E}\log W_N \geq t\sqrt{N}\right] \leq \exp(-ct^2). \tag{1.2}$$

As a consequence

$$\lim_{N\to\infty} \frac{1}{N}\log W_N = p(\beta) - \lambda(\beta). \tag{1.3}$$

Proof. First let us prove the superadditivity statement. Recall (1.23), one has by definition

$$W_{N+M} = W_N \sum_{x\in\mathbb{Z}^d} \mu_N(S_N = x)\log W_M(N,x), \tag{1.4}$$

so that

$$\mathbb{E}\log W_{N+M} = \mathbb{E}\log W_N + \mathbb{E}\log \sum_{x\in\mathbb{Z}^d} \mu_N(S_N = x)\log W_M(N,x). \tag{1.5}$$

Using Jensen's inequality

$$\mathbb{E}\log\left(\sum_{x\in\mathbb{Z}^d}\mu_N(S_N=x)W_M(N,x)\right)\geq\mathbb{E}\left[\sum_{x\in\mathbb{Z}^d}\mu_N(S_N=x)\log W_M(N,x)\right]$$
$$=\sum_{x\in\mathbb{Z}^d}\mathbb{E}\left[\mu_N(S_N=x)\right]\mathbb{E}[\log W_M(N,x)]. \quad (1.6)$$

Where last equality is derived by using independence in the environment ω. On can conclude using translation invariance of the environment and $\mathbb{E}[\log W_M(N,x)]=\mathbb{E}[\log W_M]$.

For the concentration result, we are going to use the fact that ω has bounded distribution. Let us consider the filtration defined in Theorem 1.2.1

$$\mathcal{F}_N=\sigma(\omega_{n,x},n\leq N), \quad (1.7)$$

and the martingale sequence

$$M_k^N=(\mathbb{E}[\log W_N \mid \mathcal{F}_k])_{k\in[0,N]}. \quad (1.8)$$

To show concentration we show that the martingale here has bounded increments and then use Azuma's concentration inequality for martingale. Boundedness of increments is quite intuitive given that the environment takes bounded values only, but to write a proof of it, one needs a small technical artifice: Let $\tilde{\omega}$ be an independent copy of the environment (of law $\tilde{\mathbb{E}}$) ω and define ω^k to be the interpolated environment

$$\omega_{n,x}^k:=\begin{cases}\omega_{n,x} \text{ if } n\leq k,\\ \tilde{\omega}_{n,x} \text{ if } n\geq k.\end{cases} \quad (1.9)$$

Define

$$W_N^k:=\exp[\beta(\sum_{n=1}^N\omega_{n,S_n}^k)-N\lambda(\beta)], \quad (1.10)$$

the partition function built from ω^k. From this definition

$$M_k^N=\tilde{\mathbb{E}}\log W_N^k. \quad (1.11)$$

One has

$$\frac{W_N^{k+1}}{W_N^k}=\mu_N^k[e^{\beta(\omega_{k+1,S_{k+1}}-\tilde{\omega}_{k+1,S_{k+1}})}]. \quad (1.12)$$

which belongs to the interval $[e^{-2\beta K},e^{2\beta K}]$ so that

$$|M_{k+1}^N-M_k^N|\leq\left|\tilde{\mathbb{E}}\log\frac{W_N^{k+1}}{W_N^k}\right|\leq 2\beta K. \quad (1.13)$$

We conclude by deducing (1.2) from Azuma's inequality (below).

□

For the sake of completeness we present of short proof of Azuma's inequality.

Proposition 3.1.2. *[1, Azuma's inequality] Let $(M_k)_{k\in[0,N]}$ be a martingale sequence whose increments are bounded by C, one has*

$$\mathbb{P}[|M_n - M_0| \geq t] \leq 2\exp(-t^2/2NC^2). \tag{1.14}$$

Proof. Let $Y_n = M_n - M_{n-1}$ denote the martingale increments. Using convexity of the exponential and the martingale property one has

$$\mathbb{E}\left[e^{sY_n}|\mathcal{F}_{n-1}\right] \leq \mathbb{E}\left[\left(\frac{1}{2}+\frac{Y_n}{C}\right)e^{Cs} + \left(\frac{1}{2}-\frac{Y_n}{C}\right)e^{-Cs} \mid \mathcal{F}_{n-1}\right]$$
$$= \cosh(Cs) \leq \exp((Cs)^2/2). \tag{1.15}$$

Then

$$\mathbb{E}\left[e^{sM_N - M_0}\right] = \mathbb{E}\left[e^{sM_{N-1}-M_0}e^{sY_N}\right] = \mathbb{E}\left[e^{sM_{N-1}-M_0}\mathbb{E}[e^{sY_N}|\mathcal{F}_{N-1}]\right]$$
$$= \mathbb{E}\left[e^{sM_{N-1}-M_0}\right]\exp((Cs)^2/2) \leq \exp(-N(Cs)^2/2), \tag{1.16}$$

the last inequality is obtained just by iterating $N-1$ times. Thus for $t \geq 0$, and $s > 0$

$$\mathbb{P}[M_N - M_0 \geq t] \leq \mathbb{E}\left[e^{s(M_N-M_0)-st}\right] \leq \exp(N(Cs)^2/2 - st). \tag{1.17}$$

Taking $s = t/(CN)^2$, the r.h.s. is equal to $\exp(-t^2/(2NC^2))$. One gets the same bound for $\mathbb{P}[M_N - M_0 \geq -t]$ in the same fashion to finish the proof. □

Remark 3.1.3. The proof adapt to the case of ω with sub-exponential tail by standard truncation argument. The only difference being that the tail estimates for deviation from the mean are worse, see [6].

2 Strong disorder and overlap

In this section we prove Theorem 1.2.3. We restrict again to the case of bounded disorder. Before starting the proof, let us explain how $\log W_N$ can have some relation to the overlap distribution.

Note that from the definition of the polymer measure

$$\log W_n/W_{n-1} = \log \sum_{x\in\mathbb{Z}^d} \mu_{n-1}(S_n = x)e^{\beta\omega_{n,x}-\lambda(\beta)}$$
$$= \log\left(1 + \sum_{x\in\mathbb{Z}^d}\mu_{n-1}(S_n = x)(e^{\beta\omega_{n,x}-\lambda(\beta)} - 1)\right). \tag{2.1}$$

If the measure $\mu_{n-1}(S_n = \cdot)$ is well spread on $\mathbb{Z}^d$ one has some kind of averaging and $\mu_{n-1}(S_n = x)(e^{\beta\omega_{n,x}-\lambda(\beta)} - 1)$ concentrates around zero and do does $\log W_n/W_{n-1}$, whereas if $\mu_{n-1}(S_n = \cdot)$ is concentrated on a few sites only, $\log W_n/W_{n-1}$ remains with non-vanishing random fluctuation but its mean is strictly negative due to convexity.

To extract more quantitative information an idea is to develop the $\log(1+x)$ at second order in x. On has

$$\log W_{n+1}/W_n = \sum_{x\in\mathbb{Z}^d} \mu_{n-1}(S_n = x)(e^{\beta\omega_{n,x}-\lambda(\beta)} - 1) + \left(\sum_{x\in\mathbb{Z}^d} \mu_{n-1}(S_n = x)(e^{\beta\omega_{n,x}-\lambda(\beta)} - 1)\right)^2 + \dots \quad (2.2)$$

When averaging with respect to $(\omega_{n,x})_{x\in\mathbb{Z}^d}$. The first term has mean zero and the second one has mean proportional to I_n. The general idea behind the proof of Theorem 1.2.3 is to use a rigorous version of this development to be able to neglect terms that come after the second one. The first tool is the following technical Lemma.

Lemma 3.2.1. *Let U_i be a set of IID mean zero variable that takes value in $[-1+\varepsilon, \varepsilon^{-1}]$ for some positive ε and α_i a sequence of positive reals that sums to one.*

$$\begin{aligned} c^{-1}\sum_{i=1}^n \alpha_i^2 \le \mathbb{E}\log(1+\sum_{i=1}^n \alpha_i U_i) \le c\sum_{i=1}^n \alpha_i^2, \\ \log^2(1+\sum_{i=1}^n \alpha_i U_i) \le c\sum_{i=1}^n \alpha_i^2, \end{aligned} \quad (2.3)$$

where c is a constant that depends on ε.

Proof. For the first line note that

$$\mathbb{E}\log(1+\sum_{i=1}^n \alpha_i U_i) = \mathbb{E}\phi(\sum_{i=1}^n \alpha_i U_i), \quad (2.4)$$

with $\phi(u) = \log(1+u) - u$. As $\sum_{i=1}^n \alpha_i U_i \in [-1+\varepsilon, \varepsilon^{-1}]$ one has

$$c^{-1}(\sum_{i=1}^n \alpha_i U_i)^2 \le \phi(\sum_{i=1}^n \alpha_i U_i) \le c(\sum_{i=1}^n \alpha_i U_i)^2 \quad (2.5)$$

which gives the first line of inequalities. The second one is derived in the same manner using that $\ln^2(1+u) \le cu^2$ for some constant c on the range considered ($u \in [1-\varepsilon, \varepsilon^{-1}]$). □

From the above Lemma and the observation (2.1) one can get without difficulties that

$$\mathbb{E}[\log W_N] \asymp \mathbb{E}\left[\sum_{n=1}^N I_n\right]. \quad (2.6)$$

To get a more precise result one relies on a semi-martingale decomposition of the sequence $\log W_n$:

$$\log W_n = M_n + A_n \quad (2.7)$$

where

$$\Delta A_n = A_n - A_{n-1} := \mathbb{E}[\log W_n/W_{n-1} \mid \mathcal{F}_n], \quad (2.8)$$

is called the compensator and $M_n = Z_n - A_n$ is a martingale.

Lemma 3.2.2. *There exists a constant c such that*

$$c^{-1}\sum_{n=1}^{N} I_n \leq A_N \leq c\sum_{n=1}^{N} I_n. \tag{2.9}$$

Moreover,

$$A_N \to \infty \Leftrightarrow W_\infty = 0 \Rightarrow \frac{\log W_n}{A_n} \to 1. \tag{2.10}$$

Proof. Recall (2.1). Then one can apply Lemma 3.2.1, where one averages with respect to the $(\omega_{n,x})_{x\in\mathbb{Z}}$, with $\alpha_x = \mu_{n-1}(S_n = x)$ and $U_x = e^{\beta\omega_{n,x}-\lambda(\beta)} - 1$, and one gets

$$\frac{1}{c} I_n \leq \Delta A_n \leq c I_n, \tag{2.11}$$

for some constant depending on β.

This gives us (2.9). Let us control the asymptotic of the martingale part. To this purpose we want to estimate the associated **increasing process**

$$\langle M\rangle_n := \sum_{k=1}^{n} \mathbb{E}\left[(M_k - M_{k-1})^2 \mid \mathcal{F}_{k-1}\right]. \tag{2.12}$$

From the definition of M_n one has

$$\mathbb{E}\left[(M_k - M_{k-1})^2 \mid \mathcal{F}_{k-1}\right] \leq \mathbb{E}\left[(\log W_k - \log W_{k-1})^2 \mid \mathcal{F}_{k-1}\right]. \tag{2.13}$$

Moreover (2.13) combined with Lemma 3.2.1 where one averages (again) with respect to the $(\omega_{n,x})_{x\in\mathbb{Z}}$, with $\alpha_x = \mu_{n-1}(S_n = x)$ and $U_x = e^{\beta\omega_{n,x}-\lambda(\beta)} - 1$, gives

$$\mathbb{E}\left[(\log W_k / \log W_{k-1})^2 \mid \mathcal{F}_{k-1}\right] \leq c I_k, \tag{2.14}$$

so that

$$\langle M\rangle_N \leq c\sum_{n=1}^{N} I_n. \tag{2.15}$$

Hence when $\sum_{k=1}^{\infty} I_k < \infty$, A_n converges to a finite limit A_∞ and so does $\langle M\rangle_n$ a.s. so that $\log W_n$ converges to a finite limit and hence $W_\infty > 0$ a.s.

When $\sum_{n=1}^{\infty} I_n = \infty$ one can check that $\log W_N / A_N$ tends to one almost surely. For this one must check that M_n/A_n tends to zero.

Combining (2.9) and (2.15)

$$\frac{M_n}{A_n} \leq c^2 \frac{M_n}{\langle M\rangle_n}. \tag{2.16}$$

According to the martingale law of large number (see e.g. [13, Theorem 4.10 chap. 4]), the r.h.s. converges to 0 when $\langle M\rangle_n$ tends to infinity.

□

Theorem 1.2.3 is then an easy consequence of the Lemma.

3 Strong disorder at all temperature for $d = 1$

In this section, we want to prove that W_n decays exponentially fast at all temperature when $d = 1$. We do not need bounded disorder here, and for the sake of simplicity we write the proof only in the Gaussian case. The result that $p(\beta) < \lambda(\beta)$ appeared first is [7], the approach we present here, and the quantitative estimate are from [17].

Proposition 3.3.1. *For $d = 1$, there exists a constant c such that*

$$p(\beta) \leq \lambda(\beta) - c\beta^4 |\log \beta|^{-2}, \tag{3.1}$$

for all $\beta \leq 1/2$. As a consequence, very strong disorder holds at all temperature.

Proof. Consider $\theta \in (0, 1)$ fixed. One has

$$\mathbb{E} \log W_N = \theta^{-1} \mathbb{E}[\log W_N^\theta] \leq \theta \log \mathbb{E}[W_N^\theta]. \tag{3.2}$$

Hence to show that very strong disorder holds it is sufficient to show that $\mathbb{E}[W_N^\theta]$ tends to zero exponentially fast. The reason why we use this little trick is that the expectation of a non-integer power of W_N is easier to estimate that the expectation of a log.

The second step of our reasoning is to get a finite volume criterion for very strong disorder to hold: i.e. get a sufficient condition for strong disorder only involving W_N for a finite N. To do so, we decompose the partition function W_{NM} for $M \geq 0$ according to the coordinate of the path after Ni steps, $i \in \{1, \dots, M\}$. This kind of procedure is quite common in statistical mechanics and is known as coarse-graining.

Given k, x and y set

$$W_N^k(x, y) := E_x \left[\exp \left[\sum_{n=1}^{N} (\beta \omega_{S_n, n+(k-1)N} - \lambda(\beta)) \right] \mathbf{1}_{S_N = y} \right]. \tag{3.3}$$

One has

$$\begin{aligned} W_{NM} &= \sum_{(x_1, \dots, x_M) \in \mathbb{Z}^M} E \left[\exp \left[\sum_{n=1}^{NM} (\beta \omega_{S_n, n+(K-1)N} - \lambda(\beta)) \right] \mathbf{1}_{S_{iN} = x_i, \forall i \in [1, M]} \right] \\ &= \sum_{(x_1, \dots, x_M) \in \mathbb{Z}^M} \prod_{k=1}^{M} W_N^k(x_{i-1}, x_i). \end{aligned} \tag{3.4}$$

Thus using the inequality $(\sum a_i)^\theta \leq \sum a_i^\theta$ and averaging one gets

$$\mathbb{E}[W_{NM}^\theta] \leq \sum_{(x_1, \dots, x_M) \in \mathbb{Z}^M} \prod_{k=1}^{M} \mathbb{E}[(W_N^k(x_{i-1}, x_i))^\theta] = \left(\sum_{x \in \mathbb{Z}} \mathbb{E}[(W_N(0, x))^\theta] \right)^M. \tag{3.5}$$

The last equality is due to translation invariance of the environment (and a rearrangement of the sum). This gives us a first conclusion

Lemma 3.3.2. *Very strong disorder holds if*

$$\sum_{x\in\mathbb{Z}} \mathbb{E}[(W_N(0,x))^\theta] < 1 \tag{3.6}$$

for some value of N and θ. Moreover, in that case

$$p(\beta) - \lambda(\beta) \le \frac{1}{\theta N} \log \sum_{x\in\mathbb{Z}} \mathbb{E}[(W_N(0,x))^\theta]. \tag{3.7}$$

Corollary 3.3.3. *In any dimension, if the distribution of ω is unbounded one has*

$$p(\beta) < \lambda(\beta), \tag{3.8}$$

for β large enough

Proof. Apply equation (3.7) for $\theta = 1/2$ and $N = 1$. □

Our aim is to prove then that for a given temperature one can find θ and N such that (3.6) holds. First we note that if θ is sufficiently close to one $\sum_{x\in\mathbb{Z}^d}(W_N(0,x))^\theta$ is almost equal to W_N^θ. Indeed on has trivially for all $\theta < 1$

$$\begin{aligned} \sum_{\{x\in\mathbb{Z},\ |x|\le N,\ W_N(0,x)\le \frac{1}{N^2}W_N\}} (W_N(0,x))^\theta &\le 2N^{1-2\theta} W_N^\theta, \\ \sum_{\{x\in\mathbb{Z},\ |x|\le N,\ W_N(0,x)\ge \frac{1}{N^2}W_N\}} (W_N(0,x))^\theta &\le (W_N/N^2)^{\theta-1} \sum_{x\in\mathbb{Z}^d} W_N(0,x) = N^{2(1-\theta)} W_N^\theta. \end{aligned} \tag{3.9}$$

Thus choosing $\theta = \theta_N := 1 - (\log N)^{-1}$ one gets that for N large enough

$$\sum_{x\in\mathbb{Z}^d} (W_N(0,x))^\theta \le 10 W_N^\theta. \tag{3.10}$$

The rest of the job remains to estimate $\mathbb{E}[W_N^\theta]$ in an efficient way. We do it using a change of measure argument which can be summarized in the following application of Hölder inequality: consider $\tilde{\mathbb{E}}$ a probability measure for ω absolutely continuous with respect to $\mathbb{E}$

$$\mathbb{E}[W_N^\theta] = \tilde{\mathbb{E}}\left[\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\tilde{\mathbb{P}}}(\omega) W_N^\theta\right] \le \tilde{\mathbb{E}}\left[\left(\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\tilde{\mathbb{P}}}(\omega)\right)^{\frac{1}{1-\theta}}\right]^{(1-\theta)} \tilde{\mathbb{E}}[W_N]^\theta. \tag{3.11}$$

The idea is then to find a measure for which the "cost" of the change of measure

$$\tilde{\mathbb{E}}\left[\left(\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\tilde{\mathbb{P}}}(\omega)\right)^{\frac{1}{1-\theta}}\right]^{(1-\theta)}$$

is not too big (i.e. of order 1) while $\tilde{\mathbb{E}}[W_N^\theta]$ is really small. We expose the choice for the change of measure in the case where ω is a Gaussian environment (composed of IID standard centered Gaussian variable) for simplicity.

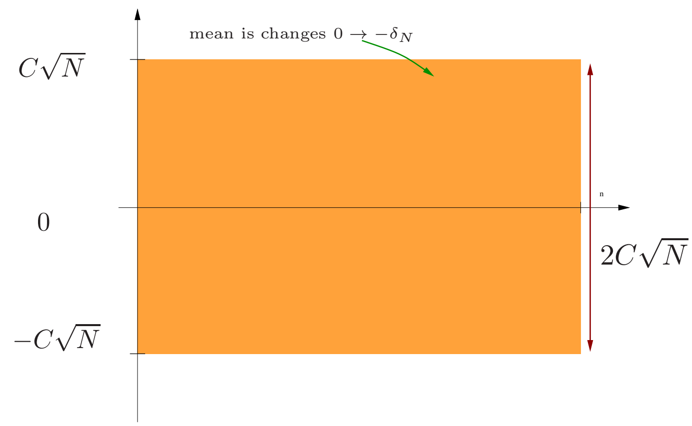

Figure 3.1: Schematic representation of the change of measure $\mathbb{P} \Rightarrow \tilde{\mathbb{P}}$. The idea is to make the sites that are in the zone that is likely to be visited by the polymer (the dark rectangle) less favorable, in order to lower the value of W_N.

Our choice is to choose $\tilde{\mathbb{P}}$ such that $\omega_{n,x}$ remains independent Gaussian variables of unit variance but the mean is changed to be equal to

$$\tilde{\mathbb{E}}[\omega_{n,x}] = -\delta_N \mathbf{1}_{(n,x)\in[0,N]\times[-C\sqrt{N},C\sqrt{N}]}, \tag{3.12}$$

for some large constant C. i.e. the mean is lowered for the sites that are typically visited by trajectory of length N (see figure 3.1).

$$\frac{\mathrm{d}\tilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}}(\omega) = \exp\left(-\sum_{(n,x)\in([0,N]\times[-C\sqrt{N},C\sqrt{N}])} \delta_N\omega_{n,x} - \delta_N^2/2\right). \tag{3.13}$$

so that

$$\tilde{\mathbb{E}}\left[\left(\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\tilde{\mathbb{P}}}(\omega)\right)^{\frac{1}{1-\theta}}\right]^{(1-\theta)} = \exp\left(\frac{CN^{3/2}\delta_N^2\theta}{(1-\theta)}\right) \le \exp(CN^{3/2}\delta_N^2 \log N). \tag{3.14}$$

This is smaller than 2 when $\delta_N = (2C)^{-1/2}N^{-3/4}(\log N)^{-1/2}$.

Now we need to bound $\tilde{\mathbb{E}}[W_N]$ for our choices of parameter. One has

$$\tilde{\mathbb{E}}[W_N] = E\left[\exp\left(-\beta\delta_N \sum_{n=1}^{N} \mathbf{1}_{|S_n|\le\sqrt{N}}\right)\right] \le P\left[\max_{n\in[0,N]} |S_n| \ge C\sqrt{N}\right] + e^{-\delta_N\beta N}. \tag{3.15}$$

The first term can be made smaller than $1/200$ by choosing C large enough. The second term is equal to $\exp\left(-(2C)^{-1/2}\beta N^{1/4}(\log N)^{-1/2}\right)$. It is also smaller than $1/200$ is N is chosen to be equal to

$$N_\beta := 10C^2\beta^{-4}|\log\beta|^2.$$

Putting everything together one gets that with our choice for N, and θ_N one has

$$\mathbb{E}[\sum_{x\in\mathbb{Z}^d} (W_N(0,x))^\theta] \le 10\mathbb{E}W_N^\theta \le e^{-1}. \tag{3.16}$$

Thus according to (3.7)

$$p(\beta) - \lambda(\beta) \leq -\frac{1}{N} = -(10C^2)^{-1}\beta^4|\log\beta|^{-2}. \tag{3.17}$$

□

Remark 3.3.4. What is crucial to make the above proof work is that the change of measure corresponding to a drift of the mean in a cylinder of diameter $\sqrt{N}$ has a cheap cost for $\delta_N \gg N^{-1}$. For this reason, it is cannot be directly extended to the case $d = 2$. However the proof that $p(\beta) < \lambda(\beta)$ for all β when $d = 2$ is based on the same ideas, but two crucial improvement are necessary: a more efficient coarse graining argument that avoid to have to chose θ close to one, and a change of measure that do not shift the mean of the ω but rather change the covariance structure in the environment. We refer to [17] for more details. Note also that the result presented in this section is sharp in the sense that up to logarithmic correction $p(\beta)-\lambda(\beta)$ is of order β^4.

Chapter 4

Superdiffusivity

In this chapter, we present a superdiffusivity result obtained for a particular model of directed polymer based on a Gaussian random walk rather than on a random walk on $\mathbb{Z}^d$. The reason why we need to change model to get a result is the following : superdiffusivity occurs from a competition between energy and entropy, i.e. going at superdiffusive distance has an entropic cost that can be compensated by an energetic gain due to fluctuation in the environment. The result presented here is from [20], but the approach of the proof is more similar to the one developed in [18], for a model of directed polymer in an environment that has long-ranged correlation.

1 Model and result

Let us introduce our model. Let $(S_n)_{n\geq 0}$ denote the random walk on $\mathbb{R}$ starting from zero and whose increments $(S_{n+1} - S_n)_{n\geq 0}$ are IID mean-zero unit variance Gaussian (call P the associated probability). Let $\omega = (\omega(n,x))_{n\geq 0, x\in\mathbb{R}}$ be our Gaussian random environment. It is a centered translation invariant Gaussian fields whose covariance function is given by

$$\mathbb{E}[\omega(n,x)\omega(n',y)] = \mathbf{1}_{n=n'}\Gamma(x-y), \tag{1.1}$$

where $\Gamma(\cdot)$ is a compact supported positive function (supported on $[-K, K]$). Equivalently $(\omega(n,\cdot))_{n\geq 0}$ is a sequence of IID translation invariant Gaussian fields on $\mathbb{R}$ with covariance function given by $\Gamma(x-y)$.

Our polymer measure for chain of size N at inverse temperature β is a measure absolutely continuous with respect to P which modifies the law of the first N increments of S. Its Radon-Nikodym derivative with respect to P is given by

$$\frac{\mathrm{d}\mu_N^\beta}{\mathrm{d}P} := \frac{1}{Z_N^{\beta,\omega}} e^{\beta\sum_{n=1}^N \omega(n,S_n)}, \tag{1.2}$$

where

$$Z_N^{\beta,\omega} := E[e^{\beta\sum_{n=1}^N \omega(n,S_n)}], \tag{1.3}$$

is the partition function.

For this model, one can prove that the trajectories S have a superdiffusive behavior under μ_N or more precisely

Theorem 4.1.1 (From [20]). *For all $\xi < 3/5$*

$$\lim_{N\to\infty} \mu_N(\forall n \in [0,N], |S_n| \le N^\xi) = 0, \tag{1.4}$$

in probability.

2 Heuristics

Let us explain the idea underlying the proof that is : computing a lower bound for the fluctuation for the log of the partition function restricted in a box of diameter L^ξ and compare it to to the energetic cost of traveling at a distance L^ξ.

Set

$$\mathcal{A}_N^\xi := \{S \mid \forall n \in [0,N], |S_n| \le N^\xi\}, \tag{2.1}$$

and define

$$\bar{Z}_N := E[e^{\beta \sum_{n=1}^N \omega(n,S_n)} \mathbf{1}_{S\in\mathcal{A}_N^\xi}]. \tag{2.2}$$

We want to show that $\log \bar{Z}_N$ has typical fluctuation of order at least $N^{\frac{1-\xi}{2}}$. We consider χ_N to be the integrated field in the box $\mathcal{C}_N^\xi$

$$\chi_N = \int_{[-(N^\xi+K),N^\xi+K]} \sum_{n=1}^N \omega(n,S_n)\mathrm{d}x. \tag{2.3}$$

For any path in $\mathcal{A}_N^\xi$ one has

$$\mathbb{E}\left[(\sum_{n=1}^N \omega(n,S_n))\chi_N \right] = N \int \Gamma(x)\mathrm{d}x. \tag{2.4}$$

Define V a Gaussian field on $\mathcal{A}_N^\xi$

$$V(S) := \sum_{n=1}^N \omega(n,S_n) - a_N\chi_N, \tag{2.5}$$

with

$$a_N := \frac{N}{\mathrm{Var}\,\chi_N} \int_{[-K,K]} \Gamma(x)\mathrm{d}x. \tag{2.6}$$

By (2.4), the field $(V(S))_{S\in\mathcal{A}_N^\xi}$ is independent of χ_N. Now we note that

$$\log \bar{Z}_N = \beta a_N \chi_N + \log E[e^{\beta V(S)}\mathbf{1}_{S\in\mathcal{A}_N^\xi}]. \tag{2.7}$$

with the second term being independent of χ_N. Thus the fluctuation of $\log \bar{Z}_N$ are at least the one of $a_N\chi_N$ which has variance equal to

$$\frac{N^2}{\mathrm{Var}\,\chi_N}\left(\int_{[-K,K]} \Gamma(x)\mathrm{d}x\right)^2 = (1+o(1))\frac{1}{2}N^{1-\xi}\int \Gamma(x)\mathrm{d}x. \tag{2.8}$$

Hence, on a heuristic level, by visiting a box lying at a distance N^{ξ} from the origin, there might be an energetic reward of $N^{\frac{1+\xi}{2}}$ just due to environmental fluctuation. This is to be compared with the entropic cost for our walk moving L^{ξ} away from the origin which is equal to

$$\log P(S_N \geq N^{\xi}) \asymp N^{2\xi-1}. \tag{2.9}$$

Thus the entropic cost is lower than the potential energetic gain if $\xi < 3/5$. There is some further work needed to do to make this argument fully rigorous.

3 Proof

For the sake of simplicity we restrict ourselves to proving the following simpler statement:

Proposition 4.3.1. *For any ε one has for N large enough, with probability larger than $1/2 - \varepsilon$*

$$\mu_N(\forall n \in [0, N], |S_n| \leq N^{\xi}) \leq \varepsilon. \tag{3.1}$$

The overall idea is to compare the contribution to the partition function of two set of paths:

$$\begin{aligned} \mathcal{A}_N^{(\xi,1)} &:= \{S \mid \forall n \in [N/2, N], |S_n| \leq N^{\xi}\}, \\ \mathcal{A}_N^{(\xi,2)} &:= \{S \mid \forall n \in [N/2, N], S_n \in [2N^{\xi}, 4N^{\xi}]\}. \end{aligned} \tag{3.2}$$

We set

$$\bar{Z}_N^{(1)} = E[e^{\beta \sum_{n=1}^{N} \omega(n,S_n)} \mathbf{1}_{S \in \mathcal{A}_N^{(\xi,1)}}] \text{ and } \bar{Z}_N^{(2)} = E[e^{\beta \sum_{n=1}^{N} \omega(n,S_n)} \mathbf{1}_{S \in \mathcal{A}_N^{(\xi,2)}}].,\tag{3.3}$$

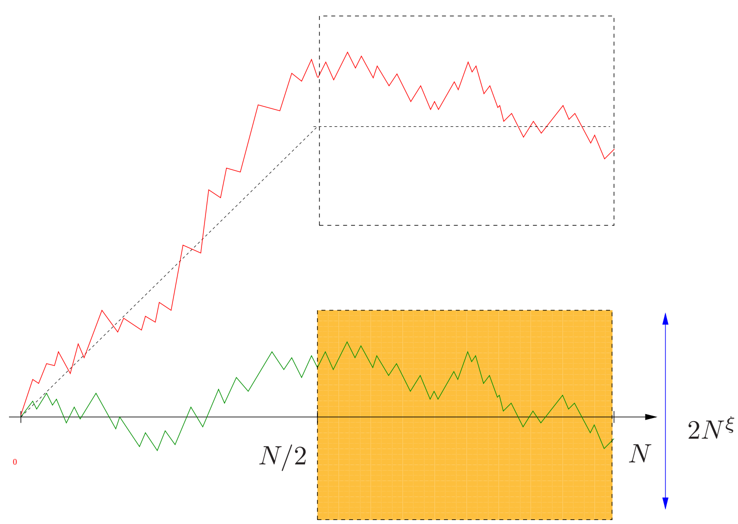

Figure 4.1: Schematic representation of two paths in $\mathcal{A}_N^{(\xi,1)}$ and $\mathcal{A}_N^{(\xi,2)}$, the second one is just obtained by adding θ_n^N to the first one. The change of measure $\mathbb{E} \to \tilde{\mathbb{E}}$ leaves the environment unchanged in the dark rectangle but makes the mean of ω higher in the clear one.

Our aim is to show that with probability almost 1/2

$$\mu_N(\forall n \in [0,N], |S_n| \le N^\xi) \le \mu_N(\mathcal{A}_N^{(\xi,1)}) = \frac{\bar{Z}_N^{(1)}}{Z_N} \le \frac{\bar{Z}_N^{(1)}}{\bar{Z}_N^{(2)}} \le \varepsilon. \tag{3.4}$$

The first step is to modify slightly the couple $(\bar{Z}_N^{(1)}, \bar{Z}_N^{(2)})$ to obtain a pair of exchangeable variables. Set

$$\tilde{\omega} = (\tilde{\omega}^N(n,x))_{n\ge 0, x\in\mathbb{R}} = (\omega^N(n, x+\theta_n^N))_{n\ge 0, x\in\mathbb{R}}, \tag{3.5}$$

where

$$\theta_n^N = \begin{cases} 6nN^{\xi-1} \text{ if } n \le N/2, \\ 3N^\xi \text{ if } N \ge N/2. \end{cases} \tag{3.6}$$

and consider

$$\tilde{Z}_N^{(1)} = E[e^{\beta \sum_{n=1}^N \tilde{\omega}(n,S_n)} \mathbf{1}_{S\in\mathcal{A}_N^{(\xi,1)}}]. \tag{3.7}$$

as $\tilde{\omega}$ and ω have the same distribution, $\tilde{Z}_N^{(1)}$ and $\bar{Z}_N^{(1)}$ also have. Furthermore

Lemma 4.3.2. *The two following statements are true*

(i)

$$\bar{Z}_N^{(2)} \ge \tilde{Z}_N^{(1)} e^{-15N^{2\xi-1}}, \tag{3.8}$$

(ii) $(\tilde{Z}_N^{(1)}, \bar{Z}_N^{(1)})$ *are exchangeable so that with probability* 1/2

$$\tilde{Z}_N^{(1)} \ge \bar{Z}_N^{(1)}. \tag{3.9}$$

Proof. Let $\tilde{P}$ denote the law of a random walk with Gaussian increment $(S_n - S_{n-1})$ of mean $6N^{\xi-1}$. One has

$$\tilde{Z}_N^{(1)} = E\left[e^{\beta \sum_{n=1}^N \omega(n,S_n+\theta_n^N)} \mathbf{1}_{(S_n+\theta_n^N)_{n\ge0}\in\mathcal{A}_N^{(\xi,2)}}\right] = \tilde{E}\left[e^{\beta \sum_{n=1}^N \omega(n,S_n)} \mathbf{1}_{S\in\mathcal{A}_N^{(\xi,2)}}\right]. \tag{3.10}$$

Remark that

$$\frac{\mathrm{d}\tilde{P}}{\mathrm{d}P}(S) = \exp\left(6S_{N/2}N^{\xi-1} - 9N^{2\xi-1}\right) \le \exp(15N^{2\xi-1}), \quad \forall S \in \mathcal{A}_N^{(\xi,2)}, \tag{3.11}$$

so that

$$\tilde{Z}_N^{(1)} \le \exp(15N^{2\xi-1}) E[e^{\beta \sum_{n=1}^N \omega(n,S_n)} \mathbf{1}_{S\in\mathcal{A}_N^{(\xi,2)}}], \tag{3.12}$$

which gives (i)

For the second point, it is sufficient to note that ω and $\tilde{\omega}$ are exchangeable. □

Now we prove that the variable $\log \tilde{Z}_N^{(1)} - \log \bar{Z}_N^{(1)}$ has large fluctuations

Lemma 4.3.3. *For every* ε*, for* N *large enough, one has*

$$\mathbb{P}\left[\log \tilde{Z}_N^{(1)} \ge \log \bar{Z}_N^{(1)} + N^{\frac{1-\xi}{2}-2\varepsilon}\right] \le 1/2 - \varepsilon. \tag{3.13}$$

Proof. We prove first this inequality under a modified measure $\tilde{\mathbb{P}}$ under which ω is a field with the same covariance but for which the mean is not uniformly zero anymore. More precisely, we choose $\tilde{\mathbb{P}}$ such that

(i) $\tilde{\mathbb{E}}[\omega(n,x)] = 0$, for $n \le N/2$,

(ii) $\tilde{\mathbb{E}}[\omega(n,x)] = 0$, for $(n,x) \in [N/2, N] \times [-N^\xi, N^\xi]$,

(iii) $\tilde{\mathbb{E}}[\omega(n,x)] = N^{-(1+\xi)/2-\varepsilon} \int \Gamma(x)\mathrm{d}x$, for $(n,x) \in [N/2, N] \times [2N^\xi, 4N^\xi]$.

To do this we simply define $\tilde{\mathbb{P}}_N$ by

$$\frac{\mathrm{d}\tilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}}(\omega) := \exp(X_N - \mathrm{Var}(X_N)/2). \tag{3.14}$$

where

$$X_N(\omega) := \frac{1}{N^{(1+\xi)/2+\varepsilon}} \int_{[2N^\xi - K, 4N^\xi + K]} \sum_{n=N/2+1}^{N} \omega(n,x)\mathrm{d}x. \tag{3.15}$$

The variance of X_N under $\mathbb{P}$ equal to $(1+o(1))N^{-2\varepsilon} \int \Gamma(x)\mathrm{d}x$. For this reason, the density $\mathrm{d}\tilde{\mathbb{P}}_N/\mathrm{d}\mathbb{P}$ tends to one in probability when $N \to \infty$ so that the total variation distance between $\tilde{\mathbb{P}}_N$ and $\mathbb{P}$ vanishes.

It is not to difficult to check that under the tilted measure $\tilde{\mathbb{P}}$

$$\tilde{\mathbb{E}}[\omega(n,x)] = \mathbb{E}[X_N \omega(n,x)] =: \delta_N(n,x). \tag{3.16}$$

We can then remark $\delta_N(n,x)$ satisfies the wanted assumption $(i)-(iii)$. Note that if ω has law $\mathbb{P}$, the process $\hat{\omega} = \omega + \delta_N$ has probability distribution $\tilde{\mathbb{P}}$.

For this reason and as

$$\forall S \in \mathcal{A}_N^{(\xi,1)}, \ \sum_{n=1}^{N} \delta_N(n, S_n) = 0, \quad \text{and} \quad \sum_{n=1}^{N} \delta_N(n, S_n + \theta_n^N) = N^{(1-\xi)/2-\varepsilon} \int \Gamma(x)\mathrm{d}x/2, \tag{3.17}$$

we have

$$\tilde{\mathbb{P}}[\log \tilde{Z}_N^{(1)} \ge \log \bar{Z}_N^{(1)} + N^{(1-\xi)/2-\varepsilon}/2] = \mathbb{P}[\log \tilde{Z}_N^{(1)} \ge \log \bar{Z}_N^{(1)}] = \frac{1}{2}. \tag{3.18}$$

As the total variation distance between $\tilde{\mathbb{P}}_N$ and $\mathbb{P}$ goes to one when $N \to \infty$ so that for N large enough

$$\mathbb{P}[\log \tilde{Z}_N^{(1)} \ge \log \bar{Z}_N^{(1)} + N^{(1-\xi)/2-\varepsilon}/2] \ge 1/2 - \varepsilon. \tag{3.19}$$

□

We can now conclude the proof of Proposition 4.3.1. With probability larger the $1/2-\varepsilon$ one has

$$\log \bar{Z}_N^{(2)} - \log \bar{Z}_N^{(1)} \ge \log \tilde{Z}_N^{(1)} \ge \log \bar{Z}_N^{(1)} - 15N^{2\xi-1} + N^{(1-\xi)/2-\varepsilon}/2 \ge N^{(1-\xi)/2-2\varepsilon}, \tag{3.20}$$

provided that

$$\frac{1-\xi}{2} - \varepsilon > 2\xi - 1$$

(which occurs if $\xi < 3/5$ and ε small) and that N is large enough. Hence from (3.4), if (3.20) holds

$$\mu_N(\forall n \in [0, N], |S_n| \leq N^{\xi}) \leq \frac{\bar{Z}_N^{(1)}}{\bar{Z}_N^{(2)}} \leq \exp(-N^{(1-\xi)/2-2\varepsilon}). \tag{3.21}$$

Remark 4.3.4. To prove the stronger result (that the probability going out a box of width L^{ξ}, $\xi < 3/5$ goes to one) on needs to reason with N different sets of paths instead of two. The proof becomes a bit more complicated in the notation but the general ideas can go through. See [20] or [18].

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