Lectures on Measure Theory

General recommendations.

- These lectures assume that the audience is familiar with measure theory.
- The videos do not replace the books. I suggest to choose one among the many listed at the end of these notes and to read the corresponding sections before or after the videos.
- After the statement of a result, interrupt the video and try to prove the assertion. It is the only way to understand the difficulty of the problem, to differentiate simple steps from crucial ones, and to appreciate the ingenuity of the solution. Sometimes you find an alternative proof of the result.
- In many exercises in Taylor's book, you should replace semi-ring, ring and σ-ring by semi-algebra, algebra and σ-algebra (exceptions will be mentioned).
- You can speed-up or slow-down the video. By pressing settings at the bottom-right corner, you can modify the playback speed.
- Send me an e-mail if you find a mistake which is not reported in these notes.
- If you typed in latex, with no personal definitions nor the use of special packages, solutions to some exercises proposed below, send the file. Hopefully, I'll create a note with solutions to the exercises, acknowledging the authors of the solutions.
- A note about the methodology. I ask the students to view the video(s) before the class. In the first part of the lecture, I recall the content of the video. Sometimes, I ask one of the students to replace me. Occasionally, the student is randomly chosen. This is the opportunity for the students to ask questions on the content of the class. In the second part of the lecture, I present some of the applications included in the "Further Readings" topic.

November 21, 2022

Lecture 1: A non-measurable set

Summary. This lecture is based on [Taylor, Section 4.4].

Content and Comments.

0:00 Let $\mathcal{P}(\mathbb{R})$ be the family of subsets of $\mathbb{R}$. We prove that it is not possible to define a function $\lambda : \mathcal{P}(\mathbb{R}) \to \mathbb{R}_+ \cup \{+\infty\}$ such that

- $\lambda(\,(a, b]\,) = b - a$ for all $a < b$;
- $\lambda(A + x) = \lambda(A)$ for all $A \in \mathcal{P}(\mathbb{R})$, $x \in \mathbb{R}$;
- $\lambda(\cup_{j\geq 1} A_j) = \sum_{j\geq 1} \lambda(A_J)$ for all countable family A_j of disjoint subsets of $\mathbb{R}$.

The proof uses the axiom of choice. There are model in set theory without the axiom of choices in which all subsets of $\mathbb{R}$ are measurable. See Solovay, Robert M. (1970), "A model of set-theory in which every set of reals is Lebesgue measurable", Annals of Mathematics, Second Series, 92 (1): 1–56, doi:10.2307/1970696.

Further Readings.

The Wikipedia page on "Vitali set" is very well made. That on "Banach–Tarski paradox" offers a nice complement to the subjet by showing that *additive* measures that extend volume cannot be defined on $\mathcal{P}(\mathbb{R}^3)$.

Lecture 2: Classes of subsets and set functions

Summary. This lecture is based on [Taylor, Sections 1.5 and 3.1].

Content and Comments.

- 0:00 Definition of a semi-algebra $\mathcal{S}$ of subsets of a set Ω. Note that Taylor considers semi-rings. The difference is that in the definition of semi-algebra one requires the set Ω to belong to the class of sets, while this is not required in the definition of semi-rings.
- 4:02 Example of semi-algebras.
- 9:12 Definition of an algebra $\mathcal{A}$ of subsets of a set Ω.
- 12:12 Definition of a σ-algebra $\mathcal{F}$ of subsets of a set Ω. At [12:36], I say semi-algebra. I meant, of course, σ-algebra.
- 14:17 Intersections of arbitrary algebras is an algebra. The same holds for σ-algebras. This is the Lemma of [Taylor, Sections 1.5].
- 21:20 Algebra generated by a class $\mathcal{C}$ of subsets of Ω.
- 29:29 Lemma: A characterization of the elements of the algebra $\mathcal{A} = \mathcal{A}(\mathcal{S})$ generated by a semi-algebra $\mathcal{S}$. [Taylor, Theorem 1.4].
- 49:18 Definition of additive functions $\mu : \mathcal{C} \subset \mathcal{P}(\Omega) \to \mathbb{R}_+ \cup \{+\infty\}$.
- 52:34 Let $\mu : \mathcal{C} \to \mathbb{R}_+ \cup \{+\infty\}$ be additive function. If there exists $A \in \mathcal{C}$ such that $\mu(A) < \infty$, then $\mu(\varnothing) = 0$.
- 54:39 Let $\mu : \mathcal{C} \to \mathbb{R}_+ \cup \{+\infty\}$ be an additive function. It $E \subset F$, then $\mu(E) \le \mu(F)$.
- 58:44 Examples of additive functions. Discrete measures.
- 1:01:04 Definition of σ-additive functions $\mu : \mathcal{C} \subset \mathcal{P}(\Omega) \to \mathbb{R}_+ \cup \{+\infty\}$.
- 1:03:33 Example of an additive function $\mu : \mathcal{C} \subset \mathcal{P}((0,1)) \to \mathbb{R}_+ \cup \{+\infty\}$ which is additive and not σ-additive.

Further Readings.

A. [Billingsley, Section 1.2] has many examples. In contrast with the lectures, this reference focuses on finite measures.

B. At the end of [Taylor, Section 2.5], the notion of Borel σ-álgebra is introduced (this the σ-álgebra generated by open sets in a topological space). It is shown in particular that the Borel σ-álgebra coincides with that generated by the semi-álgebre of semi-open intervala.

In all exercises in Taylor's book below, you should replace semi-ring, ring and σ-ring by semi-algebra, algebra and σ-algebra.

Recommended exercises.

*a. Prove [Taylor, Theorems 1.3 and 1.6],

b. [Taylor, Section 1.5], exercises 1, 2, 4, 5, 7, 10.

*c. Let $\mathcal{S}$ be a semi-algebra of subsets of a set X. Show that $\varnothing$ belongs to $\mathcal{S}$.

*d. Fill the details of Examples 1–6 in [Taylor, Section 3.1].

*e. Prove [Taylor, Theorem 3.1].

*f. Check which of the Examples 1–6 are σ-additives.

g. [Taylor, Section 3.1], exercises 5, 6.

Suggested exercises.

a. If $\mathcal{F}$ is an algebra (or a σ-algebra) then $\#\mathcal{F}$ is a power of 2.
b. [Taylor, Section 1.5], exercises 3, 6, 9.
c. [Taylor, Section 3.1], exercises 1, 2, 3, 4.
d. Fill the details of Examples 2 – 6 of [Billingsley, Section 1.2].
e. Prove [Munroe, Theorems 10.2 and 10.3].

Lecture 3: Set functions

Summary. This lecture is based on [Taylor, Sections 3.1 and 3.3].

Content and Comments.

- 0:00 Definition of continuous set functions (from below and from above).
- 5:41 Comment on the hypothesis that $\mu(E_n) < \infty$ in the definition of continuity from above.
- 7:17 Lemma: $\mathcal{A}$ algebra of sets, $\mu : \mathcal{A} \to \mathbb{R}_+ \cup \{+\infty\}$. If μ is σ-additive then it is continuous. If μ is continuous from below, then it is σ-additive. If μ is finite and continuous from above at $\varnothing$, then it is σ-additive. [Taylor, Theorem 3.2].
- 35:21 Comment on the hypothesis that μ is finite in the lemma above.
- 40:26 Theorem: Let $\mathcal{S} \subset \mathcal{P}(\Omega)$ be a semi-algebra of subsets of Ω and $\mu : \mathcal{S} \to \mathbb{R}_+ \cup \{+\infty\}$ an additive set function. There exists a unique extension of μ to the algebra $\mathcal{A}(\mathcal{S})$ generated by the semi-algebra $\mathcal{S}$. [Taylor, Theorem 3.4].
- 1:00:43 In the previous theorem, if the set function $\mu : \mathcal{S} \to \mathbb{R}_+ \cup \{+\infty\}$ is σ-additive, then the extension is also σ-additive. [Taylor, Theorem 3.4].

Further Readings.

A. [Billingsley, Section 10.1] has many examples.

Recommended exercises.

a. [Taylor, Section 3.1], exercises 10, 11, 14.
b. [Taylor, Section 3.3], exercises 3, 4.
c. Fill the details of examples 2, 3 and 4 in [Billingsley, Section 10.1].

Suggested exercises.

a. [Taylor, Section 3.1], exercises 7 (Here note that the results are false for σ-algebras or algebras), 8, 9.
b. [Taylor, Section 3.3], exercises 1, 2.
c. [Billingsley, Section 10.1], exercises 1, 2.

Lecture 4: Carathéodory theorem

Summary. This lecture is based on on [Taylor, Sections 4.1].

Content and Comments.

- 0:00 Summary of the lecture. How to extend a measure ν defined on an algebra $\mathcal{A}$ to the σ-algebra generated by this algebra.
- 5:26 Definition of the set function (which will be shown to be an outer measure) $\pi^* : \mathcal{P}(\Omega) \to \mathbb{R}_+ \cup \{+\infty\}$.
- 8:04 Definition of an outer measures.
- 10:26 Proof that π^* is an outer measure.
- 22:07 definition of the measurable sets $\mathcal{M}$.
- 24:08 The class $\mathcal{M}$ is a σ-algebra which contains the algebra $\mathcal{A}$. In particular it contains $\mathcal{F}(\mathcal{A})$, the σ-algebra generated by $\mathcal{A}$. This is part of [Taylor, Theorem 4.1].
- 1:00:51 π^* restricted to $\mathcal{M}$ is σ-additive. This is part of [Taylor, Theorem 4.1].
- 1:14:41 Uniqueness of the extension on $\mathcal{F}(\mathcal{A})$ provided ν is σ-finite. [Taylor, Theorem 4.2].
- 1:20:12 Definition of monotone classes.
- 1:23:00 Intersection of monotone classes is a monotone class. The monotone class generated by a family of sets.
- 1:26:19 The monotone class generated by an algebra coincides with the σ-algebra generated by the algebra. In particular, a monotone class which contains an algebra also contains the σ-algebra generated by the algebra. This is how this result will be applied.
- 1:27:21 Proof of the uniqueness of the extension on $\mathcal{F}(\mathcal{A})$. In fact under the same condition, there is uniqueness of the extension on $\mathcal{M}$ (see the recommended exercises e. and f. below).
- 1:41:31 Remarks on Carathéodory theorem.

Further Readings.

A. [Taylor, Sections 4.1] introduces the concept of regular outer measures and inner measures. It applies the construction to the case of the Lebesgue measure.

Recommended exercises.

a. [Taylor, Section 4.1], exercises 4, 7, 8, 10,

*b. [Taylor, Section 4.1], exercise 5, 11 (This exercise shows that the hypothesis that ν is σ-finite is needed for the uniqueness of the extension), 12.

c. [Billingsley, Section 3], exercise 5.

d. Show that $\pi^*(A) = \inf \sum_{i \geq 1} \nu(E_i)$ where infimum is taken over sequences of with $E_i \in \mathcal{A}$ for all i, $A \subset \bigcup_{i \geq 1} E_i$ AND $E_i \cap E_j = \emptyset$ (that is: disjoint sequences).

e. (π^* is the "largest" extension) Given ν a σ additive function on $\mathcal{A}$, let μ be a measure that extends ν on $\mathcal{M}$. Show that for any $A \in \mathcal{M}$ we have $\mu(A) \leq \pi^*(A)$. If $\pi(\Omega)$ is finite deduce that we necessarily have $\mu(A) = \pi^*(A)$.

f. Using the previous exercise, show that π^* is also the only extension when μ is σ-finite.

Suggested exercises.

a. [Taylor, Section 4.1], exercises 1, 2, 3, 6, 9. (exercises 2 and 9 need the concept of regular outer measure, not yet seen)
b. [Billingsley, Section 3], exercise 2.
c. In the case where $\nu(\Omega) < \infty$ show that $A \in \mathcal{M}$ if

$$\pi^*(A) + \pi^*(\Omega \setminus A) = \pi^*(\Omega).$$

This exercise illustrates that measurable sets are in fact the ones for which the outer and *inner* measure coincide, where the inner measure π_* is defined by $\pi_*(A) = \pi^*(\Omega) - \pi^*(\Omega \setminus A)$. (see [Taylor, pp 75]). $\pi^*(A)$ and $\pi_*(A)$ corresponds to the largest and smallest possible value that that the measure of A could assume in an extension of μ to a σ-algebra that includes A. In that sense $\mathcal{M}$ corresponds to the "largest" σ-algebra such that the extension is unique.

d. If E_n is a sequence of elements of $\mathcal{A}$ such that $\nu(E_n) < \infty$ and $\Omega := \bigcup_{n \geq 1} E_n$. Show that $A \in \mathcal{M}$ if $\pi^*(E_n \cap A) + \pi^*(E_n \setminus A) = \pi^*(E_n)$ for every $n \geq 1$.

We are using the notation of lecture 4. $\mathcal{A}$ is an algebra and π^* the measure constructed from a σ-additive ν using the Caratheodory Theorem.

Lemma 0.1. *Given $A \subset \Omega$ an arbitrary subset. There exists $B \in \mathcal{F}(\mathcal{A})$ such that*

$$A \subset B \quad \text{and} \quad \pi^*(A) = \pi^*(B).$$

Proof. Note that from monotonicity of π^*, $A \subset B$ implies $\pi^*(A) \leq \pi^*(B)$. Using the definition of $\pi^*(A)$, for any n we can find a sequence $(B_{i,n})_{i\geq 0}$ such that

$$A \subset \bigcup_{i\geq 1} B_{i,n} \quad \text{and} \quad \sum_{i=1}^{n} \nu(B_{i,n}) \leq \pi^*(A) + n^{-1}$$

We set

$$B_n := \bigcup_{i\geq 1} B_{i,n} \text{ and } B := \bigcap_{n\geq 1} B_n$$

With our assumption A is a subset of B_n for every n and hence a subset of B. Furthermore since $B \subset B_n$ we have for any $n \geq 1$

$$\pi^*(B) \leq \pi^*(B_n) = \pi^*\left(\bigcup_{i\geq 1} B_{i,n}\right) \leq \sum_{i\geq 1} \pi^*(B_{i,n}) = \sum_{i\geq 1} \nu(B_{i,n}) \leq \pi^*(A) + \frac{1}{n}.$$

□

When A is a measurable subset ($A \in \mathcal{M}$) we can improve the statement

Proposition 0.2. *Consider $A \in \mathcal{M}$, and assume that can be written in the form*

$$A = \bigcup_{n\geq 1} A_n \quad \text{with} \quad A_n \in \mathcal{M},\ \pi^*(A_n) < \infty.$$

Then there exists $B^*, B_* \in \mathcal{F}(\mathcal{A})$ such that

$$B_* \subset A \subset B^* \quad \text{and} \quad \pi^*(B^* \setminus B_*) = 0.$$

In particular $\pi^*(A) = \pi^*(B^*) = \pi^*(B_*)$.

The assumption that A can be written as a union of events of finite measure is satisfied whenever ν is σ-finite.

Proof. Let us start with the particular case $\pi^*(A) < \infty$. Using the lemma there exists $B^* \in \mathcal{F}$ with $B^* \subset A$ such that $\pi^*(B) = \pi^*(A)$. Hence we have

$$\pi^*(B^* \setminus A) = \pi^*(B) - \pi^*(A) = 0.$$

Applying the Lemma again to $B^* \setminus A$, there exists $C \in \mathcal{F}$ such that $(B^* \setminus A) \subset C$ and $\pi^*(C) = 0$. We set $B_* = B^* \setminus C$. It is immediate to check that $B_* \subset A$, and we have $\pi^*(B^* \setminus B_*) \leq \pi^*(C) = 0$.

If $\pi^*(A) = \infty$, we apply the result to all A_n (let B^*_n and $B_{*,n}$ denote the sets in $\mathcal{F}$ which frame A_n). We set $B^* = \bigcup_{n\geq 1} B^*_n$ and $B_* = \bigcup_{n\geq 1} B_{*,n}$. We have $B^*, B_* \in \mathcal{F}$ and (the first inequality is by inclusion, the second by subadditivity)

$$\pi^*(B^* \setminus B_*) \leq \pi^*\left(\bigcup_{n\geq 1} (B^*_n \setminus B_{*,n})\right) \leq \sum_{n\geq 1} \pi^*(B^*_n \setminus B_{*,n}) = 0.$$

□

Lecture 5: Monotone Class Theorem

Summary. This lecture presents the proof of Theorem 1.5. [Taylor, Section 1.5]

Content and Comments.

- 0:00 Recall of the definition of a monotone class and simple consequences.
- 6:19 Main result of the lecture: The monotone class generated by an algebra is a σ-algebra ($\mathcal{M}(\mathcal{A}) = \mathcal{F}(\mathcal{A})$).
- 10:06 Beginning of the main technical step of the proof. Introduction of the set $\mathcal{G}(E)$ for $E \in \mathcal{A}$ (the video displays $E \subseteq \mathcal{A}$ but this is a typo) and proof that $\mathcal{G}(E) = \mathcal{M}(\mathcal{A})$ for $E \in \mathcal{A}$.
- 18:20 Continuation of the main technical step: Proof that $\mathcal{G}(E) = \mathcal{M}(\mathcal{A})$ for $E \in \mathcal{M}(\mathcal{A})$.
- 26:50 Proof that $\mathcal{M}(\mathcal{A})$ is an algebra using the work done on $\mathcal{G}(E)$.
- 30:45 Proof that $\mathcal{M}(\mathcal{A})$ is a σ-algebra (easy).The proof shows in fact that any algebra which is a monotone class is a σ-algebra

Further Readings. A result which is sometimes used to complement or replace the Monotone Class Theorem is the so called $\lambda - \pi$ Theorem. Both results are proved in [Billingsley, Chater 1 Section 3].

Lecture 6: The Lebesgue Measure I

Summary. This lecture is based on the argument presented for the proof of Theorem 3.7 in [Taylor, Section 3.4] in the case $k = 2$.

Content and Comments.

- 0:00 Presentation of the main objective of the lecture: Extending the notion of the length μ, defined on the semi-algebra $\mathcal{S}$ of half-open intervals, using the Carathéodory Theorem (and the easier extension result of additive set function from semi-algebra to algebra)
- 3:42 Proof that $\mu(A) \geq \sum_{j\geq 1} \mu(A_j)$ if $A = \sum_{j\geq 1} \mu(A_j)$. This part of the proof uses the unique extension ν of μ to the algebra $\mathcal{A}(\mathcal{S})$ via additivity.
- 8:50 Proof that $\mu(A) \leq \sum_{j\geq 1} \mu(A_j)$ if A is of the form $[a, b)$.
- 18:30 Proof of $\mu(A) \leq \sum_{j\geq 1} \mu(A_j)$ in the remaining cases (that is, the elements of $\mathcal{S}$ which are of infinite length, since there is nothing the prove in the case of the empty set).

Recommended exercises.

a. Let us consider a closed interval $A := [a, b]$ and $(C_i)_{i=1}^{\infty}$ be a sequence of open intervals with $C_i := (c_i, d_i)$. Assume that $A \subset \bigcup_{i\geq 1} C_i$. Show (using only compactness and σ-additivity on semi-open intervals) that $b - a \leq \sum_{i=1}^{\infty}(d_i - c_i)$. The proof starting at 8:50 is slightly more difficult than this exercise because we are dealing with semi-open intervals (which is of crucial importance for the use of Caratheodory's Theorem). For this reason some surgery with epsilons is necessary to recover open and closed intervals

b. Check that the proof work also to construct Lebesgue measure on $\mathbb{R}^d$ for $d \geq 2$.

c. Given $x \in \mathbb{R}$ and $A \subset \mathbb{R}$, let $A + x := \{x + y \ : \ y \in A\}$. Letting λ^* denote the Lebesgue exterior measure show that $\lambda^*(A + x) = \lambda^*(A)$ for every A and x (and thus that the same is true for Lebesgue measure if $A \in \mathcal{M}$).

d. Given $\alpha > 0$ and $A \subset \mathbb{R}$, let $\alpha A := \{\alpha y \ : \ y \in A\}$. Show that $\lambda^*(\alpha A) = \alpha\lambda^*(A)$ for every A and α.

e. Let $(x_n)_{n\geq 1}$ be an enumeration of $\mathbb{Q}$ (a sequence in which each rational number appears one and only one time). Consider the set $A := \bigcup_{n\geq 1}(x_n - 2^{-n}, x_n + 2^{-n})$. Prove that A is an open dense set and that A is not equal to $\mathbb{R}$.

Suggested exercises.

a. Let $A \subset \mathbb{R}$. Let λ^* denote the Lebesgue exterior measure. Show that $\lim_{\varepsilon\to 0} \lambda^*(A \cup (A + \varepsilon)) = \lambda^*(A)$.

b. Let $A \subset [0, 1)$ and $(\epsilon_n)_{n\geq 0}$ be a sequence of positive numbers tending to zero such that $A \cap (A + \epsilon_n) = \emptyset$ for every n (recall that $A + x := \{y + x \ : \ x \in A\}$). Let λ^* denote the Lebesgue exterior measure. Show that either $\lambda^*(A) = 0$ or A is non measurable.

Lecture 7: The Lebesgue Measure II

Summary. This lecture is based on the proof of Theorem 3.7 presented in [Taylor, Section 3.4] (case $k = 1$).

Content and Comments.

- 0:00 Objective of the lecture: proving that the length defined on the algebra $\mathcal{A}(\mathcal{S})$ is continuous at $\emptyset$ and hence σ additive.
- 4:12 Presentation of the strategy by contradiction using compactness. Given a sequence (E_k) such that $\bigcup_{k\geq 1} E_k = \emptyset$, but with $\mu(E_k) \geq 2\delta$ for all k, a sequence of non-empty compact subsets $G_k \subset E_k$ is constructed suing induction. The sets G_k are of the form $\overline{F}_k$ where $F_k \in \mathcal{A}(\mathcal{S})$.
- 8:05 Details of the construction of F_1 for (Step 1).
- 13:02 Details of the construction of F_2 for $k = 2$ (Step 2) and definition of the induction step.
- 25:44 The induction step, construction of G_{k+1} using G_k.
- 36:22 Recapitulation of the argument and conclusion.

Recommended exercises.

a. Check that the proof work also to construct Lebesgue measure on $\mathbb{R}^d$ for $d \geq 2$.

b. Let $A \subset [0, 1]$ be the non-measurable subset considered in Lecture 1 and let λ^* denote the exterior Lebesgue measure. Show that $\lambda^*(A) > 0$.

Suggested exercises.

a. Let $A \subset [0, 1]$ be the non-measurable subset considered in Lecture 1 and let λ^* denote the exterior Lebesgue measure. Show that $\lambda^*([0, 1] \setminus A) = 1$.

Lecture 8: Complete measure

Summary. This lecture is based on [Taylor, Section 4.2]

Content and Comments.

- 0:00 Definition of a complete measure, and presentation of the main aim of the lecture: enlarging a extending the σ-algebra and extending a measure to make the pair complete.
- 6:28 Definition of $\overline{\mathcal{F}}$ and proof that it is a σ-algebra.
- 20:07 Definition of the extension $\bar{\mu}$ and proof that it does not depend on the specific decomposition $A \cup N$.
- 27:00 Verification that $\bar{\mu}$ coincides with μ on $\mathcal{F}$.
- 28:25 Proof that $\bar{\mu}$ is σ additive.
- 34:00 Proof that $(\bar{\mu}, \overline{\mathcal{F}})$ is complete.
- 39:30 Proof of the uniqueness of $\bar{\mu}$.
- 47:30 Proof that the outer measure π^* used in Lecture 4 is complete.

Recommended exercises.

a. Show that $B \in \overline{\mathcal{F}}$ if and only if there exists $A, H \in \mathcal{F}$ and N such that $B = A \cup N$, $N \subset H$, $\mu(H) = 0$ AND $A \cap H = \emptyset$.

b. Consider two σ-algebras $\mathcal{F}$ and $\mathcal{G}$ such that that $\mathcal{F} \subset \mathcal{G}$ and a measure μ defined on $\mathcal{G}$ (we also write μ for the restriction of this measure to $\mathcal{F}$). Show that if $\mathcal{G}$ is μ complete then we have $\overline{\mathcal{F}}_\mu \subset \mathcal{G}$.

c. The last part of the lecture proves that $(\pi^*, \mathcal{M})$ is complete. If μ is a σ-finite measure on $\mathcal{F}$, it is also true that $\mathcal{M}$ is the completition of $\mathcal{F}$, that is $\mathcal{M} = \overline{\mathcal{F}}$ (prove it). Note that this implies that if μ is sigma finite, the outer-measure π^* is the unique extension of μ to the σ-algebra $\mathcal{M}$.

d. Exercise 1 and 2 in [Taylor, Section 4.1] (Exercise 2 provides a counter example to c. when σ additivity does not hold).

Lecture 9: Approximation Theorems

Summary. This lecture is based on [Taylor, Section 4.3]

Content and Comments.

- 0:00 First approximation Theorem: Any *measurable* set with finite measure can be approximated with a precision ϵ by an element of the algebra. The proof is presented only for $A \in \mathcal{F}$, but the assumption is not used.
- 0:07 "We have seen in previous lectures ...": I could not find the stated claim in previous videos. But this is a consequence of the fact proved at the end of [Taylor, Section 4.1] (π^* is a *regular* outer measure). The statement is also valid for non measurable A. A proof is given in this document (after Lecture 4)
- 3:54 Proof of the first approximation Theorem.
- 13:49 Observation that the approximation theorem extends to the completed measure. σ-finite is not required here. Since from exercise b. in Lecture 8 we have $\overline{\mathcal{F}} \subset \mathcal{M}$, this observation is not extending the Theorem)
- 16:30 Definition of a regular measure on a topological space.
- 21:25 If a measure μ is regular then $\mathcal{F} \subset \overline{\mathcal{B}}_\mu$ (the completed Borel σ-algebra.
- 27:10 Second approximation Theorem: The Lebesgue measure is a regular measure and its proof.
- 52:35 Concluding remark: exact approximation by $\mathcal{F}_\sigma$ and $\mathcal{G}_\delta$ sets.

Further Readings.

A. [Taylor, Section 4.3] mentions that the second approximation Theorem is valid in the larger context of *metric outer measure*.

Recommended exercises.

a. Check that the proof of the second approximation Theorem is valid in $\mathbb{R}^d$ for $d \geq 2$.
b. Prove the statement made at 13:49 in the lecture and the concluding remark (not looking at the book) is valid in $\mathbb{R}^d$ for $d \geq 2$.
c. Exercise 1 [Taylor, Section 4.3]

Suggested exercises.

a. Exercise 2,3,4 [Taylor, Section 4.3].

Lecture 10: Measurable and integrable functions

Summary. This lecture is based on [Taylor, Section 5.1-5.2-5.3]

Content and Comments.

- 0:00 Presentation of natural property that one would expect for a notion of integral (Note that that the third property is NOT satisfied by the Riemann integral, cf. final remark below, which in itself is a sufficient reason to look for a better notion, [Taylor, Section 5.1] provide a more detailed discussion)
- 3:00 Comparison (of the ideas behind) the construction of Lebesgue integral with that of Riemann integral.
- 11:30 Definition of measurable functions taking value in $\mathbb{R} \cup \{-\infty, +\infty\}$ (Note that there is a notion of measurability for applications taking value in an arbitrary set X equipped with a σ-algebra $\mathcal{G}$, see Taylor [Taylor, Section 6.5] where these are mentioned as *measurable transformation*).
- 17:05 Presentation of a simple criterion to check measurability [Taylor, Theorem 5.1] (the proof presents in fact a more general result, a particular case of which is the presented lemma since it can be used for an arbitrarily family of set $\mathcal{G}$ that generates $\mathcal{F}$. The criterion presented is in fact valid for functions f taking values in $\mathbb{R}$. If one allows for the values $\pm\infty$ one must also check that $f^{-1}(\{\infty\}) \in \mathcal{F}$).
- 32:20 Introduction of the notion of *simple function* and proof that they are measurable.
- 38:50 Definition of the integral of (non-negative) simple function (and verification that it does not depends on the representation chosen).
- 51:29 Presentation of the roadmap to define integration.
- 57:20 Final remark: even some simple functions are not Riemann integrable. (note that the example given is the increasing limit of Riemann integrable functions.)

Further Readings.

A. [Taylor, Section 9.4] present an alternative approach to the construction of integration, where integral is constructed directly, without constructing Lebesgue measure beforehand.

Recommended exercises.

a. Show that if $(E_i)_{i=1}^n$ is a finite sequences of measurable sets (not necessarily pairwise disjoint) and $(c_i)_{i=1}^n$ are real numbers then $\sum_{i=1}^n c_i \mathbf{1}_{E_i}$ is a simple function.
b. Show that f is a simple function if and only if it satisfies the following condition: f is measurable and the set $\{f(x) \ : \ x \in \Omega\}$ is finite.
c. Show that the set of simple functions forms and $\mathbb{R}$-vector space.
d. [Taylor, Section 5.2] Exercises 1 and 6.
e. [Taylor, Section 5.3] Exercise 2.

Lecture 11: Measurable Functions

Summary. This lecture is based on [Taylor, Section 5.2]

Content and Comments.

- 0:00 Presentation of the first objective of the class: show that measurability is preserved by some basic operations.
- 1:30 of [Taylor, Theorem 5.3] is presented (stability under sum, product etc...) and proved. The main tricks are: using [Taylor, Theorem 5.1], that is, checking only the pre-image of semi-infinite intervals, and use the rational numbers to write some events as countable union/intersection cf. 7:30.
- 20:23 Presentation of [Taylor, Theorem 5.4] about conservation of measurability under countable sup, inf, $\limsup$ and $\liminf$.
- 26:21 Important comments about sup and inf over uncountable sets (not necessarily measurable).
- 30:00 Important observation: continuous function on Ω are measurable with respect to the σ-algebra generated by the collection of open sets, Borel σ-algebra.
- 32:25 Measurability of $f : \Omega \to \overline{\mathbb{R}}$ depends on the choice of σ-algebra on Ω Note that the σ-algebra considered on $\overline{\mathbb{R}}$ is always $\overline{\mathcal{B}}$, the Borel σ-algebra.
- 34:42 Definition of *almost sure* properties. "almost surely" is used when considering probability measure, while "almost everywhere" is more common in other contexts.
- 37:00 If $f : \mathbb{R} \to \overline{\mathbb{R}}$ is Lebesgue measureable and f coincides with g a.e. then g is Lebesgue measurable. The statement is not true for Borel measurable functions. Here the statement extends (with the same proof) to $f : \Omega \to \mathbb{R}$ which is $\mathcal{F}$ measurable: if f and g coincide μ-almost everywhere AND $\mathcal{F}$ is μ-complete (this is the only assumption of the Lebesgue σ-algebra which is used) then g is $\mathcal{F}$ measurable.
- 43:30 The composition of measurable functions is measurable. Here it is really important that the σ-algebra considered for g is the Borel σ-algebra.

Further Readings.

A. The more general notion of measurable transformation in [Taylor, Section 6.5] allows to generalize some of the properties above, in particular that concerning composition.

Recommended exercises.

a. Exercise 2, 7 (can assume $\Omega = \mathbb{R}$)[Taylor, Section 5.2]
b. Let $f : \mathbb{R} \to \bar{\mathbb{R}}$ be a Lebesgue measurable function. Show that there exists g which is Borel measurable such that $f = g$ almost everywhere (or show that if f is $\Omega \to \mathbb{R}$ is $\bar{\mathcal{F}}_\mu$ measurable (measurable with respect to the completed σ-algebra), then there exists g which is $\mathcal{F}$ measurable such that $f = g$, μ-almost everywhere).

Suggested exercises.

a. Exercises 8, 9 (can assume $\Omega = \mathbb{R}$)[Taylor, Section 5.2]

Lecture 12: Definition of the integral

Summary. This lecture is based on [Taylor, Section 5.2] and [Taylor, Section 5.3].

Content and Comments.

- 0:00 Plan of the lecture: First, define the integral of positive measurable function via increasing sequences of approximation by simple function, then define integral of a measurable function by decomposition into positive and negative part.
- 8:02 Important technical Lemma: "Any positive measurable function is the increasing limit of a sequence of simple functions". This is [Taylor, Theorem 5.2]
- 9:30 Explicit description of a sequence f_n converging to f, and proof that it is a simple function.
- 15:33 Proof that f_n converges to f.
- 23:00 Proof that $f_n \leq f_{n+1}$ An alternative argument is provided in Exercise (a) below.
- 38:35 Definition of integral of positive measurable function. As in Section 5.3 (2) in [Taylor].
- 40:10 Proof that for positive simple functions $I(f) \leq I(g)$ if $f \leq g$.
- 47:00 Checking that the definition of $I(f)$ given at 38:35 does not depend on the sequence.
- 48:45 Reduction of the problem to the inequality (Lemma) $g \leq I(f)$ when $g \leq f$ and g is simple.
- 53:15 Proof of the lemma when $g = c\mathbf{1}_E$
- 1:05:50 Completition of the Lemma.
- 1:10:50 Definition of an integrable function and of its integral (via decomposition into positive and negative parts).

Recommended exercises.

a. Consider the set $A_n := \{k2^{-n} \ : \ k \in \{1, \dots, n2^n\}\}$. Show that $A_n \subset A_{n+1}$ and that if f_n is the sequence introduced at [9:30] above we have for every x

$$f_n(x) = \max\{y \in A_n \ : y \leq f(x)\}.$$

Deduce from this that $f_n(x) \leq f_{n+1}(x)$.

b. Exercises 4 and 5 in [Taylor, Section 5.2].

c. Exercises 4,5 and 6 in [Taylor, Section 5.3].

Suggested exercises.

a. Exercise 3 in [Taylor, Section 5.2]

Lecture 13: Integral of simple functions

Summary. This lecture is based on [Taylor, Section 5.3].

Content and Comments.

0:00 The goal of this lecture is to show that the integration of simple function is linear. It is shown that for non-negative f and g, and $c \geq 0$, $I(f+g) = I(f) + I(g)$ and $I(cg) = cI(g)$. This lecture can be studied before Lecture 12. Additivity of the integral gives a proof of the statement made in Lecture 12 at 40:10.

Recommended exercises.

a. Show that if f is simple then $|f|$ is simple. Set

$$\mathcal{L}_{\text{simple}}(\mu) := \{f \text{ simple functions on } \Omega \text{ satisfying } I(|f|) < \infty\}.$$

Show that $I_S(\Omega)$ is a vector space, that $I(f)$ is well defined for $f \in \mathcal{I}_S(\Omega)$ and that $f \mapsto I(f)$ is a linear application from $\mathcal{I}_S(\Omega)$ to $\mathbb{R}$.

Lecture 14: Property of the integral

Summary. This lecture is based on [Taylor, Section 5.4].

Content and Comments.

- 0:00 The aim of the lecture is to prove some of the statements in [Taylor, Theorem 5.5].
- 0:50 First observation. If f and g are non-negative then $f \leq g$ implies $I(f) \leq I(g)$. As a consequence, f integrable implies that $f\mathbf{1}_A$ integrable (proof completed at 13:00).
- 7:05 Proof of "linearity" of the integral for non-negative functions ($I(f+g) = I(f) + I(g)$, $I(cf) = cI(f)$ for f, g non-negative and $c \geq 0$).
- 15:35 If $\mu(E) = 0$ and f is measurable then $I(f\mathbf{1}_E) = 0$ The lecture also assume that f is integrable but this is not necessary.
- 20:34 Integrability is stable via addition, multiplication and restriction to a measurable set Of course here f and g integrable does not imply that $f + g$ is well defined because of $\pm\infty$ conflicts. Here it is implicitely assumed that $f + g$ is well defined. If A and B are disjoint the integral on $A \cup B$ is the sum of the integrals on A and B.
- 31:00 If f is integrable then $|f| < \infty$ almost surely.
- 38:11 Proof of the linearity of the integral. This is perhaps the less "direct" proof of this lecture. The trick is to reduce the statement to something about positive integrals.

Recommended exercises.

a. Exercise 3 in [Taylor, Section 5.3].
b. Exercises 1,2, 5 and 6 in [Taylor, Section 5.4].

Suggested exercises.

a. Exercises 3 and 4 in [Taylor, Section 5.4].

Lecture 15: Property of the integral

Summary. This lecture is based on [Taylor, Section 5.4].

Content and Comments.

0:00 The aim of the lecture is to prove the remaining the statements in [Taylor, Theorem 5.5], starting with $|\int f \mathrm{d}\mu| \leq \int |f| \mathrm{d}\mu$. This inequality can be interpreted as a continuous version of the triangle inequality).
1:34 Proof of $\int cf \mathrm{d}\mu = c \int f \mathrm{d}\mu$
8:41 Monotonicity of the integral.
12:30 Non-negative functions with zero integral are equal to zero almost everywhere.
19:00 Two functions which coincide almost-everywhere have the same integral.
22:40 A measurable function whose absolute value is smaller than an integrable function is integrable.
24:40 Special case of the previous property.

Recommended exercises.

a. Show that

$$\mathcal{L}_1(\mu) := \{f, \Omega \to \mathbb{R} : f \text{ is measurable and } \int |f| \mathrm{d}\mu < \infty\}$$

is a vector space and that $f \mapsto \int f \mathrm{d}\mu$ is a linear application on that vector space.

Suggested exercises.

a. Exercise 8, 9 in [Taylor, Section 5.4]

Lecture 16: Theorem on the convergence of integrals

Summary. This lecture is based on [Taylor, Section 5.4]

Content and Comments. The three main theorems of this lecture (Monotone convergence, Fatou, and Dominated convergence) are amongst the most important result in the class..

- 0:00 Statement and proof of the Monotone convergence Theorem [Taylor, Theorem 5.6].
- 19:20 Small extension of the assumption: $f_n \geq 0$ can be replaced by $f_n \geq g$ with g integrable.
- 20:30 Construction of the measure μ_f by integrating a positive function f against a reference measure μ.
- 27:45 Absolute continuity and preview of the Random-Nikodym Theorem The function g is called the *density* of μ with respect to ν. Note that the Random-Nikodym Theorem is only valid if ν is a σ-finite measure.
- 31:25 It is possible that for two measure we have neither $\nu \ll \mu$ nor $\mu \ll \nu$.
- 36:00 Presentation of uniform integrability and proof that any integrable function is also uniformly integrable The notion of uniform integrability can be applied to a set of integrable function see [Taylor, Section 6.4], in that case uniform integrability is something stronger than integrability.
- 42:20 Fatou's Lemma [Taylor, Theorem 5.7]: presentation, proof, remark about the positivity assumption, (it can be replaced by $f_n \geq g$, with g integrable like for monotone convergence) and corollary for negative functions.
- 52:07 Statement and proof of the dominated convergence Theorem [Taylor, Theorem 5.8]. Note cf. [Taylor, Theorem 7.6] that the convergence $f_n \to f$ can be assumed to hold only in measure or a.e.

Further Readings.

A. Amongst important consequences of the dominated convergence Theorem is the possibility (under the right assumption) to exchange the position of derivatives and integrals (see [Bogachev, Corollary 2.8.7]). We provide some details on the next page.

Recommended exercises.

a. Show that in the dominated convergence Theorem, on can assume that the assumptions $\lim_{n\to\infty} f_n(x) = f(x)$ and $|f_n(x)| \leq g(x)$ are valid for μ-almost everywhere (instead of everywhere).
b. Show that the relation $\ll$ of absolute continuity between measure is an order relation (that is: prove reflexivity and transitivity).
c. Exercises 6,7 and 11 [Taylor, Section 5.4]

Suggested exercises.

a. Exercises 10, 13 in [Taylor, Section 5.4]

Consider an open interval (a,b) and $(\Omega,\mathcal{F},\mu)$ a measured space. Let $f:(a,b)\times\Omega\to\mathbb{R}$ be a function such that for every $t\in(a,b)$, the function $f(t,\cdot)$ defined on Ω is integrable. We define

$$J(t):=\int_\Omega f(t,x)\mu(\mathrm{d}x).$$

Corollary 0.3. *The following results hold true*

(i) *If for μ almost every x, $t\mapsto f(t,x)$ is continuous on (a,b) there exists g_1 integrable such that for every fixed $t\in(a,b)$,*

$$|f(t,x)|\le g_1(x),\quad \mu-a.e$$

then $J(t)$ is continuous on (a,b).

(ii) *If for μ almost every x, $t\mapsto f(t,x)$ is differentiable on (a,b) and there exists g_2 such that ,*

$$\sup_{t\in(a,b)}\left|\frac{\partial f(t,x)}{\partial t}\right|\le g_2(x),\quad \mu-a.e.$$

then $J(t)$ is differentiable on (a,b) and

$$J'(t)=\int_\Omega\frac{\partial f(t,x)}{\partial t}\mu(\mathrm{d}x)$$

Proof. The first point is a direct consequence of the dominated convergence Theorem. To prove continuity at a fixed t, it is sufficient to show that for every sequence t_n such that $t_n\to t$ we have $\lim J(t_n)=J(t)$. To show this result, we set

$$E:=\{x\ :\forall n\ge1,|f_n(x)|\le g_1(x)\}\cap\{\lim_{n\to\infty}f_n(x)\text{ exists and is equal to }f(t,x)\}.$$

We have

$$\mu(E^{\mathrm{c}})\le\sum_{n\ge1}\mu\left(\{x\ :\ |f(t_n,x)|\le g_1(x)\}\right)+\mu\left(\lim_{n\to\infty}f_n(x)\ne f(t,x)\}\right)=0.$$

We can apply the dominated convergence Theorem (we have $|f_n|\le g_1$). This yields

$$\lim_{n\to\infty}J(t_n)=\lim_{n\to\infty}\int_\Omega f_n\mathrm{d}\mu=\int_{\Omega`}f(t,x)\mathrm{d}\mu=J(t)$$

For the second point we must prove that with the same setup (assuming that $t_n\ne t$ for every n)

$$\lim_{n\to\infty}\int_\Omega\frac{f(t,x)-f(t_n,x)}{t-t_n}\mu(\mathrm{d}x)=\int_\Omega\lim_{n\to\infty}\frac{f(t,x)-f(t_n,x)}{t-t_n}\mu(\mathrm{d}x).$$

We can replace $\int_\Omega$ by $\int_E$ where

$$E:=\left\{x\ :t\mapsto f(t,x)\text{ is differentiable on }(a,b)\text{ and }\sup_{t\in(a,b)}\left|\frac{\partial f(t,x)}{\partial x}\right|\le g_2(x)\right\}$$

since by assumption $\mu(E^{\mathrm{c}})=0$. Hence we try to apply the dominated convergence Theorem to the sequence h_n defined by

$$h_n(x):=\frac{f(t,x)-f(t_n,x)}{t-t_n}\mathbf{1}_E.$$

Using the mean value theorem, for $x \in E$ and $n \geq 1$ there exists $s_n(x) \in (a,b)$ such that

$$h_n(x) = \frac{\partial f(s_n(x), x)}{\partial t}.$$

As a consequence we have $|h_n(x)| \leq g_2(x)$ and hence by dominated convergence

$$\lim_{n\to\infty} \int_E \frac{f(t,x) - f(t_n,x)}{t - t_n} = \int_E \lim_{n\to\infty} \frac{f(t,x) - f(t_n,x)}{t - t_n} \mu(\mathrm{d}x) \qquad (0.1)$$

which is the desired result. □

Lecture 17: Product measures

Summary. This lecture is based on [Taylor, Section 6.1 and 6.2].

Content and Comments.

0:00 The main goal of the lecture is given, two measure μ_1 and μ_2 on two spaces $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$, construct a natural measure on the Cartesian space $\Omega_1 \times \Omega_2$ equipped with a natural σ-algebra. Although direct construction exists, products can be used to define the Lebesgue measure on $\mathbb{R}^d$. The setup is introduced, and the requirement concerning the value of μ on rectangle sets is introduced.

4:40 Formal definition of "rectangles" and proof that they form a semi-algebra As mentioned in the lecture, if $\Omega_1 = \Omega_2 = \mathbb{R}$, the notion does not correspond to Euclidean rectangles.

12:40 Introduction of the σ-algebra $\mathcal{F}_1 \star \mathcal{F}_2$. It can also be denoted by $\mathcal{F}_1 \otimes \mathcal{F}_2$ and is usually called the product σ-algebra. The next step is check all the assumptions necessary to apply Caratheodory's Theorem.

15:32 Start by showing that if $A \in \mathcal{F}_1 \star \mathcal{F}_2$, then its one dimensional sections A_x and A^y are in $\mathcal{F}_2$ and $\mathcal{F}_1$ respectively.

36:55 Proof that μ is additive (on the set of rectangles)

51:50 Small adaptation of the previous argument to show that μ is σ-additive.

53:55 Final considerations concerning uniqueness: μ is sigma finite on $\Omega_1 \times \Omega_2$ if (and only if, unless one of the measure is uniformly zero) μ_1 and μ_2 are sigma finite.

Further Readings.

A. [Taylor, Section 6.1] introduce the concept of product algebra (similar to product σ-algebra but only with additivity)

B. [Taylor, Section 2.4] give some further introduction to product spaces.

Recommended exercises.

a. Prove the result below using two different methods:
   (i) A direct adaptation of the proof above.
   (ii) With an induction on k starting with $k = 2$.

   Given $(\Omega_i, \mathcal{F}_i, \mu_i)$, $i = 1, \ldots, k$, a finite sequence of measured space there exist a unique measure μ on $\prod_{i=1}^k \Omega_i$ equipped with the σ-algebra $\bigotimes_{i=1}^k F_i$ generated by rectangles $E_1 \times \cdots \times E_k$, $E_i \in \mathcal{F}_i$, such that

$$\mu(E_1 \times \cdots \times E_k) = \prod_{i=1}^k \mu_i(E_i).$$

b. Exercise 5 and 7 in [Taylor, Section 6.1]

c. Exercise 1 in [Taylor, Section 6.2]

Suggested exercises.

a. Let (X_1, d_1) and (X_2, d_2) be two separable metric spaces and let $\mathcal{B}_1$ and $\mathcal{B}_2$ denote the corresponding Borel σ-algebras. Set $X = X_1 \times X_2$ and equip it with the distance $d([x_1, x_2], [y_1, y_2]) = d_1(x_1, y_1) + d_2(x_2, y_2)$, and let $\mathcal{B}$ the Borel σ-algebras associated with (X, d) (the Borel σ-algebra is that generated by open sets). Show that $\mathcal{B}_1 \otimes \mathcal{B}_2 \subset \mathcal{B}$, and that if X_1 and X_2 are separable then we have equality.

b. Exercise 8 and 9 in [Taylor, Section 6.2]
c. Exercise 3 in [Taylor, Section 6.2]

Property of Lebesgue measure in $\mathbb{R}^d$

Using directly Caratheodory Theorem on the semi-algebra of half-open rectangles (cf. Exercises of Lecture 5/6) or product measure, one can define the Lebesgue measure λ on $\mathbb{R}^d$. We let $\mathcal{L}$ denote the Lebesgue σ-algebra on $\mathbb{R}^d$ (for both construction, this is the σ-algebra $\mathcal{M}$ of measurable sets coming from Caratheodory Theorem).

The solution of the following exercises can be found in [Bogachev]. More precisely [Bogachev, Theorem 1.7.3] and [Bogachev, Corollary 3.6.4].

Recommended exercises.

a. Prove that the Lebesgue measure on $\mathbb{R}^d$ is invariant by translation, that is: for every $a \in \mathbb{R}^d$ and $A \in \mathcal{L}$, $\lambda(A + a) = \lambda(A)$.
b. Prove that for $A \in \mathcal{L}$ and $c \in \mathbb{R}$, $\lambda(cA) = |c|^d \lambda(A)$.
c. Prove that if $A \in \mathcal{L}$ and $g \in GL(\mathbb{R}^d)$ is a linear endomorphism of $\mathbb{R}^d$ then the image set $g(A)$ is also Lebesgue measurable (one can treat separately the case when g is non-invertible and when g is invertible).
d. Prove that if $g \in O(\mathbb{R}^d)$ is an orthogonal transformation of $\mathbb{R}^d$, then $\lambda(g(A)) = \lambda(A)$.
e. Prove that if $g \in GL(\mathbb{R}^d)$ then $\lambda(g(A)) = |\det(g)|\lambda(A)$.

Lecture 18 : Measure on a countable product of spaces

Summary. This lecture is based on [Taylor, Section 6.1 and 6.2]

Content and Comments.

- 0:00 Presentaton of the objective of the lecture: constructing a product measure for a countably infinite product of probability measures (the result corresponds to [Taylor, Theorem 6.3]).
- 4:20 Introduction of the collection of cylinder sets $\mathcal{C}$, and proof that it is a semi-algebra (this is the last Lemma [Taylor, Section 6.1])
- 17:53 Introduction of the set function μ on $\mathcal{C}$ which is to be extended to a measure.
- 21:10 Proof that μ is additive on $\mathcal{C}$.
- 29:31 Claim and proof that the extension of μ to the algebra $\mathcal{A}$ is continuous from above at $\emptyset$. This is the main difficulty in the proof
- 1:18:00 Conclusion.

Further Readings.

A. There exists several alternative approach to countable product measure. When $\Omega_i = \{0, 1\}$ for every i and $\mu_1 = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$, the result can be obtained using the Lebesgue measure and dyadic decomposition (see [Billingsley, Theorem 5.4]). While very specific, this case has some importance, since from it, the case $\Omega_i = \mathbb{R}$ and μ_i arbitrary can be deduced (see [Billingsley, Theorem 20.4]).

B. The result can be achieved using Kolmogorov's extension Theorem which is conceptually simple but has a very technical proof. The extension Theorem is presented in [Taylor, Section 6.6] and the way to obtain product measures using it is explained in [Billingsley, Section 36], see in particular [Billingsley, Example 36.2] (Kolmogorov's Extension Theorem also allows to define uncountably infinite product distribution although but most practical application concern the countable case anyways).

Recommended exercises.

a. Exercise 10 [Taylor, Section 6.1]
b. Exercises 2,4 [Taylor, Section 6.2].

Suggested exercises.

a. Exercise 5 [Taylor, Section 6.2]

Lecture 19 : Fubini's Theorem

Summary. This lecture is based on [Taylor, Section 6.3]

Content and Comments.

- 0:00 Presentation of Fubini's Theorem (the integral of f over $\Omega_1 \times \Omega_2$ under the product measure $\mu_1 \otimes \mu_2$ can be obtained by first integrating with respect to μ_1 and then w.r.t. μ_2 or *vice versa*) and summary of the main steps to prove it.
- 4:41 First step necessary to check that the iterated integral is well defined: If f is $\mathcal{F}_1 \otimes \mathcal{F}_2$-measurable, then $f_x = f(x, \cdot)$ is $\mathcal{F}_2$ measurable for all x.
- 9:14 Second step necessary to check that the iterated integral is well defined, prove that $x \mapsto \int f(x,y)\mathrm{d}\mu_2(\mathrm{d}y)$ is $\mathcal{F}_1$ measurable, starting with indicater function. If $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$ then the functions $x \mapsto \mu_2(E_x)$ is $\mathcal{F}_1$ measurable. First it is checked that the property is satisfied for rectangles, and then shown, with a monotone class argument, that the class of sets satisfying the property is a σ-algebra. The argument is first exposed for finite measures and an extra step is needed to extend the result to the σ-finite case
- 35:13 Proof of Fubini's Theorem for indicator functions combining the previous method, with the use of the monotone convergence Theorem.
- 52:05 Proof of Fubini's Theorem (including the measurability of the integral w.r.t. μ_2) in the case of positive functions. This is called Tonelli's Theorem. It is first proved for simple function and then extended via monotone convergence.
- 1:07:30 The case of integrable f is discussed. The problem is that $\int f(x,y)\mu_2(\mathrm{d}y)$ may not be defined on a set of negligible measure.
- 1:11:38 Modification of the statement so that it makes sense, and proof (it follows from the positive case using linearity).
- 1:21:48 Final remark: An integrability criterion for measureable functions on $\Omega_1 \times \Omega_2$. A measurable function f is $\mu_1 \otimes \mu_2$ integrable if (and only if) the iterated integral of $|f|$ w.r.t. to first μ_1 and then μ_2 is finite.

Recommended exercises.

a. Prove the k-dimensional version of Fubini's Theorem using two methods (as in the first Recommended exercise of Lecture 17).

b. Let $(\Omega_1, \mathcal{F}_1, \mu_1)$ and $(\Omega_2, \mathcal{F}_2, \mu_2)$ two σ-finite measured space, $f \in \Omega_1 \times \Omega_2 \to \mathbb{R}_+$, $\mathcal{F}_1 \otimes \mathcal{F}_2$ measurable. Let us define g on $\Omega_2 \times \Omega_1$ by $g(x,y) := f(y,x)$. Show that g is $\mathcal{F}_2 \otimes \mathcal{F}_1$ measurable and that

$$\int_{\Omega_1 \times \Omega_2} f \mathrm{d}\mu_1 \otimes \mu_2 = \int_{\Omega_2 \times \Omega_1} g \mathrm{d}\mu_2 \otimes \mu_1.$$

c. Exercises 2,3,4,5 [Taylor, Section 6.3]

Suggested exercises.

a. Let $(\Omega_1, \mathcal{F}_1, \mu_1)$ and $(\Omega_2, \mathcal{F}_2, \mu_2)$ two complete measured space. Consider $\overline{\mathcal{F}_1 \otimes \mathcal{F}_2}$ the $\mu_1 \otimes \mu_2$-completition of the σ-algebra $\mathcal{F}_1 \otimes \mathcal{F}_2$. Let f be an $\overline{\mathcal{F}_1 \otimes \mathcal{F}_2}$ measurable function which is such that
$$\int |f| \mathrm{d}\mu_1 \otimes \mu_2 = 0.$$
Show that the following sets
$$E := \{x \in \Omega_1 : f_x \text{ is not } \mathcal{F}_2 \text{ measurable }\},$$
$$F := \{x \notin E : \int |f(x,y)| \mu_2(\mathrm{d}y) \neq 0\},$$
belong to $\mathcal{F}_1$ and that $\mu_1(E) = \mu_1(F) = 0$.

b. Keep the assumption of exercise a. and consider f an $\overline{\mathcal{F}_1 \otimes \mathcal{F}_2}$ measurable function which is integrable. Show that the sets
$$E' := \{x \in \Omega_1 : g_x \text{ is not } \mathcal{F}_2 \text{ measurable }\},$$
$$F' := \{x \notin E' : \int |g(x,y)| \mu_2(\mathrm{d}y) < \infty\}.$$
belong to $\mathcal{F}_1$ and that and that $\mu_1(E) = \mu_1(F) = 0$. Show that the function $h(x)$ defined by
$$h(x) := \begin{cases} \int g(x,y) \mu_2(\mathrm{d}y) & \text{on } (E' \cup F')^{\mathrm{c}}, \\ 0 & \text{on } E \cup F. \end{cases}$$
is measurable and that $\int h \mathrm{d}\mu_1 = \int f \mathrm{d}\mu_1 \otimes \mu_2$.

c. Exercise 1 (try to justify the existence of the set E) and 6 [Taylor, Section 6.3]

Lecture 20: Hahn-Jordan Theorem

Summary. This lecture is based on [Taylor, Section 3.2]

Content and Comments.

0:00 Presentation of the definition σ-additive set functions with value in $\mathbb{R}$. The Hahn Jordan Theorem (appearing later in the video) states that any σ-additive $\mathbb{R}$-valued set functions (we call it *signed* measure) can be written as the difference of two (positive) measure with disjoint support. (The definition of σ-additive for $\mu : \mathcal{F} \to \bar{\mathbb{R}}$ is never explicitly stated but is the following: (i) $\mu(\emptyset) = \emptyset$, (ii) If $(A_i)_{i\geq 1}$ is a sequence of paiwise disjoint sets in $\mathcal{F}$ then either $\sum_{i\geq 1} (\mu(A_i))_+ < \infty$ or $\sum_{i\geq 1} (\mu(A_i))_- < \infty$ and $\mu(\bigcup_{i\geq 1} A_i) = \sum_{i\geq 1} (\mu(A_i))_+ - \sum_{i\geq 1} (\mu(A_i))_-$ (the assumption implies that the case $\infty - \infty$ never presents itself so that the sum is always well defined.

6:30 First of serveral techical lemmas concerning signed measure. This one is about the measure of E and F when $E \subset F$.

10:33 Second technical Lemma: there cannot be two sets with $+\infty$ and $-\infty$ measure.

16:21 Third technical Lemma: continuity from above/below for signed measure. (Note that the idea is essentially the same as in the positive case).

30:06 Hahn-Jordan Theorem [Taylor, Theorem 3.3]. Remarks about nonuniqueness for P and N.

35:50 Presentation of the main steps of the proof.

41:14 Proof of step 1, the minimal measure α is larger than $-\infty$. (Note here that $\alpha \leq 0$ since additivity implies that $\mu(\emptyset) = 0$). The goal is to contain a sequence of sets (A_k), (B_k) and much details are given about the first step.

52:33 Construction of A_k, B_k for $k \geq 2$.

1:01:20 Key remark about bifurcations: Bifurcation at k and $j \geq k+2$ implies $B_{k+1} \cap B_j = \emptyset$.

1:05:50 Conclusion of step 1, splitting in two cases (finitely or infinitely many bifurcations).

1:10:36 Step 2: there exists a set with minimal measure.

1:16:20 Conclusion.

Recommended exercises.

a. Prove that the decomposition $\mu = \mu_+ - \mu_-$ where $\mu_+ \perp \mu_-$ (this means that there exists $A \in \mathcal{F}$ such that $\mu_+(A) = \mu_-(A^c)$ cf. the next lecture) is unique.

b. Using the notation of the previous exercise, one sets $|\mu| = \mu_+ + \mu_-$. Show that for $A \subset \mathcal{F}$

$$|\mu|(A) = 0 \quad \Leftrightarrow \quad (\forall B \in \mathcal{F}, B \subset A,\ \mu(B) = 0)\,.$$

Lecture 21: Radon-Nikodym Theorem

Summary. This lecture is based on [Taylor, Section 6.4]

Content and Comments.

- 0:00 Recalling the concept of absolute continuity and introducing that of singularity, for two measures ν and μ (see [Taylor, Equation (6.4.2)] for the written definition in the case ν is a signed measure). (Note that for positive measure $\perp$ is a symetric relation while $\ll$ is not).
- 8:40 The Radon Nykodym Theorem [Taylor, Theorem 6.7]: Given a positive measure μ, a signed measure ν can be decomposed into the sum of a part which is absolutely continuous with respect to μ and a part which is singular with respect to μ. The continuous part is of the form μ_f (recall the notation in Lecture 16 [20:30], here note that f is not necessarily integrable, but we have $\int_\Omega f_- \mathrm{d}\mu < \infty$, so that the integral always makes sense).
- 11:20 Sketch of the proof.
- 16:42 Step 1: Assuming that μ and ν are finite, construction of a function g that maximizes $\mu_g(\Omega)$ integral among all those such that $\mu_g(A) \le \nu(A)$ for all $A \in \mathcal{F}$.
- 27:58 Step 2: Showing that $\nu_2 = \mu - \mu_g$ is singular with respect to μ (completing the proof when μ and ν are finite).
- 41:30 Extension to the σ-finite case.
- 50:33 Extenstion to the case where ν is a signed measured.For this case it is quite important to notice that since $\theta_2(\Omega) < \infty$, $\int f_2^1 \mathrm{d}\mu < \infty$. This makes $\int_A f_1^1 \mathrm{d}\mu - \int_A f_2^1 \mathrm{d}\mu$ well defined in every case (and this can be taken as a definition for $\int_A f^1 \mathrm{d}\mu$)
- 57:40 Uniqueness of the decomposition.
- 1:05:13 Last comments. Uniqueness of the function f (called the density or Radon-Nikodym derivative) modulo modification on sets of μ-measure 0.

Recommended exercises.

a. Let f and g be integrable functions. Show that $\mu_f = \mu_g$ (these are measures defined $\mu_f(A) := \int_A f \mathrm{d}\mu$) are equal if and only if $f = g$, μ-a.e.

b. Let μ and ν be two signed measure on $\mathbb{R}$, we let $|\mu|$ and $|\nu|$ denote their total variation (cf. Exercise b. of Lecture 20). Show that the two following properties are equivalent (in the both statements A, B, C and D are elements of $\mathcal{F}$)

$$\exists A, \quad |\mu|(A) = |\nu|(A^{\mathrm{c}}) = 0.$$

$$\exists B,\ \forall C \subset B,\ \forall D \subset B^{\mathrm{c}},\ \mu(C) = \nu(D) = 0.$$

Either property can be taken as the definition of $\mu \perp \nu$ in the case of two signed measures.

c. Using the notation of the previous exercise show that the two following properties are equivalent

$$\forall A, \quad |\mu|(A) = 0 \quad \Rightarrow \quad |\nu|(A) = 0.$$

$$\forall B,\ (\forall C \subset B,\ \mu(C) = 0) \ \Rightarrow\ (\forall D \subset B,\ \nu(D) = 0)\,.$$

Either property can be taken as the definition of $\nu \ll \mu$ in the case of two signed measures.

d. Exercises 1,5,6,7,8 and 9 in [Taylor, Section 6.4].

Suggested exercises.

a. Exercise 2,3,4,10 and 11 [Taylor, Section 6.4].

Lecture 22: Almost sure and almost uniform

Summary. This lecture is based on [Taylor, Section 7.1]

Content and Comments.

- 0:00 The objective of the lecture is the introduction of different type of convergences for sequences of measurable function. 1- Almost-everywhere convergence 2-Almost-everywhere uniform convergence 3-Almost uniform convergence.
- 0:58 σ-complete here is a typo and should be μ-complete
- 1:12 Introduction of equivalence classes of functions which coincide a.e. This is of fundamental importance! The corresponding quotient space is the right framework to have uniqueness of limit.
- 4:43 Recalling pointwise convergence.
- 6:00 Almost sure convergence: definition and proof that this is a *class property*.
- 14:18 Uniqueness of the limit *as an equivalence class*.
- 17:18 Example of a function that converges a.e. but not pointwise. Note that in fact the example given has a pointwise limit: the function that is equal to 0 on $[0, 1)$ and equal to 1 at 1. This is not always the case (see exercise below).
- 20:00 Uniform convergence and a.e. uniform convergence.
- 24:31 Introduction of the essential supremum and basic properties. In the video the definition is given only for $|f|$ but it is be the same for f.
- 42:17 Proof that the essential supremum of $|f - g|$ induces a distance (it is fact a norm) on the equivalence class of functions (which is a vector space, see below) and that uniform convergence a.e. corresponds to the convergence for the topology induced by this distance.
- 1:06:37 Observation: The set $\mathcal{L}^\infty$ of class of functions such that $\operatorname{ess\,sup}|f| < \infty$ is a vector space, for which $\operatorname{ess\,sup}|f|$ is a norm (proved but not said).
- 1:09:00 Convergence can be a.e. without being uniform.
- 1:11:20 Definition of almost uniform convergence.
- 1:14:58 Discussing easy implications between notions of convergence.
- 1:20:20 Presentation of Egorov's Theorem [Taylor, Theorem 7.1]: Almost uniform convergence is equivalent to almost sure convergence for finite measure (statement and proof).

Recommended exercises.

a. Let $(\Omega, \mu, \mathcal{F})$ be a measured space with $\mathcal{F}$ being μ-complete. Let $\mathcal{L}^0$ be the vector space of real valued measurable functions (the values $\pm\infty$ are not authorized).

$$\mathcal{H} := \{f \in \mathcal{L}^0 : \ \mu(\{x : |f(x)| > 0\}) = 0\}.$$

Show that $\mathcal{H} \subset \mathcal{L}^0$, and that $\mathcal{H}$ is a vector space. (The set of *class of functions* $\mathcal{M}$ which are finite a.e. corresponds to the quotient vector space $L^0(\mu) := \mathcal{L}^0/\mathcal{H}$).

b. Exercises 1, 2 (assuming that μ is σ-finite) 3, 4, 7 in [Taylor, Section 7.1].

Suggested exercises.

a. Exercises 5, 6 and 8 in [Taylor, Section 7.1].

Lecture 23: Convergence in measure

Summary. This lecture is based on [Taylor, Section 7.2]

Content and Comments.

- 0:00 The aim of the lecture is to introduce another meaningful notion of convergence for (class of)-measurable functions: convergence in measure.
- 1:58 Example of a sequence converging in measure but not almost-surely.
- 13:05 Observations: We have uniqueness of the limit (in the space of equivalence class) and convergence is a class property.
- 21:00 Convergence in measure implies that a.e. convergence holds along a subsequence. (Illustration with an example and proof).
- 36:35 In the case of finite μ, almost-sure convergenge implies convergence in measure. This is not true if $\mu(\Omega) = \infty$.

Further Readings.

A. In [Taylor, Theorem 7.2] it is shown that the space $L^0(\mu)$ is complete for the topology of convergence in measure.

Recommended exercises.

a. Let $(\Omega, \mu, \mathcal{F})$ be a measured space with $\mathcal{F}$ being μ-complete and such that $\mu(\Omega) < \infty$. We define d_0 on $L_0(\mu)$ (cf. previous Lecture) by

$$d_0(f, g) = \int \min(|f - g|, 1) \mathrm{d}\mu.$$

Show that d_0 is a metric and that $f_n \to f$ in measure if and only if

$$\lim_{n\to\infty} d_0(f, f_n) = 0.$$

b. Let $(\Omega, \mu, \mathcal{F})$ be a measured space with $\mathcal{F}$ being μ-complete. We define d_1 on $L_0(\mu)$ (cf. previous Lecture) by

$$d_1(f, g) = \inf_{\delta > 0} \left\{ \mu(|f - g| > \delta) + \delta \right\}.$$

That is show that d_1 is a metric and that $f_n \to f$ in measure if and only if

$$\lim_{n\to\infty} d_1(f, f_n) = 0.$$

c. Exercises 3, 4, 7 in [Taylor, Section 7.2]

Suggested exercises.

a. We say that a sequence of measurable functions (f_n) converges *locally in measure* to f if for every $F \subset \Omega$ such that $\mu(F) < \infty$ and every $\varepsilon > 0$ we have

$$\lim_{n\to\infty} \mu(\{x \in F \ : \ |f_n(x) - f(x)| > \varepsilon\}) = 0.$$

Show that if the measure μ is σ-finite then the two following statements are equivalent

(i) (f_n) converges locally in measure to f.

(ii) From every subsequence of (f_n) we can extract a subsubsequence that converges almost surely to f.

b. Exercises 1, 2, 5, 6,7 in [Taylor, Section 7.2].

Lecture 24: Hölder and Minkowski inequalities

Summary. This lecture is based on [Taylor, Section 7.4]

Content and Comments.

- 0:00 The aim of the lecture is to prove very useful inequalites between intergral. The first which is presented is Hölder inequality.
- 4:52 Proof of Hölder inequality, and discussion about convex conjugation. The proof of Young's inequality: $xy \leq \frac{1}{p}x^p + \frac{1}{q}y^q$ could be achieved directly by studying the variation of the function $x \mapsto \frac{1}{p}x^p + \frac{1}{q}y^q - xy$, but the connection with Legendre Transform is an important cultural point and gives a generalization of the inequality.

  A more direct path to Hölder inequality can be to first to prove it in the special case $\|f\|_p = \|g\|_q = 1$ (using Young inequality) and then to apply it to $\bar{f} = f/\|f\|_p$ and $\|g\|_q = g/\|g\|_q$ to conclude (treating the case $f = 0$ and $g = 0$ separately).
- 29:05 Proof of the special case $p = 1$.
- 33:15 Minkowski Inequality. This is the triangle inequality in the normed vector space $L^p(\mu)$.
- 44:20 Observation that $\|f\|_p$ is a pre-norm on the vector space $\mathcal{L}^p(\mu)$ of functions such that $|f|^p$ is integrable. (In the definition it is important to define $f : \Omega \to \mathbb{R}$, because the values $\pm\infty$ are problematic to define the sum).

Further Readings.

A. For the definition of L^p spaces as quotient of vector spaces, we refer to [Bogachev, Chapter 4] (Note that in [Bogachev], like in these comments the quotient space is denoted by L^p and $\mathcal{L}^p$ is used for the corresponding space of measurable function. This convention is the most used)

Recommended exercises.

a. Proof the Minkowski inequality for $p = 1$.
b. Consider $L^p(\mu)$ to be the quotient vector space $\mathcal{L}^p(\mu)/\mathcal{H}$. Show that $\|\cdot\|_p$ define a norm on $L^p(\mu)$.
c. Exercises 1,3 and 5 in [Taylor, Section 7.4]

Suggested exercises.

a. Exercises 2, 4 and 6 in [Taylor, Section 7.4].

Lecture 25: L^p Spaces

Summary.

This lecture is based on [Taylor, Section 7.3]

Content and Comments.

0:00 Introduction of L^p as a vector space and definition of L^p convergence.
6:00 Examples of functions which go to zero almost surely, or even uniformly, but do not converge in L^p.
10:57 Introduction of Cauchy sequence, and statement of the main theorem of the lecture [Taylor, Theorem 7.3]: L^p is a complete space (every Cauchy sequence converge). Since L^p it is a normed vector space, it is a Banach space.
12:38 First step of the proof of the Theorem, extraction of a subsequence of f_n that converges almost surely to some limit f.
34:55 Second step: The limit f is in L^p.
41:58 Proof that $f_n \to f$ in L^p.

Further Readings.

A. Section [Bogachev, Sections 4.2 to 4.3] present some interesting properties of L^p spaces that are not mentioned in this course. The density of the set of compactly supported function and separability of L^p for the Lebesgue measure on $\mathbb{R}^d$ and $p < \infty$, and the Hilbert structure of L^2.

Recommended exercises.

a. Prove that convergence in L^p implies convergence in measure.
b. Prove that $f_n \to f$ in L^p implies $\lim_{n\to\infty} \|f_n\|_p = \|f\|_p$
c. For $f, g \in L^2$, let us define

$$\langle f, g\rangle = \int fg \, \mathrm{d}\mu.$$

Show that that $\langle \cdot, \cdot \rangle$ is defines a scalar product on $L^2(\mu)$ and that the associated norm in $\| \cdot \|_2$. Note that in that case, Hölder inequality simply corresponds to the Cauchy-Schwartz inequality for $\langle \cdot, \cdot \rangle$.
d. Exercise 9 [Taylor, Section 7.3]. Also: Prove that if $1 \le p < q < r \le \infty$ then $L^p \cap L^r \subset L^q$.
e. Consider $f \in L^p(\mu)$, $p < \infty$. Consider the set

$$\Omega_f := \{x \in \Omega \ : \ |f(x)| > 0\}.$$

Show that the restruction of μ to Ω_f is σ-finite, or in other words, that there exists a sequence of sets $(A_n)_{n\ge 0}$ such that $\bigcup_{n\ge 0} A_n = \Omega_f$ and $\mu(A_n) < \infty$. Find a counter-example to the above property when $p = \infty$.

Lecture 26: From convergence in measure to convergence in L^p

Summary. This lecture is based on [Taylor, Section 7.3]

Content and Comments.

- 0:00 The aim of the lecture is to provide a necessary additional condition so that convergence in measure implies convergence in L^p. This start with the definition of equicontinuity at $\emptyset$.
- 3:30 Second definition: Uniform absolute continuity.
- 5:42 Recalling previous examples to show that neither is equicontinuous at $\emptyset$.
- 14:00 Main result of the lecture [Taylor, Theorem 7.5]: convergence in measure, plus equicontinuity at the empty set of the family of measures $|f_n|^p \mathrm{d}\mu$ implies convergence in L^p (The proof presented in the video assumes σ-finiteness of μ, but this is not actually necessary: one can consider the restriction of μ to the union of the supports of f_n, $\Omega' := \{x : \exists n, |f_n(x)| > 0\}$ Exercise e. of the previous lesson shows that the restriction of μ to Ω' is σ-finite.
- 16:00 Main technical lemma: Equicontinuity at $\emptyset$ plus absolute continuity w.r.t. μ, implies Uniform absolute continuity
- 25:00 First step of the proof: use of σ-finiteness to reduce oneself to a finite subset of Ω.
- 27:54 Step 2: Proof that the sequence is Cauchy in L^p.
- 42:00 Step 3: Conclusion using completeness in L^p (the claim that L^p convergence implies convergence in measure does not appear in previous videos from what I could check but is given in [Taylor, Theorem 7.4]).
- 43:00 Final observations : Two criteria to verify equicontinuity at the emptyset.

Further comments. An alternative path to prove the result it to start with proving that equicontinuity at $\emptyset$ and convergence in measure to f implies that f belongs to L^p (using Fatou, it is sufficient to prove that $\|f_n\|_p$ is bounded, this is a suggested and quite challenging exercise). Then one can prove directly that $\int |f_n - f|^p \mathrm{d}\mu$ converges to zero instead of proving Cauchyness.

Recommended exercises.

a. Find a direct (and short) proof of each of the corollaries presented at the end of the section.

b. Prove that if $(f_n)_{n\geq 0}$ converges in L^p then $(|f_n|^p)_{n\geq 0}$ is uniformly integrable.

c. Prove that if $f_n \to f$ in measure and $\|f_n\|_p \to \|f\|_p$, then $f_n \to f$ in L^p. Tip: Use Fatou's Lemma for the positive function

$$g_n = 2(|f_n|^p + |f|^p) - |f - f_n|^p.$$

While the above result might look like much more powerful tool than [Taylor, Theorem 7.5] to check convergence in L^p. This is not the case: In practice, it can happen that f_n is shown to converge in measure by some abstract argument and that nothing is known about the limit, so that $\|f\|_p$ cannot be computed.

d. Show that if the family of measures $\{\nu_i, i \in I\}$ is uniformly absolute continuous w.r.t. μ, then for all i, $\nu_i \ll \mu$.

e. Show that if $\nu(\Omega) < \infty$ and $\nu \ll \mu$, then $\{\nu\}$ is is uniformly absolute continuous w.r.t. μ. Find a counter example when $\nu(\Omega) = \infty$.
e. Show that in the first example mentioned at [5.42] $|f_n|^p \mathrm{d}\mu$ is not uniformly absolutely continuous w.r.t. Lebesgue, while the second one is.
f. Exercises 1, 2, 3, 4, 5 in [Taylor, Section 7.3]

Suggested exercises.

a. Using Exercise (d) of the previous lesson, show that [Taylor, Theorem 7.5] is valid even when μ is not σ-finite. (It seems that neither Taylor nor the video mentions this assumption in the theorem but both use it in the proof)
b. Exercises 6, 7, 8 and 10 in [Taylor, Section 7.3]

Lecture 27: Dual of L^p

Summary. This lecture is based on [Taylor, Section 8.5]

Content and Comments.

- 0:00 The aim of this lecture is to prove that when $1 \le p < \infty$ and μ is σ-finite the dual space associated with L^p is isometric to L^q where $q \in (1, \infty]$ is the conjugate of p when. (The result also true for $p \in (1, \infty)$ without the σ-finiteness assumption). It starts by introducing the notion of bounded linear operator (see exercise below).
- 11:37 Proof that for $p \in (1, \infty)$ any function in L^q corresponds to a linear form on L^p with the same norm. (This is also true when $p = 1$ if μ is *semi-finite*, see exercise below)
- 32:54 Statement of the main result: the dual space of L^p is isometric to L^q for $p \in [1, \infty)$ when μ is σ-finite.
- 36:20 Starting with the case $\mu(\Omega) < \infty$. Step 1: Using indicator functions, show that an element T of the dual space corresponds to measure which is absolutely continuous w.r.t. μ, and hence (Step 2) to a function g.
- 58:35 Step 3 : Proof that $T(f) = \int fg \mathrm{d}\mu$ for every $f \in L^p$.
- 1:16:13 Step 4: $g \in L^q$.
- 1:28:00 Step 5: Extension to the σ-finite case.

Further Readings.

A. [Bogachev, Section 4.4] contains a developed discussion concerning the cases not covered by the main Theorem in this lecture: the case of $p = \infty$ (the construction of linear forms on L^∞ which are not in L^1 requires the axiom of choice via the use of Hahn-Banach Theorem), extension to the non σ-finite case for $p \in (1, \infty)$.

B. The case of $p = 1$ and μ non σ-finite is extremely delicate. Necessary and sufficient condition to have $L^1(\mu)^\star = L^\infty(\mu)$ or $L^1(\mu)^\star \subset L^\infty(\mu)$ are given in [Fremlin, Theorem 243G].

Recommended exercises.

a. Let $(V, \|\cdot\|)$ be a normed vector space let $T : V \to \mathbb{R}$ be a linear application. Show that T is continuous if and only if

$$\|T\| := \sup_{\substack{u \in V \\ \|u\| \le 1}} |T(u)| = \sup_{\substack{u \in V \\ \|u\| = 1}} |T(u)| < \infty.$$

Show that the set of continuous linear application forms a vector space, and that $\|T\|$ defined above is a norm on that vector space. Continuous linear application are commonly referred to *linear forms*. The set of linear forms on V, equipped with the norm above, is called the *topological dual* of V (often denoted as $V^\star$).

b. Exercises 1 and 2 in [Taylor, Section 8.5].

Suggested exercises.

a. We say that a measure μ on $(\Omega, \mathcal{F})$ is *semifinite* if for every $A \in \mathcal{F}$ such that $\mu(A) = \infty$ there exists $B \in \mathcal{F}$, $B \subset A$ which satisfies $\mu(B) \in (0, \infty)$. Show that if μ is semi-finite and $f \in L^\infty(\mu)$, then the linear form T_f on $L^1(\mu)$ defined by $T_f(g) = \int fg \, \mathrm{d}\mu$ satisfies $\|T_f\| = \|f\|_\infty$.

Lecture 28: Vitali's covering lemma

Summary. This lecture is based on [Taylor, Section 9.1]

Content and Comments.

0:00 The main objective of the lecture is to present a technical lemma [Taylor, Theorem 9.1] which allows to extract from the covering of a set E by intervals of arbitrarily small size, a family by *disjoint* intervals which almost covers E (in the sense that only a set of zero measure is left uncovered. This result will be then used for practical application in the next lecture.

6:10 Preliminary to the proof: reduction to the case where E is a bounded set, considering the intersection with $(k, k+1)$, $k \in \mathbb{Z}$ and showing that the covering of E contains a covering of $E \cap (k, k+1)$ by intervals contained in $(k, k+1)$ satisfying the same main property.

19:20 Construction of the covering by disjoint intervals $(I_n)_{n\geq 0}$ by induction.

34:50 Proof that the sequence of disjoint interval previously constructed covers almost all the set E. The main argument is that $E \setminus \bigcup_{j=1}^{N} I_N$ has to be covered by $\bigcup_{j\geq N+1}^{N} K_j$ where K_n is an interval which is 5 times the length if I_n.

51:22 First remark: Relaxation of the assumption that the invervals in the covering must be closed.

55:55 Second remark: Considering E bounded, we can cover most of it (that is leave a piece of exterior measure smaller than ε) with a *finite* number of disjoint intervals.

Recommended exercises.

a. Exercises 8 and 9 in [Taylor, Section 9.1]

Lecture 29: Differentiability of functions of bounded variations

Summary. This lecture is based on on [Taylor, Section 9.1]

Content and Comments.

- 0:00 The aim of the lecture is to show that every function with bounded variation is differentiable almost everywhere. It starts with the case of monotone increasing function. First the definition of differentiability is recalled using four notions of upper/lower derivative on the left/right. And the main statement of the lecture (which is [Taylor, Theorem 9.2]). (Here is would be natural to include in the definition the fact that the limit is finite! see exercise a.).
- 7:00 Step 1: decomposition of the set E^c into a countable union of sets $E_{s,t}$.
- 12:21 Step 2: Construction of an approximate covering of $E_{s,t}$ by a finite number of intervals $(I_j)_{j=1}^M$. on which the growth rate of f is smaller than s (using Vitali's Covering Theorem). (at [16:08] when defining the covering, h_k should simply be h, the same is true for r_k at [31:20])
- 25:00 Step 3: Construction of an approximate covering of $E_{s,t} \cap (I_j)_{j=1}^M$ by intervals $(J_k)_{k=1}^M$ on which the growth rate of f is larger than t.
- 41:15 Final step: Using the property of the two covering, one obtains a contradiction concerning the sum of the increments of F along the interval J_k, unless $\lambda(E) \leq 2\varepsilon$.
- 51:30 Remaining part of [Taylor, Theorem 9.2]: the increment of a non-decreasing function over an interval is smaller than the integral of its derivative.
- 59:30 Introduction of the notation of total variation (and of bounded variation function).
- 1:08:00 The increment of $f(b) - f(a)$ is the difference between the positive and negative total variation on this interval.
- 1:16:31 As a consequence of the previous item, bounded variation function on an interval can be written as the difference of two non-increasing functions and are thus differentiable almost-everywhere.

Further Readings. The lecture provides a condition for function to be differentiable almost everywhere but does not provide examples of functions that are almost nowhere differentiable. Of course, it is quite easy to find non-continuous examples such that $\mathbf{1}_{\mathbb{Q}}$, but continuous examples also exists the first of which was discovered by Weierstrass. We refer to [Falconer, Example 11.3] for a short proof that the Weierstrass function is nowhere differentiable.

Recommended exercises.

a. Using Vitali's Lemma show that if f is increasing the set

$$E := \left\{ x \ : \ \limsup_{h \to 0+} \frac{f(x+h) - f(x)}{h} = \infty \right\}$$

has measure zero. Note that alternatively, this fact can be deduced from the inequality $\int_a^b f'(x) \mathrm{d}x = f(b) - f(a)$.

b. Show that Monotone functions have bounded variation and that the set of functions with bounded variations on $[a, b]$ form a vector space (in particular the difference of two monotone function has bounded variation).

c. Exercise 2 in [Taylor, Section 9.2]

Lecture 30: Absolutely continuous functions

Summary. This lecture is based on [Taylor, Section 9.2].

Content and Comments.

- 0:00 The main goal of this lecture is to provide a characterization of the bounded total variation functions which coincide a.s. with the integral of their derivative. The introduction of the lecture outline of property that function of the type $F(x) = C + \int_a^x f(u)\mathrm{d}u$ have.
- 9:10 Definition of an absolutely continuous function (which is the property previously outlined).
- 11:55 Statements of the two main results presented in the lecture (which together form [Taylor, Theorem 9.3 and 9.4]): For $F(x)$ defined above $F' = f$, and reciprocally any function which is absolutely continuous can be written as $C + \int_a^x f(u)\mathrm{d}u$ for some F.
- 14:49 First step of the proof of Theorem 1: reduce to the case where f is non-negative using linearity.
- 18:04 Second step: Proof that $\int_a^x F'\mathrm{d}\lambda = F(x) - F(a) = \int_a^x f\mathrm{d}\lambda$.
- 40:29 Step 3: Proof that $\int_A F'\mathrm{d}\lambda = F(x) - F(a) = \int_A f\mathrm{d}\lambda$ for every Borel set A using a monotone class argument.
- 48:25 First step of the proof of Theorem 2: Proof that any absolutely continuous function has bounded variation.
- 1:02:54 Second step: Considering the function $f(x) - G(x) = f(x) - f(a) - \int_a^x f'\mathrm{d}\lambda$, one reduces the proof to the case of function with zero derivative.
- 1:09:45 Proof that an absolutely continuous function with zero derivative is constant.

Further Readings.

A. The notion of "set density" discussed at the end of [Taylor, Section 9.2] and brings

Recommended exercises.

a. Let F a function which is almost everywhere differentiable and such that F' is integrable on $[a, b]$. Show that if $F(x) - F(a) = \int_a^x F'(u)\mathrm{d}u$ for every $x \in [a, b]$, then F is absolutely continuous on $[a, b]$. This shows that being absolutely countinuous is equivalent to satisfying the fundamental theorem of calculus.

b. Exercises 1 and 6 [Taylor, Section 9.2].

Suggested exercises.

a. Exercises 3 and 4 [Taylor, Section 9.2].

Lecture 31: Decomposition of distribution functions

Summary. This lecture is based on [Taylor, Section 9.3]

Content and Comments.

- 0:00 The main objective of this lecture is to prove that any right-continuous increasing function F can be decomposed into the sum of three parts (which are unique up to an additive constant) a jump function F_1, a continuous function with zero derivative a.e. F_2 and an absolutely continuous function F_3. This is [Taylor, Theorem 9.5]
- 1:50 Introduction of the notion distribution function F_μ for a Borel measure μ on $\mathbb{R}$ and main properties of F_μ. (Here the correct assumption to take on μ is not σ-finiteness but local finiteness! Indeed there is absolutely no reason for $\mu((0,x])$ to be finite if one only assumes σ-finiteness (see comments below and exercise c.). The assumption in $\mathcal{M}$ is $\mu((-\infty,x]) < \infty$ for every $x \in \mathbb{R}$, or equivalently μ is locally finite AND $\mu((-\infty,0])) < \infty$.
- 15:02 Description of the inverse transformation, which, to a right-continuous function, associates a measure.
- 20:06 Introduction of the notion of atom for a measure and proof that the set of atoms for a locally finite measure μ is countable (Here the proof is given for locally finite measures on $\mathbb{R}$ but deep down the argument does not use anything beyond σ-finiteness and is valid for a σ finite measure on an arbitrary space Ω.) .
- 31:45 Introduction of the notion of jump function and proof that they correspond (with the previous transformation) to *discrete* or *purely atomic* measures.
- 44:33 Proof that any measure can be decomposed into a discrete part and a part with no atoms.
- 51:35 Remark: Measures with no atoms corresponds to continuous (increasing) functions.
- 57:45 Measure which are absolutely continuous w.r.t. Lebesgue corresponds to functions which are absolutely continuous.
- 1:03:16 Definition of singular functions (continuous functions whose derivative is equal to zero a.e.) and proof that they correspond to non-atomic measures which are singular w.r.t. Lebesgue.
- 1:08:50 Here $\nu \le \mu_F$ is a shorthand notation to say that $\nu(A) \le \mu_F(A)$ for every Borel set A.
- 1:21:43 Every increasing continuous function can be decomposed into a sum of an absolutely continuous function and that of singular function.
- 1:24:15 Conclusion: Statement of the main Theorem (which follows from the combination of the Lemmas proved in the lecture).

Further Comments.

Given Ω a topological space equipped with its Borel σ-algebra, a measure μ on Ω is said to be *locally finite* if for every $x \in \Omega$ there exists an open neighborhood of x, N_x such that $\mu(N_x) < \infty$. Local finiteness implies in particular that the measure of every compact set is finite. In the case where $\Omega = \mathbb{R}$ a Borel measure μ on $\mathbb{R}$ is finite if and only if the measure $\mu([a,b]) < \infty$ for every $a < b$.

Further Readings.

A. The notions surveyed in this lecture are very much related to that of Lebesgue-Stieljes integral introduced in [Taylor, Section 4.5]

Recommended exercises.

a. Prove the claim that for a general topological space Ω, μ locally finite implies that $\mu(K) < \infty$ for any compact K.
b. Prove that a measure μ on $\mathbb{R}$ which is locally finite is σ-finite.
c. Let μ be the Borel measure defined by $\mu(A) := \int_A \frac{1}{|x|}\mathrm{d}x$. Show that μ is σ-finite but not locally finite. Prove that the same is true for ν defined by $\nu(A) = \#\{A \cap \mathbb{Q}\}$
d. Prove that for $F \in \mathcal{F}$ the functional defined by $\mu_F((a, b]) = F(b) - F(a)$ on the semi-algebra of half-open intervals is σ-additive (one can use the method of Lecture 6 or that of Lecture 7 but in both cases some adaptation is required).
e. Prove that the decomposition is unique if (using the notation introduced in the video) one assumes F, F_1, F_2, $F_3 \in \mathcal{F}$.
f. Prove that main theorem remains valid if one does not assume that $F(-\infty) = 0$, and that we have unicity of the decomposition if one assumes that $F_1(0) = 0$ and $F_2(0) = 0$.
g. Exercises 1 and 2 [Taylor, Section 9.3] (An easier version of exercise 2 is given in the next lecture)

Suggested exercises.

a. Exercise 3 [Taylor, Section 9.3].

Lecture 32: Cantor ternary set and function

Summary. This lecture is based on [Taylor, Sections 2.7 and 4.4]

Content and Comments.

0:00 The aim of the lecture is to present the construction of: (1) a set which is infinite uncountable (and furthermore: of the same cardinality as $\mathbb{R}$ with zero measure (2) a function which is continuous and increases from 0 to 1 while having zero derivative almost everywhere. These are respectively the Cantor ternary set and the Cantor function (or Cantor staircase). The lecture start with the definition of the Cantor set as well as a proof that it has zero measure.

24:40 Informal presentation of the Cantor staircase function.

29:10 Definition of the Cantor staircase function based on the development in base 3 of x and proof that the expression does not depend on the choice of the development (there are two choices for the development of numbers that are multiples of 3^{-k}).

40:00 Proof that the Cantor function is monotone increasing (in reality non-decreasing).

49:36 Proof that the Cantor function is Hölder continuous with exponent $(\log 2)/(\log 3)$.

1:22:40 Proof that the Cantor function is piecewise constant on each of the intervals of the complement of the Cantor set (The claim that f is constant on E_N made at [1:26:08] is clearly a slip of the tongue).

Further Readings.

A. Both Cantor set and the Cantor function present a degree of *self-similarity*, in the sense that a zooming on a small part of the set (or of the graph of the function), an observer recover a structure similar to the whole set one started with. Such object have called *fractals*. An introduction to the topic which is accessible to master's student is provided in [Falconer].

Recommended exercises.

a. Let g_1 be the function defined on $[0,1]$ as

$$g_1(u) := \begin{cases} 3u/2 & \text{if } u \in [0,1/3], \\ 1/2 & \text{if } u \in (1/2,2/3), \\ (3u-1)/2 & \text{if } u \in [2/3,1]. \end{cases} \tag{0.2}$$

Set for $n \geq 1$

$$g_{n+1}(u) := \begin{cases} \frac{1}{2} g_n(3u) & \text{if } u \in [0,1/3], \\ 1/2 & \text{if } u \in (1/2,2/3), \\ \frac{1}{2}\left(1 + g_n(3u-2)\right) & \text{if } u \in [2/3,1]. \end{cases} \tag{0.3}$$

(1) Show by induction that g_n is continuous.
(2) Show that $\|g_n - g_{n+1}\|_\infty \leq \frac{1}{2}\|g_n - g_{n-1}\|_\infty$.
(3) Deduce from the previous result that $\|g_n - g_{n+1}\|_\infty \leq 2^{-n+1}$.
(4) Prove that g_n converges to a limit g which is continuous.
(5) Prove that the limit $g(0) = 0$, $g(1) = 1$, and that has zero derivative on the complement of the Cantor set.
(6) Prove that g is the Cantor staircase function.

b. Let $\bar{g}$ denote the Cantor staircase function extended to $\mathbb{R}$

$$\bar{g}(u) := \begin{cases} 0 & \text{if } u \leq 0 \\ g(u) & \text{if } u \in [0,1], \\ 1 & \text{if } u \geq 1, \end{cases} \tag{0.4}$$

where g is the Cantor function. Let $(q_n)_{n\geq 1}$ be an enumeration of $\mathbb{Q}$ (a sequence in which each rational number appears only once). We define

$$h(u) := \sum_{n\geq 1} 2^{-n}\bar{g}(u - q_n).$$

Show that h is continuous, strictly increasing and that $h'(u) = 0$ a.e.

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