Statistical Mechanics

Percolation exercise sheet

22 de janeiro de 2018

1 Percolation in one dimension

(1) Consider percolation on $\mathbb{Z}$. Give an explicit expression for $\theta_n(p) := \mathbb{P}_p[0 \leftrightarrow \{-n, n\}]$ and deduce from it the value of $p_c(\mathbb{Z})$.

(2) Consider now percolation on the graph $G_k := \mathbb{Z} \times [\![1, k]\!]$ for $k \geqslant 2$ (the subgraph of $\mathbb{Z}^2$ induced by the set of vertices $\mathbb{Z} \times [\![1, k]\!]$). Provide an upper bond for $\mathbb{P}_p\left(0 \leftrightarrow \{-n, n\} \times [\![1, n]\!]\right)$. Deduce from it the value of $p_c(G_k)$.

2 The law of large number for the infinite cluster

In this exercise we want to show that for edge percolation in $\mathbb{Z}^d$ if $p > p_c$ we have the following convergence in probability

$$\lim_{N \to \infty} \frac{1}{(2N+1)^d} \sum_{x \in \Lambda_N} \mathbf{1}_{\{x \leftrightarrow \infty\}} = \theta(p).$$

where $\Lambda_N := [\![-N, N]\!]^d$. Set $F(N, \omega) := \sum_{x \in \Lambda_N} \mathbf{1}_{\{x \leftrightarrow \infty\}}$.

(1) Compute the expectation of $F(N, \omega)$ as a function of $\theta(p)$.

(2) Prove that

$$\lim_{|x| \to \infty} \mathbb{P}_p\left[\{0 \leftrightarrow \infty\} \cap \{x \leftrightarrow \infty\}\right] = \theta(p)^2$$

Indication: prove upper and lower bounds separately, for the upper-bound, try to reduce to the case of independent events.

(3) Prove that for any $\varepsilon > 0$, for $N \geqslant N_0(\varepsilon)$ we have

$$(2N+1)^d \left[\theta(p) - \theta(p)^2\right] \leqslant \mathrm{Var}_{\mathbb{P}_p}(F(N, \omega)) \leqslant \varepsilon N^{2d}$$

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(4) Show that for any $\delta > 0$ we have

$$\lim_{N \to \infty} \mathbb{P}\left[\left|\frac{F(N, \omega)}{(2N+1)^d} - \theta(p)\right| > \delta\right] = 0. \tag{1}$$

3 Galton-Watson tree conditioned to extinction and skeleton tree

Consider the Galton Watson process with offspring distribution given by $\mathbb{P}[X_{1,1} = k] = p_k$ where $(p_k)_{k \geqslant 1}$ satisfies

$$p_0 > 0 \quad \text{and} \quad \sum_{k \geqslant 0} k p_k = \mu > 1.$$

We let $G_X(s) := \sum_{k \geqslant 0} s^k$ denote the associated characteristic function, and $\eta \in (0,1)$ denote the probability of extinction $G_X(\eta) = \eta$.

3.1 Preliminary

Let us consider the following two functions defined on $[0,1]$.

$$G_1(s) = \frac{1}{\eta} G_X(\eta s) \text{ and } G_2(s) = \frac{1}{1-\eta} G_X(\eta + (1-\eta)\eta_s)$$

(1) Justify that G_1 and G_2 have a power series development which converges on the interval $[0,1]$, and find q_k and r_k (give the explicit expression in terms of $(p_k)_{k \geqslant 0}$ and η) such that for all $s \in [0,1]$

$$G_1(s) := \sum_{k=0}^{\infty} q_k s^k \quad \text{and} \quad G_2(s) := \sum_{k=0}^{\infty} r_k s^k. \tag{2}$$

(2) Justify that $(q_k)_{k \geqslant 0}$ and $(r_k)_{k \geqslant 0}$ correspond to probability distribution on $\mathbb{N}$, and give an expression for the mean of the associated variables.

3.2 Tree conditioned to extinction

(1) Compute the probability that the root has k children and that the process eventually extincts, that is the event

$$\{\lim_{n \to \infty} Z_n = 0\} \cap \{Z_1 = k\}.$$

Deduce from it the distribution of Z_1 conditioned to extinction.

(2) Using the random walk representation, prove that conditioned to extinction, the process is a subcritical Gatson-Watson process offspring distribution $(q_k)_{k \geqslant 0}$.

3.3 Skeleton tree

(1) Compute the probability that among the offsprings of the root, k have infinite descent (and the same probability conditioned on non-extinction).

(2) Conditioning the branching process to non-extinction, let us consider

$$\widetilde{Z}_n := \text{ number of individual at generation } n \text{ which have an infinite line of descent }.$$

Show that $\widetilde{Z}_n$ is a Galton-Watson process with offspring distribution $(r_k)_{k \geqslant 0}$.

4 Connectivity threshold for the Erdös Renyie random graph

Consider the percolation process on the complete graph with vertex set $[\![1,n]\!]$, and with parameter p. We let $P_{n,p}$ denote the associated probability. We let $\mathcal{G}(\omega)$ the induced graph. We say that a vertex x is isolated if he is the only element of its connected component $\mathcal{C}(x)=\{x\}$. Let A_i denote the event $\{i \text{ is an isolated vertex in } \Gamma(\omega)\}$

4.1 Lower bound

(1) Compute the probability of A_i as a function of p.

(2) Compute the covariance of $\mathbf{1}_{A_i}$ and $\mathbf{1}_{A_j}$ for $i \neq j$.

(3) Show that given $\varepsilon > 0$ for $p = p_n^1 = (1-\varepsilon)\log n/n$ we have

$$\lim_{n\to\infty} P_{n,p_n^1}(\text{The graph}\mathcal{G}(\omega) \text{ is not connected}) = 1. \tag{3}$$

4.2 Upper bound

(1) For $i_1,\dots,i_k$ distinct, we let $B(i_1,\dots,i_k)$ be the event

$$\{\text{there is no open edge linking } \{i_1,\dots,i_k\}\text{to } [\![1,n]\!]\setminus\{i_1,\dots,i_k\}\}.$$

Compute the probability of $B(i_1,\dots,i_k)$ as a function of p,n and k.

(2) Deduce from the first item that

$$P_{n,p}(\mathcal{G}(\omega)\text{is not a connected graph }) \leqslant \sum_{k=1}^{n/2}\binom{n}{k}(1-p)^{k(n-k)}. \tag{4}$$

(3) Conclude that for $p_n^2 = (1+\varepsilon)\log n/n$

$$\lim_{n\to\infty} P_{n,p_n^2}(\mathcal{G}(\omega)\text{is a connected graph }) = 1. \tag{5}$$

On might use that $\binom{n}{k} \leqslant \left(\frac{en}{k}\right)^k$.