# Probabilidade I. 2017.1

2nd Exercise Sheet.

Due: 5/April/2017

The solutions of at least four of the following exercises should be presented until next Tuesday. Write your solutions as concise and clear as possible.

**1.** Let $X, Y, Z$ be positive independent random variables with a common density $\lambda$. Let $F(t) = \mathbf{P}(X \in (0, t])$. Show that the probability that the polynomial $Xt^2 + Yt + Z$ has real roots is $\int_0^\infty \int_0^\infty F(t^2/4s)\lambda(t)\lambda(s)dsdt$.

**2.** Let $X_1, X_2, ...$ be independent Bernulli variables with the same success probability $p \in (0, 1)$. Define for each $k \in \mathbb{N}$ the time of $k$th success by:

$$T_k := \inf\{n \geq 1 : X_1 + ... + X_n \geq k\}$$

with $T_k = \infty$ if $X_1 + ... + X_n < k$ for all $n$.

- Show that $T_k$ is a random variable for all $k$.
- Show that

$$\mathbf{P}(T_k = n) = \frac{(n-1)!}{(k-1)!(n-k)!}p^k(1-p)^{n-k}$$

  and

$$\mathbf{P}(T_k \leq n) = \sum_{j=k}^{n} \frac{n!}{j!(n-j)!}p^j(1-p)^{n-j}.$$

- Prove $T_k < \infty$ a.s. and conclude that $\lim_{n\to\infty} X_1 + ... + X_n = \infty$ a.s.

**3.** Let

$$\Omega = \{\omega = (\omega_1, \omega_2, ...) : \omega_i \in \{0, 1\} \text{ for all } i \in \mathbb{N}\}$$

be the space of sequences of 0's and 1's and

$$X_n : \Omega \to \{0, 1\}$$

such that $X_n(\omega) = \omega_n$, the canonical $n^{th}$ projection. Let $\mathcal{F} = \sigma(X_1, X_2, ...)$ be the canonical $\sigma$-algebra and let $\mathbf{P}$ be a probability measure on $(\Omega, \mathcal{F})$ such that $X_1, X_2, ...$ are i.i.d. Let $\tau : \Omega \to \Omega$ be the shift operator

$$\tau((\omega_1, \omega_2, ...)) = (\omega_2, \omega_3, ...).$$

Let $A \in \mathcal{F}$ such that $\tau^{-1}(A) = A$

- Give an example of a non-trivial event $A \in \mathcal{F}$ satisfying $\tau^{-1}(A) = A$.
- Let $B \in \sigma(X_1, X_2, ...X_n)$. Show that $A$ and $B$ are independent.
- Show that for all $C \in \mathcal{F}$, $A$ and $C$ are independent.
- Show that $\mathbf{P}(A) \in \{0, 1\}$.

**4.** Let $\Omega$ be a countable set. The **total variation distance** between two probability distributions $\mu$ and $\nu$ on $\Omega$ is defined by

$$||\mu - \nu||_{TV} = \max_{A \subset \Omega} |\mu(A) - \nu(A)|.$$

A **coupling** of $\mu$ and $\nu$ is a pair of random variables $(X, Y)$ defined on a single probability space such that the marginal distribution of $X$ is $\mu$ and the marginal distribution of $Y$ is $\nu$. That is, a coupling $(X, Y)$ satisfies $\mathbf{P}(X = x) = \mu(x)$ and $\mathbf{P}(Y = y) = \nu(y)$ for all $x, y \in \Omega$.

- Show that
$$||\mu - \nu||_{TV} = \frac{1}{2} \sum_{x \in \Omega} |\mu(x) - \nu(x)|.$$
- Show that, if $(X, Y)$ is a coupling of $\mu, \nu$ then
$$||\mu - \nu||_{TV} \leq \mathbf{P}(X \neq Y).$$
- Show that
$$||\mu - \nu||_{TV} = \inf\{\mathbf{P}(X \neq Y) : (X, Y) \text{ coupling of } \mu, \nu\}$$
by constructing a particular coupling that attains the infimum.

**5.** Compute the minimum $n$ such that the probability that there are two people that share the same birthday among a group of $n$ people, is at least $\frac{1}{2}$. What if we want the probability to be at least $\frac{99}{100}$. Assume independence of birthday between different people.

**6.**

- Let $X$, $Y$ random variables that satisfy $X \leq Y$ almost surely. Show that the cumulative distribution $F_X, F_Y$ satisfy, for all $x \in \mathbb{R}$
$$F_X(x) \geq F_Y(x)$$

- Let $\mu$, $\nu$ two probability measures on $\mathbb{R}$ satisfying, for all continuous non-decreasing limited function $g : \mathbb{R} \to \mathbb{R}$,

  $$\int g(x)\mu(dx) \leq \int g(x)\nu(dx).$$

  Show that the cumulative distribution satisfy, for all $x \in \mathbb{R}$,

  $$F_\mu(x) \geq F_\nu(x).$$

- Show the converse of the previous item.

- Also related to the previous item, show that it is possible to construct a coupling (X, Y), defined on the same probability space $\Omega$, such that $X$ has distribution $\mu$, $Y$ has distribution $\nu$, and $X \geq Y$ almost surely.