Probabilidade I. 2017.1

1st Exercise Sheet.

The solutions of at least six of the following exercises should be presented until next Friday. Write your solutions as concise and clear as possible.

Due: 24/March/2017

1.Let $\{A_n\}_{n\geq 1}$ a sequence of events in a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Show the following properties

- $\mathbf{P}\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mathbf{P}\left(A_n\right)$
- $\mathbf{P}\left(\bigcap_{k=1}^{n} A_k\right) \geq 1 - \sum_{k=1}^{n} \mathbf{P}\left(A_k^c\right)$
- If $\mathbf{P}\left(A_k\right) \geq 1 - \epsilon$ for $k = 1, ..., n$, then $\mathbf{P}\left(\bigcap_{k=1}^{n} A_k\right) \geq 1 - n\epsilon$.
- $\mathbf{P}\left(\bigcap_{k=1}^{\infty} A_k\right) \geq 1 - \sum_{k=1}^{\infty} \mathbf{P}\left(A_k^c\right)$
- If $\mathbf{P}\left(A_n\right) = 0$ for all $n = 1, 2, ...$ then $\mathbf{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = 0$.
- If $\mathbf{P}\left(A_n\right) = 1$ for all $n = 1, 2, ...$ then $\mathbf{P}\left(\bigcap_{n=1}^{\infty} A_n\right) = 1$.

2. Show that if $\{A_n\}_{n\geq 1}$ and $\{B_n\}_{n\geq 1}$ are sequences of events in $(\Omega, \mathcal{F}, \mathbf{P})$ such that $\mathbf{P}\left(A_n\right) \to 1$ and $\mathbf{P}\left(B_n\right) \to p$, when $n \to \infty$, then $\mathbf{P}\left(A_n \cap B_n\right) \to p$.

3. Let Ω be a non-empty set.

- Show that if $\mathcal{A}$ and $\mathcal{B}$ are σ-algebras of Ω, then $\mathcal{A} \cap \mathcal{B}$ is also a σ-algebra.
- Show that if $\{\mathcal{A}_i\}_{i\in I}$ are σ-algebras of Ω, where I is a non-empty index set, then $\cap_{i\in I}\mathcal{A}_i$ is also a σ-algebra.
- Let $\mathcal{C}$ be a class of subsets of Ω. Show that there exists at least one σ-algebra that contains $\mathcal{C}$.
- How would you define the smallest σ-algebra that contains a fixed class of subsets $\mathcal{C}$?

4. Let $\{A_n\}_{n\geq 1}$ a sequence of events in a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Define

$$\limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k,$$

$$\liminf_{n\to\infty} A_n = \bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} A_k.$$

- Show that
$$\liminf_{n\to\infty} A_n \subset \limsup_{n\to\infty} A_n$$
.

- Show that if
$$\mathbf{P}(\limsup_{n\to\infty} A_n \setminus \liminf_{n\to\infty} A_n) = 0$$
then the probability of the symmetric difference
$$\mathbf{P}(A\Delta A_n) \to 0$$
as $n \to \infty$ (where A is either $\limsup_{n\to\infty} A_n$ or $\liminf_{n\to\infty} A_n$)

- Show that the converse is true: if the probability of the symmetric difference vanishes, then $\limsup_{n\to\infty} A_n$ and $\liminf_{n\to\infty} A_n$ are equal up to some event of probability 0.

5. For each one of the following experiments describe a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ that suits the model

- Select of a point, at random, from the unit square
$$\{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, 0 \leq y \leq 1\}.$$

- We take a card, at random, from a deck of cards (52 cards), observe the number and then put the card back in the deck. We repeat this until we see an ACE. We record the number of times we perform the operation.

- From an urn containing 5 red balls, 9 black balls and 1 white ball, we pick one ball, at random, observe the color and put the ball back in the urn. We repeat this operation 15 times. We record the number of times each color occurs.

6. Given $n \in \mathbb{N}$, let $[n] := \{1, 2, ..., n\}$. Let $\Omega_n = \{\sigma : [n] \to [n] : \sigma \text{ is bijective}\}$. Let $\mathbf{P}_n$ be the uniform probability measure on $(\Omega_n, \mathcal{P}(\Omega_n))$.

- Compute $p_n := \mathbf{P}_n\left(\sigma(i) = i, \text{ for some } i \in [n]\right)$.

- Show that $\lim_{n\to\infty} p_n = 1 - \frac{1}{e}$.

7. Let $\{A_n\}_{n\geq 1}$ a sequence of events in a probability space $(\Omega, \mathcal{F}, \mathbf{P})$

- Show that if $\{A_1, ..., A_k\}$ are independent, so are $\{A_1, ..., A_{k-1}, A_k^c\}$.
- Show that if $\{A_n\}_{n\geq 1}$ is a collection of independent events, so is $\{B_n\}_{n\geq 1}$, where B_n is either A_n or A_n^c.

8. Let $\Omega := \{0,1\}^n = \{\omega : \{1, ..., n\} \to \{0,1\} : \omega \text{ is a function}\}$ and for each $i = 1, ..., n$ consider $A_i = \{\omega : \omega(i) = 1\}$. Let $P, \hat{P}$ two probability measures on $(\Omega, \mathcal{P}(\Omega))$ such that $P(B) = \hat{P}(B)$ for all sets B given by intersections of A_i. Show that $P = \hat{P}$.