The answer to the question can be written in either English or Portuguese and should be returned by the 3rd of march.

We consider the Ising model with plus boundary condition on $\mathbb{Z}^2$ associated with inverse temperature $\beta > 0$ and magnetic field $h \in \mathbb{R}$ in the box $\Lambda_N := [\![1, N]\!]$. The probability of a configuration $\sigma \in \Omega_N = \{-1,1\}^{|\Lambda_N|}$ is given by

$$\mu^+_{N,\beta,h}(\sigma) := \frac{1}{Z^+_{N,\beta,h}} e^{\beta \sum_{\{x,y\}\in\mathcal{E}^b_N} \sigma_x\sigma_y + h\sum_{x\in\Lambda_N}\sigma_x},$$

where

$$Z^+_{N,\beta,h} := \sum_{\sigma\in\Omega_N} e^{\beta \sum_{\{x,y\}\in\mathcal{E}^b_N} \sigma_x\sigma_y + h\sum_{x\in\Lambda_N}\sigma_x}.$$

with the convention that $\sigma\restriction_{\partial\Lambda_N}\equiv 1$. Recall that $\mathcal{E}^b_N$ is the set of edges with at least one end in Λ_N. We use the notation $\langle\cdot\rangle^+_{N,\beta,h}$ for expectation with respect to $\mu^+_{N,\beta,h}$.

We set $\psi(\beta,h) = \lim N^{-2}\log Z_{N,\beta,h}$ and recall that for every $\beta>0$, $\psi(\beta,h)$ is infinitely differentiable everywhere, except, possibly, at zero. We set $m^*(\beta) = \lim_{h\to 0+}\partial_h\psi(\beta,h)$.

1 A few inequalities

1. Show that for any $x\in\Lambda_N$, $\beta>0$ and $h>0$.

$$\langle\sigma_x\rangle^+_{N,\beta,h} \le \tanh(4\beta+h),$$

   Deduce from it that $m^*(\beta)\le\tanh(4\beta)$.

2. Prove that for any $h\in\mathbb{R}$

$$\left|\langle\sigma_x\rangle^+_{N,\beta,h}\right| \le \tanh(4\beta+|h|),$$

3. Show that for any $h\in\mathbb{R}$.

$$\partial_h\langle\sigma_x\rangle^+_{N,\beta,h} \ge 1-\left(\langle\sigma_x\rangle^+_{N,\beta,h}\right)^2.$$

   Deduce from it that $\psi(\beta,h)$ is strictly convex in h, and that for $h>0$

$$\langle\sigma_x\rangle^+_{N,\beta,h} \ge \tanh(h).$$

4. Show that when $h>0$

$$\partial_\beta\langle\sigma_x\rangle^+_{N,\beta,h} \ge (1-\langle\sigma_x\rangle^+_{N,\beta,h})\sum_{y\sim x}\langle\sigma_y\rangle^+_{N,\beta,h}.$$

   Deduce from it in particular that $\beta\mapsto\partial_h\psi(\beta,h)$ is strictly increasing when $h>0$.

2 Deviation for the magnetization

We defined the magnetization and renormalized magnetization by

$$M_N := \sum_{x\in\Lambda_N}\sigma_x \quad\text{ and }\quad m_N := \frac{1}{N^2}\sum_{x\in\Lambda_N}\sigma_x.$$

We know under $\langle\cdot\rangle^+_{N,\beta,h}$ converges to $\partial_h\psi(\beta,h)$ when $h\neq 0$ and to $m^*(\beta)$ when $h=0$. The aim of this exercise is to estimate the probability of observing an unusual magnetization.

2.1 Large deviation bounds

1. Show that for any $a \in \mathbb{R}$
$$\langle e^{aM_N} \rangle^+_{N,\beta,h} = \frac{Z^+_{N,\beta,h+a}}{Z^+_{N,\beta,h}}$$

2. Prove that for any $u > 0$, $\lambda > 0$
$$\mu^+_{N,\beta,h}(m_N \geq u) \leq e^{-N^2 \lambda u} \frac{Z^+_{N,\beta,h+\lambda}}{Z^+_{N,\beta,h}}.$$

3. Deduce from it that
$$\limsup \frac{1}{N^2} \log \mu^+_{N,\beta,h}(m_N \geq u) \leq -\max_{\lambda > 0} \left(\lambda u - \psi(\beta, h+\lambda) + \psi(\beta, h) \right).$$

4. Prove a similar bound for
$$\limsup \frac{1}{N^2} \log \mu^+_{N,\beta,h}(m_N \leq u).$$

5. From the above deduce that when $m^*(\beta) = 0$ for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ and C_ε such that for all N
$$\mu^+_{N,\beta,0}(|m_N| \geq \varepsilon) \leq C_\varepsilon e^{-\delta(\varepsilon) N^2}.$$
Derive a similar statement for $h \neq 0$.

6. When $m^*(\beta) > 0$, show that
$$\mu^+_{N,\beta,0}(|m_N| \geq m^*(\beta) + \varepsilon) \leq C_\varepsilon e^{-\delta(\varepsilon) N^2}.$$

2.2 On the probability of reversing the magnetization

We assume in this section that $h = 0$, and that the

1. Show that for any N, for any finite sequence of vertices $x_1, \dots, x_k$ in Λ_N, and any sequences $(\varepsilon_1, \dots, \varepsilon_k) \in \{-1, 1\}^k$,
$$\left(\frac{1}{1 + e^{8\beta}} \right)^k \leq \mu^+_{N,\beta}(\sigma_{x_1} = \varepsilon_1, \dots, \sigma_{x_k} = \varepsilon_k) \leq \left(\frac{e^{8\beta}}{1 + e^{8\beta}} \right)^k.$$

2. Show that for every $\delta > 0$ for all N and k sufficiently large we can find $(x_1, \dots, x_k)$ in Λ_N, and $(\varepsilon_1, \dots, \varepsilon_k) \in \{-1, 1\}^k$, such that
$$\mu^+_{N,\beta}(\sigma_{x_1} = \varepsilon_1, \dots, \sigma_{x_k} = \varepsilon_k) \leq \left(\frac{e^{8\beta}}{(1 + e^{8\beta})^2} - \delta \right)^{k/2}.$$

3. Set $\partial^- \Lambda_N := \{x \in \Lambda_N \ : \ \exists y \in \Lambda_N^{\complement}, \ x \sim y\}$. Let show that
$$\langle m_N \mid \sigma_x = -, \ \forall x \in \partial^- \Lambda_N \rangle^+_{N,\beta} = -\frac{4(N-1)}{N^2} + \frac{(N-2)^2}{N^2} \langle m_{N-2} \rangle^-_{N-2,\beta}.$$

4. Show that for an explicit constant C for every $\varepsilon > 0$ and every N sufficiently large
$$\mu^+_{N,\beta}(|m_N + m^*(\beta)| \leq \varepsilon) \geq e^{-CN}.$$

2.3 On the probability of observing an intermediate magnetization

Given $a \in [-m^*(\beta), m^*(\beta)]$. Using the ideas of the previous section as well as some new ones, show that for there exists a constant C such that for every N large enough.
$$\mu^+_{N,\beta}(|m_N - a| \leq \varepsilon) \geq e^{-CN}.$$
(if this is more simple one can restrict to the case $a = 0$)