Random Polymers

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Chapter 1

Introduction and generalities

1 Phenomenology: the wetting transition for a substrate

A polymer is interacting with a colloid or anther type of attractive substrate. We observe the following transition when the temperature varies (see Figure 1.1)

- When the temperature is low, the polymers stick to the colloids and form macroscopic aggregates.
- At high temperature, there are no aggregates.

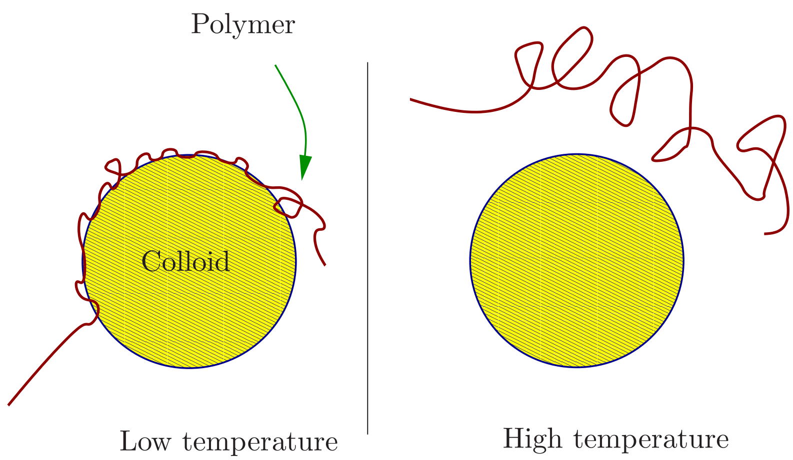

Figure 1.1: Schematic representation of the phase transition of a colloid interacting with a single polymer.

We want to model this phenomenon in the frame work of statistical mechanics with the simplest possible model. We decide to place ourselves in a two dimensional setup:

- The substrate (colloids) are represented by an half-space.
- The polymer is represented by a two dimensional path. To avoid complications due to self-interaction of the polymer we also choose the trajectory to be directed. The polymer will be the graph of a one dimensional random-walk (Figure 1.2) .

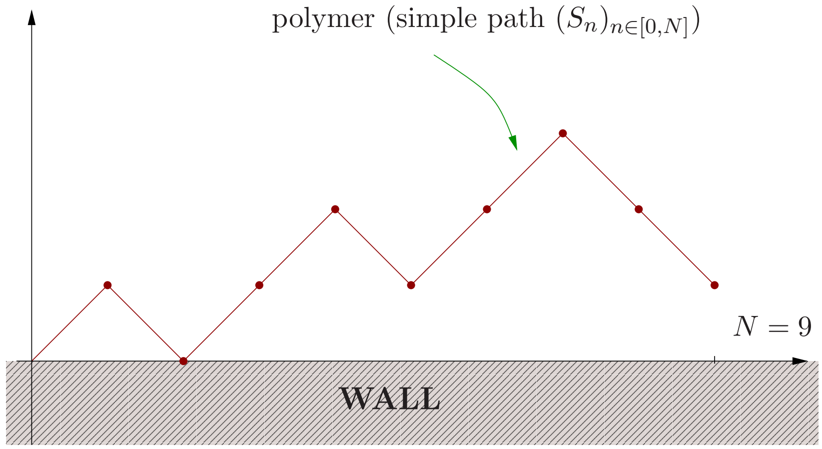

Figure 1.2: Representation of the polymer trajectory as the graph of a function from $[0, N]$ to $\mathbb{Z}_+$. The presence of the colloid is materialized by a solid wall which constrains the trajectory to remain positive.

We let $N \in \mathbb{N}$ denote the size of the system. Our state-space is the space of polymer configuration

$$\mathcal{S}_N^+ := \left\{ S = (S_n)_{n=0}^{2N} \ : \ S_0 = S_{2N} = 0, \forall n \in [\![1, 2N]\!], S_n \geq 0 \text{ and } |S_n - S_{n-1}| = 1 \right\}. \tag{1.1}$$

where for two integers a, b

$$[\![a, b]\!] = \{a, a+1, \dots, b\}. \tag{1.2}$$

The condition $S_0 = S_{2N} = 0$ are boundary conditions. They are needed for instance to have a finite state space, they might seem a bit arbitrary but in the end they do not have a strong effect of the behavior of the system. The condition $S_n \geq 0$ models the presence of a substrate in the bottom half-space.

The state of the polymer is given by a probability measure on $\mathcal{S}_N^+$. We want the interaction with the substrate to appear in the expression of the probability measure Set

$$H_N(S) := \sum_{n=1}^{2N} \mathbf{1}_{\{S_n = 0\}}. \tag{1.3}$$

Given a parameter $\beta \in \mathbb{R}$ we define μ_N^β a probability on $\mathcal{S}_N^+$

$$\mu_N^{\beta,+}(S) := \frac{1}{\widehat{Z}_N^{\beta,+}} e^{\beta H_N(S)}, \tag{1.4}$$

where

$$\widehat{Z}_N^{\beta,+} = \sum_{S \in \mathcal{S}_N^+} e^{\beta H_N(S)}. \tag{1.5}$$

If $\beta > 0$ the measure gives higher probability to paths with more contacts with the wall (this is the contrary for negative β).

For the physicists

$$\beta = -\frac{k_b E}{T}, \tag{1.6}$$

where T is the temperature, k_b is the Boltzman constant and E is the interaction energy of a monomer with the substrate. The term $e^{\beta H_N(S)}$ is often referred to as the Boltzman-weight of the state S. With a small abuse of language, in statistical mechanics β is often referred to as the inverse-temperature. If $E < 0$ (this is the case of an attractive substrate), β close to zero corresponds to high temperature, while β close to infinity corresponds to high temperature.

We would like to be able to show that there exists β_c such that

(a) For $\beta < \beta_c$, under μ_N^β, S typically does not stick to the wall (i.e. S has very few return to zeros).

(b) For $\beta > \beta_c$, under μ_N^β, S typically sticks to the wall (i.e. S has a lot of contact, possibly of order N)

This β_c will be called the critical value for the parameter β. For the moment it is difficult to predict what should be the behavior of the system when β is equal to β_c.

Note that for $N > 0$ and $S \in \mathcal{S}_N^+$ fixed, $\mu_N^{\beta,+}(S)$ is a very regular ($\mathcal{C}^\infty$) function of β, hence there is no chance to observe an abrupt transition when β varies. For this reason, statistical mechanics is interested in the behavior of the system when the size parameter tends to infinity. It turns out that to understand the behavior of $\mu_N^{\beta,+}$, it is essential to understand the asymptotic behavior of the normalizing factor $\widehat{Z}_N^{\beta,+}$ (called the *partition function*).

Let us finally introduce a variant of the model where paths can also visit the negative half-space. It models e.g. a polymer interaction of a polymer with a “crossable” interface.

$$\mathcal{S}_N := \left\{ S = (S_n)_{n=0}^{2N} \ : \ S_0 = S_{2N} = 0, \forall n \in [\![1, 2N]\!], |S_n - S_{n-1}| = 1 \right\}. \tag{1.7}$$

We have

$$\mu_N^\beta(S) := \frac{1}{\widehat{Z}_N^\beta} \exp\left(\beta H_N(S)\right), \tag{1.8}$$

where

$$\widehat{Z}_N^\beta = \sum_{S \in \mathcal{S}_N} \exp\left(\beta H_N(S)\right). \tag{1.9}$$

2 Counting paths, case $\beta = 0$

Let us first try to understand the case $\beta = 0$. We are going to show that under $\mu_N^{0,+}$, the polymer has very few contact with the walls. Note that the partition function $Z_N^{0,+}$ is simply the cardinal of $\mathcal{S}_N^+$ which we denote by $\#\mathcal{S}_N^+$. We will try to get a simple expression and an asymptotic equivalent for it. We use the notation

$$f(N) \overset{N\to\infty}{\sim} g(N),$$

to say that the ratio $f(N)/g(N)$ tends to one.

Let us first deal with the simpler case of $\mathcal{S}_N$. Note that a path in $\mathcal{S}_N$ is made of N *up* $(+1)$ steps and N down (-1) steps. Choosing a path corresponds to choosing the position of the up steps among the $2N$. Hence

$$\#\mathcal{S}_N := \binom{2N}{N} = \frac{2N!}{(N!)^2} \tag{1.10}$$

Using Stirling's formula ($n! \sim \sqrt{2\pi n}(n/e)^n$) we obtain that

$$\#\mathcal{S}_N \overset{N\to\infty}{\sim} \frac{4^N}{\sqrt{\pi N}}. \tag{1.11}$$

The case of $\#\mathcal{S}_N^+$ is a bit more delicate

Lemma 1.1. *We have*

$$\#\mathcal{S}_N^+ = \frac{1}{N+1}\binom{2N}{N}. \tag{1.12}$$

In particular

$$\#\mathcal{S}_N \overset{N\to\infty}{\sim} \frac{1}{\sqrt{\pi}N^{3/2}}4^N. \tag{1.13}$$

Proof. We decide to count the paths that make one more step and end at -1

$$\bar{\mathcal{S}}_N^+ := \Big\{ S = (S_n)_{n=0}^{2N+1} \ : \ S_0 = 0,\ S_{2N+1} = -1, \\ \forall n \in [\![0, 2N]\!], S_n \geq 0 \text{ and } |S_{n+1} - S_n| = 1 \Big\}. \tag{1.14}$$

Obviously $\#\bar{\mathcal{S}}_N^+ = \#\mathcal{S}_N^+$. We define the counterpart of $\bar{\mathcal{S}}_N^+$ without the constraint of being positive.

$$\bar{\mathcal{S}}_N := \Big\{ S = (S_n)_{n=0}^{2N+1} \ : \ S_0 = 0, S_{2N+1} = -1, \forall n \in [\![0, 2N]\!], |S_{n+1} - S_n| = 1 \Big\}. \tag{1.15}$$

Obviously we have

$$\#\bar{\mathcal{S}}_N = \binom{2N+1}{N} = \frac{2N+1}{N+1}\binom{2N}{N}. \tag{1.16}$$

Now we will prove that

$$\#\bar{\mathcal{S}}_N = (2N+1)\bar{\mathcal{S}}_N^+, \tag{1.17}$$

by constructing an explicit bijection $\bar{\mathcal{S}}_N \to [\![1, 2N+1]\!] \times \bar{\mathcal{S}}_N^+$. For $S \in \bar{\mathcal{S}}_N$ and $x \in [\![0, 2N+1]\!]$, we define $\theta_x S$ to be the path whose increments are given by a periodic shift of those of S (note that $\theta_{2N+1} = \theta_0$ simply corresponds to the identity)

$$(\theta_x S)_n = \begin{cases} (S_{n+x} - S_x) & \text{if } n \leq 2N+1-x, \\ (S_{n+x-2N-1} - 1 - S_x) & \text{if } n \geq 2N+2-x. \end{cases} \tag{1.18}$$

It is rather easy to check that $(\theta_x S)$ is still a path with ± 1 increments which ends at -1. Now we are going to check that the for any $S \in \bar{\mathcal{S}}_N$, there is only one value of x which makes $(\theta_x S)$ an element of $\bar{\mathcal{S}}_N^+$.

Given $S \in \bar{\mathcal{S}}_N$, set

$$y(S) := \inf\left\{ n \in [\![1, 2N+1]\!] \ : \ S_n = \min_{m \in [\![0,2N+1]\!]} S_m \right\}. \tag{1.19}$$

Then one can check that

$$\begin{aligned} \bar{\mathcal{S}}_N &\to [\![0, 2N]\!] \times \bar{\mathcal{S}}_N^+, \\ S &\mapsto (y(S), \theta_{y(S)} S). \end{aligned} \tag{1.20}$$

A look at Figure 1.3 will convince the reader that $\theta_{y(S)} S$ is indeed in $\bar{\mathcal{S}}_N^+$. To check that the application is bijective, we just notice that the original path can obtained from the image just by shifting the increments a second time as

$$\theta_{2N+1-y(S)} \theta_{y(S)} S = S.$$

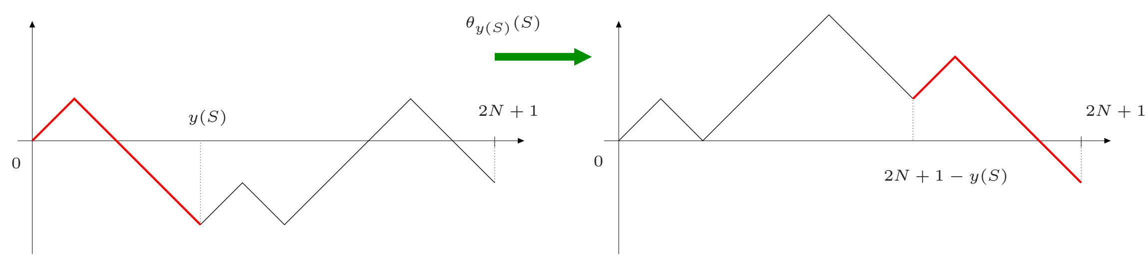

Figure 1.3: Illustration of the transformation $\theta_{2N+1-y(S)}(S)$ which to a path in $\bar{\mathcal{S}}_N$ associates one in $\bar{\mathcal{S}}_N^+$.

□

2.1 Trajectory property: the entropic repulsion phenomenon

Now we want to use the asymptotic expression we have for the partition function to get information on the trajectories. The reader might already know that a simple symmetric random walk on $\mathbb{Z}$, with N steps typically visits 0 of order $\sqrt{N}$ times. We are going to show that this remains the case under the constraint $S_{2N} = 0$.

Proposition 1.2.

$$c\sqrt{N} \le \mu_N^0 \left(H_N(S) \right) \le C\sqrt{N} \tag{1.21}$$

A more advanced study could show that these contact points are "spread on the whole interval".

We are going to show that the constraint of staying in the upper-half space has a drastic consequences on the number of contacts and their repartition.

Proposition 1.3. *The measure $\mu_N^{0,+}$ has the following property for some constant C which does not depend on N:*

(i) We have

$$\mu_N^{0,+}\left(H_N(S)\right) \le C. \tag{1.22}$$

(ii) We have

$$\lim_{N\to\infty} \mu_N^{0,+}\left(H_N(S) = 1\right) \ge \frac{1}{4}. \tag{1.23}$$

(iii) For any $a \in \mathbb{N}$

$$\mu_N^{0,+}\left(\exists n \in [\![a, N-a]\!],\ S_{2n} = 0\right) \le \frac{C}{\sqrt{a}} \tag{1.24}$$

To avoid having to deal with a spurious factor 4^N everywhere, we will now modify our definition of the partition function and of the polymer measure.

We let $\mathbf{P}$ denote the law of the simple symmetric random walk on $\mathbb{Z}$. This is the process $(S_n)_{n\ge 0}$ under which the increments

$$X_n := S_n - S_{n-1}$$

are independent and identically distributed with distribution $\mathbf{P}[X_n = \pm 1] = 1/2$. We redefine

$$Z_N^{\beta,+} := \mathbf{E}\left[e^{\beta H_N(S)} \mathbf{1}_{\{S_{2N}=0 \text{ and } S\restriction_{[\![0,2N]\!]}\ge 0\}}\right] \quad \text{and} \quad Z_N^{\beta} := \mathbf{E}\left[e^{\beta H_N(S)} \mathbf{1}_{\{S_{2N}=0\}}\right]. \tag{1.25}$$

We define $\mathbf{P}_N^{\beta,+}$ as a law on the set of infinite trajectories , which is absolutely continuous with respect to $\mathbf{P}$ and whose density is given by

$$\frac{\mathrm{d}\mathbf{P}_N^{\beta,+}}{\mathrm{d}\mathbf{P}} = \frac{1}{Z_N^{\beta,+}} e^{\beta H_N(S)} \mathbf{1}_{\{S_{2N}=0 \text{ and } S\restriction_{[\![0,2N]\!]}\ge 0\}}. \tag{1.26}$$

where $S\restriction_{[\![0,2N]\!]}$ denotes the restriction of S to the interval $[\![0,2N]\!]$. Similarly we define $\mathbf{P}_N^{\beta}$ by

$$\frac{\mathrm{d}\mathbf{P}_N^{\beta,+}}{\mathrm{d}\mathbf{P}} = \frac{1}{Z_N^{\beta}} e^{\beta H_N(S)} \mathbf{1}_{\{S_{2N}=0\}}. \tag{1.27}$$

This means that for any measurable event A

$$\mathbf{P}_N^{\beta,+}(A) := \frac{1}{Z_N^{\beta,+}} \mathbf{E}\left[e^{\beta H_N(S)} \mathbf{1}_{\{S_{2N}=0 \text{ and } S\restriction_{[\![0,2N]\!]}\ge 0\}} \mathbf{1}_A\right]. \tag{1.28}$$

Note that even though $\mathbf{P}_N^{\beta,+}$ is now formally defined on infinite trajectory, but of course we only care about the behavior of $S\restriction_{[\![0,2N]\!]}$. Note finally that

$$\mathbf{P}_N^{\beta,+}\left[S\restriction_{[\![0,2N]\!]} \in \cdot\right] = \mu_N^{\beta,+}. \tag{1.29}$$

Note that from the previous section

$$Z_N^{0,+} := 4^{-N} \# \mathcal{S}_N^+ = \frac{1}{\sqrt{\pi} N^{3/2}}. \tag{1.30}$$

We define $\mathbf{P}_N^{\beta}$ in the same manner without the positivity constraint. As there are 4^N choices for the first $2N$ steps of N it follows that

$$Z_N^{\beta,+} = 4^{-N} \widehat{Z}_N^{\beta,+} \quad \text{and } Z_N^{\beta} = 4^{-N} \widehat{Z}_N^{\beta} \tag{1.31}$$

Lemma 1.4. *There exists constants c and C such that for all $n \in \{1, \dots, N/2\}$*

$$\begin{aligned} \frac{c}{n^{3/2}} &\le \mathbf{P}_N^{0,+}(S_{2n} = 0) \le \frac{C}{n^{3/2}}, \\ \frac{c}{n^{1/2}} &\le \mathbf{P}_N^{0}(S_{2n} = 0) \le \frac{C}{n^{1/2}}. \end{aligned} \tag{1.32}$$

The result is valid also for $n \in \{N/2, \dots, N\}$ if n is replaced by $N-n$ in the right-hand side.

Proof. We have from the Markov Property from the simple random walk

$$\begin{aligned} \mu_N^{0,+}(S_{2n} = 0) &= \frac{1}{Z_N^{0,+}} \mathbf{P}\left[S_{2N} = 0, S_{2n} = 0 \text{ and } S\restriction_{[\![0,2N]\!]} \ge 0\right] \\ &= \frac{\mathbf{P}\left[S_{2n} = 0 \text{ and } S\restriction_{[\![0,2n]\!]} \ge 0\right] \mathbf{P}\left[S_{2(N-n)} = 0 \text{ and } S\restriction_{[\![0,2(N-n)]\!]}\right]}{Z_N^{0,+}} \\ &= \frac{Z_n^{0,+} Z_{N-n}^{0,+}}{Z_N^{0,+}}. \end{aligned} \tag{1.33}$$

The right-hand side of the above inequality is asymptotically equivalent to $\left(\frac{N}{(N-n)n}\right)^{3/2}$ hence the result. ☐

Proof of Proposition 1.2. It follows immediately from the first point of the Lemma. ☐

Proof of Proposition 1.3. Note that by symmetry

$$\mathbf{E}_N^{0,+}\left(H_N(S)\right) \le \sum_{k=0}^{N/2} \mathbf{P}_N^{0,+}\left(S_{2k} = 0\right) \le C \sum_{k=0}^{N/2} k^{-3/2} \le C'. \tag{1.34}$$

For the second point it is sufficient to see that

$$\mathbf{P}_N^{0,+}\left(H_N(S) = 1\right) = \mathbf{P}_N^{0,+}\left[S_1 = S_{2N-1} = 1, \text{ and } S\restriction_{[\![1,2N-1]\!]} \ge 1\right] = \frac{Z_{N-1}^{0,+}}{4 Z_N^{0,+}}. \tag{1.35}$$

For the third point we simple observe that

$$\mathbf{P}_N^{0,+}\left(\exists n \in [\![a, N-a]\!],\ S_{2n=0}\right) \le 2 \sum_{n=a}^{N/2} \mathbf{P}_N^{0,+}[S_{2n} = 0] \le C' \sum_{k=a}^{N/2} k^{-3/2} \le \frac{C}{\sqrt{a}}. \tag{1.36}$$

. ☐

Chapter 2

The free energy

In this chapter we are going to prove the existence and underline the importance of the free energy (or pressure) defined as

$$\lim_{N\to\infty} \frac{1}{N} \log Z_N^{\beta,+}.$$

1 Definition and importance

Let us, first enumerate some properties of the free energy.

Proposition 2.1. *The following statements hold:*

(i) The quantity

$$\lim_{N\to\infty} \frac{1}{N} \log Z_N^{\beta,+} = \mathrm{F}^+(\beta), \tag{2.1}$$

is well defined.

(ii) $\beta \mapsto \mathrm{F}^+(\beta)$ *is a non-decreasing convex function of* β*.*

(iii) There exists $\beta_c > 0$ *such that*

$$\begin{cases} \mathrm{F}^+(\beta) = 0, & \textit{for } \beta \le \beta_c^+, \\ \mathrm{F}^+(\beta) > 0, & \textit{for } \beta > \beta_c^+. \end{cases} \tag{2.2}$$

Remark 2.2. *We can define* $\mathrm{F}(\beta)$ *the free-energy of the polymer without the positivity constraint as*

$$\mathrm{F}(\beta) := \lim_{N\to\infty} \frac{1}{N} \log Z_N^{\beta}. \tag{2.3}$$

It enjoys the same properties and we define in the same manner

$$\beta_c := \inf\{\beta \in \mathbb{R} \ : \ \mathrm{F}(\beta) > 0\}. \tag{2.4}$$

We will prove the existence of the free energy in the next section, but let us first analyze how one can extract properties for the polymer trajectories from those of the free-energy.

Let us start with a short digression on convexity. We note that

$$\partial_\beta \log Z_N^{\beta,+} = \frac{\partial_\beta Z_N^{\beta,+}}{Z_N^{\beta,+}} = \frac{\mathbf{E}\left[H_N(S)e^{\beta H_N(S)}\mathbf{1}_{\{S_{2N}=0 \text{ and } S\restriction_{[\![0,2N]\!]}\geq 0\}}\right]}{Z_N^{\beta,+}} = \mathbf{E}_N^{\beta,+}(H_N(S)). \tag{2.5}$$

A second differentiation gives

$$\partial_\beta^2 Z_N^{\beta,+} = \mathbf{E}_N^{\beta,+}((H_N(S)^2) - \left[\mathbf{E}_N^{\beta,+}(H_N(S))\right]^2 \geq 0. \tag{2.6}$$

Hence in particular $\log Z_N^{\beta,+}$ is convex β.

Lemma 2.3. *If a sequence of differentiable convex function of $\mathbb{R}$, f_N tends to f then f is convex and at all point where f is derivable (all $\mathbb{R}$ but possibly finitely many points) f_N' converges and*

$$\lim_{N\to\infty} f_N'(x) = f'(x). \tag{2.7}$$

Proof. The fact that f is convex is obvious so we concentrate on proving the convergence of f_N' to f'. Let us assume that f is differentiable at x. Fixing $\varepsilon > 0$ we can find $y_0(\varepsilon)$ such that for all $y \in (x, y_0)$

$$\lim_{N\to\infty} f_N(x) - f_N(y) = f(y) - f(x) \leq (y-x)(f'(x)+\varepsilon). \tag{2.8}$$

By convexity of f_N we have for all $y \geq x$

$$f_N'(x) \leq \frac{f_N(y) - f_N(x)}{y-x}$$

and hence

$$\limsup_{N\to\infty} f_N'(x) \leq (f'(x)+\varepsilon). \tag{2.9}$$

We get an analogous inequality (in the other direction) for the lim inf by considering $y_1 \leq x$ such that conclude

□

This statement is enough to ensure that $\mathrm{F}^+(\beta)$ is convex (and non-decreasing) and we have

$$\partial_\beta \mathrm{F}^+(\beta) = \lim_{N\to\infty} \frac{1}{N}\mathbf{P}_N^{\beta,+}(H_N(S)), \tag{2.10}$$

whenever the r.h.s. us well defined This implies the following

- When $\beta > \beta_c$, the expected number of contact with the wall $\mu_N^{\beta,+}(H_N(S))$ is of order N, we say we are in the localized phase.
- When $\beta > \beta_c$, the expected number of contact with the wall $\mu_N^{\beta,+}(H_N(S)) = o(N)$, we say we are in the delocalized phase.

2 Proof of the existence of the free-energy

First we are going to prove that for every M and N we have

$$Z_{N+M}^{\beta,+} \geq Z_N^{\beta,+} Z_M^{\beta,+} \tag{2.11}$$

We have

$$\begin{aligned} &\mathbf{E}\left[e^{\beta H_{N+M}(S)} \mathbf{1}_{\{S_{2(N+M)}=0 \text{ and } S\restriction_{[\![0,2(N+M)]\!]}\geq 0\}}\right] \\ &\geq \mathbf{E}\left[e^{\beta H_{N+M}(S)} \mathbf{1}_{\{S_{2N}=0,\ S_{2(N+M)}=0 \text{ and } S|_{[\![0,2(N+M)]\!]}\geq 0\}}\right] \\ &= \mathbf{E}\left[e^{\beta H_N(S)} \mathbf{1}_{\{S_{2N}=0 \text{ and } S\restriction_{[\![0,2N]\!]}\geq 0\}} \right. \\ &\times \mathbf{E}\left[e^{\beta \sum_{n=2N+1}^{2(N+M)} \mathbf{1}_{\{S_n=0\}}} \mathbf{1}_{\{S_{N+M}=0 \text{ and } S\restriction_{[\![2N,2(N+M)]\!]}\geq 0\}} \middle| (S_n)_{n\in[\![0,2N]\!]}\right]\Bigg]. \end{aligned} \tag{2.12}$$

By the Markov property for S we have, whenever $S_{2N}=0$

$$\mathbf{E}\left[e^{\beta \sum_{n=2N+1}^{2(N+M)} \mathbf{1}_{\{S_n=0\}}} \mathbf{1}_{\{S_{N+M}=0 \text{ and } S|_{[\![2N,2(N+M)]\!]}\geq 0\}} \middle| (S_n)_{n\in[0,2N]}\right] = Z_M^{\beta,+}. \tag{2.13}$$

Hence (2.11) holds. We will use this fact to prove the existence of the free energy.

Lemma 2.4. *Let u be a super-additive sequence of real numbers. That is one which satisfies*

$$\forall N, M \in \mathbb{N} \quad u_{N+M} \geq u_N + u_M \tag{2.14}$$

then u_N/N converges and

$$\lim_{N\to\infty} \frac{u_N}{N} = \sup_{N\geq 1} \frac{u_N}{N}. \tag{2.15}$$

As $\log Z_N^{\beta,+}$ is super additive, cf. (2.11), we have the existence of the free-energy. Furthermore we have

$$\textsc{f}^+(\beta) = \sup_N \frac{1}{N} \log Z_N^{\beta,+}. \tag{2.16}$$

We have

$$Z_N^{\beta,+} \geq e^{\beta} \mathbf{P}\left[S_{2N}=0 \text{ and } S|\restriction_{[\![1,2N-1]\!]} \geq 1\right] = e^{\beta} \frac{1}{4} Z_{N-1}^{0,+}. \tag{2.17}$$

As $Z_{N-1}^{0,+} \sim N^{-3/2}/\sqrt{\pi}$, this implies that

$$\lim_{N\to\infty} \frac{1}{N} \log Z_N^{\beta,+} \geq 0. \tag{2.18}$$

As trivially $Z_N^{\beta,+} \leq 1$ for $\beta \leq 0$ this implies

$$\forall \beta < 0, \quad \textsc{f}^+(\beta) = 0. \tag{2.19}$$

On the other hand, by considering the contribution of the path with N contacts we have

$$Z_N^{\beta,+} \geq 4^{-N} e^{N\beta} \tag{2.20}$$

and hence

$$\lim_{N\to\infty} \frac{1}{N} \log Z_N^{\beta,+} \geq \beta - 2\log 2. \tag{2.21}$$

As a result we have

$$0 \leq \beta_c \leq 2\log 2. \tag{2.22}$$

3 Proof of Fekété's Lemma

Assume for simplicity that

$$\sup_{n\geq 1} \frac{u_n}{n} = l < \infty \tag{2.23}$$

We fix $\varepsilon > 0$ and let n_0 be such that

$$u_{n_0} \geq n_0(l - \varepsilon/2).$$

Now for $m \in \mathbb{N}$ we let

$$m = n_0 p + q, \tag{2.24}$$

denote the Euclidean division of m by n_0 (p and $q \leq n_0 - 1$ are natural integers). Using the sub-additivity assumption, we have

$$u_m \leq u_{n_0} + u_{m-n_0} \leq u_{n_0} + u_{n_0} + u_{m-2n_0} \cdots = p u_{n_0} + u_q \tag{2.25}$$

Hence

$$\frac{u_m}{m} \geq \frac{p}{m} u_{n_0} + \frac{1}{m} u_q = \frac{1}{n_0} u_{n_0} + \frac{1}{m}\left(-\frac{q}{n_0} u_{n_0} + u_q\right). \tag{2.26}$$

Set

$$K := \max_{1\leq i\leq n_0} |u_i|. \tag{2.27}$$

For $m \geq 4K\varepsilon^{-1}$ we have

$$\frac{u_m}{m} \geq \frac{u_{n_0}}{n_0} - \varepsilon/2 \geq l - \varepsilon. \tag{2.28}$$

□

4 Identifying β_c

To conclude this chapter let us finally determine where the localization transition occurs.

Proposition 2.5. *We have* $\beta_c^+ = \log 2$.

Let us now try to identify where the phase transition occurs. First let us show that the condition $S_n \geq 0$ can be removed from the partition function at the cost of shifting the value of β. With this in mind we only need to show that the critical point for $\textsc{f}$ (cf. Remark 2.2) is 0.

Lemma 2.6. *We have for all* $\beta \in \mathbb{R}$

$$Z_N^{\beta,+} = Z_N^{\beta - \log 2}, \tag{2.29}$$

as a consequence we have $\textsc{f}^+(\beta) = \textsc{f}(\beta - \log 2)$ *and* $\beta_c^+ = \beta_c + \log 2$.

Proof. We prove the equality for the original partition functions $\widehat{Z}$ Let $\mathcal{S}_{N,k}$, $\mathcal{S}_{N,k}^+$, be the set of paths in $\mathcal{S}_N$ resp. $\mathcal{S}_N^+$ with $H_N(S) = k$ There is a natural bijection

$$\mathcal{S}_{N,k} \to \mathcal{S}_{N,k}^+ \times \{-1, 1\}^k \tag{2.30}$$

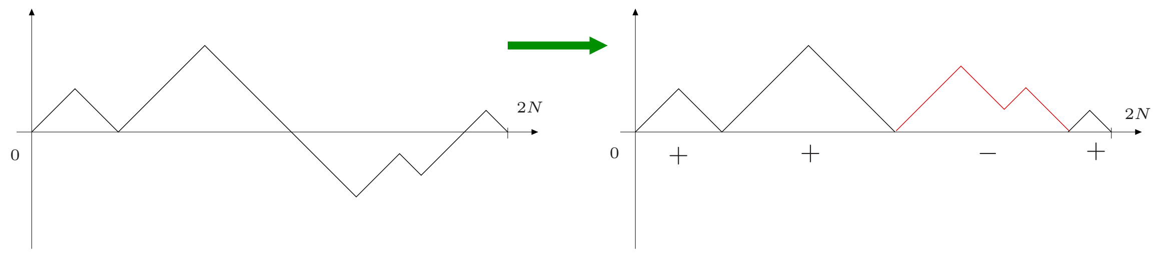

Figure 2.1: Illustration for the bijection $\mathcal{S}_{N,k} \to \mathcal{S}^+_{N,k} \times \{-1,1\}^k$: the original path can be recovered by flipping the excursions that receive $-$ signs.

It goes as follows (see Figure 2.1): to a path in $S \in \mathcal{S}_{N,k}$ we associate the sequence of sign of the k excursions away from zero and the path reflected above the axis $(|S_n|)_{n=0}^N$.

Hence

$$\widehat{Z}_N^{\beta,+} = \sum_{k=1}^N e^{\beta k} \# \mathcal{S}^+_{N,k} = \sum_{k=1}^N e^{\beta k} 2^{-k} \# \mathcal{S}_{N,k} = \widehat{Z}_N^{\beta - \log 2}. \tag{2.31}$$

□

Let us now show that $\beta_c = 0$. First we notice that

$$Z_N^0 = \mathbf{P}[S_{2N} = 0] \leq 1$$

The we have $\textsc{f}(0) \leq 0$ and thus $\beta_c \geq 0$.

To conclude the proof we need to show that $\textsc{f}(\beta) > 0$ whenever $\beta > 0$. As $\mathbf{P}_N^0(S_{2n} = 0) \geq cn^{-1/2}$ we have

$$\partial_\beta \log Z_N^\beta |_{\beta=0} = \mathbf{P}_N^0 \left[H_N(S) \mid S_{2N} = 0 \right] \geq c\sqrt{N}. \tag{2.32}$$

Using convexity we have, for any $\beta > 0$

$$\log Z_N^\beta \geq \log Z_N^0 + c\beta\sqrt{N}. \tag{2.33}$$

Using the fact that

$$Z_N^0 = \mathbf{P}[S_{2N} = 0] \geq cN^{-1/2}$$

we have for N sufficiently large, for all $\beta \geq \log 2$,

$$\textsc{f}(\beta) = \frac{1}{N} \log Z_N^\beta \geq \frac{1}{N} \left(c\beta\sqrt{N} - \log N \right). \tag{2.34}$$

Choosing $N = \beta^{-2} |\log \beta|^3$, we obtain that for $\beta > 0$.

$$\textsc{f}(\beta) \geq c' \beta^2 |\log \beta|^{-3/2}. \tag{2.35}$$

Hence we can conclude that $\beta_c = 0$ and thus that $\beta_c^+ = 0$.

Chapter 3

Pinning based on a renewal

1 From random walk to renewal process

In the previous chapter, the polymer measure $\mathbf{P}_N^\beta$ was presented as a modification of the law of a simple random walk. However the only object whose distribution is affected be the change is the set of the return time to zero

$$\mathcal{T} := \{n \geq 0 \;:\; S_{2n} = 0\}.$$

Indeed it can immediately be checked that the distribution of the excursions of S away from zero are the same under $\mathbf{P}_N^\beta$ and $\mathbf{P}$.

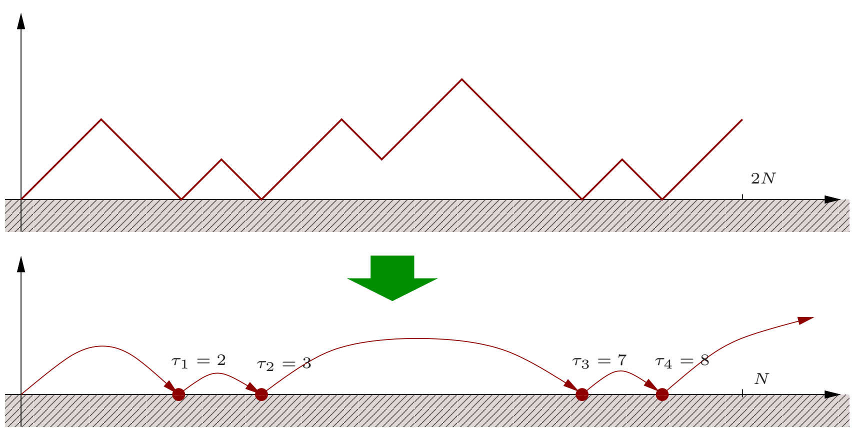

Figure 3.1: The sequence τ of returned time to zero extracted from a trajectory S.

Given a simple random walk trajectory $(S_n)_{n\geq 0}$ starting from 0, we define (see Figure 3.1) the sequence τ of its return times to zero by

$$\begin{aligned} \tau_0 &:= 0 \\ \tau_{k+1} &:= \frac{1}{2}\min\{n > 2\tau_k \mid S_n = 0\}. \end{aligned} \tag{3.1}$$

The choice of dividing by two is made so that the increments of τ can take arbitrary values in $\mathbb{N}$. With some small abuse of notation we will write $n \in \tau$ instead of $n \in \mathcal{T}$.

Within this setup, the polymer measure $\mathbf{P}_N^\beta$ density with respect to $\mathbf{P}$ (see (1.27) in the previous chapter) can be rewritten as

$$\frac{\mathrm{d}\mathbf{P}_N^\beta}{\mathrm{d}\mathbf{P}} = \frac{1}{Z_N^\beta} \exp\left(\sum_{n=1}^N \beta \mathbf{1}_{\{n\in\tau\}}\right) \mathbf{1}_{\{N\in\tau\}}, \tag{3.2}$$

and the partition function can be rewritten as

$$Z_N^\beta = \mathbf{E}\left[\exp\left(\sum_{n=1}^N \beta \mathbf{1}_{\{n\in\tau\}}\right)\right]. \tag{3.3}$$

Remark 3.1. *In fact $\mathbf{P}_N^{\beta,+}$ can also be defined in this manner, we just have to replace $\mathbf{P}$ by*

$$\mathbf{P}^+ := \mathbf{P}\left[\ldots \ |\forall n \in \mathbb{N},\ S_n \geq 0\right]. \tag{3.4}$$

Some moderate effort is required for the definition since we are conditioning with respect to an event of zero probability.

One might want to try to generalize this setup to an arbitrary renewal processes τ.

Definition 3.2. *A renewal process τ is an increasing sequence with value in $\mathbb{N}\cup\{\infty\}$ satisfying*

(i) $\tau_0 = 0$

(ii) $(\tau_{k+1} - \tau_k)_{k\geq 0}$ *are IID variables with value in* $\mathbb{N}\cup\{\infty\}$.

From now on, we forget about random walks: $\mathbf{P}_N^\beta$ will be the definition of our polymer measure, where $\mathbf{P}$ is the law of an arbitrary renewal process τ. We set

$$H_N(\tau) := \sum_{n=1}^N \mathbf{1}_{\{n\in\tau\}} \tag{3.5}$$

We are interested in renewals for which there exists $\alpha > 0$, and a constant $C_K > 0$ such that

$$\mathbf{P}[\tau_1 = n] := K(n) \overset{n\to\infty}{\sim} C_K n^{-(1+\alpha)}. \tag{3.6}$$

For simplicity we will also assume that $K(n) > 0$ for every $n \in \mathbb{N}$.

We do not require τ_1 to be almost surely-finite, and hence the sum of the $(K(n))_{n\geq 1}$ is not necessarily equal to one. We introduce the notation.

$$K(\infty) = 1 - \sum_{n=1}^\infty K(n) \geq 0. \tag{3.7}$$

When $K(\infty) > 0$ we say that the renewal is *terminating* (or almost surely terminating). In that case the sequence τ has almost surely only finitely many terms and $\#\{n\ :\ \tau_n < \infty\}$ is a Geometric random variable of parameter $(1 - K(\infty))$.

The case $\alpha = 1/2$ corresponds to the simple random walk in dimension 1 (as can be deduced from Lemma 1.1 in Chapter 1). We do not prove the existence of the free-energy as the proof from the previous chapter can straightforwardly be generalized.

Proposition 3.3. *The limit*

$$\textsc{f}(\beta) := \lim_{N\to\infty} \frac{1}{N} \log \mathbf{E}\left[\exp\left(\sum_{n=1}^{N} H_N(\tau)\right) \mathbf{1}_{\{N\in\tau\}}\right] := \lim_{N\to\infty} \frac{1}{N} \log Z_N^\beta, \tag{3.8}$$

is defined and is a non-negative, non-decreasing, convex function of β.

Moreover

$$\beta_c := \inf\{\beta \in \mathbb{R} \ : \ \textsc{f}(\beta) > 0\} \in [0, \infty)$$

is finite and we have

$$\lim_{N\to\infty} \frac{1}{N} \mathbf{E}_N^\beta [H_N(\tau)] \begin{cases} = 0 \ \text{if } \beta < \beta_c, \\ > 0 \ \text{if } \beta > \beta_c. \end{cases} \tag{3.9}$$

2 Solving the model

The above setup is more-general than the random-walk model, it is also more easy to work with as we will see. Let b be defined as the only solution of

$$e^\beta \sum_{n=1}^{\infty} K(n) e^{-nb} = 1, \tag{3.10}$$

if there is one, and $b = 0$ if not.

Note that $\sum_{n\geq 1} e^{-nb} K(n) < \infty$ only for $b \geq 0$. It is continuous decreasing and takes value in $(0, 1 - K(\infty))$. Hence Equation (3.10) admits a solution if and only if

$$\beta \geq -\log(1 - K(\infty)). \tag{3.11}$$

We set

$$\widetilde{K}(n) = e^\beta K(n) e^{-nb}, \tag{3.12}$$

and as $\sum_{n=1}^{\infty} \widetilde{K}(n) \leq 1$, one can associate to it a renewal process $\widetilde{\tau}$ of law $\widetilde{\mathbf{P}}$ ($\widetilde{\tau}_0 = 0$, and the increments of $\widetilde{\tau}$ are IID and with distribution given by $\widetilde{K}(\cdot)$). We have

$$Z_N^\beta = \sum_{k=1}^{N} \sum_{\substack{l_1, l_2, \dots, l_k \\ \sum_{i=1}^k l_i = n}} \prod_{i=1}^{k} e^\beta K(l_i) = \sum_{k=1}^{N} \sum_{\substack{l_1, l_2, \dots, l_k \\ \sum_{i=1}^k l_i = n}} \prod_{i=1}^{k} e^{bl_i} \widetilde{K}(l_i) = e^{bN} \widetilde{\mathbf{P}}\left(N \in \widetilde{\tau}\right). \tag{3.13}$$

When $b = 0$, it is not hard to see that as

$$\widetilde{K}(n) = e^\beta K(n) \leq \widetilde{\mathbf{P}}\left(N \in \widetilde{\tau}\right) \leq 1 \tag{3.14}$$

and thus

$$N^{-1}(\beta + \log K(n)) \leq N^{-1} \log Z_N^\beta \leq 0.$$

For $b > 0$, we have

$$\widetilde{\mathbf{E}}[\widetilde{\tau}_1] = e^\beta \sum_{n\geq 1} n K(n) e^{-bn} < \infty. \tag{3.15}$$

Hence one can use the Discrete Renewal Theorem (proved in the next chapter) which asserts that asymptotically each point is equally likely to be in the renewal.

Theorem 3.4 (Erdös-Feller-Pollard '49). *If τ is a renewal process whose associated inter-arrival distribution K satisfies*

$$\forall n \geq 0, K(n) > 0, \qquad and \qquad K(\infty) = 0. \tag{3.16}$$

then we have

$$\lim_{N\to\infty} \mathbf{E}[N \in \tau] = \frac{1}{\mathbf{E}\left[\tau_1\right]}. \tag{3.17}$$

Hence we have

$$N^{-1} \log Z_N^\beta = b + N^{-1} \log \widetilde{\mathbf{P}}\left(N \in \widetilde{\tau}\right), \tag{3.18}$$

and the second term on the right-hand side tends to zero. In every case we have

$$\lim_{N\to\infty} \frac{1}{N} \log Z_N^\beta = b. \tag{3.19}$$

We have thus identified the free-energy.

Proposition 3.5. *The free-energy is given by*

$$\textsc{f}(\beta) := \begin{cases} 0 & \text{if } \beta \leq -\log(1-K(\infty)) \\ \textsc{g}^{-1}(\beta) & \text{if } \beta \geq -\log(1-K(\infty)) \end{cases} \tag{3.20}$$

Where where G *is the function*

$$\begin{aligned} \mathbb{R}_+ &\to [-\log(1-K(\infty)), \infty) \\ b &\mapsto -\log\left(\sum_{n\geq 1} e^{-nb} K(n)\right). \end{aligned} \tag{3.21}$$

Note also that this computation gives a complete description of $\mathbf{P}_N^\beta$. Indeed if A is an event for the renewal τ which depends only on $\tau \cap [0, N]$, repeating (3.13) gives

$$\mathbf{E}\left[e^{\beta H(\tau)} \mathbf{1}_{\{N\in\tau\}} \mathbf{1}_A\right] = e^{bN} \widetilde{\mathbf{E}}\left[\mathbf{1}_{\{N\in\widetilde{\tau}\}} \mathbf{1}_A\right]. \tag{3.22}$$

And hence after dividing by the partition function we obtain $\widetilde{\tau} \cap [0, N]$ under

$$\mathbf{P}_N^\beta[A] = \widetilde{\mathbf{P}}[A \mid \widetilde{\tau} \in N]. \tag{3.23}$$

One can use this fact to prove many sharp statements on the trajectories

Proposition 3.6. *When the free-energy* $\textsc{f}(\beta)$ *is positive, one has*

(i) For any $\varepsilon > 0$ we have

$$\lim_{N\to\infty} \mathbf{P}_N^\beta \left[H_N(\tau) \in [(\textsc{f}'(\beta) - \varepsilon)N, (\textsc{f}'(\beta) + \varepsilon)N]\right] = 0. \tag{3.24}$$

(ii) Let $L_N(\tau)$ be the length of the longest jump of τ in the interval $[0, N]$, for any ε we have

$$\lim_{N\to\infty} \mathbf{P}_N^\beta \left(L_N(\tau) \in [(\textsc{f}(\beta)^{-1} - \varepsilon) \log N, (\textsc{f}(\beta)^{-1} + \varepsilon) \log N]\right) = 0. \tag{3.25}$$

Proof. Note that for event concerning only the renewal on segment $[0, N]$ we have

$$\mathbf{P}_N^\beta[A] = \widetilde{\mathbf{P}}[A \mid N \in \widetilde{\tau}] \le \frac{\widetilde{\mathbf{P}}[A]}{\widetilde{\mathbf{P}}[N \in \widetilde{\tau}]}. \tag{3.26}$$

As $b > 0$ by assumption, the denominator converges due to the Renewal Theorem, and thus is uniformly bounded away from zero, it is in fact sufficient to show that our statements hold for the measure $\widetilde{\mathbf{P}}$. Under $\widetilde{\mathbf{P}}$ the mean length of a of a jump is

$$\widetilde{\mathbf{E}}[\tau_1] = \sum_{n\ge 1} n e^\beta e^{-bn} K(n) = \frac{\sum_{n\ge 1} n e^\beta e^{-bn} K(n)}{\sum_{n\ge 1} e^\beta e^{-bn} K(n)} = \partial_b \mathrm{G}(b) = [\mathrm{F}'(\beta)]^{-1}. \tag{3.27}$$

where the last equality comes from the formula for the derivative of an inverse. Hence by the law of large number we have

$$\begin{aligned} &\lim_{N\to\infty} \widetilde{\mathbf{E}}\left[\tau_{(\mathrm{F}'(\beta)-\varepsilon)N} \ge N\right] = 0, \\ &\lim_{N\to\infty} \widetilde{\mathbf{E}}\left[\tau_{(\mathrm{F}'(\beta)+\varepsilon)N} \le N\right] = 0. \end{aligned} \tag{3.28}$$

For the second point note that there are at most N jumps in $[\![0, 2N]\!]$ hence

$$L_N(\widetilde{\tau}) \le \max_{1\le k\le N} (\widetilde{\tau}_k - \widetilde{\tau}_{k-1}). \tag{3.29}$$

Hence we have

$$\widetilde{\mathbf{P}}[L_N(\widetilde{\tau}) \ge t] \le N\widetilde{\mathbf{P}}[\widetilde{\tau}_1 \ge t] \le N \left(\sum_{n\ge t} e^{-\beta} e^{-bn} K(n) \right) \le \frac{Ne^{-bt}}{1-e^{-b}}. \tag{3.30}$$

Obviously for $t \ge (b^{-1} + \varepsilon) \log N$ we get something negligible on the right-hand side. On the other hand we have for any δ we have

$$\widetilde{\mathbf{P}}[L_N(\widetilde{\tau}) \le t] \le \mathbf{P}\left[\max_{1\le k\le \delta N} (\widetilde{\tau}_k - \widetilde{\tau}_{k-1}) \le t \right] + \mathbf{P}[\widetilde{\tau}_{\delta N} \ge N]. \tag{3.31}$$

The second term is negligible if one chooses $\delta = \mathrm{F}'(\beta)/2$. As for the first one it is equal to

$$\left(1 - \sum_{n\ge t} e^\beta e^{-bn} K(n)\right)^{\delta N} \le \exp\left(-\delta N \sum_{n\ge t} e^\beta e^{-bn} K(n)\right). \tag{3.32}$$

We have for t sufficiently large

$$\delta N \sum_{n\ge t} e^{-\beta} e^{-bn} K(n) \ge \delta N e^\beta e^{-bt} K(t) \ge \frac{\delta N C_K e^\beta e^{-bt}}{2t^{1+\alpha}}. \tag{3.33}$$

for $t = (b^{-1} - \varepsilon) \log N$, this is larger than $N^{b^{-1}\varepsilon/2}$ and we can conclude. □

3 The free-energy

Proposition 3.7. *The following statements hold:*

(i) *The free energy is real-analytic on* $\mathbb{R} \setminus \{\beta_c\}$.

(ii) *The critical point is given by* $\beta_c = -\log(1 - K(\infty))$

(iii) *Moreover if* $K(n) \stackrel{N\to\infty}{\sim} C_K n^{-(1+\alpha)}$, *with* $\alpha \in (0, \infty) \setminus 1$ *we have*

$$\textsc{f}(\beta) \stackrel{\beta\to\beta_c^+}{\sim} C'_K (\beta - \beta_c)^{\max(\alpha^{-1},1)} \tag{3.34}$$

where

$$C'_K = \begin{cases} \left(\frac{C_K \Gamma(1-\alpha)}{\alpha(1-K(\infty))}\right)^{-1/\alpha}, & \text{if } \alpha < 1, \\ (1 - K(\infty)) \left(\sum_{n\geq 1} nK(n)\right)^{-1}, & \text{if } \alpha > 1, \end{cases} \tag{3.35}$$

where

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1} \mathrm{d}x$$

is Euler's Gamma function.

(iv) *We have*

$$\lim_{N\to\infty} \textsc{f}(\beta) - \beta = \log K(1). \tag{3.36}$$

Note that $\textsc{f}(\beta)$ is the limit of $\frac{1}{N} \log Z_N^\beta$ which is a sequence of analytic functions. The appearance of a point-of non analyticity in the limit is the signature of a phase transition. We say that the phase transition is of order n if the $(n-1)$-th derivative of the free energy is continuous. Note that here the transition is of first order for $\alpha > 1$ but of second order for $\alpha < 1$. In fact the asymptotic of $\textsc{f}(\beta)$ seems to indicate that the order of the phase transition should be $\lceil \alpha^{-1} \rceil$.

Proof. The value of β_c is given by the remark above (3.11). It is obvious that the function is analytic on $(-\infty, \beta_c)$. On the interval (β_c, ∞) the free energy given by the inverse of the function $\textsc{g}$ defined in (3.21) which is real analytic. Without loss of generality one can restrict to the case $\beta_c = 0$ (i.e. $K(\infty) = 0$), by changing the renewal K for $\widehat{K}$ defined by

$$\widehat{K}(n) = \frac{K(n)}{1 - K(\infty)}. \tag{3.37}$$

If one defines $\widehat{\textsc{g}}$ using (3.21) with $\widehat{K}(n)$ we see that

$$\widehat{\textsc{g}}(b) = \textsc{g}(b) - \log(1 - K(\infty)).$$

Thus the transform $K \to \widehat{K}$ has the effect of performing an horizontal translation in the free-energy diagram which shifts β_c to zero. Hence from now on assume that $\beta_c = 0$.

Let us compute the asymptotic at zero of the function given by (3.20). It is straight forward to check that $\textsc{g}(b)$ tends to 0 when b goes to zero, and thus the asymptotic behavior of $\textsc{f}$ can be obtained by studying that of $\textsc{g}$ and inverting it. We thus need to prove that

$$\mathrm{G}(b) \sim \begin{cases} \left(\frac{C_K \Gamma(1-\alpha)}{\alpha}\right) b^\alpha, \text{ if } \alpha < 1, \\ \left(\sum_{n\geq 1} nK(n)\right) b \text{ if } \alpha > 1, \end{cases} \tag{3.38}$$

The log quantity is equivalent to

$$1 - \sum_{n\geq 1} e^{-nb} K(n) = \sum_{n\geq 1} (1-e^{-nb}) K(n). \tag{3.39}$$

In the case $\alpha > 1$, the sum can be rewritten as

$$b \sum_{n\geq 1} \frac{(1-e^{-nb})}{nb} nK(n). \tag{3.40}$$

Then applying the dominate convergence theorem for the sum above, we can check that

$$\lim_{b\to 0} \sum_{n\geq 1} \frac{(1-e^{-nb})}{nb} nK(n) = \sum_{n\geq 1} nK(n) < \infty. \tag{3.41}$$

We can then treat the case of $\alpha < 1$. We note that

$$\frac{\mathrm{G}(b)}{C_K b^\alpha} = b \sum_{n\geq 1} \frac{1-e^{-nb}}{(nb)^{1+\alpha}} \frac{K(n) n^{1+\alpha}}{C_K}. \tag{3.42}$$

Apart for the last term $K(n)n^{1+\alpha}/C_K$ which tends to one, this looks very much like a Riemann sum for the function

$$g_\alpha(x) := \frac{1-e^{-x}}{x^{1+\alpha}}.$$

It is a simple exercise to check that

$$\lim_{b\to 0} b \sum_{n\geq 1} g_\alpha(bn) = \int_0^\infty g_\alpha(x) \mathrm{d}x.$$

We also have to care about a correcting term. Given $\varepsilon > 0$ one sets C_1 and C_2 which satisfies $\frac{K(n)n^{1+\alpha}}{C_K} \leq C_1$ for all n and,

$$\left| \frac{K(n)n^{1+\alpha}}{C_K} - 1 \right| \leq \varepsilon \tag{3.43}$$

for all $n \geq C_2$. We have

$$b \left| \sum_{n\geq 1} (1-e^{-nb})(nb)^{1+\alpha} \left(\frac{K(n)n^{1+\alpha}}{C_K} - 1 \right) \right| \leq C_1 b \sum_{1\leq n\leq C_2} g_\alpha(bn) + \varepsilon b \sum_{n\geq C_2} g_\alpha(bn). \tag{3.44}$$

The first term is smaller than $b^{1-\alpha} C_3$ for some constant C_3 (it is a finite sums of terms which are all of that order) and the second is smaller than

$$2\varepsilon \int g_\alpha(x) \mathrm{d}x.$$

We only need to check that

$$\int g_\alpha(x)\mathrm{d}x = \frac{\Gamma(1-\alpha)}{\alpha}. \tag{3.45}$$

but this is readily checked by an integration by part $1 - e^{-x} \to e^{-x}$, $x^{-(1+\alpha)} \to \alpha^{-1}x^{-\alpha}$.

For the last item, first note that the lower bound is easy: by considering the renewal trajectories which only make jumps of size one we have

$$Z_N^\beta \geq K(1)^N e^{N\beta}. \tag{3.46}$$

Note that it implies in particular that $\textsc{f}(\beta)$ tends to infinity. Hence we have

$$e^{-\beta} = \sum_{n\geq 1} K(n) e^{-\textsc{f}(\beta)n} \overset{\beta\to\infty}{\sim} K(1)e^{-\textsc{f}(\beta)} \tag{3.47}$$

which proves the result. □

4 The free-boundary condition

We set

$$Z_N^{\beta,\mathrm{f}} = \mathbf{E}\left[e^{\beta H_N(\tau)}\right] \tag{3.48}$$

and let $\mathbf{P}_N^{\beta,\mathrm{f}}$ denote the pinning measure with no constraint for the end point

$$\frac{\mathrm{d}\mathbf{P}_N^{\beta,\mathrm{f}}}{\mathrm{d}\mathbf{P}}(\tau) = e^{\beta H_N(\tau)}. \tag{3.49}$$

We want to show here briefly that the change in the boundary condition does not change the free-energy.

Proposition 3.8.

$$\lim_{N\to\infty} \frac{1}{N} \log Z_N^{\beta,\mathrm{f}} = \textsc{f}(\beta). \tag{3.50}$$

Proof. Note that

$$Z_N^{\beta,\mathrm{f}} \geq Z_N^\beta, \tag{3.51}$$

hence we only have to prove that

$$\lim_{N\to\infty} \frac{1}{N} \log Z_N^{\beta,\mathrm{f}} \leq \textsc{f}(\beta). \tag{3.52}$$

For a renewal τ we let

$$L_N(\tau) := \inf\{n \leq N \mid n \in \tau\}. \tag{3.53}$$

We have

$$\mathbf{E}\left[e^{\beta H_N(\tau)} \mathbf{1}_{\{L_N(\tau)=a\}}\right] = Z_a^\beta \left(\sum_{n=N-a+1}^{\infty} K(n) + K(\infty)\right). \tag{3.54}$$

Hence if one sets

$$\bar{K}(n) := 1 - \sum_{m=1}^{n} K(m)$$

we have

$$Z_N^{\beta,\mathrm{f}} = \sum_{a=0}^{N} Z_a^\beta \bar{K}(N-a) \leq \sum_{a=0}^{N} e^{\textsc{f}(\beta)a} \bar{K}(N-a) \leq N e^{\textsc{f}(\beta)N}. \tag{3.55}$$

Hence the result. □

Of course more than the free-energy, one would like to know if $\mathbf{P}_N^{\beta,\mathrm{f}}$ and $\mathbf{P}_N^\beta$ look alike. A close look at (3.55) shows that when $\textsc{f}(\beta) > 0$, the main contrition to the partition function is given by by the a that are close to N and thus that the measures are very similar.

5 The sub-critical and the critical case

We know that when $\beta > \beta_c$ partition function is asymptotically equivalent to $e^{\textsc{f}(\beta)}/\widetilde{\mathbf{E}}[\widetilde{\tau}_1]$. Now let us give some result concerning the case $\beta = \beta_c$ and $\beta \leq \beta_c$.

Proposition 3.9. *(i) When $\beta < \beta_c$ we have*

$$Z_N^\beta \overset{N\to\infty}{\sim} \frac{e^\beta K(n)}{(1-\rho)^2} \tag{3.56}$$

where $\rho = e^{\beta-\beta_c} < 1$. As a consequence we have

$$\mathbf{E}_N^\beta[H_N(\tau)] < \infty. \tag{3.57}$$

(ii) For all $\alpha \in (0,1)$ we have

$$Z_N^{\beta_c} \overset{N\to\infty}{\sim} \frac{(1-K(\infty))\sin(\pi\alpha)}{C_K \pi} N^{\alpha-1}. \tag{3.58}$$

In particular there exists constants such that

$$cN^\alpha \leq \mathbf{E}_N^{\beta_c}[H_N(\tau)] \leq CN^\alpha. \tag{3.59}$$

The expected number of contact in (3.57) and (3.59) can be deduced from the asymptotic of the partition function with the following formula valid for all $n \leq N$

$$\mathbf{E}_N^\beta[n \in \tau] = \frac{Z_n^\beta Z_{N-n}^\beta}{Z_N^\beta}. \tag{3.60}$$

The asymptotic of the partition function in the critical point is a quite delicate computation due to Doney [3], and requires a few pages of computation. We focus on the sub-critical case. Recall that

$$Z_N^\beta = \widetilde{P}[N \in \widetilde{\tau}], \tag{3.61}$$

where the renewal $\widetilde{\tau}$ has inter-arrival distribution given by

$$\widetilde{K}(n) = e^\beta K(n),$$

which satisfies

$$\sum_{n\geq 0} \widetilde{K}(n) = e^{\beta-\beta_c} < 1. \tag{3.62}$$

Let us write $e^{\beta_c}K(n) = q(n)$ and recall $e^{\beta-\beta_c} = \rho < 1$. Finally we let $*$ denote the convolution operator

$$q^{*n} := \sum_{\substack{l_1,\dots,l_k \\ \sum_{i=1}^k l_i}} q(l_1)\dots q(l_k). \tag{3.63}$$

We have

$$\widetilde{P}[N \in \widetilde{\tau}] = \sum_{k=1}^{N} \widetilde{K}^{*k}(n) = \sum_{k\geq 1} \rho^k q^{*k}(N). \tag{3.64}$$

To compute the asymptotic equivalent of the sum we use the following results

Lemma 3.10. *Let q be a positive function on $\mathbb{N}$ such that*

$$\sum_{n\geq 1} q(n) = 1 \quad \text{and } q(n) \overset{n\to\infty}{\sim} C_q n^{-(1+\alpha)}.$$

Then for any $k \geq 0$

$$\lim_{n\to\infty} \frac{q^{*k}(n)}{q(n)} = k. \tag{3.65}$$

Moreover there exists $c > 0$ such that for all n and k

$$q^{*k}(n) \leq k^c q(n) \tag{3.66}$$

We have, by dominated convergence

$$\lim_{n\to\infty} \sum_{n\geq 1} \rho^k \frac{q^{*k}(n)}{q(n)} = \sum_{n\geq 1} k\rho^k = \frac{\rho}{(1-\rho)^2}. \tag{3.67}$$

and hence

$$Z_N^{\beta} \overset{N\to\infty}{\sim} \frac{\rho q(n)}{1-\rho)^2} = \frac{e^{\beta}K(n)}{(1-\rho)^2}, \tag{3.68}$$

which is the desired result.

Proof of Lemma 3.10. We prove the two points by induction on k. In both cases the result is trivial for $k = 1$ so we just need to perform the induction step.

We start with (3.66). Let us assume that the statement holds for all n and all $k < 2m$ we have

$$\frac{q^{*2m}(n)}{q(n)} \leq 2\sum_{j=1}^{\lfloor n/2\rfloor} q^{*m}(j)\frac{q^{*m}(n-j)}{q(n)} \leq 2m^c \sum_{j=1}^{\lfloor n/2\rfloor} q^{*m}(j)\frac{q(n-j)}{q(n)}. \tag{3.69}$$

Using the assumption we have for the tail distribution of $q(\cdot)$, there exists c_1 such that for all n

$$\max_{1\leq j\leq n/2} \frac{q(n-j)}{q(n)} \leq c_1.$$

Hence

$$\frac{q^{*2m}(n)}{q(n)} \leq 2c_1 m^c \sum_{j=1}^{\lfloor n/2 \rfloor} q^{*m}(j) \leq 2c_1 m^c \leq (2m)^c, \tag{3.70}$$

if c has been chosen such that $2^c \geq 2c_1$. The case $k = 2m+1$ works in the same manner with the same c.

For $k \geq 2$ we have

$$\frac{q^{*k+1}(n)}{q(n)} = \sum_{m=1}^{n-1} \frac{q^{*k}(m)(n-m)}{q(n)} = \sum_{m=1}^{\lfloor n/2 \rfloor} q^{*k}(m) \frac{q(n-m)}{q(n)} + \sum_{m=1}^{\lceil n/2 \rceil - 1} q(m) \frac{q^{*k}(n-m)}{q(n)}. \tag{3.71}$$

For m fixed $\frac{q(n-m)}{q(n)}$ and $\frac{q^{*k}(n-m)}{q(n)}$ tend to one and k respectively and the ratio are uniformly bounded (cf. (3.66)) on $m = 1, \dots, \lceil n/2 \rceil$. Hence by dominated convergence, the two sums above converge respectively to 1 and k.

□

6 Back to the random walk case

Let us see how the formula of the free-energy gives in the case of the random walk pinning. Recall that we have in that case

$$K(n) = \mathbf{P}[\tau_1 = n] = \frac{2\#\mathcal{S}_{n-1}^+}{4^n} \overset{n \to \infty}{\sim} \frac{n^{-3/2}}{2\sqrt{\pi}}. \tag{3.72}$$

Thus K satisfies (3.6) with $\alpha = 1/2$ and $C_K = \frac{1}{2\sqrt{\pi}}$. Note also that with the change of variable $x = u^2/2$ we obtain

$$\Gamma(1/2) = \int_0^\infty x^{-1/2} e^{-x} \mathrm{d}x = \sqrt{2} \int_0^\infty e^{-u^2} \mathrm{d}u = \sqrt{\pi}. \tag{3.73}$$

Hence we have in that case

$$\textsc{f}(\beta) \overset{\beta \to 0+}{\sim} \beta^2. \tag{3.74}$$

Hence from Lemma 2.6 we have

$$\textsc{f}^+(\beta) \overset{\beta \to 0+}{\sim} (\beta - \log 2)^2.$$

Moreover we can show that $\textsc{f}^+(\beta) > 0$ corresponds to a physical localization of the trajectories

Proposition 3.11. *For any $\beta > \beta_+$ there exists a constant C_β such that*

$$\forall n \in [\![0, N]\!], \quad \mathbb{P}[S_n \geq h] \leq C_\beta e^{-2\textsc{f}(\beta) h}. \tag{3.75}$$

In particular

$$\lim_{N \to \infty} \mathbf{P}\left[\max_{n \in [\![0,N]\!]} S_n \geq \frac{\log N}{2\textsc{f}(\beta)}(1+\varepsilon)\right] = 0. \tag{3.76}$$

Proof. Given n let r_n and l_n be the last resp. first point of contact before and after n. We have

$$S_n \leq \min(n - l_n, r_n - n). \tag{3.77}$$

Hence

$$\mathbf{P}_N^{\beta,+}(S_n \geq h) \leq \mathbf{P}^{\beta,+}[l_n \leq n-h \ ; \ r_n \geq n+h]. \tag{3.78}$$

The contribution of the right hand side to the partition function can be computed by decomposing over all the possible values for n. It is equal to

$$Z_N^{\beta,+}\mathbf{P}^{\beta,+}[l_n \leq n-h \ ; \ r_n \geq n+h] = \sum_{\substack{l \leq n-h \\ r \geq n+h}} Z_r^{\beta,+} K(l-r) e^{\beta} Z_{N-r}^{\beta,+} \leq e^{\beta} e^{\mathrm{F}(\beta)(N-(l-r))}. \tag{3.79}$$

Hence

$$\mathbf{P}^{\beta,+}[l_n \leq n-h \ ; \ r_n \geq n+h] \leq \frac{1}{\widetilde{P}[N \in \widetilde{\tau}]} \sum_{\substack{l \leq n-h \\ r \geq n+h}} e^{-\mathrm{F}(\beta)(l-r)} \leq C_\beta e^{-\mathrm{F}(\beta)2h}. \tag{3.80}$$

□

Chapter 4

The renewal Theorem

1 Reformulation of the result

It comes out to be practical to rewrite the result in a less probabilistic setting. Let $(f_n)_{n\geq 1}$ be a sequence of positive number that sums to one: $\sum_{n\geq 0} f_n = 1$. Set $\mu := \sum_{n\geq 1} n f_n \in [1,\infty]$. We define u to be the sequence recursively defined by the relation

$$\begin{cases} u_0 := 1, \\ u_n := \sum_{k=1}^{n} f_n u_{n-k}. \end{cases} \tag{4.1}$$

The second line in (4.1) is called the renewal equation, and the reader can check that this is the one satisfied by $\mathbf{P}[n \in \tau]$, if one sets $f_n = K(n)$. This probabilistic interpretation or a simple induction implies

$$\forall n \geq 1, \quad 0 \leq u_n \leq 1. \tag{4.2}$$

Theorem 4.1. *We have*

$$\lim_{n\to\infty} u_n = \mu^{-1}. \tag{4.3}$$

Remark 4.2. *Note that the Theorem fails to be true if one assumes the $(f_n)_{n\geq 1}$ are only non-negative. Indeed if $f_{2n} > 0$ for every n and $f_{2n+1} = 0$ then a trivial consequence of the theorem stated above is that*

$$\lim_{n\to\infty} u_{2n} = (\mu/2)^{-1} \lim_{n\to\infty} u_{2n+1} = 0. \tag{4.4}$$

It turns out that this periodicity problem is the only thing that can prevent the result to be true, and the result remains true on the assumption that there exists no integer $k \geq 2$ such that

$$\{n \in \mathbb{N} \ : \ f_n > 0\} \subset k\mathbb{N}.$$

2 Technical Lemmas

In the proof of Theorem we will need two technical results. The first one is somehow a generalization of the Bolzano-Weierstrass Theorem.

Lemma 4.3. *Let $(r_n^i)_{n\geq 0, i\geq 1}$, be a sequence indexed by two integers, which satisfies*

$$\forall n \geq 0,\ i \geq 1, \quad 0 \leq r_n^i \leq 1.$$

Then one can find an increasing sequence of integers n_k and a sequence of real numbers $l_i \in [0,1]$ such that

$$\forall i \geq 1,\ \lim_{k\to\infty} r_{n_k}^i = l_i. \tag{4.5}$$

Proof. The proof relies on what is called the diagonal argument (similar to the one used in the proof of Cantor's Theorem). Applying Bolzano-Weirstrass' Theorem to the sequence $(r_n^1)_{n\geq 0}$, one can find $l_1 \in [0,1]$ and a sequence m_k^1 such that

$$\lim_{k\to\infty} r_{m_k^1}^1 = l_1. \tag{4.6}$$

Applying Bolzano-Weierstrass Theorem to the sequence $r_{m_k^1}^2$, one can find extract from m_k^1 and increasing subsequence m_k^2 such that

$$\lim_{k\to\infty} r_{m_k^2}^2 = l_2. \tag{4.7}$$

Finally one proceeds recursively and for all $i \geq 3$ we find l_i a subsequence m_k^i of m_k^{i-1} such that

$$\lim_{k\to\infty} r_{m_k^i}^i = l_i. \tag{4.8}$$

Finally one decides to set $n_k := m_k^k$. For any given i the terms of the sequence n_k belongs to m_k^i for $k \geq i$ and hence (4.5) holds.

□

Lemma 4.4. *Let $(w_n)_{n\in\mathbb{Z}}$ be a sequence of numbers in $[0,1]$ satisfying*

$$w_n = \sum_{k=1}^{\infty} w_{n-k} f_k, \tag{4.9}$$

then, if $w_0 = 1$, all the w_n are also equal to one.

Proof. From the assumption that the f_k are positive and sum to one.

$$w_0 = \sum_{k=1}^{\infty} w_{-k} f_k, \tag{4.10}$$

implies that the w_{-k} are all equal to one. It is then immediate by recursion to obtain the result for the positive integers. □

3 Proof of the renewal Theorem

Set

$$\eta = \limsup_{n\to\infty} u_n \tag{4.11}$$

and let (u_{m_k}), $k \geq 0$ be a subsequence converging to η. Now if we consider the sequences r^i, $i \in \mathbb{Z}$,

$$r^i_n := u_{m_k+i}, \tag{4.12}$$

By Lemma 4.3 one can extract from m_k, a subsequence n_k, $k \geq 0$ along with all the u_{n_k+i}, converge simultaneously. Let η_i denote the corresponding limits. Necessarily $0 \leq \eta_i \leq \eta$ and $\eta_0 = \eta$. As we have for all $i \in \mathbb{Z}$

$$u_{n_k+i} = \sum_{j=0}^{n_k+i} f_j u_{n_{k+i-j}}, \tag{4.13}$$

passing to the limit we have

$$\eta_i = \sum_{j=0}^{\infty} f_j \eta_{i-j}. \tag{4.14}$$

and hence from Lemma 4.4 we have

$$\forall i \in \mathbb{Z}, \quad \eta_i = \eta. \tag{4.15}$$

By summing the recursive equation defining f from we have

$$\sum_{i=0}^{n} u_i = f_1 u_{n-1} + (f_1 + f_2) u_{n-2} + \cdots + (f_1 + \cdots + f_n)\, u_0 + 1. \tag{4.16}$$

And hence if one sets

$$\rho_r := \sum_{n=r+1}^{\infty} f_n = 1 - \sum_{n=1}^{r} f_n$$

we have for all N

$$\sum_{i=0}^{n} \rho_i u_{n-i} = 1. \tag{4.17}$$

Note that ρ_n tends to zero and is summable

$$\sum_{i=0}^{\infty} \rho_i = \mu. \tag{4.18}$$

Passing to the limit along the subsequence n_k and using the dominate convergence Theorem, we have

$$1 = \lim_{n\to\infty} \sum_{i=0}^{n} \rho_i u_{n-i} = \left(\sum_{i=0}^{\infty} \rho_i \eta_{-i} \right) = \mu\eta = 1, \tag{4.19}$$

and hence $\eta = \mu^{-1}$. Note that it ends the proof in the case $\mu = \infty$.

Now we can show that u_n converges to η. Let us fix ε. From the definition of η note that all terms of the sequence u are smaller than $\eta+\varepsilon$ after a certain rank. Hence for any fixed r, for n sufficiently large, one has (recall (4.17) and note that $\rho_0 = 1$)

$$1 = \sum_{i=0}^{n} \rho_i u_{n-i} \leq u_n + \sum_{i=1}^{r} \rho_n(\eta+\varepsilon) + \sum_{i=r+1}^{n} \rho_i \tag{4.20}$$

Hence choosing r such that $\sum_{i=r+1}^{\infty} \rho_i \leq \varepsilon$ we have

$$u_n + (\mu-1)(\eta+\varepsilon) + \varepsilon = (u_n - \eta) + \mu(\eta+\varepsilon) \geq 1 \tag{4.21}$$

Which implies that for all large n,

$$u_n \geq \eta - \mu\varepsilon. \tag{4.22}$$

□

Chapter 5

A weakly inhomogeneous model: periodic disorder

Until now, we have supposed that the energy reward (or penalty) for a contact with the wall was the same along the whole trajectory.

However, several physical consideration may lead us to consider models where this is not the case:

- A first reason is that polymers can be formed by combining monomers of several types (these polymers are called heteropolymer), these different monomers can be combined in a periodic pattern or a randomly.
- Secondly the substrate with which polymer interact and the polymer itself might present impurity.

For this reason we alter our previously defined polymer model to introduce inhomogeneity in the interaction. Let $(\omega_n)_{n\geq 1}$ be a sequence of real numbers. We might assume to fix ideas that ω has "zero mean"

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \omega_n = 0. \tag{5.1}$$

Given a renewal τ with inter-arrival distribution $K(\cdot)$ which satisfies (3.6) (as usual, we let $\mathbf{P}$ denote the associated probability), two real parameters β (which quantifies amplitude of variation of the reward) and h (the mean reward), $N \in \mathbb{N}$. Using the notation $\delta_n := \mathbf{1}_{\{n\in\tau\}}$. we define

$$\frac{\mathrm{d}\mathbf{P}^{\beta,\omega}_{N,h}}{\mathrm{d}\mathbf{P}}(\tau) := \frac{1}{Z^{\beta,\omega}_{N,h}} \exp\left(\sum_{n=1}^{N} (\beta\omega_n + h)\delta_n\right) \delta_N, \tag{5.2}$$

where

$$Z^{\beta,\omega}_{N,h} := \mathbf{E}\left[\exp\left(\sum_{n=1}^{N} (\beta\omega_n + h)\delta_n\right)\right]. \tag{5.3}$$

We assume without loss of generality (see the proof of Proposition 3.7) that

$$\sum_{n\geq 1} K(n) = 1.$$

Our aim remains to understand the asymptotic behavior of τ under $\mathbf{P}^{\beta,\omega}_{N,h}$ and in particular we would like study the effect of the presence of homogeneity on the localization transition. We study the system for a fixed value of β and try identify the characteristics of the phase transition when h varies.

The present chapter is focused on the case where ω is a periodic sequence . This case is referred to as "weakly inhomogeneous" since the inhomogeneity can in some sense be washed out by considering steps of size proportional to the period of ω. Our aim is to show that this periodic model is still "solvable" in the sense of Proposition 3.5. However the solution has a more complex expression and it requires more advanced tools to derive the critical behavior

We study a truly inhomogeneous version of the model in Chapter 6, where we consider ω to be given by the realization of a sequence of IID random variables.

1 Existence of the free energy in the periodic setup

For simplicity we consider only the case where the period $T(\omega)$ associated to ω is equal to 2. While the proof can easily be extended to the general case (see [5, Chapter 3]) but this restriction considerably simplifies the notation.

Up to a change for β and a translation in h, we can thus restrict to the case $\omega_n := (-1)^n$ (allowing β to be negative if must be).

Let us check that in that case, the free-energy is well defined.

Proposition 5.1. *Assuming that*

$$K(N) \overset{N\to\infty}{\sim} C_K N^{-(1+\alpha)},$$

the following limit

$$\textsc{f}(\beta,h) := \lim_{N\to\infty} \frac{1}{N} \log Z^{\beta,\omega}_{N,h}, \tag{5.4}$$

is well defined and $h \mapsto \textsc{f}(\beta,h)$ *is non-negative, non-increasing, convex, and*

$$h_c(\beta) := \max\{h \in \mathbb{R} \ : \ \textsc{f}(\beta,h) > 0\} \in [-|\beta|, |\beta|].$$

Proof. We have for any N, M

$$Z^{\beta,\omega}_{2N+M,h} \geq \mathbf{E}\left[e^{\sum_{n=1}^{2} N(\beta\omega_n+h)\delta_n}\delta_{2N} e^{\sum_{n=2N+1}^{2N+M}(\beta\omega_n+h)\delta_n}\delta_{2(N+M)}\right] = Z^{\beta,\omega}_{2N,h} Z^{\beta,\omega}_{2M,h} \tag{5.5}$$

Hence $\log Z^{\beta,\omega}_{2N,h}$ is a super-additive sequence and thus $\frac{1}{2N}\log Z^{\beta,\omega}_{2N,h}$ converges. To ensure that convergence holds also along the sequence of odd integers, it is sufficient to observe that for every $N \geq 2$

$$e^{-|\beta|+h} K(1) Z^{\beta,\omega}_{N-1,h} \leq Z^{\beta,\omega}_{N,h} \leq \left(e^{-|\beta|+h}K(1)\right)^{-1} Z^{\beta,\omega}_{N+1,h}. \tag{5.6}$$

The lower bound is obtained by considering the contribution of trajectories visiting $N-1$ and the upper bound is exactly the lower bound applied to $N+1$. Finally, using the notation $Z_{N,h} = Z^{0,\omega}_{N,h}$ for the partition function of the homogeneous system it is straightforward to observe that

$$Z^{\beta,\omega}_{N,h-|\beta|} \leq Z^{\beta,\omega}_{N,h} \leq Z^{\beta,\omega}_{N,h+|\beta|}$$

which setting $\textsc{f}(h) = \textsc{f}(0,h)$ yields

$$\textsc{f}(h-|\beta|) \leq \textsc{f}(\beta,h) \leq \textsc{f}(h+|\beta|)$$

and the corresponding inequality on h. □

Our concern is now to try to identify the value of $h_c(\beta)$. A particular point of concern is to show that $\textsc{f}(\beta,0) > 0$, that is, the system is localized when $h = 0$. This would prove that that there exists a strategy to localize even when the mean reward is zero.

Another question is to identify the critical exponent associated to $\textsc{f}(\beta,\cdot)$, and see if there is a dependence in β.

2 Writing the partition function in matrix form

We try to adapt the strategy adopted in Section 2. Recall that starting point in the homogeneous case was to express the partition function in terms of convolution of K and then to replace K by another function $\widetilde{K}$ in order to absorb the extra-factor produced by the pinning reward.

Setting by convention $Z_{0,h} = 1$ we have

$$Z^{\beta,\omega}_{N,h} = \sum_{n=1}^{N} Z^{\beta,\omega}_{N-n,h} e^h K(n) \exp\left(h + \beta(-1)^N\right). \tag{5.7}$$

The main concern with this expression is that the term $K(n)\exp\left(h+\beta(-1)^N\right)$ depends not only on n but also on the parity of $N-n$.

To find a way out of this we decide to add an additional variable in the problem, which is the parity of N. We define a line vector

$$\mathbf{Z}_{N,h} = \begin{cases} (Z^{\beta,\omega}_{N,h}, 0) \text{ if } N \text{ is even },\\ (0, Z^{\beta,\omega}_{N,h}) \text{ if } N \text{ is odd }. \end{cases} \tag{5.8}$$

and try to write (5.7) like a matrix convolution. For this sake we introduce

$$M_\beta(n) = \begin{cases} \begin{pmatrix} e^\beta K(n) & 0 \\ 0 & e^{-\beta}K(n) \end{pmatrix} \text{ if } n \text{ is even,} \\ \\ \begin{pmatrix} 0 & e^{-\beta}K(n) \\ e^\beta K(n) & 0 \end{pmatrix} \text{ if } N \text{ is odd }. \end{cases} \tag{5.9}$$

With this notation the reader can check that we have

$$\mathbf{Z}_{N,h} := e^h \sum_{n=1}^{N} \mathbf{Z}_{N-n,h} M_\beta(n). \tag{5.10}$$

Now let us introduce the convolution power of a sequence of matrices $((A(n))_{n\geq 0})$ being defined by

$$A^{*0}(n) = \mathbf{1}_0(n) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(from now on we will write I_2 for the identity matrix when more practical) and for $k \geq 1$

$$A^{*k}(n) := \sum_{\substack{(l_i,\ldots,l_k) \\ \sum_{i=1}^k l_i = N}} M_\beta(l_1) M_\beta(l_2) \cdots M_\beta(l_k). \tag{5.11}$$

A rather direct consequence of (5.10) is the following

Lemma 5.2. *We have for every $N \geq 0$*

$$\mathbf{Z}_{N,h} := \begin{pmatrix} 1 & 0 \end{pmatrix} \sum_{k=0}^{\infty} e^{kh} M_\beta^{*k}(N) \tag{5.12}$$

Proof. Let us prove the statement by induction on N. The statement is easily checked for $N = 0$, and the result follows from the fact that for $N \geq 1$.

$$\begin{aligned} \sum_{n=0}^{N-1} \mathbf{Z}_{N-n,h} M_\beta(n) &= \sum_{k=0}^{\infty} e^{(k+1)h} \sum_{n=1}^{N} M_\beta^{*k}(N-n) M_\beta(n) \\ &= \sum_{k=0}^{\infty} e^{(k+1)h} M_\beta^{*(k+1)}(N) = \sum_{k=0}^{\infty} e^{kh} M_\beta^{*k}(N). \end{aligned} \tag{5.13}$$

where in the last equality we used the fact that $M_\beta^{*0}(N) = 0$. □

3 Characteristics of the phase transition

To compute the free-energy in the homogeneous case, the idea was to cancel the term e^{kh} by introducing an inter-arrival function $\widetilde{K}$ that would include it. Here we are working with matrices, and the analogous of a renewal process, would be a Markov process which includes information about the parity.

Let us now present the matrix formulation of the condition $e^h \sum_{n=1} e^{-bn} K(n) = 1$ which appeared in the case $\beta = 0$. For $b \geq 0$ we define

$$\widehat{M}_\beta(b) := \sum_{n\geq 1} e^{-bn} M_\beta(n). \tag{5.14}$$

We let $\lambda_\beta(b)$ be the largest eigenvalue of $\widehat{M}_\beta$

Having a 2×2 matrix, we have no problem in giving an expression for $\lambda_\beta(b)$, however to justify its existence in the more general case we can rely on Perron-Frobenius theory.

Proposition 5.3 (Perron-Frobenius Theorem)**.** *Let A be an $T \times T$ matrix with positive coefficients, let $\lambda_A = \max_{\lambda \in \mathrm{Sp}(A)} \lambda$ be its spectral radius.*

(i) *Λ_A is a simple eigenfunction of A and the left and right eigenvectors associated to it have positive coordinates.*

(ii) *We have $\max_{\lambda \in \mathrm{Sp}(A) \setminus \lambda_A} |\lambda| < \lambda_A$.*

(iii) *λ_A is a strictly increasing function of every coefficient of A in the sense that if B is non-zero matrix with non negative coefficients*

$$\lambda_{A+B} > \lambda_A. \tag{5.15}$$

We also have

$$\lambda_{A+B} \leq \lambda_A + \lambda_B. \tag{5.16}$$

Remark 5.4. *The last point of the theorem can be deduced from the fact that*

$$\lambda_A = \max_{v \in (0,\infty)^n} \min_i \frac{(Av)_j}{v_j}.$$

Remark 5.5. *As a consequence of (ii), $b \mapsto \lambda_\beta(b)$ is a decreasing function. Note also that $\lambda_\beta(b)$ is a single root of a polynomial with analytic coefficients and thus it is an analytic function of b.*

We are now ready to state our result

Theorem 5.6. *Under assumption* (3.6) *we have*

$$\mathrm{F}(\beta, h) = \begin{cases} 0 & \text{if } h \leq -\log \lambda_\beta(0), \\ \mathrm{G}_\beta^{-1}(h) & \text{if } h \geq -\log \lambda_\beta(0). \end{cases} \tag{5.17}$$

where $\mathrm{G}_\beta :\ \mathbb{R}_+ \to (-\infty, -\log \lambda_\beta(0)]$, is defined by

$$\mathrm{G}_\beta(b) = -\log \lambda_\beta(b).$$

In particular $h_c(\beta) := -\log \lambda_\beta(0)$.

Note that according to the previous remarks, G_β is an analytic function in b and thus $\mathrm{F}(\beta, h)$ is analytic in h out of the critical point .

We prove the result in the next section and quickly deduce some properties from the above theorem. We define $\widehat{K}_0$ and $\widehat{K}_1$ as follows

$$\begin{aligned} \widehat{K}_0(b) &:= \sum_{n \geq 1} e^{-2nb} K(2n), \\ \widehat{K}_1(b) &:= \sum_{n \geq 0} e^{-2(n+1)b} K(2n+1). \end{aligned} \tag{5.18}$$

We let $p_0 := \widehat{K}_0(0)$ and $p_1 := \widehat{K}_1(0)$ denote the respective probabilities of observing jump of even and odd side.

Note that we have

$$\widehat{M}_\beta(b) = \begin{pmatrix} e^\beta \widehat{K}_0(b) & e^{-\beta} \widehat{K}_1(b) \\ e^\beta \widehat{K}_1(b) & e^{-\beta} \widehat{K}_0(b) \end{pmatrix}. \tag{5.19}$$

Proposition 5.7. *(i) We have* $h_c(\beta) = -\log\left(p_0\cosh(\beta) + \sqrt{\sinh(\beta)^2 p_0^2 + p_1^2}\right) < 0$, *and*

$$\mathrm{G}_\beta(b) := -\log\left(\widehat{K}_0(b)\cosh(\beta) + \sqrt{\sinh(\beta)^2\widehat{K}_0(b)^2 + \widehat{K}_1(b)^2}\right).$$

(ii) Under assumption (3.6)*, there exists a constant* C *(depending on* β*) such that for all* $b \in [0,1]$

$$C^{-1}b^{\min(1,\alpha)} \leq \mathrm{G}_\beta(b) - \mathrm{G}_\beta(0) \leq Cb^{\min(1,\alpha)}, \tag{5.20}$$

and thus there exists C *for all* $u \in [0,1]$

$$C^{-1}u^{\max(1,\alpha^{-1})} \leq \mathrm{F}(\beta, h_c(\beta)+u) \leq Cu^{\max(1,\alpha^{-1})}. \tag{5.21}$$

Proof. The respective determinant and trace of $\widehat{M}_\beta(b)$ are given by

$$\mathrm{tr}(\widehat{M}_\beta(b)) = 2\cosh(\beta)\widehat{K}_0(b) \quad \text{and } \mathrm{Det}(\widehat{M}_\beta(b)) = \widehat{K}_0(b)^2 - \widehat{K}_1(b)^2. \tag{5.22}$$

and thus $\lambda_\beta(b)$ is the largest root of the polynomial

$$X^2 - 2\cosh(\beta)\widehat{K}_0(b)X + \widehat{K}_0(b)^2 - \widehat{K}_1(b)^2$$

which yields immediately

$$\lambda_\beta(b) = \widehat{K}_0(b)\cosh(\beta) + \sqrt{\left(\cosh(\beta)^2 - 1\right)\widehat{K}_0(b)^2 + \widehat{K}_1(b)^2} \tag{5.23}$$

and thus the first point.

The second point could be proved directly by performing a Taylor expansion of (5.23) but we wish to give a proof which extends to higher values of the period $T(\omega)$ We observe that we have for $i = 0, 1$

$$p_i - \widehat{K}_i(b) \overset{b\to 0}{\sim} C_{i,K} b^{\min(1,\alpha)}. \tag{5.24}$$

And thus, using $\leq$ for the partial order induced by comparison by coordinate, there exists a constant C such that for all $b \in [0,1]$

$$C^{-1}b^{\min(1,\alpha)}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \leq \widehat{M}_\beta(0) - \widehat{M}_\beta(b) \leq Cb^{\min(1,\alpha)}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} \tag{5.25}$$

With (5.15) the lower bound asserts that the spectral radius of $\widehat{M}_\beta(0)$ is larger than that of $\widehat{M}_\beta(b) + C^{-1}b^{\min(1,\alpha)}I_2$ meaning

$$\lambda_\beta(0) \geq \lambda_\beta(b) + C^{-1}b^{\min(1,\alpha)}.$$

From (5.16), the upper bound implies that

$$\lambda_\beta(0) \geq \lambda_\beta(b) + 2Cb^{\min(1,\alpha)}.$$

which ends the proof of (5.20). Equation (5.20) implies immediately that $\mathrm{G}_\beta(b) - \mathrm{G}_\beta(0)$ is of order $b^{\min(1,\alpha)}$, which in turns implies (5.21)

□

4 Proof of Theorem 5.6

Let $b \geq 0$ be such that $\lambda_\beta(b) = e^{-h}$ if the equation has a solution and $b = 0$ if not. We set

$$\widetilde{M}(n) := e^{-bn} e^h M_\beta(n)$$

Observe that $\sum_{n\geq 0} \widetilde{M}(n) = e^h \widehat{M}_\beta(b)$ is a matrix of spectral radius one (if $b > 0$) or smaller (if $b = 0$). Using Lemma 5.2 and the fact that $Z_{N,h}^{\beta,h}$ is obtained by summing the two coordinates of $\mathbf{Z}_{N,h}$

$$Z_{N,h}^{\beta,h} = \begin{pmatrix} 1 & 0 \end{pmatrix} \sum_{k=0}^{\infty} e^{kh} M_\beta^{*k}(N) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = e^{bN} \begin{pmatrix} 1 & 0 \end{pmatrix} \sum_{k=0}^{\infty} \widetilde{M}^{*k}(N) \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{5.26}$$

To show that b is the free energy we must show that the term that remains does decay or increase exponentially

$$\lim_{N\to\infty} \frac{1}{N} \log \begin{pmatrix} 1 & 0 \end{pmatrix} \sum_{k=0}^{\infty} \widetilde{M}^{*k}(N) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0. \tag{5.27}$$

The case $b > 0$ Let us start we the localized phase. We want, as in the homogeneous case, get an interpretation of $\sum_{k=0}^{\infty} \widetilde{M}^{*k}(N)$ in terms of renewal. For this sake, let us consider $\xi = \begin{pmatrix} \xi_0 \\ \xi_1 \end{pmatrix}$ the right eigenvector of $\widetilde{M}$ associated to eigenvalue 1 and define

$$A_\xi := \begin{pmatrix} \xi_0 & 0 \\ 0 & \xi_1 \end{pmatrix}.$$

We define the matrix $\Gamma(n)$ as the conjugate of $\widetilde{M}$ by A_ξ

$$\Gamma(n) := A_\xi^{-1} \widetilde{M}(n) A_\xi = \begin{cases} K(n) e^{h-bn} \begin{pmatrix} e^\beta & 0 \\ 0 & e^{-\beta} \end{pmatrix} & \text{if } n \text{ is even,} \\ \\ K(n) e^{h-bn} \begin{pmatrix} 0 & e^{-\beta}(\xi_1/\xi_0) \\ e^\beta(\xi_0/\xi_1) & 0 \end{pmatrix} & \text{if } n \text{ is odd.} \end{cases} \tag{5.28}$$

Note that

$$\left(\sum_{n\geq 1} \Gamma(n) \right) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = A_\xi^{-1} \left(\sum_{n\geq 1} \widetilde{M}(n) \right) A_\xi \begin{pmatrix} 1 \\ 1 \end{pmatrix} = A_\xi^{-1} \left(\sum_{n\geq 1} \widetilde{M}(n) \right) \xi = A_\xi^{-1} \xi = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{5.29}$$

Thus any $i \in \{0, 1\}$ we have, $\sum_{n\geq 1} \sum_{j\in\{0,1\}} \Gamma_{i,j}(n) = 1$.

We can thus define a process $(J, \widetilde{\tau}) = (J_k, \widetilde{\tau}_k)_{k\geq 0}$: First we define a Markov process $(J, \eta)_{k\geq 0}$ with the following transition

$$\widetilde{\mathbf{P}}\left[(J_k, \eta_k) = (j, n) \mid (J_{k-1}, \eta_{k-1}) = (i, m) \right] = \Gamma_{i,j}(n). \tag{5.30}$$

This means that with the convention $j_0 = 0$ we have

$$\widetilde{\mathbf{P}}\left[\forall m \in [\![1,k]\!],\ (J_m, \eta_m) = (j_m, n_m)\right] = \prod_{m=1}^{k} \Gamma_{j_{m-1},j_m}(n_m). \tag{5.31}$$

Then we set

$$\widetilde{\tau}_k := \sum_{m=1}^{k} \eta_k.$$

Note that $(J, \widetilde{\tau})$ itself a Markov process with transition

$$\widetilde{\mathbf{P}}\left[(J_k, \widetilde{\tau}_k) = (j,n) \mid (J_{k-1}, \widetilde{\tau}_{k-1}) = (i,m)\right] = \Gamma_{i,j}(n-m). \tag{5.32}$$

For this reason, it is called a Markov renewal process. With our choice for Γ, one can check that J_k is the parity of $\widetilde{\tau}_k$.

Note that as $M(n) = A_\xi \Gamma(n) A_\xi^{-1}$ we have

$$\begin{aligned} e^{-bn} Z_{N,h}^{\beta,\omega} = \begin{pmatrix} 1 & 0 \end{pmatrix} \sum_{k=0}^{\infty} \widetilde{M}^{*k}(N) \begin{pmatrix} 1 \\ 1 \end{pmatrix} &= \begin{pmatrix} 1 & 0 \end{pmatrix} A_\xi \left(\sum_{k=0}^{\infty} \widetilde{\Gamma}^{*k}(N) \right) A_\xi^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} \xi_0 & 0 \end{pmatrix} \left(\sum_{k=0}^{\infty} \widetilde{\Gamma}^{*k}(N) \right) A_\xi^{-1} \begin{pmatrix} \xi_0^{-1} \\ \xi_1^{-1} \end{pmatrix} \end{aligned} \tag{5.33}$$

This is equivalent to

$$e^{-bn} Z_{N,h}^{\beta,\omega} = \begin{cases} \begin{pmatrix} 1 & 0 \end{pmatrix} \left(\sum_{k=0}^{\infty} \Gamma^{*k}(N) \right) \begin{pmatrix} 1 \\ 1 \end{pmatrix} & \text{if } N \text{ is even },\\ (\xi_0/\xi_1) \begin{pmatrix} 1 & 0 \end{pmatrix} \left(\sum_{k=0}^{\infty} \Gamma^{*k}(N) \right) \begin{pmatrix} 1 \\ 1 \end{pmatrix} & \text{if } N \text{ is odd.} \end{cases} \tag{5.34}$$

Now, from the definition of $\widetilde{\tau}$ is follows that

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \left(\sum_{k=0}^{\infty} \Gamma^{*k}(N) \right) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \sum_{k=1}^{\infty} \sum_{\substack{(n_1,\dots,n_k) \\ \sum_{m=1}^{k} n_m = N}} \sum_{(j_m)_{m=1}^{k} \in \{0,1\}^k} \Gamma_{j_{k-1},j_k}(n_k) = \widetilde{\mathbf{P}}[N \in \widetilde{\tau}]. \tag{5.35}$$

Now $\widetilde{\tau}$ not being a renewal we cannot apply the renewal theorem. Let us define

$$\widetilde{\tau}' := \{ n \in \mathbb{N} \ : \ 2n \in \widetilde{\tau} \}$$

Lemma 5.8. *The process $\widetilde{\tau}'$ is a recurrent renewal process,*

$$\widetilde{\mathbf{E}}[\tau_1'] < \infty$$

Proof. The first thing to prove is that the inter-arrival in τ' are IID, that is, that

$$\widetilde{\mathbf{P}}[\forall k \in [\![0,m]\!], \widetilde{\tau}_k' - \widetilde{\tau}_{k-1}' = n_k] = \prod_{k=1}^{m} \widetilde{\mathbf{P}}[\widetilde{\tau}_k' = n_k]. \tag{5.36}$$

This can be either deduced from (5.31), summing over all possible trajectories (using the fact that as $J_n = 0$ when $\widetilde{\tau}'_k = \widetilde{\tau}_n$ to factorize the sum), or by iteratively using the strong Markov property (the index T_k such that $\widetilde{\tau}'_k = \widetilde{\tau}_{T_k}$ is a stopping time for the Markov chain (J, τ)).

Let us now show that $\mathbf{P}[\widetilde{\tau}'_1 < \infty] = 1$. $\widetilde{\tau}$ being a recurrent renewal process we have to show that

$$\widetilde{\mathbf{P}}[\forall k \geq 1,\ \widetilde{\tau}_k \text{ is odd }] = \widetilde{\mathbf{P}}[\forall k \geq 1,\ J_1 = 1] = 1. \tag{5.37}$$

We have

$$\widetilde{\mathbf{P}}[\forall k \in [\![1, m]\!] \widetilde{\tau}_k \text{ is odd }] = \left(\sum_{n\geq 1} \Gamma_{0,1}(2n-1)\right) \left(\sum_{n\geq 1} \Gamma_{1,1}(2n)\right)^{m-1}, \tag{5.38}$$

which obviously goes to zero when m goes to infinity.

Finally let us check that the renewal has finite mean. We have

$$\widetilde{\mathbf{P}}[\widetilde{\tau}'_1 = n] = \Gamma_{0,0}(2n) + \sum_{k=1}^{\infty} \sum_{\substack{(l_0,\dots,l_k) \\ \sum_{i=1}^k l_i = n+1}} \Gamma_{0,1}(2l_0 - 1) \left(\prod_{i=1}^{k-1} \Gamma_{1,1}(2l_i)\right) \Gamma_{1,0}(2l_k - 1)$$

$$=: \Gamma_{0,0}(2n) + \sum_{k=1}^{\infty} \Lambda(k). \tag{5.39}$$

Note that there exists a constant such that $\Gamma_{i,j}(n) \leq Ce^{-bn}$ for every n thus all the terms appearing in the sum $\Lambda(k)$ can be bounded by $C^K e^{-bn}$. Are there are at most n^k choices for $(l_0, \dots, l_k)$ this yields

$$\Lambda(k) \leq (Cn)^k e^{-bn}.$$

This bound is sufficient for small values of k but not for large ones. However we have

$$\Lambda(k) \leq \widetilde{\mathbf{P}}[\forall i \in [\![1, k-1]\!] \widetilde{\tau}_i \text{ is odd }] \leq \left(\sum_{n\geq 1} \Gamma_{1,1}(2n)\right)^{i-2} \leq e^{-c(i-2)}. \tag{5.40}$$

Using this last bound for $k \geq \sqrt{n}$ and the other one for $k < \sqrt{n}$ we obtain for some positive constants c and C

$$\widetilde{\mathbf{P}}[\widetilde{\tau}'_1 = n] \leq \sqrt{n}(Cn)^{\sqrt{n}} e^{-bn} + Ce^{-c\sqrt{n}}. \tag{5.41}$$

from which we deduce that

$$\widetilde{\mathbf{E}}[\widetilde{\tau}'_1] = \sum_{n\geq 1} n \widetilde{\mathbf{P}}[\widetilde{\tau}'_1 = n] < \infty$$

□

Thus using the renewal Theorem, $\widetilde{\mathbf{P}}[2N \in \widetilde{\tau}]$, converges to $(\mathbf{E}[\widetilde{\tau}'_1])^{-1}$. From this it is trivial to see that $\widetilde{\mathbf{P}}[(2N+1) \in \widetilde{\tau}]$ is bounded away from zero.

Finally we obtain that there exists a constant C (depending on β and h) such that

$$C^{-1} e^{bN} \leq Z_{N,h}^{\beta,\omega} \leq C e^{bN}$$

Case $b = 0$**,** $\lambda_\beta(0) = e^{-h}$ In this case what has been said before remains valid except that τ_1' possibly has infinite mean. However this does not matter since the important part is the upper bound, since we already know that $\textsc{f}(\beta, h)$ is positive. As $\widetilde{\mathbf{P}}[N \in \widetilde{\tau}] \leq 1$ we thus obtain

$$e^{-|\beta|+h} K(n) \leq Z_{N,h}^{\beta,\omega} \leq C. \tag{5.42}$$

Case $b = 0$**,** $\lambda_\beta(0) < e^{-h}$ In that case $\sum_{n\geq 1} \widetilde{M}(n)$ has a spectral radius smaller than one but (5.28) still makes sense if one consider ξ to be the right Perron-Frobenius eigenvector. Then $\sum_{n\geq 1} \Gamma(n)$ is a sub-stochastic Matrix, and we define a process $(J, \widetilde{\tau})$ using it, with the additional assumption when $J_{k-1} = i$ the process is killed with rate

$$1 - \sum_{\substack{n\geq 1 \\ j\in\{0,1\}}} \Gamma_{i,j}(n).$$

We recover (5.42) again by noticing that $\widetilde{\mathbf{P}}[N \in \widetilde{\tau}] \leq 1$.

Chapter 6

Adding disorder into the game

1 The Poland-Scherhaga Model for DNA

2 Existence and Self averaging of the free-energy

In this chapter and the following ones, we will always assume that $K(\infty) = 0$ for simplicity.

Let us fix $h \in \mathbb{R}$ and $\beta > 0$ to be fixed parameters. Let $(\omega_n)_{n \geq 1}$ be a fixed realization of a sequence of random variables. We assume that

$$\mathbb{E}[|\omega_n|] < \infty$$

We define $\mathbf{P}_{N,h}^{\beta,\omega}$ by its density with respect to $\mathbb{P}$

$$\frac{\mathrm{d}\mathbf{P}_{N,h}^{\beta,\omega}}{\mathrm{d}\mathbf{P}} = \frac{1}{Z_{N,h}^{\beta,\omega}} \exp\left(\sum_{n=1}^{N} (\beta\omega_n + h)\delta_n\right) \delta_N \tag{6.1}$$

where

$$\delta_n := \mathbf{1}_{\{n \in \tau\}}$$

and

$$Z_{N,h}^{\beta,\omega} := \mathbf{E}\left[\exp\left(\sum_{n=1}^{N} (\beta\omega_n + h)\delta_n\right) \delta_N\right]. \tag{6.2}$$

We can define the free-energy of this system

Theorem 6.1. *The free-energy of the disordered model defined by*

$$\textsc{f}(\beta, h) := \lim_{N \to \infty} \frac{1}{N} \log Z_{N,h}^{\beta,\omega}, \tag{6.3}$$

is well defined and non-random. Moreover we have

$$\textsc{f}(\beta, h) = \lim_{N \to \infty} \frac{1}{N} \mathbb{E}\left[\log Z_{N,h}^{\beta,\omega}\right] = \sup_{N} \frac{1}{N} \mathbb{E}\left[\log Z_{N,h}^{\beta,\omega}\right] \tag{6.4}$$

For simplicity we prove the result under the assumption that (3.6) holds for some $\alpha > 0$, but the result holds in full generality. The proof can be decomposed in three steps

Lemma 6.2. *The following limit exists*

$$\textsc{f} = \lim_{N\to\infty} \frac{1}{N}\mathbb{E}\left[\log Z_{N,h}^{\beta,\omega}\right] \tag{6.5}$$

and it is equal to the sup *of the same sequence.*

Proof. According to Fékété's Lemma it is sufficient to prove that

$$\frac{1}{N}\mathbb{E}\left[\log Z_{N,h}^{\beta,\omega}\right]$$

is a super additive sequence. First note that

$$\log K(N) + \beta\omega_N + h \le \log Z_{N,h}^{\beta,\omega} \le \sum_{n=1}^{N} \beta|\omega_n| + |h|. \tag{6.6}$$

Hence that ensures that the expectation of $\log Z$ exists and that the free energy is finite.

Let θ_N denote the shift-operator on ω: the sequence $\theta_N\omega$ is defined by

$$(\theta_N\omega)_n = \omega_{n+N} \tag{6.7}$$

Using the Markov property for the renewal

$$\begin{aligned} Z_{N+M}^{\beta,h,\omega} &= \mathbf{E}\left[\exp\left(\sum_{n=1}^{N+M}(\beta\omega_n+h)\delta_n\right)\delta_{N+M}\right] \ge \mathbf{E}\left[\exp\left(\sum_{n=1}^{N+M}(\beta\omega_n+h)\delta_n\right)\delta_N\delta_{N+M}\right] \\ &= \mathbf{E}\left[\exp\left(\sum_{n=1}^{N}(\beta\omega_n+h)\delta_n\right)\delta_N\right]\mathbf{E}\left[\exp\left(\sum_{n=1}^{M}(\beta\omega_{n+N}+h)\delta_n\right)\delta_M\right] = Z_{N,h}^{\beta,\omega} Z_{M,h}^{\beta,\theta_N\omega} \end{aligned} \tag{6.8}$$

Now for a fixed N note that the distribution of ω and that of $\theta_N\omega$ are identical. Hence

$$\mathbb{E}\left[Z_{N+M}^{\beta,h,\omega}\right] \ge \mathbb{E}\left[\log Z_{N,h}^{\beta,\omega}\right] + \mathbb{E}\left[\log Z_{M,h}^{\beta,\theta_N\omega}\right] = \mathbb{E}\left[\log Z_{N,h}^{\beta,\omega}\right] + \mathbb{E}\left[\log Z_{M,h}^{\beta,\omega}\right] \tag{6.9}$$

and the result follows by super-additivity. □

Lemma 6.3. *We have almost surely*

$$\liminf_{N\to\infty} \frac{1}{N}\log Z_{N,h}^{\beta,\omega} \ge \textsc{f} \tag{6.10}$$

Proof. Given $\varepsilon > 0$, let N_0 be an integer such that

$$\frac{1}{N_0}\mathbb{E}\left[\log Z_{N_0}^{\beta,h,\omega}\right] \ge \textsc{f} - \varepsilon. \tag{6.11}$$

Now let m be an integer. By iterating the inequality (6.8) we obtain that

$$\frac{1}{N_0 m}\log Z_{N_0 m}^{\beta,h,\omega} \ge \frac{1}{m}\sum_{i=0}^{m-1}\frac{1}{N_0}\log Z_{N_0}^{h,\beta,\theta_{iM}\omega}. \tag{6.12}$$

As the ωs are IID, $\frac{1}{N_0}\log Z_{N_0}^{h,\beta,\theta_{iM}\omega}$ are also IID random variables with finite mean. Hence by the law of large number we have

$$\liminf_{m\to\infty}\frac{1}{N_0 m}\log Z_{N_0 m}^{\beta,h,\omega}\geq\frac{1}{N_0}\mathbb{E}\left[\log Z_{N_0}^{\beta,h,\omega}\right]\geq\lim_{m\to\infty}\frac{1}{m}\sum_{i=0}^{m-1}\frac{1}{N_0}\log Z_{N_0}^{h,\beta,\theta_{iM}\omega}$$
$$=\mathbb{E}\left[\log Z_{N_0}^{\beta,h,\omega}\right]\geq\textsc{f}-\varepsilon \quad (6.13)$$

For the case of general N we consider the Euclidean division $N=mN_0+q$ for $q=1,\dots,N_0-1$ fixed. From the same computation we have

$$\liminf_{N\to\infty}\frac{1}{N}\log Z_q^{\beta,h,\omega}\geq\lim_{N\to\infty}\frac{m}{N}\frac{1}{m}\sum_{i=0}^{m-1}\log Z_{N_0}^{h,\beta,\theta_{iN_0}\omega}+\lim_{N\to\infty}\frac{1}{N}\log Z_q^{h,\beta,\theta_{mN_0}\omega}. \quad (6.14)$$

The first term converges by the law of large number to $\frac{1}{N_0}\mathbb{E}\left[\log Z_{N_0}^{\beta,h,\omega}\right]\geq\textsc{f}-\varepsilon$. It remains to show that the second term converges to zero. We have

$$\left|\log Z_q^{h,\beta,\theta_{mN_0}\omega}\right|\leq|\log P(q\in N)|+\sum_{n=mN_0+1}^{mN_0+q}|\beta\omega_n+h|\,. \quad (6.15)$$

Hence

$$\max_{q\in\{0,\dots,N_0-1\}}\left|\log Z_q^{h,\beta,\theta_{mN_0}\omega}\right|\leq\beta\sum_{n=mN_0+1}^{(m+1)N_0-1}|\omega_n|+C(N_0,h). \quad (6.16)$$

and hence the r.h.s. is an l_1 variable. Hence from a corollary of the law of large numbers

$$\lim_{m\to\infty}\frac{1}{m}\left|\max_{q\in\{0,\dots,N_0-1\}}\log Z_q^{h,\beta,\theta_{mN_0}\omega}\right|=0, \quad (6.17)$$

and one can conclude. □

There is no direct sub-additivity property like (6.8) but we can cook up one at the cost of adding some extra terms in order to prove the reverse inequality

Lemma 6.4. *Assume that* (3.6) *holds. We have*

$$\limsup_{N\to\infty}\frac{1}{N}\log Z_{N,h}^{\beta,\omega}\leq\textsc{f} \quad (6.18)$$

Proof. We first prove that there exists a constant C such that for all N, β and h

$$Z_{N+M,h}^{\beta,\omega}\leq Z_{N,h}^{\beta,\omega}Z_{M,h}^{\beta,\theta_N\omega}\left(1+Ce^{-\beta\omega_N-h}\min(N^{1+\alpha},M^{1+\alpha})\right) \quad (6.19)$$

To prove this statement, we have to decompose $Z_{N+M,h}^{\beta,\omega}$ according to the first and last renewal point before N

$$Z_{N+M,h}^{\beta,\omega}=Z_{N,h}^{\beta,\omega}Z_{M,h}^{\beta,\theta_N\omega}+\sum_{\substack{0\leq a<N\\ N<b\leq N+M}}Z_{a,h}^{\beta,\omega}K(b-a)e^{\beta\omega_b+h}Z_{N+M-b,h}^{\beta,\theta_b\omega}. \quad (6.20)$$

Here and latter we take the convention that $Z_0 = 1$.

Now we want to compare the second term with $Z^{\beta,\omega}_{N,h} Z^{h,\beta,\theta_N\omega}_{M,h}$ As we have

$$\begin{aligned} Z^{\beta,\omega}_{N,h} &= \sum_{0\le a<N} e^{\beta\omega_N+h} Z^{\beta,h,\omega}_{a} K(N-a), \\ Z^{\beta,\theta_N\omega}_{M,h} &= \sum_{N<b\le N+M} K(b-N) e^{\beta\omega_b+h} Z^{\beta,\theta_b\omega}_{N+M-b,h}. \end{aligned} \tag{6.21}$$

Combining (6.20) and (6.21) we have

$$Z^{\beta,h,\omega}_{N+M} \le Z^{\beta,\omega}_{N,h} Z^{h,\beta,\theta_N\omega}_{M} \left(1 + e^{-\beta\omega_N - h} \max_{a<N<b} \frac{K(b-a)}{K(N-a)K(b-N)} \right). \tag{6.22}$$

Finally because of our assumption $K(n) \sim C_K N^{-(1+\alpha)}$ we have

$$\frac{K(b-a)}{K(N-a)K(b-N)} \le C \min(N^{1+\alpha}, M^{1+\alpha}) \tag{6.23}$$

and this finishes the proof of (6.19).

Now let us fix N_0 and write $N = N_0 m + q$ with $q \in \{1, \dots, N_0\}$. We can iterate (6.19) to obtain

$$\log Z^{\beta,\omega}_{N,h} \le \sum_{i=0}^{m-1} Z^{\beta,\theta_{iN_0}\omega}_{N_0,h} + \sum_{i=1}^{m} \log\left(1 + C e^{-\beta\omega_{iN_0} - h} N_0^{1+\alpha}\right) + \log Z^{\beta,\theta_{mN_0}\omega}_{q,h}. \tag{6.24}$$

Hence using the law of large numbers and (6.17) we obtain that

$$\limsup_{N\to\infty} \frac{1}{N} \log Z^{\beta,\omega}_{N,h} \le \frac{1}{N_0} \mathbb{E}\left[Z^{\beta,\theta_{iN_0}\omega}_{N_0,h} \right] + \frac{1}{N_0} \mathbb{E} \log\left(1 + C e^{-\beta\omega_{N_0} - h} N_0^{1+\alpha}\right) \tag{6.25}$$

The first term is always smaller than $\textsc{f}$. The second term is smaller than

$$\frac{1}{N_0} \left(\log[C N_0^{1+\alpha}] + |h| + \beta \mathbb{E}\left[|\omega_n|\right] \right).$$

which can be made arbitrarily small by choosing N_0 large. And hence the conclusion. □

3 A sufficient condition for the existence of a phase transition

Let us note first that the result of the previous section does not imply that the existence of a phase transition, as one might have

$$\forall h \in \mathbb{R}, \quad \textsc{f}(\beta, h) > 0. \tag{6.26}$$

For this reason we introduce a new assumption on ω which prevents this to happen

$$\forall \beta \in \mathbb{R}, \quad \lambda(\beta) := \log \mathbb{E}\left[e^{\beta\omega_1} \right] < \infty. \tag{6.27}$$

For convenience we will also assume

$$\mathbb{E}[\omega_1] = 0, \quad \text{and} \quad \mathbb{E}[\omega_1^2] = 0. \tag{6.28}$$

Using Jensen's inequality we have

$$\frac{1}{N}\mathbb{E}\left[\log Z_N^{\beta,h,\omega}\right] \leq \frac{1}{N}\log\mathbb{E}\left[Z_N^{\beta,h,\omega}\right]. \tag{6.29}$$

The right-hand side can be computed explicitly.

$$\mathbb{E}\left[Z_N^{\beta,h,\omega}\right] = \mathbf{E}\mathbb{E}\left[e^{\sum_{n=1}^N(\beta\omega_n+h)\delta_n}\delta_N\right] = \mathbf{E}\left[e^{\sum_{n=1}^N(\lambda(\beta)+h)\delta_n}\delta_N\right]. \tag{6.30}$$

It corresponds to the partition function of an homogeneous system with pinning parameter $\lambda(\beta)+h$.

$$\textsc{f}(\beta,h) \leq \lim_{N\to\infty}\frac{1}{N}\log Z_N^{\lambda(\beta)+h} = \textsc{f}(0,\lambda(\beta)+h). \tag{6.31}$$

We can also obtain a lower bound for $\textsc{f}(\beta,h)$. We have

$$\partial_\beta \log Z_{N,h}^{\beta,\omega} = \frac{1}{Z_{N,h}^{\beta,\omega}}\left(\sum_{n=1}^N \omega_n\delta_n\right)e^{\sum_{n=1}^N(\beta\omega_n+h)\delta_n}\delta_N, \tag{6.32}$$

and

$$\partial_\beta^2 \log Z_{N,h}^{\beta,\omega} = \mathbf{E}_{N,h}^{\beta,\omega}\left[\left(\sum_{n=1}^N \omega_n\delta_n\right)^2\right] - \mathbf{E}_{N,h}^{\beta,\omega}\left[\sum_{n=1}^N \omega_n\delta_n\right]^2 \geq 0. \tag{6.33}$$

Hence $\beta \mapsto \mathbb{E}\log Z_N^{\lambda(\beta)+h}$ is convex in β, and its derivative at zero is equal to

$$\mathbb{E}\left[\sum_{n=1}^N \omega_n\delta_n\right] = 0. \tag{6.34}$$

Thus

$$\mathbb{E}\left[\log Z_{N_h}^{\beta,\omega}\right] \geq \log Z_N^h \tag{6.35}$$

and hence

$$\textsc{f}(\beta,h) \geq \textsc{f}(0,h). \tag{6.36}$$

We can summarize these statements

Proposition 6.5. *We have for all β and h*

$$\textsc{f}(0,h) \leq \textsc{f}(\beta,h) \leq \textsc{f}(0,h+\lambda(\beta)), \tag{6.37}$$

and in particular

$$h_c(\beta) := \inf\{h \mid \textsc{f}(\beta,h) > 0\} \tag{6.38}$$

is finite and we have

$$-\lambda(\beta) \leq h_c(\beta) \leq 0. \tag{6.39}$$

Now the question if the inequalities above are sharp. A first question of interest is to wonder whether $h_c(\beta) < 0$. Even though the environment has mean zero, can one find a strategy to get a benefice of the zones with positive ω without losing too much when ω is negative.

4 The delocalized phase

Theorem 6.6. *When $h < h_c(\beta)$, there exists a constant C such that*

$$\mathbb{E}\mathbf{E}_{N,h}^{\beta,\omega}\left[\sum_{n=1}^{N}\delta_n\right] \le C\log N. \tag{6.40}$$

With a lot more efforts, one can show that $\mathbb{E}\mathbf{E}_{N,h}^{\beta,\omega}\left[\sum_{n=1}^{N}\delta_n\right] \le C$.

Proof. A simple computation gives

$$\partial_h \log Z_{N,h}^{\beta,\omega} = \mathbf{E}_{N,h}^{\beta,\omega}\left[\sum_{n=1}^{N}\delta_n\right]. \tag{6.41}$$

Hence by convexity, we have for every $u \ge 0$

$$\log Z_{N,,h+u}^{\beta,\omega} \ge \log Z_{N,h}^{\beta,\omega} + u\mathbf{E}_{N,h}^{\beta,\omega}\left[\sum_{n=1}^{N}\delta_n\right]. \tag{6.42}$$

We choose to apply the inequality for $u = h_c(\beta) - h$. We have $\mathbb{E}\log Z_{N,h_c(\beta)}^{\beta,\omega} \le N\textsc{f}(\beta, h_c(\beta)) = 0$ and hence one has

$$\mathbb{E}\mathbf{E}_{N,h}^{\beta,\omega}\left[\sum_{n=1}^{N}\delta_n\right] \le -\frac{\mathbb{E}\log Z_{N,h}^{\beta,\omega}}{h_c(\beta)-h} \ge \frac{-\log K(N)+h}{h_c(\beta)-h}. \tag{6.43}$$

In the last inequality we have used the trivial inequality

$$Z_{N,h}^{\beta,\omega} \ge K(N)e^{\beta\omega_N + h}. \tag{6.44}$$

This gives the result. □

5 The localized phase

It is almost immediate to prove that when $h > h_c(\beta)$

$$\lim_{N\to\infty}\frac{1}{N}\mathbf{E}_{N,h}^{\beta,\omega}\left[\sum_{n=1}^{N}\delta_n\right] = \partial_h\textsc{f}(\beta,h), \tag{6.45}$$

whenever the r.h.s. exists. What one would like to check then is that the function $\textsc{f}(\beta,h)$ is indeed differentiable, and that $\sum_{n=1}^{N}\delta_n$ behaves likes its mean value. The following result was prove in [7]. The proof is quite demanding and relies on estimates on correlation between the variables $(\delta_n)_{n\ge0}$.

Theorem 6.7. *The function $(\beta,h)\mapsto\textsc{f}(\beta,h)$ is infinitely derivable in β and h in the domain*

$$\{(\beta,h) \mid h > h_c(\beta)\}$$

In particular the variance of $\frac{1}{\sqrt{N}}\sum_{n=1}^{N}\delta_n$ under $\mathbf{E}_{N,h}^{\beta,\omega}$ converges to

$$\partial_h^2\textsc{f}(\beta,h). \tag{6.46}$$

Chapter 7

Disorder helping localization

1 Getting something better than the mean reward

We assume for simplicity that $K(\infty) = 0$. We are going to show that

Proposition 7.1. *For all $\beta > 0$ we have*

$$\textsc{f}(\beta, 0) > 0 \tag{7.1}$$

as a consequence

$$h_c(\beta) > 0. \tag{7.2}$$

This argument is due in part to Alexander and Sidoravicius [1]. The idea is to change the expression of the partition function so that an environment with positive mean appears artificially.

We set

$$\delta_n := \mathbf{1}_{\{n \in \tau\}} \tag{7.3}$$

and

$$\widehat{\delta}_n = \begin{cases} \mathbf{1}_{\{\{n-1, n+1\} \subset \tau\}}, & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases} \tag{7.4}$$

Finally we set

$$\widehat{\omega}_n(\beta) = \log \frac{K(1)^2 e^{\beta\omega} + K(2)}{K(1)^2 + K(2)} = \log \mathbf{E}\left[e^{\beta\omega_n\delta_n} \mid \{n-1, n+1\} \subset \tau\right]. \tag{7.5}$$

The $\widehat{\omega}_n(\beta)$ are IID and when $\beta > 0$ Jensen's inequality (strict) gives

$$\mathbb{E}[\widehat{\omega}_1(\beta)] > 0. \tag{7.6}$$

The idea is thus to identify the places where $\widehat{\delta}_n = 1$ and to say that they correspond to an energetic reward $\widehat{\omega}_n$.

Proposition 7.2. *We have for all even N*

$$Z_N^{\beta,0,\omega} = \mathbf{E}\left[e^{\sum_{n=1}^N \beta\omega_n\delta_n(1-\widehat{\delta}_n) + \sum_{n=1}^N \beta\widehat{\omega}_n(\beta)\widehat{\delta}_n} \delta_N\right]. \tag{7.7}$$

Proof. Set

$$\begin{aligned}\mathcal{T}_N &:= \{n \in [\![1,N]\!] \mid n \in \tau, \text{ and } \widehat{\delta}_n = 0\},\\ \mathcal{I}_N &:= \{n \in [\![1,N]\!] \mid \widehat{\delta}_n = 1\}.\end{aligned} \tag{7.8}$$

We have

$$\mathbf{E}\left[e^{\sum_{n=1}^N \beta\omega_n\delta_n}\delta_N \mid \mathcal{T}_n\right] = e^{\sum_{n\in\mathcal{T}_n}\beta\omega_n}\delta_N \mathbf{E}\left[e^{\sum_{n\in\mathcal{I}_n}\beta\omega_n\delta_n} \mid \mathcal{T}_n\right]. \tag{7.9}$$

Note that given $\mathcal{T}_n$ the variable $(\delta_n)_{n\in\mathcal{I}_n}$ are IID Bernouilli of parameter $K(1)^2/(K(1)^2+K(2))$ and hence

$$\mathbf{E}\left[e^{\sum_{n=1}^N \beta\omega_n\delta_n\widehat{\delta}_n} \mid \mathcal{T}_n\right] = e^{\sum_{n\in\mathcal{I}_n}\widehat{\omega}_n(\beta)}. \tag{7.10}$$

Hence

$$Z_{N,0}^{\beta,\omega} = \mathbf{E}\left[\mathbf{E}\left[e^{\sum_{n\in\mathcal{I}_n}\beta\omega_n} \mid \mathcal{T}_n\right]\right] = \mathbf{E}\left[e^{\sum_{n\in\mathcal{T}_n}\beta\omega_n + \sum_{n\in\mathcal{I}_n}\widehat{\omega}_n(\beta)}\delta_N\right] \tag{7.11}$$

□

Recall that we have for all N

$$\mathrm{F}(\beta,0) \geq \frac{\mathbb{E}\left[\log Z_{N,0}^{\beta,\omega}\right]}{N}. \tag{7.12}$$

And we have by Jensen's inequality

$$\begin{aligned}\log Z_{N,0}^{\beta,\omega} &= \log \mathbf{P}[N\in\tau] + \log \mathbf{E}\left[e^{\sum_{n=1}^N \beta\omega_n\delta_n(1-\widehat{\delta}_n)+\sum_{n=1}^N \beta\widehat{\omega}_n(\beta)\widehat{\delta}_n} \mid N\in\tau\right]\\ &\geq \log \mathbf{P}[N\in\tau] + \mathbf{E}\left[\sum_{n=1}^N \beta\omega_n\delta_n(1-\widehat{\delta}_n) + \sum_{n=1}^N \beta\widehat{\omega}_n(\beta)\widehat{\delta}_n \mid N\in\tau\right].\end{aligned} \tag{7.13}$$

As the ω_ns have mean zero, we have

$$\mathrm{F}(\beta,0) \geq \frac{1}{N}\left(\log \mathbf{P}[N\in\tau] + \mathbb{E}[\widehat{\omega}_1(\beta)]\sum_{n=1}^N \mathbf{E}\left[\widehat{\delta}_n \mid N\in\tau\right]\right). \tag{7.14}$$

We know that $\mathbb{E}[\widehat{\omega}_1(\beta)]$ is positive and does not depend on N and also that

$$\log \mathbf{P}[N\in\tau] \geq \log K(N) \geq -(2+\alpha)\log N$$

for N sufficiently large. Hence to conclude it is sufficient to show that

$$\sum_{n=1}^N \mathbf{E}\left[\widehat{\delta}_n \mid N\in\tau\right]$$

grows faster than $\log N$. We have

$$\mathbf{E}\left[\widehat{\delta}_n \mid N\in\tau\right] = \frac{\mathbf{P}[n-1\in\tau]\mathbf{P}[2\in\tau]\mathbf{P}[N-n-1\in\tau]}{\mathbf{P}[N\in\tau]}. \tag{7.15}$$

When $\alpha\in(0,1)$, knowing (cf. Proposition 3.9) that

$$\mathbf{P}[N\in\tau] \sim c'_K N^{\alpha-1}$$

we can conclude that there exists a constant c such that for all odd n

$$\mathbf{E}\left[\widehat{\delta}_n \mid N \in \tau\right] \geq c\,(\min(n, N-n))^{\alpha-1}\,. \tag{7.16}$$

Hence we have

$$\sum_{n=1}^{N} \mathbf{E}\left[\widehat{\delta}_n \mid N \in \tau\right] \geq N^{\alpha}. \tag{7.17}$$

We can show, that the sum is of order N for $\alpha > 1$ in the same manner by the use of the Renewal Theorem. As a result we have the following lower bound on the free-energy valid for all N suffiently large

$$\textsc{f}(\beta, 0) \geq \frac{1}{N}\left(\log \mathbf{P}[N \in \tau] + \mathbb{E}[\widehat{\omega}_1(\beta)]N^{\alpha}\right). \tag{7.18}$$

Choosing N large enough, one can show it is positive (and even get an explicit lower bound). □

Chapter 8

Comparison with the annealed bound: the Harris Criterion

1 The Harris Criterion

We have seen in the previous chapter that for all choices of β we have $h_c(\beta) > 0$ and hence that the inequality $\mathrm{F}(\beta, h) \geq \mathrm{F}(0, h)$ is never sharp.

The question for the inequality $\mathrm{F}(\beta, h) \leq \mathrm{F}(0, h + \lambda(\beta))$ turns out to have a more complex answer and is very much related to that of disorder relevance: if one introduces a small amount (small β) of disorder, will it change its critical properties?

Here for the pinning problem, we are interested in two properties: the position of the critical point $h_c(\beta)$ and the critical-exponent of the free-energy (which is equal to $\max(1, \alpha^{-1})$). We ask ourselves

- If $h_c(\beta) = -\lambda(\beta)$ (if the critical point is that of the annealed system).
- If $\mathrm{F}(\beta, h) \overset{h \to h_c(\beta)+}{\sim} (h - h_c(\beta))^{\nu}$ for some exponent ν. In particular, do we have $\nu = \max(1, \alpha^{-1})$?

The physicists A.B. Harris gave a suprizingly very efficient heuristic criterion to predict the effect of disorder. It depends on the value of the critical exponent for the free-energy of the pure system: If this is exponent is larger than 2 then the disorder is irrelevant, and a small amount of disorder should not change the critical properties. If on the contrary the exponent is smaller than 2, the critical properties of the disordered system should differ from that of the pure one for every β.

In the case of random pinning, this corresponds to saying that disorder is relevant for $\alpha < 1/2$ and irrelevant for $\alpha > 1/2$. Note that the criterion does not make any prediction in the case $\alpha = 1/2$ (that of the simple random walk).

2 The case $\alpha < 1/2$: disorder irrelevance

For this chapter, we set

$$h := -\lambda(\beta) + u$$

With some slight abuse of notation we will denote by $\textsc{f}(\beta, u)$ the corresponding free-energy.

Theorem 8.1. *When $\alpha < 1/2$, there exists β_0 (which depends both on K and of the law of ω) such that for all $\beta \in [0, \beta_0]$ we have*

$$h_c(\beta) = -\lambda(\beta).$$

and there exists a constant c_β such that for all $u \le 1/2$

$$\textsc{f}(\beta, u) \ge c_\beta \left(\frac{u}{|\log u|} \right)^{1/\alpha} \tag{8.1}$$

To prove this result we need to work with the free partition function, which bears no constrain for the end point N

$$\widetilde{Z}^{\beta,\omega}_{N,u} := \mathbf{E}\left[\exp\left(\sum_{n=1}^{N} (\beta\omega_n - \lambda(\beta) + u)\delta_n \right) \right]. \tag{8.2}$$

We let $\widetilde{\mathbf{P}}^{\beta,u,\omega}_{N}$ be the associated measure defined by

$$\frac{\mathrm{d}\widetilde{\mathbf{P}}^{\beta,\omega}_{N,u}}{\mathrm{d}\widetilde{\mathbf{P}}} = \exp\left(\sum_{n=1}^{N} (\beta\omega_n - \lambda(\beta) + u)\delta_n \right). \tag{8.3}$$

We let $Z^{\beta,\omega}_{N,u}$ denote the constrained partition function.

A first fundamental observation is that to prove the result, we only need to prove a bound on the contact density at the critical point.

Lemma 8.2. *There exists a constant $C(\beta)$ such that for all $u \in [0,1]$ we have*

$$\textsc{f}(\beta, u) \ge \frac{u}{N} \mathbb{E}\widetilde{\mathbf{E}}^{\beta,\omega}_{N,0}\left[\sum_{n=1}^{N} \delta_n \right] - \frac{(1+\alpha)\log N + C(\beta)}{N}. \tag{8.4}$$

The main statement to prove Theorem 8.1 is then

Proposition 8.3. *When $\alpha < 1/2$, there exists β_0 such that for all $\beta \in [0, \beta_0]$ there exists c_β such that for all $N \ge 0$*

$$\mathbb{E}\widetilde{\mathbf{E}}^{\beta,\omega}_{N,0}\left[\sum_{n=1}^{N} \delta_n \right] \ge c_\beta N^{\alpha}. \tag{8.5}$$

Note that N^α corresponds to the order of magnitude for $\mathbf{E}\left[[\sum_{n=1}^{N} \delta_n \right]$. The idea is thus to prove that $\mathbf{E}^{\beta,\omega}_{N,0}$ are not that different.

Proof of Theorem 8.1 from Proposition 8.3. Combining Lemma 8.2 and Proposition 8.3

$$\textsc{f}(\beta, u) \ge u c_\beta N^{\alpha - 1} - \frac{(1+\alpha)\log N + C}{N}. \tag{8.6}$$

Choosing $N = C'(|\log u|/u)^{1/\alpha}$ for a large C' we obtain

$$\textsc{f}(\beta, u) \ge c' u^{1/\alpha} \; |\log u|^{1-\alpha^{-1}} \tag{8.7}$$

$\square$

Proof of Lemma 8.2. By decomposing over the possible location of the last renewal point before N, we have (recall that $\bar{K}(n) = \sum_{m>n} K(n)$)

$$\begin{aligned} \widetilde{Z}^{\beta,\omega}_{N,u} &= \sum_{n=0}^{N-1} Z^{\beta,\omega}_{N,u} \bar{K}(N-n) + Z^{\beta,\omega}_{N,u}, \\ Z^{\beta,\omega}_{N,u} &= \sum_{n=0}^{N-1} Z^{\beta,\omega}_{N,u} K(N-n) e^{\beta\omega_N - \lambda(\beta)+u}. \end{aligned} \tag{8.8}$$

Hence

$$\widetilde{Z}^{\beta,\omega}_{N,u} \le Z^{\beta,\omega}_{N,u} \left(1 + \max_{n\in 1,\dots,N-1} \frac{\bar{K}(n)}{K(n)} e^{-(\beta\omega_N - \lambda(\beta)+u)}\right) \le C(\beta) N Z^{\beta,\omega}_{N,u} e^{\beta|\omega_N|}. \tag{8.9}$$

where the last inequality is valid for $u \in [0,1]$. As $\textsc{f}(\beta,u) \ge \frac{1}{N}\mathbb{E}\left[\log Z^{\beta,\omega}_{N,u}\right]$ this yields

$$\textsc{f}(\beta,u) \ge \frac{1}{N}\mathbb{E}\left[\log \widetilde{Z}^{\beta,\omega}_{N,u}\right] - \frac{\log C(\beta)N}{N}. \tag{8.10}$$

Using convexity we have

$$\log \widetilde{Z}^{\beta,\omega}_{N,u} \ge u\left(\partial_u \log \widetilde{Z}^{\beta,\omega}_{N,u}|_{u=0}\right) + \log \widetilde{Z}^{\beta,\omega}_{N,0} \ge u \widetilde{\mathbf{E}}^{\beta,\omega}_{N,0}\left[\sum_{n=1}^{N} \delta_n\right] - \log \sum_{n=N+1}^{\infty} K(n). \tag{8.11}$$

Combining (8.10) and (8.11) we obtain the following finite volume criterion. □

3 Proof of Proposition 8.3

The idea is to show first that N^α corresponds indeed to the typical number of contact under the measure $\mathbf{P}$ (Section 3.1) and then to show using a second moment computation (Section 3.2) that $\widetilde{\mathbf{P}}^{\beta,\omega}_{N,0}$ and $\mathbf{P}$ are close (Section 3.3).

3.1 A lower bound for the typical number of contact

We want to show the event

$$A_N := \left\{\sum_{n=1}^{N} \delta_n \ge \varepsilon N^\alpha\right\}$$

has a large probability under the renewal measure.

Lemma 8.4. *Given δ there exists ε such that for all N sufficiently large*

$$\mathbf{P}\left[\sum_{n=1}^{N} \delta_n \ge \varepsilon N^\alpha\right] \ge 1-\delta \tag{8.12}$$

Proof of Lemma 8.4. Given N we set $n = n(\varepsilon, N) = \varepsilon N^\alpha$. We want to prove

$$\mathbf{P}[\tau_n \ge N] \le \delta. \tag{8.13}$$

Let us set

$$\widetilde{\tau}_n := \sum_{k=1}^{N} (\tau_k - \tau_{k-1}) \mathbf{1}_{\{\tau_k - \tau_{k-1} \geq N\}}. \tag{8.14}$$

We have

$$\mathbf{P}[\tau_n \geq N] \leq \mathbf{P}[\widetilde{\tau}_n \geq N] + \mathbf{P}[\max_{k \leq n} (\tau_k - \tau_{k-1}) \geq N]. \tag{8.15}$$

The second term can be easily bounded by

$$n\mathbf{P}[\tau_1 \geq N] \leq n\bar{K}(N) \leq \varepsilon C \leq \delta/2, \tag{8.16}$$

provided that ε is chosen sufficiently small. As for the second term we have for any choice of $b > 0$.

$$\mathbf{P}[\widetilde{\tau}_n \geq N] = \mathbf{P}[e^{b\widetilde{\tau}_n} \geq e^{bN}] \leq e^{-Nb} \mathbf{E}[e^{-b\widetilde{\tau}_n}] = e^{n \log \mathbf{E}[e^{b\widetilde{\tau}_1}] - bN}. \tag{8.17}$$

Let us choose $b = 2N^{-1} \log \delta^{-1}$. We must show that

$$n \log \mathbf{E}[e^{b\widetilde{\tau}_1}] \leq \frac{1}{2} \log \delta^{-1}. \tag{8.18}$$

Indeed this would imply that

$$\mathbf{P}[\widetilde{\tau}_n \geq N] \leq \delta^{3/2}. \tag{8.19}$$

Note that if $X \in [0, A]$, $A \geq 1$ we have by convexity of the exponential

$$\log \mathbb{E}[e^X] \leq \log \mathbf{E}\left[1 + \frac{e^A - 1}{A} \mathbb{E}[X]\right] \leq \frac{e^A}{A} \mathbb{E}[X]. \tag{8.20}$$

Hence we have

$$\log \mathbf{E}[e^{b\widetilde{\tau}_1}] \leq \delta^{-2} N^{-1} \mathbf{E}[\widetilde{\tau}_1]. \tag{8.21}$$

We have

$$\mathbf{E}[\widetilde{\tau}_1] \leq \sum_{m=1}^{N} mK(m) \leq CN^{1-\alpha}. \tag{8.22}$$

Hence we have

$$n \log \mathbf{E}[e^{b\widetilde{\tau}_1}] \leq C\varepsilon\delta^{-2}, \tag{8.23}$$

and we can conclude by choosing ε small enough. $\square$

3.2 Computing the second moment

One can readily check that

$$\mathbb{E}\left[\widetilde{Z}_{N,0}^{\beta,\omega}\right] = 1. \tag{8.24}$$

We are going to show that when β is small the second moment is also bounded.

Lemma 8.5. *When* $\alpha < 1/2$ *and*

$$\lambda(2\beta) - 2\lambda(\beta) < \log\left(1 + \sum_{n=0}^{N} \mathbf{P}(n \in \tau^2)\right)$$

we gave

$$\sup_{N\geq 0} \mathbb{E}\left[(\widetilde{Z}_{N,0}^{\beta,\omega})^2\right] < \infty. \tag{8.25}$$

Proof. The second moment of $\widetilde{Z}_{N,0}^{\beta,\omega}$ can be computed explicitly. One obtains

$$\mathbb{E}\left[\left(\widetilde{Z}_{N,0}^{\beta,\omega}\right)^2\right] = \mathbf{E}^{\otimes 2}\left[\mathbb{E}\left[\exp\left(\sum_{n=1}^{N}(\beta\omega_n - \lambda(\beta))\left(\delta_n^{(1)} + \delta_n^{(2)}\right)\right)\right]\right]. \tag{8.26}$$

where $\delta_n^{(1)} = \mathbf{1}_{\{\tau^{(i)}\in N\}}$ and the τ_i are two independent renewals with law $\mathbf{P}$. It is quite straight forward to check that

$$\mathbb{E}\left[\exp\left(\sum_{n=1}^{N}(\beta\omega_n - \lambda(\beta))\left(\delta_n^{(1)} + \delta_n^{(2)}\right)\right)\right] = e^{(\lambda(2\beta)-2\lambda(\beta))\delta_n^{(1)}\delta_n^{(2)}}. \tag{8.27}$$

Hence we have

$$\mathbb{E}\left[\left(\widetilde{Z}_{N,0}^{\beta,\omega}\right)^2\right] = \mathbf{E}^{\otimes 2}\left[e^{\sum_{n=1}^{N}(\lambda(2\beta)-2\lambda(\beta))\delta_n^{(1)}\delta_n^{(2)}}\right] \leq \mathbf{E}^{\otimes 2}\left[e^{\sum_{n=1}^{\infty}(\lambda(2\beta)-2\lambda(\beta))\delta_n^{(1)}\delta_n^{(2)}}\right] \tag{8.28}$$

Now note that

$$\delta_n^{(1)}\delta_n^{(2)} = \mathbf{1}_{\{N\in\tau^{(1)}\cap\tau^{(2)}\}},$$

and it is not difficult to check that $\tau' = \tau^{(1)} \cap \tau^{(2)}$ is a renewal process. Hence the important question is: is τ' recurrent (with infinitely many contact point almost surely) or not? From Lemma 3.9 there exists a constant $\widehat{C}_K$ such that

$$\mathbf{P}[N \in \tau] = Z_{N,0} \overset{N\to\infty}{\sim} \widehat{C}_K N^{\alpha-1}. \tag{8.29}$$

Hence we have

$$\mathbf{P}^{\otimes 2}[N \in \tau'] = \mathbf{P}\left[N \in \tau\right]^2 \overset{N\to\infty}{\sim} \widehat{C}_K^2 N^{2(\alpha-1)}. \tag{8.30}$$

By the Markov property for renewals, $\sum_{n=1}^{\infty}\delta_n^{(1)}\delta_n^{(2)}$ is distributed like a geometric variable, and in particular it is a.s. infinite if and only if its expectation is finite. Because of (8.30) we have

$$\sum_{n=1}^{\infty}\mathbf{P}\left[n \in \tau\right]^2 < \infty \quad \Leftrightarrow \quad \alpha < 1/2. \tag{8.31}$$

In particular τ' is finite if and only if $\alpha < 1/2$. When τ' is finite the total number of renewal point a geometric variable. In particular we have

$$\mathbf{E}^{\otimes 2}\left[e^{\sum_{n=1}^{\infty}(\lambda(2\beta)-2\lambda(\beta))\delta_n^{(1)}\delta_n^{(2)}}\right] = \sum_{k=0}^{\infty}\mathbf{P}[\tau_1' < \infty]^k\mathbf{P}[\tau_1' = \infty]e^{k(\lambda(2\beta)-2\lambda(\beta))}. \tag{8.32}$$

As we know the mean of this geometric variable, we can deduce that

$$\frac{1}{\mathbf{P}[\tau_1' < \infty]} = 1 + \frac{1}{\sum_{n\geq 1} \mathbf{P}\left[n \in \tau\right]^2}. \tag{8.33}$$

Hence we can conclude that

$$\sup_N \mathbb{E}\left[\left(\widetilde{Z}_{N,0}^{\beta,\omega}\right)^2\right] < \infty \quad \Leftrightarrow \quad \lambda(2\beta) - 2\lambda(\beta) \leq \log\left(1 + \frac{1}{\sum_{n\geq 1} \mathbf{P}\left[n \in \tau\right]^2}\right). \tag{8.34}$$

□

We can set

$$\beta_0 := \sup\left\{\beta \ : \ \lambda(2\beta) - 2\lambda(\beta) \leq -\log K'(\infty)\right\}, \tag{8.35}$$

and prove Proposition 8.3 under the assumption $\beta < \beta_0$. Note that we can have $\beta_0 = \infty$ if ω is almost surely bounded, and that $\beta_0 > 0$ provided $\alpha < 1/2$.

3.3 Comparing measure using second moment estimates

We introduce a tool called Paley-Zygmund inequality, which allows to control the probability for Z to be small if one knows its second moment.

Lemma 8.6 (Paley-Zygmund inequality)**.** *Let Z be a positive random variable. We have for every $\theta \in (0,1)$*

$$\mathbb{P}[Z \geq \theta\mathbb{E}[Z]] \geq (1-\theta)^2 \frac{\mathbb{E}[Z]^2}{\mathbb{E}[Z^2]}. \tag{8.36}$$

Proof. Using the Cauchy-Schwartz inequality we have

$$\mathbb{E}[Z] \leq \theta\mathbb{E}[Z] + \mathbb{E}[Z\mathbf{1}_{\{Z\geq\theta\mathbb{E}[Z]\}}] \leq \theta\mathbb{E}[Z] + \sqrt{\mathbb{E}[Z^2]}\sqrt{\mathbf{P}[Z \geq \theta\mathbb{E}[Z]]}. \tag{8.37}$$

Rearranging the inequality gives the result. □

We want to use this to show that the probability $\widetilde{\mathbf{P}}_N^{\beta,0,\omega}$ is not too different from $\mathbf{P}$.

Lemma 8.7. *If*

$$\sup_{N\geq 0} \mathbb{E}\left[(\widetilde{Z}_{N,0}^{\beta,\omega})^2\right] < C \tag{8.38}$$

then there exists a constant $\delta_\beta > 0$such that for all N for all event A satisfying

$$\mathbf{P}(A) \geq 1 - \delta \tag{8.39}$$

we have

$$\mathbb{E}\left[\widetilde{\mathbf{P}}_{N,0}^{\beta,\omega}(A)\right] \geq \delta. \tag{8.40}$$

Proof. Note

$$\widetilde{\mathbf{P}}^{\beta,\omega}_{N,0}(A^c) \le 2\mathbf{E}\left[\mathbf{1}_{A^c} \exp\left(\sum_{n=1}^N (\beta\omega_n - \lambda(\beta) + u)\delta_n\right)\right] + \mathbf{1}_{\{\widetilde{Z}_N < 1/2\}}. \tag{8.41}$$

Hence by averaging with respect to ω we obtain

$$\widetilde{\mathbf{P}}^{\beta,\omega}_{N,0}(A^c) \le 2\delta + \mathbb{P}\left[\widetilde{Z}^{\beta,\omega}_{N,0} < 1/2\right]. \tag{8.42}$$

The Paley-Zygmund inequality guaranties that the second term is smaller than

$$1 - \frac{1}{4\mathbb{E}[(\widetilde{Z}^{\beta,\omega}_{N,0})^2]} \le 1 - 1/(4C). \tag{8.43}$$

Hence the result holds if $\delta < 1/(12C)$. □

Proof of Proposition 8.3. When $\beta \le \beta_0$, using (8.5), one can set a δ_β for which Lemma 8.7 is valid. We use it for the event

$$A_N := \mathbf{P}\left[\sum_{n=1}^N \delta_n \ge \varepsilon N^\alpha\right].$$

for ε sufficiently small (cf. Lemma 8.4).

The we can conclude that

$$\widetilde{\mathbf{E}}^{\beta,\omega}_{N,0}\left[\sum_{n=1}^N \delta_n\right] \ge \varepsilon N^\alpha \widetilde{\mathbf{P}}^{\beta,\omega}_{N,0}\left[\sum_{n=1}^N \delta_n \ge \varepsilon N^\alpha\right] \ge \delta\varepsilon N^\alpha. \tag{8.44}$$

□

4 For $\alpha \in (1/2, 1)$: a bound on $h_c(\beta)$

4.1 Extracting the right statement from the previous proof

Let us now investigate how the results of the previous Section can be used to say something about the case $\alpha > 1/2$. The combination of Lemma 8.7 (or rather its proof) and Lemma 8.4 we deduce that we can find $\varepsilon > 0$ such that for all N and β which satisfy

$$\mathbb{E}[(\widetilde{Z}^{\beta,\omega}_{N,0})^2] \le 10, \tag{8.45}$$

we have

$$\mathbb{E}\left[\widetilde{\mathbf{P}}^{\beta,\omega}_{N,0}\left(\sum_{n=1}^N \delta_n \ge \varepsilon N^\alpha\right)\right] \ge \frac{1}{120}. \tag{8.46}$$

Hence by Lemma 8.2 we can find a constant C such that

$$\textsc{f}(\beta, u) \ge u\frac{\varepsilon}{120} N^{\alpha-1} - \frac{(1+\alpha)\log N + C}{N}. \tag{8.47}$$

In particular

$$\textsc{f}(\beta, u) > 0 \tag{8.48}$$

for

$$u > \frac{120(1+\alpha)\log N + C}{N^\alpha}. \tag{8.49}$$

Hence we have

Lemma 8.8. *There exists a constant C such that for all $\beta \le 1$ we have, and all $\alpha > 1/2$ we have*

$$u_c(\beta) \le \frac{120(1+\alpha)\log N_\beta + C}{N_\beta^\alpha} \tag{8.50}$$

where

$$N_\beta := \max\left\{ N \mid \mathbb{E}[(\widetilde{Z}_{N,0}^{\beta,\omega})^2] \le 10 \right\}. \tag{8.51}$$

The question is now, how can one estimate N_β.

4.2 General remarks on how to compute the variance

Recall that from previous computation we have

$$\mathbb{E}\left[\widetilde{Z}_{N,0}^{\beta,\omega})^2\right] = \mathbf{E}\left[e^{[\lambda(2\beta)-2\lambda(\beta)]H_N(\tau')}\right] \tag{8.52}$$

where τ' is the renewal obtain by the intersection of two independent copies of τ, and

$$H_N(\tau') := \sum_{n=1}^N \mathbf{1}_{\{n\in\tau'\}}. \tag{8.53}$$

Note that for $\alpha > 1/2$, τ' is recurrent from Lemma 3.9 there is a constant c' such that

$$\mathbf{P}[n \in \tau'] \stackrel{n\to\infty}{\sim} c' n^{2(\alpha-1)}. \tag{8.54}$$

We will use this information to estimate N_β the length at which the variance starts to blow up (it corresponds to the length at which $\mathbf{P}_N^{\beta,0,\omega}$ and $\mathbf{P}$ start to look different).

First let us notice that as $\lambda(\beta) \sim \beta^2/2$ there exists positive constants c_1 and c_2 such that for all $\beta \le 1$ we have

$$c_1\beta^2 \le \lambda(2\beta) - 2\lambda(\beta) \le c_2\beta^2. \tag{8.55}$$

Hence we can reduce to treating the Gaussian case (where $[\lambda(2\beta) - 2\lambda(\beta)] = \beta^2$) and the result will remain true for other environments after a multiplication by a constant. Our aim is to prove the following result.

Proposition 8.9. *Set*

$$N_\beta := \max\left\{ N \mid \mathbf{E}\left[e^{\beta^2 H_N(\tau')}\right] \le 10 \right\}. \tag{8.56}$$

(i) For $\alpha \in (1/2, 1)$, there exists a constant c (which depends on K) such that for every $\beta \in (0,1]$

$$N_\beta \ge c\beta^{\frac{2}{2\alpha-1}}. \tag{8.57}$$

(ii) For $\alpha = 1/2$, there exists a constant c (which depends on K) such that for every $\beta \in (0,1]$

$$N_\beta \ge e^{-c\beta^{-2}}. \tag{8.58}$$

Note that this result is in fact optimal up to the choice of the constant c. Indeed by convexity one has

$$\mathbf{E}\left[e^{\beta^2 H_N(\tau')}\right] \geq \exp\left(\beta^2 \mathbf{E}\left[H_N(\tau')\right]\right). \tag{8.59}$$

As from (8.54) $\mathbf{E}[H_N(\tau')]$ is of order $N^{2\alpha-1}$ for $\alpha \in (1/2,1)$ and of order $\log N$ when $\alpha = 1/2$, one sees readily that the laplace transform blows up when N is much larger than the bound given in Proposition 8.9.

Combining this result with Lemma 8.8 we obtain

Theorem 8.10. *There exists a constant C_K (depending the the interarrival law) which is such that for all $\beta \in (0,1)$ we have*

$$u_c(\beta) \leq \begin{cases} C_K |\log \beta| \beta^{\frac{2\alpha}{(1-2\alpha)}} & \text{if } \alpha \in (1/2,1), \\ e^{-C_K \beta^{-2}} & \text{if } \alpha = 1/2. \end{cases} \tag{8.60}$$

Our job is now to prove Proposition 8.9 and thus to find a way to control the laplace transform of $H_N(\tau')$.

4.3 Getting a bound on the Laplace transform of $H_N(\tau')$ in terms of that of τ'_1

Note that if X is a random variable with integer values and f is an increasing function such that $f(X)$ has finite expectation. Then

$$\mathbb{E}[f(X)] := f(0) + \sum_{k=1}^{\infty} [f(k) - f(k-1)] \mathbb{P}[X \geq k]. \tag{8.61}$$

In particular

$$\mathbf{E}\left[e^{\beta^2 H_N(\tau')}\right] \leq 1 + (e^{\beta^2} - 1) \sum_{k=1}^{N} e^{\beta^2 k} \mathbf{P}\left[H_N(\tau') \geq k\right]. \tag{8.62}$$

Now we have to find a good bound on

$$\mathbf{P}\left[H_N(\tau') \geq k\right] = \mathbf{P}[\tau'_k \leq N]. \tag{8.63}$$

We have for any $x > 0$

$$\mathbf{P}[\tau'_k \leq N] = \mathbf{P}[e^{-x\tau'_k} \geq e^{-xN}] \leq e^{xN} \mathbf{E}[e^{-x\tau'_k}] \leq \exp\Big(k \log \mathbf{E}[e^{-x\tau'_1}] + xN\Big) \tag{8.64}$$

and hence

$$\log \mathbf{P}[\tau'_k \leq N] \leq \max_{x>0} \Big(k \log \mathbf{E}[e^{-x\tau'_1}] + xN\Big). \tag{8.65}$$

As we do not have directly access to the distribution of τ'_1 we use the following trick to estimate its Laplace transform

Lemma 8.11. *We have*

$$1 - \mathbb{E}[e^{-x\tau'_1}] = \frac{1}{\sum_{n\geq 0} e^{-nx} \mathbf{P}[n \in \tau']} \tag{8.66}$$

Proof. We have

$$\sum_{n\geq 0} e^{-nx}\mathbf{P}[n\in\tau'] = 1+\sum_{k=1}^{\infty}\sum_{n\geq 1} e^{-nx}\mathbf{P}[\tau'_k = n]$$
$$= 1+\sum_{k=1}^{\infty}\mathbf{E}\left[e^{-x\tau'_k}\right] = \sum_{k\geq 0}\left(\mathbf{E}[e^{-x\tau'_1}]\right)^k = \frac{1}{1-\mathbb{E}[e^{-x\tau'_1}]}. \quad (8.67)$$

□

4.4 Wrapping up the case $\alpha\in(1/2,1)$

Lemma 8.12. *We have*

$$\sum_{n\geq 0} e^{-nx}\mathbf{P}[n\in\tau'] \overset{x\to 0+}{\sim} c''x^{1-2\alpha}, \quad (8.68)$$

where

$$c'' := c'\Gamma(2\alpha-1)$$

for c' *given by* (8.54).

Proof. We have

$$\sum_{n\geq 0} e^{-nx}\mathbf{P}[n\in\tau'] = 1+c'x^{1-2\alpha}\left(x\sum_{n\geq 1} e^{-nx}(nx)^{2(\alpha-1)}\right)\frac{\mathbf{P}[n\in\tau']}{c'n^{2(1-\alpha)}}. \quad (8.69)$$

As the quotient on the left-hand side tends to one, we can infer that the sum is asymptotically equivalent to

$$c'x^{1-2\alpha}\left(x\sum_{n\geq 1} e^{-nx}(nx)^{2(\alpha-1)}\right). \quad (8.70)$$

We conclude by noticing that the Riemann sum in the bracket converges to

$$\int e^{-y}y^{2(\alpha-1)}\mathrm{d}y = \Gamma(2\alpha-1) \quad (8.71)$$

□

As $\log Z\sim Z-1$ for Z around one, Lemma 8.11-8.12 combined allow to conclude thus that

$$\log\mathbb{E}[e^{-x\tau'_1}] \overset{x\to 0+}{\sim} -(c'')^{-1}x^{2\alpha-1}. \quad (8.72)$$

In particular one can find c_1 such that for all $x\in[0,1]$

$$\mathbb{E}[e^{-x\tau'_1}] \leq -c_1x^{2\alpha-1}. \quad (8.73)$$

Hence from (8.65) we obtain that

$$\log\mathbf{P}[\tau'_k\leq N] \leq \max_{x\in[0,1]}\left(xN-c_1kx^{2\alpha-1}\right). \quad (8.74)$$

By taking the differentiating over x we obtain that the max on the r.h.s. is attained for

$$x_{\max} = \left(\frac{2c_1(\alpha-1)k}{N}\right)^{\frac{1}{2(1-\alpha)}},$$

which is always smaller than 1 if c_1 is small enough. For an adequate choice of c_2, we thus have

$$\log \mathbf{P}[\tau'_k \le N] \le -c_2 \left(\frac{k}{N^{2\alpha-1}}\right)^{\frac{1}{2(1-\alpha)}}. \tag{8.75}$$

which implies

$$\sum_{k=1}^{N} e^{\beta^2 k}\mathbf{P}[\tau'_k \le N] \le \beta^{-2}e + \sum_{k=\beta^{-2}+1}^{N} \exp\left(\beta^2 k - c_2\left(\frac{k}{N^{2\alpha-1}}\right)^{\frac{1}{2(1-\alpha)}}\right). \tag{8.76}$$

We want to show that for all $k \in \{\beta^{-2}, \dots, N\}$

$$c_2\left(\frac{k}{N^{2\alpha-1}}\right)^{\frac{1}{2(1-\alpha)}} \ge 2\beta^2 k. \tag{8.77}$$

As the ratio between the two quantities is monotone in k, it is sufficient to check this for the smallest value $k = \beta^{-2}$. This gives

$$N \le c_3 \beta^{\frac{2}{2\alpha-1}}. \tag{8.78}$$

with

$$c_3 = \left(\frac{c_2}{2}\right)^{\frac{1}{2(1-\alpha)}}.$$

With this choice one has

$$\mathbf{E}\left[e^{\beta^2 H_N(\tau')}\right] \le 1 + \left(e^{\beta^2}-1\right)\left[e\beta^{-2} + \sum_{k=\beta^{-2}+1}^{N} e^{-k\beta^2}\right] \le 2 + 2e \le 10. \tag{8.79}$$

and hence we have proved

$$N_\beta \ge c_3 \beta^{\frac{2}{2\alpha-1}}. \tag{8.80}$$

4.5 The case $\alpha = 1/2$

We assume in this section that $\alpha = 1/2$ and thus (8.54) becomes

$$\mathbf{P}[\tau'_1 = n] \overset{n\to\infty}{\sim} c' n^{-1} \tag{8.81}$$

This yields the following estimate on the Laplace transform

Lemma 8.13. *We have*

$$\sum_{n\ge 0} e^{-nx}\mathbf{P}[n \in \tau'] \overset{x\to 0+}{\sim} c'|\log x|, \tag{8.82}$$

As a consequence

$$\log \mathbf{E}[e^{-x\tau'_1}] \overset{x\to 0+}{\sim} (c' \log x)^{-1}. \tag{8.83}$$

Proof. Note that the second point follows from the first by Lemma 8.11. The sum is equivalent to

$$c' \sum_{n\geq 1} e^{-nx}\frac{1}{n} = c' \sum_{n=1}^{\lceil x^{-1}\rceil} \frac{1}{n} + x \sum_{n\geq \lceil x^{-1}\rceil} \left(e^{-nx} - \mathbf{1}_{\{n\leq x^{-1}\}}\right)\frac{1}{nx}. \tag{8.84}$$

The first term is equivalent to $c'|\log x|$. The second is a Riemann sum which converges to

$$\int_0^\infty \frac{e^{-y} - \mathbf{1}_{y\in[0,1]}}{y}\mathrm{d}y.$$

□

Hence as for the case $\alpha > 1/2$ we have for an adequate choice of c_2, for all $k \geq 1$

$$\log \mathbf{P}[\tau'_k \leq N] \leq \max_{x\in[0,1]} \left(xN + c_1 k(\log x)^{-1}\right) \leq -c_2 k \left(\log \frac{N}{c_1 k}\right)^{-1}. \tag{8.85}$$

Thus we have

$$\sum_{k=1}^{N} e^{\beta^2 k}\mathbf{P}[\tau'_k \leq N] \leq \sum_{k=1}^{N} \exp\left(\beta^2 k - c_2 k \left(\log \frac{N}{c_1 k}\right)^{-1}\right). \tag{8.86}$$

As before we want to choose N such that for all $k \geq 1$ we have

$$c_2 \left(\log \frac{N}{c_1 k}\right)^{-1} \geq 2\beta^2. \tag{8.87}$$

Of course it is sufficient to deal with the case $k = 1$ and to choose N satisfying

$$N \leq c_1 e^{-\frac{c_2}{2\beta^2}} \tag{8.88}$$

With this choice we have

$$\sum_{k=1}^{N} e^{\beta^2 k}\mathbf{P}[\tau'_k \leq N] \leq \sum_{k=1}^{N} e^{-k\beta^2} \leq \frac{1}{e^{\beta^2} - 1}, \tag{8.89}$$

and hence

$$\mathbf{E}[(\widetilde{Z}^{\beta,\omega}_{N,0})^2] \leq 1 + (e^{\beta^2} - 1)\sum_{k=1}^{N} e^{\beta^2 k} \leq 2. \tag{8.90}$$

This implies

$$N_\beta \geq c_1 e^{-\frac{c_2}{2\beta^2}}. \tag{8.91}$$

Chapter 9

Disorder relevance

1 Shift of the critical point and smoothing of the free-energy curve

The aim of this chapter is to prove a caracteristic feature of the relevant disorder regime: a shift of the quenched critical point with respect to the annealed one.

Theorem 9.1. *When $\alpha > 1/2$, the quenched and annealed critical point differ, for all $\beta > 0$.*

Furthermore for all $\alpha \in (1/2, \infty) \setminus \{1\}$ we have there exist constants c and C such that for all $\beta \in (0,1)$

$$c\beta^{\min\left(\frac{2\alpha}{1-2\alpha},2\right)} \le h_c(\beta) + \lambda(\beta) \le C\beta^{\min\left(\frac{2\alpha}{1-2\alpha},2\right)}. \tag{9.1}$$

These two results are deep ones and are technical to prove and we shall in these notes prove weaker versions. Note that the lower-bound on the critical point shift was (up to log correction) proved in the previous chapter.

2 Either smoothing or critical point shift holds

2.1 The result

The aim of this chapter is to prove that both smoothing of the critical curve and critical point shift holds. However, for didactical purpose, we choose to prove first that at least one of the two holds. More precisely we are going to show that

Proposition 9.2. *For all $\beta > 0$ there exist $c_\beta > 0$ such that*

$$\textsc{f}(\beta, u) \le c_\beta u^2 \tag{9.2}$$

Note that first, if we had $h_c(\beta) = -\lambda(\beta)$ the inequality would coincide with (10.1). Hence this proposition asserts that if we do not have a shift of the critical point, we have a smoothing of the free energy curve.

Second, the result holds for all values of $\alpha > 0$. However, it does not say much in the case $\alpha > 1/2$ as the annealed bound gives for small α

$$\textsc{f}(\beta, u) \le \textsc{f}(0, u) \le cu^{1/\alpha}. \tag{9.3}$$

2.2 Fractional moment

Let us transform the problem of estimating the expection of a $\log Z_{N,u}^{\beta,\omega}$ which is a difficult one, to that of estimating the expectation of a non-integer moment of $Z_{N,u}^{\beta,\omega}$ Consider $\theta \in (0,1)$. We have

$$\mathbb{E}\left[\log Z_{N,u}^{\beta,\omega}\right] = \frac{1}{\theta}\mathbb{E}\left[\log (Z_{N,u}^{\beta,\omega})^{\theta}\right] \le \frac{1}{\theta}\log \mathbb{E}\left[(Z_{N,u}^{\beta,\omega})^{\theta}\right]. \tag{9.4}$$

Hence the free-energy can be bounded above as follows

$$\textsc{f}(\beta,u) \le \liminf_{N\to\infty} \frac{1}{N\theta}\log \mathbb{E}\left[(Z_{N,u}^{\beta,\omega})^{\theta}\right]. \tag{9.5}$$

In this section we choose use the inequality in the case $\theta = 1/2$. The question is then, how can we estimate

$$\mathbb{E}\left[\sqrt{Z_{N,u}^{\beta,\omega}}\right].$$

2.3 Change of measure

Let us introduce $g(\omega)$ an arbitrary positive function of the environment. By Cauchy-Schwartz inequality we have

$$\mathbb{E}\left[\sqrt{Z_{N,u}^{\beta,\omega}}\right] = \mathbb{E}\left[\sqrt{Z_{N,u}^{\beta,\omega}g(\omega)}g(\omega)^{-1/2}\right] \le \sqrt{\mathbb{E}\left[Z_{N,u}^{\beta,\omega}g(\omega)\right]}\sqrt{\mathbb{E}[(g(\omega)^{-1}]}. \tag{9.6}$$

If the quenched and annealed free-energy differ it is that $Z_{N,u}^{\beta,\omega}$ is typically much smaller (exponentially smaller in N) than its expectation $\mathbb{E}[Z_{N,u}^{\beta,\omega}]$. Hence the expectation $\mathbb{E}[Z_{N,u}^{\beta,\omega}]$ is typically supported by an event of very small probability on which $Z_{N,u}^{\beta,\omega}$. The idea is thus to choose a function $g(\omega)$ which penalizes this event a lot while $\mathbb{E}[(g(\omega)^{-1}]$ does not grow too fast. Note that if $\mathbb{E}[g(\omega)] = 1$ one can consider g as the probability density of a new measure $\widetilde{\mathbb{P}}$ with

$$\mathrm{d}\widetilde{\mathbb{P}}/\mathrm{d}\mathbb{P}(\omega) = g(\omega).$$

We have

$$\mathbb{E}\left[\sqrt{Z_{N,u}^{\beta,\omega}}\right] \le \sqrt{\widetilde{\mathbb{E}}\left[Z_{N,u}^{\beta,\omega}\right]}\sqrt{\mathbb{E}[(g(\omega)^{-1}]}. \tag{9.7}$$

2.4 The choice for $\widetilde{\mathbb{P}}_N$

One thing which is clear is that $\widetilde{\mathbb{P}}_N$ must depend on N and has to be a change of measure which only concerns $(\omega_n)_{n=1}^N$, as modifying the law of the rest will only make the cost $\mathbb{E}[(g_N(\omega)^{-1}]$ higher while it will not give any benefits. Another remark is that it seems reasonable that $g(\omega)$ should be increasing in ω.

For a moment let us assume that the ω_n are standard Gaussians (we will explain later how the proof adapts to other cases). Let us choose $\widetilde{\mathbb{P}}_N$ to be the law under which $(\omega_n)_{n=1}^N$ are standard Gaussians with mean $-\delta$. We have

$$\frac{\mathrm{d}\widetilde{\mathbb{P}}_N}{\mathrm{d}\mathbb{P}}(\omega) = g_N(\omega) = \exp\left(-\left(\sum_{n=1}^N \delta_n\omega_n\right) - N\delta^2/2\right). \tag{9.8}$$

Hence we have

$$\begin{aligned}\mathbb{E}[g_N(\omega)^{-1}] &= e^{N\delta^2},\\ \widetilde{\mathbb{E}}_N\left[Z_{N,u}^{\beta,\omega}\right] &= \mathbb{E}\left[Z_N^{\beta,u-\beta\delta,\omega}\right] = Z_N^{u-\delta\beta}.\end{aligned} \tag{9.9}$$

Hence we have

$$\frac{1}{N}\mathbb{E}\left[\log Z_{N,u}^{\beta,\omega}\right]\frac{2}{N}\log\mathbb{E}\left[\sqrt{Z_{N,u}^{\beta,\omega}}\right] \le \frac{1}{N}\left(\log\widetilde{\mathbb{E}}\left[Z_{N,u}^{\beta,\omega}\right] + \log\mathbb{E}[(g(\omega)^{-1}]\right)$$
$$= \frac{1}{N}\log Z_N^{u-\delta\beta} + \delta^2. \tag{9.10}$$

Hence passing to the limit we obtain

$$\textsc{f}(\beta,u) \le \textsc{f}(0,u-\delta\beta) + \delta^2. \tag{9.11}$$

Note first that this implies that,

$$\forall u > 0, \quad \textsc{f}(\beta,u) < \textsc{f}(0,u-\delta\beta)$$

(it is sufficient to choose δ sufficiently small). Second, choosing $\delta = u\beta^-1$ we obtain

$$\textsc{f}(\beta,u) \le \frac{u^2}{\beta^2}. \tag{9.12}$$

Let us explain now how to treat the case of general ω. We have to replace the shift of the ωs by an exponential tilt, and choose

$$g_N(\omega) = \exp\left(-\left(\sum_{n=1}^N \delta_n\omega_n\right) - N\lambda(-\delta)\right). \tag{9.13}$$

We have then

$$\begin{aligned}\mathbb{E}[g_N(\omega)^{-1}] &= e^{N\lambda(-\delta)+\lambda(\delta)},\\ \widetilde{\mathbb{E}}\left[Z_{N,u}^{\beta,\omega}\right] &= Z_N^{u-\lambda(\beta)+\lambda(\beta-\delta)-\lambda(-\delta)}.\end{aligned} \tag{9.14}$$

where the second equation is due to

$$\widetilde{\mathbb{E}}\left[e^{\beta\omega_n-\lambda(\beta)+u}\right] = \mathbb{E}\left[e^{(\beta-\delta)\omega_n-\lambda(-\delta)-\lambda(\beta)+u}\right] = e^{\lambda(\beta-\delta)-\lambda(-\delta)-\lambda(\beta-\delta)+u} \tag{9.15}$$

This yields

$$\textsc{f}(\beta,u) \le \textsc{f}\left[0, u+\lambda(\beta-\delta)-\lambda(-\delta)-\lambda(\beta-\delta)\right] + \lambda(\delta)+\lambda(-\delta). \tag{9.16}$$

Let us choose $\delta(u,\beta)$ such that

$$\lambda(\beta-\delta)-\lambda(-\delta)-\lambda(\beta) = u. \tag{9.17}$$

It is not difficult to check that this equation has one unique solution at least for u small enough as the right hand side is increasing in δ (by convexity of $\lambda(\cdot)$)

Note that as ω are centered and of unit variance $\lambda(x) \sim x^2/2$, hence when u tends to zero we have $\delta(u,\beta) \sim u/\lambda'(\beta)$ and thus

$$\textsc{f}(\beta,u) \le \lambda(\delta(u,\beta)) + \lambda(-\delta(u,\beta)) \overset{u\to 0+}{\sim} \frac{u^2}{(\lambda'(\beta))^2}. \tag{9.18}$$

For the rest of the chapter, for simplicity we may assume that the environment in Gaussian to avoid technicalities.

3 Critical point shift

3.1 Finite volume criterion

Note that by super-additivity we know that from super-additivity

$$\mathbb{E}[\log Z_{N,u}^{\beta,\omega}] > 0 \quad \Rightarrow \textsc{f}(\beta, u) > 0. \tag{9.19}$$

However, we do not have yet such a criterion to show that $\textsc{f}(\beta, u) = 0$ only observing a system of finite size N. We will exhibit such a criterion by using fractional moments. For now just recall that from (9.5)

$$\liminf_{N\to\infty} \frac{1}{N} \log \mathbb{E}\left[(Z_{N,u}^{\beta,\omega})^\theta\right] = 0 \quad \Rightarrow \textsc{f}(\beta, u) = 0. \tag{9.20}$$

for some $\theta \in (0, 1)$. In this section we derive a simple recursive inequality for $A_N := \mathbb{E}\left[(Z_{N,u}^{\beta,\omega})^\theta\right]$, which will allows to prove (9.20) only by checking the value of finitely many A_N.

$$Z_{N,u}^{\beta,\omega} = \sum_{n=0}^{N-1} Z_{N,u}^{\beta,\omega} K(N-n) e^{\beta\omega_N - \frac{\beta^2}{2} + u}. \tag{9.21}$$

Hence using the inequality

$$\left(\sum_{i\in\mathcal{I}} a_i\right) \le \sum_{i\in\mathcal{I}} a_i^\theta. \tag{9.22}$$

valid for any $\theta \in (0, 1)$, and averaging we obtain that

$$A_N \le \sum_{n=0}^{N-1} A_n (K(N-n))^\theta e^{-\frac{\beta^2}{2}\theta(1-\theta)+\theta u} \tag{9.23}$$

Lemma 9.3. *If $\theta < 1$ and u satisfies*

$$\rho := e^{\theta u - \frac{\theta(1-\theta)\beta^2}{2}} \sum_{n=1}^{\infty} (K(n))^\theta \le 1, \tag{9.24}$$

then the sequence A_N is bounded.

Proof. From (9.23) one has for all $N \ge 1$

$$A_N \le \sum_{n=0}^{N-1} (K(N-n))^\theta e^{\theta u - \frac{\theta(1-\theta)\beta^2}{2}} \max(A_0, A_1, \ldots, A_{N-1}) \le \rho \max(A_0, A_1, \ldots, A_{N-1}). \tag{9.25}$$

Hence by immediate induction we have $A_N \le A_0 = 1$ and the result is proved. □

As a consequence we have the following

Proposition 9.4. *If ω are Gaussian variables then for any $\alpha \in (0, \infty)$, there exists $\beta_c(K)$ such that for all $\beta > \beta_c$*

$$u_c(\beta) > 0.$$

Proof. If we fix $\theta \in \big((1+\alpha)^{-1}, 1\big)$, (e.g $\theta = (1+\alpha/2)^{-1}$) we have

$$\eta := \sum_{n=1}^{\infty} (K(n))^{\theta} < \infty. \tag{9.26}$$

Hence we have $\rho \leq 1$ provided that

$$\beta \geq \sqrt{\frac{2(\log \eta + \theta u)}{\theta(1-\theta)}}. \tag{9.27}$$

and in particular the above inequality is valid for some $u > 0$ sufficiently small as soon as

$$\beta > \sqrt{\frac{2 \log \eta}{\theta(1-\theta)}}. \tag{9.28}$$

□

4 Critical point shift at all disorder intensity ($\alpha > 1/2$)

Theorem 9.5. *If K satisfies* (3.6) *with $\alpha > 1/2$ then every $\beta > 0$ there exists $u_\beta > 0$ such that*

$$\mathrm{F}(\beta, u_\beta) = 0.$$

We give a complete proof for the statement in the case $\alpha \in (1/2, 1)$ which is the more difficult and explain how to adapt it to the case $\alpha > 1$ Note that in Lemma 9.6 the choice of k and θ can be made arbitrary.

The tool we use to prove the result is the following generalized version of Lemma 9.3

Lemma 9.6. *If given $\theta \in (0,1)$, $k \in \mathbb{N}$, $\beta > 0$ and $u \in \mathbb{R}$ we have*

$$\rho_k := e^{-\frac{\beta^2 \theta(1-\theta)}{2} + u} \sum_{m=0}^{k} A_m \sum_{n=k-m+1}^{\infty} K^{\theta}(n) \leq 1 \tag{9.29}$$

then

$$\mathrm{F}(\beta, u) = 0.$$

Proof. For this, for a fixed $k > 0$ we modify our decomposition (9.21) to make partition function of size up to k appear. For

$$Z^{\beta,\omega}_{N,u} = \sum_{n=0}^{N-k-1} \sum_{m=0}^{k} Z^{\beta,\omega}_{n,u} K(N-n-m) e^{\beta\omega_{N-m} - \frac{\beta^2}{2} + u} Z^{\beta,u,\theta_{N-m}\omega}_{m}. \tag{9.30}$$

Using

$$\left(\sum_{i \in \mathcal{I}} a_i\right) \leq \sum_{i \in \mathcal{I}} a_i^{\theta},$$

and averaging we obtain

$$A_N \leq \sum_{n=0}^{N-k-1} \sum_{m=0}^{k} A_n K^\theta(N-n-m) e^{-\frac{\beta^2\theta(1-\theta)}{2}+u} A_m \tag{9.31}$$

Note that for $N \geq k+1$ we have from (9.31)

$$\begin{aligned} A_N \leq \max(A_0, A_1, \dots, A_{N-k-1}) e^{-\frac{\beta^2\theta(1-\theta)}{2}+u} \sum_{m=0}^{k} \sum_{n=0}^{N-k-1} K^\theta(N-n-m) A_m \\ \leq \max(A_0, A_1, \dots, A_{N-k-1}) \rho_k. \end{aligned} \tag{9.32}$$

Hence by an immediate induction we have for every $N \geq k+1$

$$A_N \leq A_0 = 1. \tag{9.33}$$

□

The challenge is then choose an appropriate value of k and to find a proper way to bound A_m for $m \leq k$. In what follows, we always assume that

$$u \leq \frac{\beta^2\theta(1-\theta)}{2}$$

and thus it is sufficient to find an upper bound on

$$\sum_{m=0}^{k} A_m \sum_{n=k-m+1}^{\infty} K(n)^\theta \tag{9.34}$$

It will be simpler in the computation to consider θ close to one. This is in order to make $K(n)^\theta$ very close to $K(n)$ (and will also be used to estimate $A(m)$) We set for the rest of the proof

$$\theta = \theta_k = 1 - (\log k)^{-1}.$$

It turns out from the computation that the good value to chose for k is $k := u^{-1/\alpha}$. From now on we keep theses parameters fixed, and we aim to prove that $\rho_k \leq 1$ provided that u is sufficiently small.

Our first task is to show that $\sum_{n=k-m+1}^{\infty} K(n)^\theta$ is of the same order as $\sum_{n=k-m+1}^{\infty} K(n)$.

Lemma 9.7. *There exists a constant C such that if k is sufficiently large $m \leq k$*

$$\sum_{n \geq m+1} K(n)^{\theta_k} \leq C\mathbf{P}[\tau_1 > m] \tag{9.35}$$

Proof. We split the sum into two parts $n \leq k^3$ and $n \geq k^3$. For the first part, using assumption (3.6) we have

$$\sum_{n=m+1}^{k^3} K(n)^\theta \leq \left(\max_{r \leq k^3} K(r)^{(\theta-1)}\right) \left(\sum_{n=m+1}^{\infty} K(n)\right). \tag{9.36}$$

According to (3.6) the second factor is of order $m^{-\alpha}$. And concerning the first factor, it is smaller than

$$(ck - 3(1+\alpha))^{1-\theta} \leq 2e^{3(1+\alpha)}$$

.

Now if k is large enough, we have $(1+\alpha)\theta \geq (1+\alpha/2)$. Hence from (3.6)

$$\sum_{n \geq k^3+1} K(n)^\theta \leq \sum_{n \geq k^3+1} C n^{-(1+\alpha/2)} \leq C k^{-3\alpha/2}. \tag{9.37}$$

which is of a smaller order than the first term.

□

Thus we can prove prove that $\rho_k \leq 1$ and thus that Theorem 9.5 holds, if we show the following.

Proposition 9.8. *Given $\eta > 0$ one can choose such that if a (in the definition of u_β) is sufficiently small, then for all $\beta \in [0, M]$.*

$$\sum_{m=0}^{k} A_m \mathbf{P}[\tau_1 > k - m] < \eta$$

What we are going to prove is actually that A_m is much smaller than $\mathbb{E}[Z_{m,0}^{\beta,\omega}]$ for most values of m.

Lemma 9.9. *Given $\eta > 0$ and $\varepsilon >: 0$ if u is chosen sufficiently small We have for all $m \in [\![k^\varepsilon, k]\!]$*

$$A_m \leq \eta \mathbf{P}[m \in \tau] \tag{9.38}$$

On the other hand, if u is sufficiently small then for all $m \leq k^\varepsilon$ we have

$$A_m \leq 2\mathbf{P}[m \in \tau] \tag{9.39}$$

Proof of Proposition 9.8. We have

$$\begin{aligned}\sum_{m=0}^{k-1} & A_m \mathbf{P}[\tau_1 > m] \\ & \leq 2 \sum_{m=0}^{k^\varepsilon} \mathbf{P}[m \in \tau] \mathbf{P}[\tau_1 > k - m] + \eta \sum_{m=k^\varepsilon+1}^{k} \mathbf{P}[m \in \tau] \mathbf{P}[\tau_1 > k - m] \\ & \leq \eta + 2(k^\varepsilon + 1) P[\tau_1 > k/2] \leq 2\eta. \end{aligned} \tag{9.40}$$

The first is obtained by noticing that $\sum_{m=1}^{k} \mathbf{P}[m \in \tau] \mathbf{P}[\tau_1 \geq k - m] = 1$ (as the terms in the sum corresponds to the probability distribution of the last renewal point before k). The last inequality is valid for ε small if k is chosen sufficiently large. □

5 Proving Lemma 9.9

5.1 The case of small m

Note that we have

$$A_m \leq \mathbf{E}[Z_{m,u}^{\beta,\omega}]^\theta = Z_{m,u}^\theta \leq e^{\theta m u}\mathbf{P}[m \in \tau]^\theta. \tag{9.41}$$

The upper bound is obtained just by observing that the number of contact is smaller than m. We notice that if $m \leq k^\varepsilon$, as $u = k^{-\alpha}$ the first term is smaller than $\sqrt{2}$ if u is small enough. And we can conclude by observing that for the value of m considered

$$\mathbf{P}[m \in \tau]^{1-\theta} \leq K(m)^{1-\theta} \leq \sqrt{2}.$$

5.2 The change of measure

To show that A_m is small the idea is to perform a change of measure similar to that used in Section 2.3. We introduce

$$g_m(\omega) := e^{\sum_{n=1}^m \varepsilon\omega_n - \frac{m\varepsilon^2}{2}} \tag{9.42}$$

where $\varepsilon := \varepsilon_{k,m} = \frac{1}{\sqrt{m\log k}}$. We define $\widetilde{\mathbb{P}}_m$ as the measure whose density with respect to $\mathbb{P}$ is $\mathrm{d}\widetilde{\mathbb{P}}_m/\mathrm{d}\mathbb{P}(\omega) = g_m(\omega)$. Using Hölder inequality we have

$$\mathbf{E}\left[\left(Z_{m,u}^{\beta,\omega}\right)^\theta\right] = \mathbf{E}\left[\gamma(\omega)^{-\theta}\left(g(\omega)Z_{m,u}^{\beta,\omega}\right)^\theta\right] \leq \mathbf{E}\left[g_m(\omega)^{-\frac{\theta}{1-\theta}}\right]^{1-\theta}\mathbf{E}\left[g_m(\omega)Z_{m,u}^{\beta,\omega}\right]^\theta. \tag{9.43}$$

To conclude we need to show that the first term is not too large and that the second is much smaller than $\mathbf{P}[m \in \tau]$.

Lemma 9.10. *We have*

$$\mathbf{E}\left[g(\omega)^{-\frac{\theta}{1-\theta}}\right]^{1-\theta} \leq 2. \tag{9.44}$$

Proof. We have

$$\mathbf{E}\left[g(\omega)^{-\frac{\theta}{1-\theta}}\right] = \mathbf{E}\left[e^{-\frac{\theta}{1-\theta}\sum_{n=1}^m \varepsilon\omega_n + \frac{\theta}{1-\theta}\frac{m\varepsilon^2}{2}}\right] \leq e^{\frac{\theta\varepsilon^2 m}{2(1-\theta)^2}}. \tag{9.45}$$

Hence

$$\mathbf{E}\left[g(\omega)^{-\frac{\theta}{1-\theta}}\right]^{1-\theta} \leq e^{\frac{\varepsilon^2 m}{2(1-\theta)}} \leq e^{1/2}. \tag{9.46}$$

□

Lemma 9.11. *Given $\eta > 0$, if u is chosen sufficiently small, for all $m \geq k^\varepsilon$*

$$\mathbf{E}\left[g_m(\omega)Z_{m,u}^{\beta,\omega}\right] = \widetilde{\mathbb{E}}_m[Z_{m,u}^{\beta,\omega}]. \tag{9.47}$$

Note that under $\widetilde{\mathbb{P}}_m$ the first m ωs are Gaussian with mean $-\varepsilon_{k,m}$ and hence

$$\widetilde{\mathbb{E}}_m[Z_{m,u}^{\beta,\omega}] = \mathbb{E}\left[Z_{m,u-\varepsilon\beta}^{\beta,\omega}\right] = Z_{m,u-\varepsilon\beta}. \tag{9.48}$$

Now the important observation, is that with our choice $u = k^{-\alpha}$, given $A > 0$ arbitrary, we have for u sufficiently small

$$u - \varepsilon\beta = k^{-\alpha} - \frac{\beta}{\sqrt{m|\log k|}} \leq -Am^{-\alpha}$$

Hence to prove (9.11), we just need an estimate on

$$Z_{m,-Am^{-\alpha}} \geq Z_{m,\beta-u\varepsilon}.$$

5.3 Estimating pure partition function close to the critical point

Lemma 9.12. *For any η there exists m_0 and A such that for every $m \geq m_0$*

$$Z_{m,-Am^{-\alpha}} \leq \eta\mathbf{P}[m \in \tau]. \tag{9.49}$$

Note that

$$Z_{m,-Am^{-\alpha}} = \mathbf{P}[m \in \tau]\mathbf{E}\left[e^{-Am^{-\alpha}\sum_{n=1}^{m}\delta_n} \mid m \in \tau\right]. \tag{9.50}$$

Our objective is to show that the second term is small. We have

$$\mathbf{E}\left[e^{-Am^{-\alpha}\sum_{n=1}^{m}\delta_n} \mid m \in \tau\right] \leq e^{-A\varepsilon} + \mathbf{P}\left[\sum_{n=1}^{m}\delta_n \leq \varepsilon m^{\alpha} \mid m \in \tau\right]. \tag{9.51}$$

Hence we are left to prove

Lemma 9.13.

$$\lim_{\varepsilon\to 0}\limsup_{m\to\infty}\mathbf{P}\left[\sum_{n=1}^{m}\delta_n \leq \varepsilon m^{\alpha} \mid m \in \tau\right] = 0 \tag{9.52}$$

Note that we already proved the statement for the unconditional measure. Hence we just need to show that the conditioning $m \in \tau$ is not that much of a problem. We admit the following result (for a proof see [2]).

Lemma 9.14. *Let A be any event that can be expressed in terms of $\tau \cap [0, m/2]$. We have*

$$\mathbf{P}\left[A \mid m \in \tau\right] \leq C\mathbf{P}\left[A\right], \tag{9.53}$$

where the constant C does not depend on m.

Using this Lemma we obtain that

$$\mathbf{P}[\sum_{n=1}^{m}\delta_n \leq \varepsilon m^{\alpha} \mid m \in \tau] \leq \mathbf{P}[\sum_{n=1}^{m/2}\delta_n \leq \varepsilon m^{\alpha} \mid m \in \tau] \leq C\mathbf{P}[\sum_{n=1}^{m/2}\delta_n \leq \varepsilon m^{\alpha}]. \tag{9.54}$$

And by Lemma 8.4 the r.h.s can be made arbitrarily small by choosing ε small.

5.4 The case $\alpha > 1$

To perform the proof in the case $\alpha > 1$, it is sufficient to consider $k = u^{-1}$ in the computation, and to show that the pinning parameter after the change of measure is larger than Ak^{-1}. Note that using the renewal Theorem, we have $\mathbf{P}[m \in \tau]$ converges to a positive constant in this case, and thus Lemma 9.14 is trivial.

Chapter 10

Smoothing of the free-energy

1 The result

This result is valid for all α but bears more meaning when $\alpha > 1/2$

Theorem 10.1. *For any $\beta > 0$, there exists a constant c_β such that*

$$\mathrm{F}(\beta, h) \leq c_\beta (h - h_c(\beta))^2. \tag{10.1}$$

Moreover, there exists a constant C such that for $\beta \leq 1$ we can choose $c_\beta = C\beta^{-2}$.

The result implies in particular that

$$\liminf_{h \to h_c(\beta)} \frac{\log \mathrm{F}(\beta, h)}{\log(h - h_c(\beta))} \geq 2. \tag{10.2}$$

In particular the value of this exponent differ from that corresponding to $\beta = 0$.

$$\lim_{h \to h_c(\beta)} \frac{\log \mathrm{F}(0, h)}{\log(h - h_c(0))} = \max(1, \alpha^{-1}). \tag{10.3}$$

We are going to restrict the proof to the Gaussian case. The base of the proof is to say that if the free-energy is too big, then one can find a localization strategy at the critical point $h_c(\beta)$.

The ideas for the localization strategy is to visit only regions where the ω takes higher values.

2 The contribution and frequency of rare stretches

In this chapter we use the notation $u = h - h_c(\beta)$ Let us consider $u > 0$, fix $\varepsilon > 0$. From the definition of the free-energy, there exists $N_0(u, \varepsilon)$ which is such that for all $N > N_0$

$$\mathbb{P}\left[\log Z_N^{\beta, h_c(\beta)+u} \geq N\left(\mathrm{F}(\beta, h_c(\beta) + u) - \varepsilon\right)\right] \geq 1/2. \tag{10.4}$$

Now we want to get a lower bound for the probability for the event

$$A_N := \left\{\log Z_N^{\beta, h_c(\beta), \omega} \geq N\left(\mathrm{F}(\beta, h_c(\beta) + u) - \varepsilon\right)\right\}. \tag{10.5}$$

Lemma 10.2. *For any $\varepsilon, \beta, u > 0$, one can find $N_1 \geq N_0$ such that for all $N \geq N_1$*

$$\mathbb{P}[A_N] \geq \exp\left(-\left(\frac{u^2}{2\beta^2} + \varepsilon\right) N\right). \tag{10.6}$$

Proof. Set

$$g_N(\omega) := e^{\sum_{n=1}^N (u/\beta)\omega_n - \frac{u}{2\beta^2}}. \tag{10.7}$$

and set

$$\frac{\mathrm{d}\widetilde{\mathbb{P}}_N}{\mathrm{d}\mathbb{P}}(\omega) = g_{N_0}(\omega). \tag{10.8}$$

We have (10.4)

$$\widetilde{\mathbb{P}}_N(A_N) \geq 1/2 \tag{10.9}$$

On the other hand we have for all $\theta < 1$

$$\widetilde{\mathbb{P}}_N(A) = \mathbb{E}\left[g_N(\omega)\mathbf{1}_A\right] \leq (\mathbb{P}[A])^\theta\, \mathbb{E}\left[g_N(\omega)^{\frac{1}{1-\theta}}\right]^{1-\theta}. \tag{10.10}$$

As we have

$$\mathbb{E}\left[g_N(\omega)^{\frac{1}{1-\theta}}\right]^{\frac{1-\theta}{\theta}} = e^{-\frac{Nu^2}{2\beta^2(1-\theta)}}, \tag{10.11}$$

by choosing θ sufficiently close to zero (e.g $N^{-1/2}$), we get

$$\mathbb{P}[A_N] \geq 2^{-(1/\theta)} e^{-\frac{N_0 u^2}{2\beta^2(1-\theta)}} \geq \exp\left(-\left(\frac{u^2}{2\beta^2} + \varepsilon\right) N\right). \tag{10.12}$$

□

In the rest of this chapter, we are going to use the Lemma for a partition function which includes environment at the first point but not at the last

$$Z^\omega_{[a,b]} := \mathbf{E}\left[e^{\sum_{n=0}^{b-a-1}(\omega_{a+n}+h)\delta_n}\delta_{b-a}\right] \tag{10.13}$$

3 The strategy

We split the system in stretches of size $N_2 \geq N_1$, and we consider a system of size mN_2 for $m \geq 1$. We want the renewal to visit only the regions $[(i-1)N_2, iN_2]$ which are such that

$$\log Z^\omega_{[(i-1)N_2, iN_2]} \geq N_2\left(\mathrm{F}(\beta, h_c(\beta) + u) - \varepsilon\right). \tag{10.14}$$

We set $T_0 = 0$ and

$$T_k := \inf\left\{ i \geq T_{k-1} + 2 \mid \log Z^\omega_{[(i-1)N_2, iN_2]} \geq N_2\left(\mathrm{F}(\beta, h_c(\beta) + u) - \varepsilon\right)\right\}. \tag{10.15}$$

Setting

$$I_m := \max\{k \mid T_k \leq N - 2\}, \tag{10.16}$$

we have (using our strategy) and setting by convention

$$Z_{N_2 m}^{\beta,h_c(\beta),\omega} \geq \prod_{k=1}^{I_m} K((T_k - T_k - 1 - 1)N_2) Z^{\omega}_{[(i-1)N_2, iN_2]} e^{\beta\omega_{N_2 T_k} + h_c(\beta)} \times K(N_2(m - T_{I_m}))e^{\beta\omega_{N_2 m}+h_c(\beta)}. \quad (10.17)$$

Using the definition of T_i this implies

$$\log Z_{N_2 m}^{\beta,h_c(\beta),\omega} \geq |I_m| N_2 \left(\mathrm{F}(\beta, h_c(\beta)+u) - \varepsilon\right) + \sum_{k=1}^{I_m} \left[(\beta\omega_{N_2 T_k} + h_c(\beta)) + \log K((T_k - T_k - 1 - 1)N_2)\right] + \log K(N_2(m - T_{I_m})) + (\beta\omega_{N_2 m} + h_c(\beta)). \quad (10.18)$$

Note that the increment of T_k are IID variables (this comes from the fact the law of $(\omega_n)_{n\geq (T_i+1)N}$ is independent of T_i and of i) and that the $\omega_{N_2 T_k}$ are also IID For this reason as I_m goes to infinity almost surely

$$\lim_{m\to\infty} \frac{1}{|I_m|} \sum_{k=1}^{I_m} \left[(\beta\omega_{N_2 T_k} + h_c(\beta)) + \log K((T_k - T_k - 1 - 1)N_2)\right] = h_c(\beta) + \mathbb{E}[\log\left(K(T_1 - 1)N_2\right)]. \quad (10.19)$$

Now not also that by the law of large numbers we have

$$\lim_{m\to\infty} \frac{|I_m|}{m} = \mathbb{P}[A_{N_2}]. \quad (10.20)$$

Hence we have

$$0 = \mathrm{F}(\beta, h_c(\beta)) \geq \mathbb{P}[A_{N_2}] \left(\frac{1}{N_2} \left(h_c(\beta) + \mathbb{E}\left[\log\left(K(T_1-1)N_2\right)\right]\right) + \mathrm{F}(\beta, h_c(\beta)+u) - \varepsilon \right). \quad (10.21)$$

Hence we have

$$\mathrm{F}(\beta, h_c(\beta)+u) \leq \frac{1}{N_2} \left(-\mathbb{E}\left[\log K\left((T_1-1)N_2\right)\right] - h_c(\beta)\right) + \varepsilon. \quad (10.22)$$

Hence we are left with estimating

$$-\frac{1}{N_2}\mathbb{E}\left[\log\left(K(T_1-1)N_2\right)\right]. \quad (10.23)$$

Note that $T_1 - 1$ is a geometric variable of parameter $p_{N_2} := \mathbb{P}[A_{N_2}]$
We will use the following Lemma

Lemma 10.3. *Let T_N be a geometric variable of parameter $p_N := \mathbb{P}[A_N]$. We have we have*

$$\liminf_{N\to\infty} \frac{1}{N}\mathbb{E}\left[\log\left(K(T_N N)\right)\right] \geq -(1+\alpha)\frac{u^2}{2\beta^2}. \quad (10.24)$$

Proof. Set $p = p_{N_2}$ we have

$$\begin{aligned}\mathbf{P}_p\left[\log K\left(T_N N\right)\right] &= \sum_{n\geq 1} p(1-p)^{n-1}\log K(nN) \\ &= -(1+\alpha)\sum_{n\geq 1} p(1-p)^{n-1}\log(nN) + \sum_{n\geq 1} p(1-p)^{n-1}\log\left((nN)^{1+\alpha}K(nN)\right). \quad (10.25)\end{aligned}$$

Note that the second sum is bounded uniformly in N_2 from the assumption on K. The first sum is equal to

$$\log N - \log p + \sum_{n\geq 1} p(1-p)^{n-1}\log pn, \tag{10.26}$$

and we have

$$\lim_{p\to 0} p\sum_{n\geq 1}\sum_{n\geq 1} p(1-p)^{n-1}\log pn = \int_0^\infty \log te^{-t}\mathrm{d}t. \tag{10.27}$$

in particular the sum is uniformly bounded in p As we have

$$\liminf_{N\to\infty} \frac{\log p_N}{N} \geq -\frac{u^2}{2\beta^2}, \tag{10.28}$$

we can conclude. □

Hence choosing N_2 sufficiently large we have

$$\textsc{f}(\beta, h_c(\beta)+u) \leq (1+\alpha)\frac{u^2}{2\beta^2} + 2\varepsilon. \tag{10.29}$$

As ε is arbitrary, this finishes the proof.

4 Consequences on number of contact at the critical point

Proposition 10.4. *There exists a constant C_β such that for all $N \geq 1$*

$$\mathbb{E}\mathbf{E}_N^{\beta,\omega,h_c(\beta)}\left[\sum_{n=1}^N \delta_n\right] \leq C_\beta\sqrt{N\log N} \tag{10.30}$$

Recall that $\mathbf{E}[H_N(\tau)] = N^\alpha$, hence for $\alpha > 1/2$ we obtain that the number of contact at the critical point is much smaller in the critical case.

Proof. As we have seen before we have for all u

$$\textsc{f}(\beta, h_c(\beta)+u) \geq \frac{u}{N}\mathbb{E}\mathbf{E}_N^{\beta,\omega,h_c(\beta)}\left[\sum_{n=1}^N \delta_n\right] + \frac{1}{N}\mathbb{E}\log Z_N^{\beta,\omega,h_c(\beta)}. \tag{10.31}$$

Hence

$$\mathbb{E}\mathbf{E}_N^{\beta,\omega,h_c(\beta)}\left[\sum_{n=1}^N \delta_n\right] \leq \frac{N\textsc{f}(\beta,h_c(\beta)+u)}{u} - \frac{\mathbb{E}\log Z_N^{\beta,\omega,h_c(\beta)}}{u} \leq \frac{(1+\alpha)Nu}{2\beta^2} - \frac{\log K(N) - h_c(\beta)}{u}. \tag{10.32}$$

Choosing $u = N^{-1/2}(\log N)^{1/2}$ we obtain the result.

□

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