POLYNOMIAL COMPLETION OF SYMPLECTIC JETS AND SURFACES CONTAINING INVOLUTIVE LINES

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Abstract. Motivated by work of Dragt and Abell on accelerator physics, we study the completion of symplectic jets by polynomial maps of low degrees. We use Andersén-Lempert Theory to prove that symplectic completions always exist, and we prove the degree bound conjectured by Dragt and Abell in the physically relevant cases. However, we disprove the degree bound for 3-jets in dimension 4. This follows from the fact that if Σ is the disjoint union of $r = 7$ involutive lines in $\mathbb{P}^3$, then Σ is contained in a degree $d = 4$ hypersurface, *i.e.*, the restriction morphism $\iota : H^0(\mathbb{P}^3, \mathcal{O}(4)) \to H^0(\Sigma, \mathcal{O}(4))$ has a nontrivial kernel (Todd). We give two new proofs of this fact, and finally we show that if $(r, d) \neq (7, 4)$ then the map ι has maximal rank.

1. Introduction

Throughout this article we let ω denote the standard symplectic form on $\mathbb{C}^{2n}$, *i.e.*, $\omega(z) = \sum_{j=1}^{n} dz_j \wedge dz_{n+j}$. If $\Omega \subset \mathbb{C}^{2n}$ is a domain, and $F : \Omega \to \mathbb{C}^{2n}$ is an analytic map, we say that F is a symplectomorphism if $F^*\omega = \omega$. If $a \in \Omega$ and F merely satisfies $(F^*\omega - \omega)(z) = O(\|z - a\|^d)$, then we say that F is symplectic to order d at a. If F is also a degree d polynomial map, then we call F a symplectic d-jet at a. In this article we consider the completion of symplectic jets

A linear subspace $\Lambda \subset \mathbb{C}^{2n}$ is said to be Lagrangian if it has dimension n and $\omega|_\Lambda = 0$. We will say that a linear subspace $H \subset \mathbb{P}^{2n-1}$ is involutive if $\pi^*(H) \cup \{0\}$ is Lagrangian, where π is the projection $\pi : \mathbb{C}^{2n} \setminus \{0\} \to \mathbb{P}^{2n-1}$. For integers (n, d) we set $N(n, d) := \binom{n+d-1}{d}$, the dimension of the space of homogeneous polynomials of degree d in $\mathbb{C}^n$. We also write $M(2n, d) := \lceil N(2n, d)/N(n, d) \rceil$. We denote by $Aut_{Sp}\mathbb{C}^{2n}$ the group of symplectic automorphisms of $\mathbb{C}^{2n}$.

In this article we will prove the following results, motivated by accelerator physics.

Theorem 1. *Let P be a symplectic d-jet at the origin in $\mathbb{C}^{2n}$. Then there exists a polynomial map $F \in Aut_{Sp}\mathbb{C}^{2n}$ such that $(F - P)(z) = O(\|z\|^d)$.*

Theorem 2. *Let P be a symplectic d-jet at the origin in $\mathbb{C}^6$ with $2 \leq d \leq 11$. Then there exists a polynomial map $F \in Aut_{Sp}\mathbb{C}^6$ with $deg(F) \leq d^{M(6,d+1)}$ such that $(F - P)(x) = O(\|z\|^d)$.*

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The proof of Theorem 1 uses Andersén-Lempert theory, and follows closely the work of Forstnerič [6]. Theorem 2 follows from the same proof and the following:

Theorem 3. *Let $3 \leq d \leq 12$ and let Σ be a disjoint union of $r = M(6,d)$ involutive planes in general position in $\mathbb{P}^5$. Then the restriction map*

$$H^0(\mathbb{P}^5, \mathcal{O}(d)) \to H^0(\Sigma, \mathcal{O}(d))$$

has maximal rank.

Recall that the global sections of $\mathcal{O}(d)$ correspond to homogeneous polynomials of degree d. Theorem 3 proves a conjecture by Dragt and Abell [1] on bounding the degree of symplectic completions in the cases relevant for accelerator physics. The proof of Theorem 3 uses computer algebra and is described in Section 4. We note that the proofs of Theorems 1, 2 and 3 are constructive, and give algorithms for finding the symplectic completions.

However, the conjecture does not hold in general:

Theorem 4. *(Todd) Let Σ be a collection of 7 disjoint involutive lines in $\mathbb{P}^3$. Then Σ is contained in a quartic.*

We give two new proofs of this result. In Section 4 we prove Theorem 4 by using computer algebra, and in Section 5 by exhibiting the sought quartic as the tangency locus of two foliations tangent to the contact structure and the 7 involutive lines.

Finally we give a complete version of Theorem 3 in dimension 3:

Theorem 5. *Let Σ be a disjoint union of r involutive lines in general position in $\mathbb{P}^3$ and $d \geq 0$ be an integer. If $(r,d) \neq (7,4)$ then the restriction map*

$$H^0(\mathbb{P}^3, \mathcal{O}(d)) \to H^0(\Sigma, \mathcal{O}(d))$$

has maximal rank.

Our proof in Section 6 follows very closely the work of Hartshorne and Hirschowitz [9] who proved the result for non-involutive lines.

Motivation from Accelerator Physics. Let us briefly outline the motivation from accelerator physics for the problems considered in this article, for more details see [1]. Consider a circular particle accelerator where two groups of particles circulate a large number of times in opposite directions, before hitting each other at a given location. In order to control this collision, a precise understanding of the orbits of the particles is needed. Instead of following the continuous flows of these orbits, we can take a cross-section of the accelerator and compute the map G that, given a current intersection of the orbits with this cross-section, computes the next intersection. It turns out that this is a symplectic map in 6 real variables.

Let us assume that the map G has an ideal orbit. After rescaling we may assume that $G(0) = 0$. We also assume that G is given by a convergent power series expansion. We can then try to measure the first so many coefficients of this expansion. In practice the power series of G is measured up to and including degree 11. Denote the degree 11-jet of G by P. Then in general P is only symplectic to order 11, and the iterative behavior of P can be distinctly different from the behavior of G. The idea is now to find a new symplectomorphism F that has exactly the same 11-jet as P. It turns out that even though the higher order terms of F are unrelated to the higher order terms of G, the iterative behavior of F may approximate the iterative behavior of G better than P does.

The goal is therefore to not only prove that P has a symplectic completion, but also to find a method for actually finding a symplectic completion of P, and preferably a completion with a relatively small complexity so that a large number of iterations of F can be computed. In this article we find symplectic completions given by polynomials with relatively small degrees.

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2. Background: A question of Dragt and Abell

Throughout this article we denote vectors in $\mathbb{C}^{2n}$ by boldface letters $\mathbf{a}$ when we want to emphasize that we do not regard them as variables, and we let $\mathbf{a} \cdot z$ denote the product $\mathbf{a} \cdot z = \sum_{j=1}^{2n} \mathbf{a_j} \cdot z_j$. We let J denote the symplectic involution

$$J(z) = (-z_{n+1}, ..., -z_{2n}, z_1, ..., z_n).$$

We denote by $V_{n,d}$ the vector space of d-homogenous polynomials in n variables.

Recall the definition of a *kick-map*; a holomorphic automorphism of the simple form

$$G(z) = (z_1 + g_1(z_{n+1}, ..., z_{2n}), \cdots, z_n + g_n(z_{n+1}, ..., z_{2n}), z_{n+1}, \cdots, z_{2n}). \tag{1}$$

If all the g_j's are d-homogenous polynomials we call G a d-homogenous kick-map. A d-homogenous kick-map is symplectic if and only if the matrix

$$(\partial g_j / \partial z_{n+i})_{1 \leq i,j \leq n} \tag{2}$$

is symmetric, *i.e.*, if the differential form $\sum_{j=1}^{n} g_j(z) dz_{n+j}$ is closed, hence also exact. In particular we may identify the space of symplectic kick-maps with the space of $(d+1)$-homogenous polynomials in n variables. We can now try to find symplectic completions by considering conjugations $L \circ G \circ L^{-1}$ of symplectic kick-maps by symplectic linear maps, and compositions of these. This reduces to finding a suitable basis for the space $V_{n,d+1}$ (see Section 4). Since we will mostly be working with potentials of the Q_d's we will from now on write d instead of $d+1$.

Related to this, Dragt and Abell pose the following problem [1]: given $d \in \mathbb{N}$, find the least number of linear maps $L_j \in Aut_{Sp}\mathbb{C}^{2n}$, $j = 1, ..., M$, such that any $P \in V_{2n,d}$ can be written as

$$P = \sum_{j=1}^{M} Q_j \circ L_j, \tag{3}$$

where the Q_j's are d-homogenous polynomials in the variables $(z_{n+1}, ..., z_{2n})$. They conjecture that one always can achieve (3) with $M = M(2n, m)$, and they note that in the physically relevant cases the dimension $2n$ equals 6, and the degree d is at most 12. Recall that $d = 12$ corresponds to symplectic jets of degree 11. Dragt and Abell claim to have found linear maps (L_j) that confirm the conjecture for degrees up to 6.

To see how this conjecture is related to Theorem 3 we use the following fundamental fact.

2.1. **Lemma.** *Let $\Sigma \subset \mathbb{C}^n$ be any subset, and let $d \in \mathbb{N}$. Then the elements in*

$$E = \{(\mathbf{a} \cdot z)^d : \mathbf{a} \in \Sigma\}$$

form a basis for the vector space $V_{n,d}$, if and only if there does not exist a d-homogenous polynomial P which vanishes on Σ. In particular, if $\Sigma = \mathbb{C}^n$ then E spans $V_{n,d}$.

Proof. The elements z^α with $|\alpha| = d$, is a basis for the vector space $V_{n,d}$. Expressed in this basis we have that $(\mathbf{a} \cdot z)^d = \sum_\alpha b(\alpha)\mathbf{a}^\alpha z^\alpha$ where the coefficients $b(\alpha)$ are the multi-binomial coefficients corresponding to α. Assume that Λ is a non-zero linear form that anilates all elements of the form $(\mathbf{a} \cdot z)^m$, and write

$$\Lambda(\mathbf{a} \cdot z)^m = \sum_\alpha \lambda_\alpha \cdot b(\alpha)\mathbf{a}^\alpha \equiv 0.$$

Considering $\mathbf{a}^\alpha$ as a basis for $V_{n,d}$ we get a d-homogenous polynomial that vanishes on Σ. Conversely, any such polynomial gives rise to a linear form on $V_{n,d}$ which vanishes on E. □

Now assume that Σ is a disjoint union of $M(2n, d)$ involutive subspaces of $\mathbb{P}^{2n-1}$ and assume that the restriction $H^0(\mathbb{P}^{2n-1}, \mathcal{O}(d)) \to H^0(\Sigma, \mathcal{O}(d))$ is injective. Then we think of Σ as a collection $\Sigma_1, ..., \Sigma_{M(2n,d)}$ of Lagrangian subspaces of $\mathbb{C}^{2n}$, and according the previous lemma the set E forms a basis for $V_{n,d}$. For each Σ_j pick $L_j \in Aut_{Sp}\mathbb{C}^{2n}$ such that $(L_j^{-1})^T$ maps Σ_j to $\{z \in \mathbb{C}^{2n} : z_1 = \cdots = z_n = 0\}$. Now the elements $(\mathbf{a} \cdot z)^d = ((L_j^{-1})^T \mathbf{a} \cdot L_j z)^m$, with $\mathbf{a} \in \Sigma_j$ and $j = 1, ..., M(2n, d)$, form a basis for $V_{2n,d}$.

3. Proof of Theorem 1

We start by making the following observation.

3.1. **Lemma.** *Consider a polynomial map $P : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$,*

$$P(z) = z + P_d(z) + O(\|z\|^{d+1}),$$

Then $(P^*\omega - \omega)(z) = O(\|z\|^d)$ if and only if $d(P_m \lrcorner \omega) = 0$, i.e., *if and only if P_d is Hamiltonian regarded as a vector field on $\mathbb{C}^{2n}$.*

Proof. We have

$$\begin{aligned}
P^*\omega(u,v) &= \omega(P_*u, P_*v) \\
&= \omega(u + (P_m)_*u + O(\|z\|^d), v + (P_d)_*v + O(\|z\|^d)) \\
&= \omega(u,v) + \omega(u, (P_d)_*v) + \omega((P_d)_*u, v) + O(\|z\|^d) \\
&= \omega(u,v) - v[(P_d \lrcorner \omega)(u)] + u[(P_d \lrcorner \omega)(v)] + O(\|z\|^d) \\
&= \omega(u,v) + d(P_d \lrcorner \omega)(u,v) + O(\|z\|^d)
\end{aligned}$$

The last equality is the well-known formula for the differential of a 1-form, and the next to last equality is an easy computation. The second term is of degree $d-1$ in z and must therefore be zero, *i.e.*, $d(P_d \lrcorner \omega) = 0$ □

We will now prove the key lemma for proving Theorem 1.

3.2. **Lemma.** *The vector space of d-homogenous Hamiltonian vector fields on $\mathbb{C}^{2n}$ is spanned by vector fields of the form $(J\mathbf{a} \cdot z)^d \cdot \mathbf{a}$.*

Proof. Let $P_d(z)$ be a hamiltonian vector field. Then there exists a $(d+1)$-homogenous polynomial $Q(z)$ such that $P_d(z) = J\nabla Q(z)$. By Lemma 2.1 we may write

$$Q(z) = \sum_{j=1}^{N} c_j \cdot (\mathbf{a}^j \cdot z)^{m+1}$$

from which the result follows. □

Proof of Theorem 1: We are given a symplectic d-jet $P(z)$. We will prove the result by induction on k for $2 \leq k \leq d$. Since $DP(0) \in Aut_{Sp}\mathbb{C}^{2n}$ we may match P to order one.

Assume next that we have found $F_k \in Aut_{Sp}\mathbb{C}^{2n}$ with $(F_k - P)(z) = O(\|z\|^k)$ for $2 \leq k < d$. Write

$$G_k(z) := P \circ F_k^{-1}(z) = z + P_{k+1}(z) + O(\|z\|^{k+2}).$$

Since G_k is a symplectic d-jet and $d \geq k+1$, it follows by Lemma 3.1 that P_{k+1} is Hamiltonian, and by Lemma 3.2 we may write

$$P_{k+1}(z) = \sum_{j=1}^{N} c_j \cdot (J\mathbf{a}^j \cdot z)^m \cdot \mathbf{a}^j.$$

Define $S_j(z) := z + c_j \cdot (J\mathbf{a}^j \cdot z)^m \cdot \mathbf{a}^j$. Then $S_j \in Aut_{Sp}\mathbb{C}^n$ for all j, and the composition $H_{k+1} := S_N \circ \cdots \circ S_1$ matches G_k to order $k+1$. Then $F_{k+1} := H_{k+1} \circ F_k$ matches P to order $k+1$. □

4. Proofs of Theorems 2, 3 and 4 using computer algebra

Let $P(z) = z + P_d(z)$ be a homogenous symplectic d-jet at the origin. The algorithm in the proof of Theorem 1 is likely to produce a symplectic completion F whose degree is too large to be useful in practice. More specifically, the degree of F will depend on the choice of basis $\{(J\mathbf{a}^j \cdot z)^{d+1}\}_{j=1}^{N(2n,d+1)}$. To illustrate this, consider two vectors $\mathbf{a}$ and $\mathbf{b}$, and the composition of two shears:

$$S_b \circ S_a(z) = z + (J\mathbf{a} \cdot z)^d \cdot \mathbf{a} + (J\mathbf{b} \cdot (z + (J\mathbf{a} \cdot z)^d \cdot \mathbf{a}))^d \cdot \mathbf{b} \tag{4}$$

Composing more maps, we see that we run the risk of producing a degree $d^{N(2n,d+1)}$ polynomial map while trying to match a d-jet. However, inspecting (4) we note that if $J\mathbf{b} \cdot \mathbf{a} = 0$, *i.e.*, if $\mathbf{a}$ and $\mathbf{b}$ lie in a *Lagrangian* subspace of $\mathbb{C}^{2n}$, then the degree of $S_2 \circ S_1$ is d (recall that an n-dimensional subspace $\Lambda \subset \mathbb{C}^{2n}$ is Lagrangian if $\omega|_\Lambda = 0$, or, equivalently, if $J\mathbf{a} \cdot \mathbf{b} = 0$ for all $\mathbf{a}, \mathbf{b} \in \Lambda$). Hence, to keep the growth of degree down when composing our shear maps, we should choose the maximal number of vectors $\mathbf{a}^j$ from the same Lagrangian subspaces. For any given Lagrangian $\Lambda \subset \mathbb{C}^{2n}$, it follows from Lemma 2.1 that we may find vectors $\mathbf{a}^j \in \Lambda$, $j = 1, ..., N(n, d+1)$, such that the vectors $(\mathbf{a}^j \cdot z)^{d+1}$ are linearly independent. It follows that we need at least $M(2n, d+1) = \lceil N(2n, d+1)/N(n, d+1) \rceil$ Lagrangian subspaces. Note that $M(2n, d+1)$ is exactly the number appearing in the Conjecture of Dragt and Abell.

For simplicity of notation we will from now on write d instead of $d+1$. This will not create confusion as we will no longer need to consider the jet P.

4.1. **Definition.** If the d-homogenous polynomials $\{(\mathbf{a} \cdot z)^d\}_{\mathbf{a} \in \Lambda_j, 1 \leq j \leq M(2n,d)}$ span the vector space of all d-homogenous polynomials then we say that the Lagrangian subspaces (Λ_j) *span degree* d.

We first discuss our approach to proving Theorem 3 for general dimension n, and later restrict to $2n = 6$. Choose $M(2n, d)$ Lagrangian subspaces $\Lambda_j \subset \mathbb{C}^{2n}$ at random, and for each j choose vectors $\mathbf{a}^{k,j} \in \Lambda_j$, for $1 \leq k \leq N(n, d)$. Expand the d-homogenous polynomials $(\mathbf{a}^{k,j} \cdot z)^d$ in a convenient basis, and call the corresponding vectors $v_{k,j}$. We form the matrix whose rows are the vectors $v_{k,j}$ and then compute the rank. If the rank turns out to be $N(2n, d)$ we have proved Theorem 3 for degree d. If however the rank turns out to be less than $N(2n, d)$ we cannot draw conclusions; it might merely be caused by a bad choice of Lagrangian spaces and vectors.

If two Lagrangian subspaces Λ_1 and Λ_2 intersect non-trivially then it is clear that the matrix formed by the vectors $(\mathbf{a}^{k,j} \cdot z)^d$ cannot have maximal rank. Therefore we should consider collections of Lagrangian subspaces $\Lambda_1, \ldots, \Lambda_k$ that satisfy

$$\Lambda_i \cap \Lambda_j = \{0\},$$

for all $i \neq j$. It turns out that there exists a convenient form for such collections.

4.2. **Lemma.** *Let $\Lambda_1, \ldots, \Lambda_k$ be Lagrangian subspaces of $\mathbf{C}^{2n}$ with the property that*

$$\Lambda_i \cap \Lambda_j = \{0\},$$

for any $i \neq j$*. Then, after a suitable linear symplectic change of coordinates the Lagrangian subspaces* (Λ_j) *are spanned by the matrices*

$$L_1 = [I, 0], L_2 = [0, I],$$

and

$$L_j = [I, A_j]$$

for $j = 3, \dots k$*. Moreover, the* $n \times n$ *matrices* A_j *are symmetric, have non-zero determinant, and the same holds true for the matrices* $(A_i - A_j)$ *for distinct* $i, j \geq 3$*.*

Proof. First note that we can change coordinates so that Λ_1 is induced by $[I, 0]$. Now suppose that Λ_2 is induced by the matrix $[A, B]$. Using the fact that the intersection of Λ_1 and Λ_2 is $\{0\}$ we get that $\det B \neq 0$. Hence after changing coordinates with the linear symplectomorphism given by

$$(z, w) \to (z - AB^{-1}w, w)$$

the subspace Λ_2 will be given by the matrix $[0, B]$, or equivalently $[0, I]$, while Λ_1 is still given by $[I, 0]$.

Using that $\Lambda_j \cap \Lambda_2 = \{0\}$ for $j \geq 3$ immediately gives that Λ_j is spanned by a matrix of the form $[I, A_j]$. The fact that the subspace is Lagrangian forces the matrix A_j to be symmetric. The determinant of A_j must be non-zero or else Λ_j intersects Λ_1 in at least a line. Similarly $\det(A_j - A_i) \neq 0$ or else Λ_j and Λ_i intersect in at least a line. □

Working with Lagrangian subspaces that are generated by matrices of this form has several advantages. One is that, at least for small degree d and dimension $2n$, we can explicitly describe by computer for which matrices (A_j) the Lagrangian subspaces can generate a matrix T of maximal rank. When $2n = 4$ and $d = 3$ the matrix T is a 20×20 square matrix. A symbolic computation of its determinant in Mathematica gives the following.

4.3. **Lemma.** *Consider five Lagrangian subspaces* (Λ_j) *in* $\mathbb{C}^4$ *generated by* $[I, 0]$*,* $[0, I]$*,* $[I, A]$*,* $[I, B]$ *and* $[I, C]$*, where* A, B, C *are symmetric* 2×2 *matrices that satisfy the conditions in Lemma* 4.2*. Let us write* $A = (a_{kl})$*,* $B = (b_{kl})$*, and* $C = (c_{kl})$*. Then* $\Lambda_1, \dots, \Lambda_5$ *span degree* 3 *if and only if*

$$\det \begin{bmatrix} a_{11} & b_{11} & c_{11} \\ a_{12} & b_{12} & c_{12} \\ a_{22} & b_{22} & c_{22} \end{bmatrix} \neq 0.$$

We were unable to find similar formulas for larger degrees. We shall see below that there is no collection of $M(4, 4)$ Lagrangian subspaces that spans degree 4. For degrees 5 and larger our computer was unable to run the symbolic computation.

4.4. **Proof of Theorem 2.** An advantage of Lagrangian subspaces of the form described in Lemma 4.2 is that it allows us to easily work with integer coefficients. If there exists a collection (Λ_j) that spans degree d, then degree d is spanned for collections (Λ_j) in a Zariski open subset. Hence maximal rank will also be attained for matrices (A_j) with integer coefficients. Working with integer coefficients will turn out to be very helpful.

Let $max \in \mathbb{N}$. Using a computer we randomly generate symmetric matrices (A_j) for $j = 1, \ldots, M(2n, d)$, where each upper diagonal entry is chosen independently from the interval $[1, max] \subset \mathbb{N}$. In order to choose the vectors $\mathbf{a}^{j,k}$ from the Lagrangian subspace Λ_j, we randomly choose an $n \times N(n, d)$ matrix X, again with each entry chosen independently from $[1, max]$, and define

$$\begin{bmatrix} \mathbf{a}^{j,1} \\ \vdots \\ \mathbf{a}^{j,M} \end{bmatrix} = X \cdot [I, A_j] \,.$$

We write each homogeneous polynomial $(\mathbf{a}^{j,k} \cdot z)^d$ as a vector with respect to the same monomial basis to obtain the matrix

$$T = T(\{\mathbf{a}^{j,k}\} = \begin{bmatrix} \vdots \\ (\mathbf{a}^{j,k} \cdot z)^d \\ \vdots \end{bmatrix}, \tag{5}$$

where T again has integer coefficients. Hence, at least in theory, a computer is able to compute the rank of T. If this rank is maximal then we have found a collection (Λ_j) that spans degree d.

We were able to run this program in Mathematica for dimension $2n = 6$ and $d \leq 12$. It turns out that as the degree d grows, not only does the size of T grow, but also the necessary interval $[1, max]$. Hence the coefficients occurring in T grow rapidly, and the program quickly becomes too large to run even on the strongest computer.

We were able to deal with larger degrees by computing the rank of T modulo suitably chosen primes. Again if this rank is maximal then we know for sure that the collection (Λ_j) spans degree d. This allowed us to find a collection (Λ_j) that spans degrees up to 12, without having to run our program on a special computer. It is likely that by running the program on a supercomputer, spanning collections can be found for slightly larger degrees.

For completeness, we print the Mathematica code used to prove Theorem 3 for degree $d = 12$. The code is similar for lower degrees.

```
makeL[max_]:=Module[{matrix},
matrix = Table[RandomInteger[{1, max}], {j, 1, 3}, {i, 1, 3}];
matrix[[1, 2]] = matrix[[2, 1]];
matrix[[1, 3]] = matrix[[3, 1]];
matrix[[2, 3]] = matrix[[3, 2]];
Transpose[Flatten[{IdentityMatrix[3], matrix}, 1]]
]
tensor[{a_, b_, c_, d_, e_, f_}]:=
Evaluate[List@@Expand[(a + b + c + d + e + f)^12]]
makematrix[max_]:=Module[{matrix2, matrix3},
matrix2 =  Table[RandomInteger[{1, max}], {j, 1, 91}, {i, 1, 3}];
matrix3 = Flatten[Table[matrix2.makeL[max], {i, 1, 68}], 1];
```

```
Thread[tensor[matrix3]]
]
T = makematrix[20000];
Do[Print[MatrixRank[T, Modulus → Prime[i]]], {i, 100, 150}]
```

After running the program for one night the ranks of the first 26 were computed:
6188, 6181, 6180, 6187, 6186, 6183, 6188, 6187, 6186, 6184, 6187, 6185, 6184, 6185, 6185, 6184, 6186, 6186, 6187, 6187, 6184, 6185, 6184, 6183, 6186, 6185

Notice that the required rank 6188 was obtained twice.

4.5. **Proof of Theorem 4.** While we will give an explicit proof of Theorem 4 in the next section, we will now outline how the same result can be obtained using computer algebra. As explained above, Theorem 4 follows if it can be shown that the determinant of the matrix T is zero for all choices of the matrices A_3 through A_7. One easily sees that the determinant of T is a polynomial of degree 50 in the 15 independent entries of the symmetric 2×2 matrices $A_3, \ldots, A_7$. Hence to conclude that $P = 0$ it is sufficient to evaluate P on a grid with 51^{15} entries. Unfortunately this is far beyond the scope of current-day computers.

The size of the grid can be significantly reduced using the following observations. Most importantly, after dividing by

$$Q = \prod_{j=3,\ldots 7} \det(A_j) \prod_{i<j} \det(A_j - A_i),$$

which is a product of 15 distinct prime factors, the remaining polynomial P/Q has only degree either 4 or 5 in each of the variables separately. Secondly P and Q are invariant under an action of the group $S_2 \times S_5$. Using these observations we constructed an easily programmable grid (on which $Q \neq 0$) with only $489,742,800$ entries, and evaluated P on this grid using a computer. As expected, P turned out to vanish identically, proving Theorem 4.

5. Seven Involutive Lines

It is well-known that every line in $\mathbb{P}^3$ is contained in a one-parameter family of hyperplanes; that 3 pairwise disjoint lines in $\mathbb{P}^3$ determine a unique quadric containing them; and that 5 pairwise disjoint lines in $\mathbb{P}^3$ are contained in a cubic surface if and only if there exists another line intersecting all of them. Less well-known is the following result of Cayley: given six lines $\ell_1, \ldots, \ell_6$ in general position in $\mathbb{P}^3$ then there exists a quartic line complex $\mathfrak{Q}$ (depending on the six lines) such that the union of $\ell_1, \ldots, \ell_6$ and a seventh line ℓ_7 is contained in a quartic surface if and only ℓ_7 belongs to $\mathfrak{Q}$, see [3, §107]. Motivated by Cayley's result, the problem of giving conditions on seven lines such that they are contained in a quartic surface received considerable attention in the late 1920s and early 1930s. Perhaps the deepest results on the subject are due to Todd [11]. He proves that there exists a quartic surface containing seven pairwise disjoint lines if and only if there exists a rational curve of degree 19 meeting six of the lines in twelve points and the seventh in ten points. Moreover, if the seven lines belong to a linear line complex (in particular if the seven lines are all involutive) then he proves that there is always a quartic surface containing them. We could not follow in detail the arguments of Todd, but we present a simple proof of his result below. For our purposes it is convenient to state it as follows.

Theorem 6 (Todd)**.** *Let $\Lambda = \{\Lambda_1, ..., \Lambda_7\} \subset \mathbb{C}^4$ be a set of seven Lagrangian planes. Then Λ is contained in a quartic.*

Notice that Theorem 6 implies that seven lines in an arbitrary line complex are contained in a quartic. Indeed, up to projective automorphisms there are only two linear line complexes: the one defined by the projectivazation of Lagrangian planes, and a degeneration of it consisting of all lines which intersect another given line. The result for lines in an arbitrary linear line complex follows from the semi-continuity of the dimension of the linear system of quartics containing seven lines when the lines vary.

5.1. **Proof of Theorem 6.** Given seven involutive lines in $\mathbb{P}^3$ we seek a quartic containing them. We will do this by finding a non-trivial section $s \in H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(4))$ such that the zero locus of s contains them.

Let $X = \sum_{j=1}^4 z_j \frac{\partial}{\partial z_j}$ be the Euler (radial) vector field. Then $\ell \subset \mathbb{P}^3$ is involutive if and only the 1-form $\sigma := \omega \lrcorner X$ vanishes identically when pulled-back to $\pi^*\ell$. If we twist the dual of the Euler exact sequence ([7, page 409])

$$0 \to \Omega^1_{\mathbb{P}^3} \to \bigoplus_{i=0}^{3} \mathcal{O}_{\mathbb{P}^3}(-1) \to \mathcal{O}_{\mathbb{P}^3} \to 0,$$

by $\mathcal{O}_{\mathbb{P}^3}(2)$ we see that σ can be interpreted as a section $\sigma \in \Gamma(\mathbb{P}^3, \Omega^1_{\mathbb{P}^3}(2))$. Thus a line ℓ is involutive if and only if it is tangent to the subbundle $\mathcal{C} := Ker(\sigma) \subset T\mathbb{P}^3$, which is called a contact distribution. Considering the exact sequence

$$0 \to \mathcal{C} \to T\mathbb{P}^3 \to \mathcal{O}_{\mathbb{P}^3}(2) \to 0 \tag{6}$$

we obtain

$$det(\mathcal{C}) \simeq \det(T\mathbb{P}^3) \otimes \mathcal{O}_{\mathbb{P}^3}(-2) \simeq \mathcal{O}_{\mathbb{P}^3}(2). \tag{7}$$

We want a lower estimate for the dimension of $H^0(\mathbb{P}^3, \mathcal{C}(1))$, the space of vector fields with coefficients in $\mathcal{O}_{\mathbb{P}^3}(1)$ which are tangent to the distribution. For this we consider the exact sequence (6) twisted by $\mathcal{O}_{\mathbb{P}^3}(1)$. Clearly we have the inequality

$$\dim H^0(\mathbb{P}^3, \mathcal{C}(1)) \geq \dim H^0(\mathbb{P}^3, T\mathbb{P}^3(1)) - \dim H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3)).$$

The vector space $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))$ is nothing but the vector space of cubic homogeneous polynomials in 4 variables, and therefore has dimension 20. To determine the dimension $H^0(\mathbb{P}^3, T\mathbb{P}^3(1))$, first twist the Euler exact sequence [7, page 409] to obtain

$$0 \to \mathcal{O}_{\mathbb{P}^3}(1) \to \bigoplus_{i=0}^{3} \mathcal{O}_{\mathbb{P}^3}(2) \to T\mathbb{P}^3(1) \to 0,$$

and then look at the long exact sequence in cohomology. Since $h^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)) = 0$ by [8, Theorem 5.1 of Chapter III], we have that

$$\dim H^0(\mathbb{P}^3, T\mathbb{P}^3(1)) = 4 \dim H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)) - \dim H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)) = 40 - 4 = 36\,.$$

We conclude that $\dim H^0(\mathbb{P}^3, \mathcal{C}(1)) \geq 16$.

We now want to show that at least two linearly independent sections of $H^0(\mathbb{P}^3, \mathcal{C}(1))$ leave the seven lines invariant. For this we note that $v \in H^0(\mathbb{P}^3, \mathcal{C}(1))$ leaves a line ℓ invariant if and only if it is in the kernel of the map

$$H^0(\ell, \mathcal{C}|_\ell(1)) \to H^0(\ell, \mathcal{N}(1)), \tag{8}$$

where $\mathcal{N} = \mathcal{C}|_\ell / T\ell$. Since both $T\ell$ and $\det \mathcal{C}|_\ell$ are isomorphic to $\mathcal{O}_\ell(2)$ we see that $\mathcal{N}$ is the trivial line bundle on ℓ, i.e., $\mathcal{N} = \mathcal{O}_\ell$. So (8) is the same as

$$H^0(\ell, \mathcal{C}|_\ell(1)) \to H^0(\ell, \mathcal{O}_\ell(1)).$$

Since $\dim H^0(\ell, \mathcal{O}_\ell(1)) = 2$ it follows that the codimension of the space of vector fields leaving ℓ invariant is at most 2; hence the vector space leaving the total of seven lines invariant has dimension greater than or equal to two. So let $v_1, v_2 \in H^0(\mathbb{P}^3, \mathcal{C}(1))$ be linearly independent leaving the seven lines invariant. If $v_1 \wedge v_2$ is not identically zero then $v_1 \wedge v_2 \in H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(4))$ is the sought quartic containing the seven lines.

Aiming at contradiction let us assume that $v_1 \wedge v_2 = 0$. Thus $v_1 = f v_2$ for some non-constant rational function $f \in \mathbb{C}(\mathbb{P}^3)$. It follows that v_1 vanishes on the hypersurface $Z = \{f = 0\}$ while v_2 vanishes on the hypersurface $P = \{f = \infty\}$. After dividing v_1 by the equation of Z we get a section $w \in H^0(\mathbb{P}^3, \mathcal{C}(-a))$ for $a = \deg Z - 1$. Since $H^0(\mathbb{P}^3, T\mathbb{P}^3(k)) = 0$ is zero for $k < -1$ (look at the Euler sequence) we conclude that $\deg Z \in \{1, 2\}$.

If $\deg Z = 2$ then we obtain a non-zero section $w \in H^0(\mathbb{P}^3, \mathcal{C}(-1))$. But this leads to contradiction since $H^0(\mathbb{P}^3, \mathcal{C}(-1)) = 0$ because the map induced by (6)

$$H^0(\mathbb{P}^3, T\mathbb{P}^3(-1)) \longrightarrow H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)),$$
$$\sum_{j=1}^{4} \lambda_j \frac{\partial}{\partial z_j} \mapsto \lambda_2 z_1 - \lambda_1 z_2 + \lambda_4 z_3 - \lambda_3 z_4$$

is clearly an isomorphism.

If $\deg Z = 1$ then Z is a hyperplane containing at most one of the involutive lines (the lines are disjoint) and $w \in H^0(\mathbb{P}^3, \mathcal{C})$ is a global holomorphic vector field tangent to the contact distribution leaving invariant six distinct lines. The global holomorphic vector fields on $\mathbb{P}^3$ are in one to one correspondence with the elements of $\mathfrak{sl}(4, \mathbb{C})$. To every 4×4 matrix $A = (a_{ij})$ of trace zero we associate the vector field $w_A = \sum_{i,j=1}^{4} a_{ij} z_i \frac{\partial}{\partial z_j}$. Under this identification, the singular points of w_A correspond to the eigenvectors of A, and the lines left invariant by w_A correspond to the two-dimensional subspaces left invariant by A. Therefore the matrix corresponding to w leaves invariant 6 generic two-dimensional subspaces of $\mathbb{C}^4$, and because of that must be a multiple of the identity. Since it has trace zero, it follows that $w = 0$. Contradiction. □

6. Hartshorne–Hirschowitz Theorem for involutive lines

Here we are interested in the question: What is the dimension of the linear system of surfaces of degree d containing a given finite set of involutive lines? If we consider a union Σ of r pairwise disjoint lines in $\mathbb{P}^3$ then the degree d surfaces in $\mathbb{P}^3$ containing Σ can be identified with the projectivization of the kernel of the restriction morphism

$$H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d)) \longrightarrow H^0(\Sigma, \mathcal{O}_\Sigma(d)) .$$

Hartshorne and Hirschowitz [9, 10] proved that a general set of lines imposes independent conditions on the linear system of degree d surfaces [9], in other words if the r lines in Σ are in general position and $d \geq 0$, then the restriction morphism above is a linear map of maximal rank.

Theorem 6 implies that the same does not hold for $d = 4$ and seven involutive lines in general position, but it turns out that this is the only forbidden pair.

Theorem 7. *Let Σ be a union of r involutive lines in $\mathbb{P}^3$ in general position and $d \geq 0$ be an integer. If $(r, d) \neq (7, 4)$ then the restriction map*

$$H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d)) \to H^0(\Sigma, \mathcal{O}_\Sigma(d))$$

is of maximal rank.

Despite the existence of the exceptional case, the proof of Theorem 7 contains no novelty when compared with the proof of Hartshorne-Hirschowitz result presented in [9], except for some minor extra complications coming from the speciality of the intersection of involutive lines with quadrics. Indeed, the proof presented in this section follows [9] word-by-word most of the time.

The linear system of hypersurfaces in $\mathbb{P}^5$ containing arbitrary number of planes (involutive or not) does not seem to be studied so far, cf. [4].

6.1. **Special quadrics.** Let $Q \subset \mathbb{P}^3$ be a quadric. The restriction of the contact form $\omega \in H^0(\mathbb{P}^3, \Omega^1_{\mathbb{P}^3}(2))$ determines a foliation on $Q = \mathbb{P}^1 \times \mathbb{P}^1$. The induced foliation $\mathcal{F}_Q$ on a general Q will have normal bundle equal to $N\mathcal{F}_Q = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2, 2)$ and trivial tangent bundle. In this case, one or two lines from each ruling of Q will be involutive. We will say that a smooth quadric $Q \subset \mathbb{P}^3$ is a *special quadric* if all the lines of one of the rulings of Q are involutive. This is equivalent to requiring that three lines of one of the rulings are involutive.

The intersections of involutive lines with a special quadric are described by a classical result of Chasles (cf. [5, Theorem 10.2.10]): for any given special quadric Q, there exists an automorphism $\sigma : \mathbb{P}^1 \to \mathbb{P}^1$ such that the involutive lines are exactly one of the rulings (say the vertical ruling) of Q, or intersect the horizontal ruling of Q at heights given by an orbit of σ.

6.2. **Reduction.** The proof of Theorem 7 will follow step-by-step the proof of the analogous statement for lines (instead of involutive lines) established by Hartshorne and Hirschowitz. The main part of the proof consists of an induction argument of the statement (H_d) below, for $d \neq 4$.

Statement (H_d): Let

$$r = \left\lfloor \frac{1}{d+1}\binom{d+3}{3} \right\rfloor \text{ and } q = (d+1)\left(\frac{1}{d+1}\binom{d+3}{3} - r\right) .$$

Then there exists a scheme $\Sigma \subset \mathbb{P}^3$ given by the union of r involutive lines and q points contained in another involutive line such that the restriction map

$$H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d)) \to H^0(\Sigma, \mathcal{O}_\Sigma(d))$$

is bijective.

6.3. **Lemma.** *The assertions $(H_1), (H_2), (H_3), (H_6)$, and (H_7) hold true.*

Proof. The assertions $(H_1), (H_2)$, and (H_3) are elementary. Assertions (H_6) and (H_7) have been checked with the help of computer algebra, analogously to Section 4. □

6.4. **Inductive steps.**

6.5. **Proposition.** *If $d = 0 \mod 3$ and $d \geq 3$ then (H_{d-2}) implies (H_d).*

Proof. Write $d = 3k$, $k \geq 1$. To prove (H_d) we have to find Y, a union of $r = \frac{(k+1)(3k+2)}{2}$ involutive lines, not contained in a surface of degree d. We consider $Y = Y' \cup Y''$ where Y' is the union of $2k+1$ involutive lines contained in a quadric Q (all belonging to the same family of lines in Q), and Y'' is a union of $\frac{k(3k+1)}{2}$ involutive lines intersecting Q transversely.

Suppose there exists a surface F of degree d containing Y. If F does not contain Q, then its intersection with Q is a curve of type $(3k, 3k)$ containing Y' and the $k(3k+1)$ points of $Y'' \cap Q$. Therefore $F \cap Q$ is the union of Y' and a curve C' of bidegree $(k-1, 3k)$ containing the points of $Y'' \cap Q$. Since $h^0(Q, \mathcal{O}_Q(k-1, 3k) = k(3k+1)$ we expect that there exists no such curve C' if the points of Y'' are in general position. Lemma 6.6 below guarantees that this is the case, and implies that our surface F must contain Q. Taking Q out of F we obtain a surface of degree $d-2$ containing Y''. If (H_{d-2}) is true then we can start with Y'' not contained in any surface of degree $d-2$, and conclude that (H_d) holds true. □

6.6. **Lemma.** *If $k \geq 1$, Y'' is a union of $\frac{k(3k+1)}{2}$ involutive lines in general position, and Q is any smooth quadric, then there is no curve of bidegree $(k-1, 3k)$ in Q containing $Y'' \cap Q$.*

Proof. The set of $\frac{k(3k+1)}{2}$ involutive lines in general position satisfying the conclusion of the lemma is open, and we only need to prove that it is non-empty.

If k is even then we can choose the $\frac{k(3k+1)}{2}$ involutive lines in such way that they intersect Q in $k(3k+1)$ points distributed over k distinct involutive lines of bidegree $(1, 0)$ on Q, each of these involutive lines containing $3k+1$ points. A curve of bidegree $(k-1, 3k)$ containing all these points would also have to contain the k involutive lines, which is impossible for a curve of this bidegree. This contradiction proves the lemma when k is even.

If k is odd then we can distribute the $k(3k+1)$ points of intersection with Q over $(k-1)$ involutive lines with each of them containing $3k+1$ points, and the extra $3k+1$ can be assumed to lie on $3k+1$ distinct lines of the non-involutive ruling of q. Clearly, a curve of bidegree $(k-1, 3k)$ cannot pass through all these points, and the lemma follows for k odd. □

6.7. **Proposition.** *If* $d = 2 \mod 3$ *and* $d \geq 3$ *then* (H_{d-2}) *implies* (H_d).

Proof. Write $d = 3k+2$, $k \geq 1$. Now we have to find Y, a union of $r = \frac{(k+1)(3k+6)}{2}$ involutive lines and $q = k+1$ points contained in another involutive line, so that Y is not contained in a surface of degree d. Set Y' equal to the union of $2k+2$ involutive lines contained in a quadric Q and $k+1$ points contained in another involutive line of Q, with all the $2k+3$ lines contained in the same family.

If k is even then set Y'' equal to the union of $\frac{(k+1)(3k+2)}{2}$ involutive lines not in Q, such that the $(k+1)(3k+2)$ points of intersection of these lines with Q can be written as a disjoint union $A \cup B$ such that

(1) the set A consists of $k(3k+3)$ points contained in k distinct involutive lines $\ell_1, \dots, \ell_k$ of the same family mentioned before; and each of these k lines contains exactly $3k+3$ of these points;
(2) the set B consists of the other $2k+2$ points and together with the $k+1$ points of Y', form a set of $3k+3$ points that do not contain two points belonging to the same line belonging to the other family of lines of Q.

If S is a surface of degree d containing $Y = Y' \cup Y''$, then either S contains Q, or the intersection of S with Q will be equal to the union of the lines in Y' together with a curve of bidegree $(k, 3k+2)$ containing the union of A, B, and the $k+1$ points in Y'. A curve Γ of bidegree $(k, 3k+2)$ containing A must contain $\ell_1, \dots, \ell_k$. Therefore $\Gamma - \cup_{i=1}^{k} \ell_i$ is a curve of bidegree $(0, 3k+2)$ containing the points of B and the $k+1$ points in Y'. Since no two of these points are contained in a line of the other family we arrive at a contradiction, showing that S must contain Q. Since this holds for this particular choice of Y'', it also holds true for a generic choice of Y''. Hence we conclude the proof of the proposition when d is even.

Suppose now that d (and thus also k) is odd, and set Y'' equal to the union of $\frac{(k+1)(3k+2)}{2}$ involutive lines not in Q, such that the $(k+1)(3k+2)$ points of intersection of these lines with Q can be written as a disjoint union $A \cup B \cup C$ such that

(1) the set A consists of $(k-1)(3k+3)$ points contained in $k-1$ distinct involutive lines $\ell_1, \dots, \ell_{k-1}$ of the same family mentioned before; and each of these $k-1$ lines contains exactly $3k+3$ of these points;
(2) the set B consists of $2k+2$ points contained in the line ℓ_0;
(3) the set C consists of $(2k+2)+k+1 = 3k+3$ points which do not contain two points belonging to the same line belonging to the other family of lines of Q.

If S is a surface of degree d containing $Y = Y' \cup Y''$ and not containing the quadric Q then the intersection of S with Q will be equal to the union of the lines in Y' together with a curve Γ of bidegree $(k, 3k+2)$ containing the union of A, B, C and the $k+1$ points in Y'. The curve Γ must contain the $k-1$ lines $\ell_1, \dots, \ell_{k-1}$, as well as the line ℓ_0, since each of these lines contains $3k+3$ points and Γ has bidegree $(k, 3k+2)$. It follows that $\Gamma \setminus \cup_{i=0}^{k-1} \ell_i$ is the union of $(3k+2)$ points of the other family and cannot contain all the points of C. As before, we arrived at a contradiction which shows that S contains Q, and allow us to conclude. □

When $d = 1 \mod 3$ the reasoning of Hartshorne and Hirschowitz uses schemes with nilpotents elements and also the notion of *residual schemes*. If H and Y are subschemes of $\mathbb{P}^3$ then $Z = res_H Y$, the residual scheme of Y in H, is the subscheme of $\mathbb{P}^3$ with defining ideal given the kernel of the natural morphism

$$\mathcal{O}_{\mathbb{P}^3} \longrightarrow Hom(I_H, \mathcal{O}_Y)\,.$$

If H is a surface of degree d then the residual scheme fits into the exact sequence

$$0 \to \mathcal{O}_Z(-d) \to \mathcal{O}_Y \to \mathcal{O}_{Y\cap H} \to 0\,.$$

Let us consider Y_ε, $\varepsilon \in \mathbb{C}^*$, the family of disjoint lines $y = z = 0$ and $x = z - \varepsilon = 0$ on $\mathbb{C}^3$. For any $\varepsilon \in \mathbb{C}^*$ the two lines are involutive with respect to the contact structure determined by $xdy - ydx + dz$. The ideal of Y_ε is

$$(y, z) \cap (x, z - \varepsilon) = (xy, xz, y(z - \varepsilon), z(z - \varepsilon))\,.$$

The flat limit when $\varepsilon \to 0$ is the ideal (xy, xz, yz, z^2). It represents the union of the lines $x = z = 0$ and $y = z = 0$ together with an immersed point at the origin. If Y is the associated scheme and H is the plane $y = x$, then the ideal of the scheme $H \cap Y$ is $(y - x, x^2, xz, z^2)$. Thus $H \cap Y$ is a subscheme of H supported at one point $p \in H$ and with structural ring equal to $\mathcal{O}_{H,p}/\mathfrak{m}^2_{H,p}$. It is a scheme of length three or, in other words, a triple point in H.

The residual scheme has ideal (z, xy) which is a reduced degenerate conic at the plane $z = 0$.

6.8. **Proposition.** *If $d = 3k + 1$ with $k \geq 3$ then $(H_{d-4}) \implies (H'_{d-2}) \implies (H_d)$, where (H'_{d-2}) is the following*

Statement (H'_{d-2}): *there exists a scheme $Y \subset \mathbb{P}^3$ which is the union of $\frac{(k-1)(3k-2)}{2}$ involutive lines and $2k$ involutive reduced degenerate conics having their singular points at a special quadric Q, and such that the natural morphism*

$$\rho(d - 2) : H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d - 2)) \to H^0(Y, \mathcal{O}_Y(d - 2))$$

is bijective.

Proof. We will start by proving that (H'_{d-2}) implies (H_d). We have to find $\frac{(k+1)(3k+4)}{2}$ involutive lines such that $\rho(d)$ is bijective. As the condition is open it suffices to produce one specialization of a union of disjoint involutive lines having the sought property. We start by choosing a special quadric Q and will take $Y = Y' \cup Y''$, where Y' is the union of $2k + 1$ involutive lines on the involutive ruling of Q, and Y'' as the union of $\frac{(k-1)(3k-2)}{2}$ involutive lines in general position together with $2k$ involutive degenerate conics having nilpotent elements at singular points, which are limits of pairs of disjoint involutive lines and have their singular points at Q.

As discussed above, the residual intersection of Y'' with Q is nothing but Y''_{red}. We have an exact diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(\mathcal{O}_{\mathbb{P}^3}(d-2)) & \longrightarrow & H^0(\mathcal{O}_{\mathbb{P}^3}(d)) & \longrightarrow & H^0(\mathcal{O}_Q(d)) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \rho(d-2)} & & \downarrow{\scriptstyle \rho(d)} & & \downarrow{\scriptstyle \alpha(d)} & & \\ 0 & \longrightarrow & H^0(\mathcal{O}_{Y''_{red}}(d-2)) & \longrightarrow & H^0(\mathcal{O}_Y(d)) & \longrightarrow & H^0(\mathcal{O}_{Y\cap Q}(d)) & \longrightarrow & 0 \end{array}$$

The hypothesis (H'_{d-2}) implies that the leftmost vertical arrow is bijective. Lemma 6.9 implies that $\alpha(d)$ is bijective if $d = 3k+1$ with $k \geq 3$, and so the rightmost arrow is also bijective. Hence the same holds true for the middle arrow, and we have that (H_d) holds true.

Let us now verify that (H_{d-4}) implies (H'_{d-2}). We want to find a scheme Y which is the union of $\frac{(k-1)(3k-2)}{2}$ involutive lines and $2k$ degenerate conics with singular points on the special quadric Q such that $\rho(3k-1)$ is bijective. We will choose Y with one of the lines of each degenerate conic belonging to the involutive fibration of Q, and one further simple line belonging to the same fibration. The other $2k$ lines belonging to the degenerate conics and the remaining $\frac{(k-1)(3k-2)}{2} - 1$ involutive lines will be chosen in general position. Therefore the residual scheme $Y'' = res_Q Y$ is formed by $2k + \frac{(3k^2-5k)}{2} = \frac{k(3k-1)}{2}$ involutive lines in general position.

Let us consider the diagram below.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(\mathcal{O}_{\mathbb{P}^3}(d-4)) & \longrightarrow & H^0(\mathcal{O}_{\mathbb{P}^3}(d-2)) & \longrightarrow & H^0(\mathcal{O}_Q(d-2)) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \rho(d-4)} & & \downarrow{\scriptstyle \rho(d-2)} & & \downarrow{\scriptstyle \alpha(d-2)} & & \\ 0 & \longrightarrow & H^0(\mathcal{O}_{Y''_{red}}(d-4)) & \longrightarrow & H^0(\mathcal{O}_Y(d-2)) & \longrightarrow & H^0(\mathcal{O}_{Y\cap Q}(d-2)) & \longrightarrow & 0 \end{array}$$

On the one hand the leftmost vertical morphism $\rho(d-4) = \rho(3k-1)$ is bijective, by hypothesis. On the other hand the intersection of Y with Q is the union of $2k+1$ involutive lines and $2k + (3k^2 - 5k)$ points on Q. The $2k$ points coming from the intersection of smooth points of the conics are in general position in Q, and the remaining $(3k^2 - 5k)$ points are constrained only by Chasles Theorem, see Subsection §6.1 on special quadrics. If k is even then we can place $(3k^2 - 6k) = 3k(k-2)$ points over $k-2$ involutive lines each containing $3k$ points. This suffices to ensure that an element in the kernel of $\alpha(d-2) = \alpha(3k-1)$ is a product of $3k-1$ involutive lines and $3k-1$ non-involutive lines. But we can choose the remaining $3k$ points not contained in any set of $3k-1$ non-involutive lines. This shows that $\alpha(d-2)$ is bijective for a general Y when d is even. Similarly, if d is odd then we can place $(3k^2 - 5k) = 3k(k-2) + k$ over $k-1$ involutive lines in such a way that $k-2$ involutive lines contains $3k$ points, and the remaining involutive line contains k points in general position. If we place the $2k$ points coming from the degenerate conics in general position on this last line, the injectivity of $\alpha(d-2)$ follows as before. To summarize, for a general Y the morphism $\alpha(d-2)$ is also bijective. It follows that the middle vertical morphism is also bijective, i.e., (H'_{d-2}) holds true. □

6.9. **Lemma.** *The map $\alpha(3k+1)$ is bijective when $k \geq 3$.*

Proof. Notice that $Y \cap Q$ consists of $2k+1$ involutive lines on the involutive ruling of Q, since $Y' \subset Q$, and $(k-1)(3k-2)+4k$ simple points together with $2k$ triple points. If $\alpha(d)$ is not bijective, there exists a curve of bidegree $(d,d)=(3k+1,3k+1)$ containing schematically all these points, triple points, and lines.

To prove that $\alpha(d)$ is bijective for a general Y'', it suffices, by semi-continuity, to prove it for a special Y'' which we now proceed to construct.

If k is even and $k \geq 4$ then we start by choosing 4 involutives lines belonging to the involutive ruling of Q and distinct from the $2k+1$ involutive lines in Y', say ℓ_1, ℓ_2, ℓ_3, and ℓ_4. In each of the ℓ_1 and ℓ_2 we place k triple points and choose the corresponding degenerate involutive conics of Y'' in such way that each of them intersect each of the lines ℓ_3 and ℓ_4 in one point. We put $k+2$ simple points in ℓ_1 with the corresponding involutive lines intersecting ℓ_2. And finally we place $k+2$ simple points in ℓ_3 with the corresponding involutive lines intersecting ℓ_4. If we set A equal to the union of these $2k$ degenerate conics and $2(k+2)$ involutive lines, then the length of the restriction of $A \cap Q$ to each of the lines ℓ_i, $i=1,\dots,4$, is $3k+2$. Now let B be a union of $\frac{(k-4)(3k+2)}{2}$ involutive lines such that $B \cap Q$ is contained in $k-4$ lines of the involutive ruling and each of these contains $3k+2$ points; and let C be a union of general $\frac{k}{2}+1$ involutive lines. If we choose $Y'' = A \cup B \cup C$ then a curve C of bi-degree $(d,d)=(3k+1,3k+1)$ containing $Y \cap Q = (Y' \cup Y'') \cap Q$ must contain the $2k+1$ lines in Y', the 4 involutive lines $\ell_1, \dots, \ell_4$, and the $(k-4)$ involutive lines containing the support of B. It follows that a hypothetical curve C containing $Y \cap Q$ must be a product of $3k+1$ involutive lines and $3k+1$ non-involutive lines of Q.

If we further assume that no two of the $2k$ triple points in $A \cap Q$ have support on the same line of the non-involutive ruling, and we assume the same for the $k+1$ points of intersection of C with Q, then it follows that such a hypothetical curve cannot exists, since it would pass through the support of $2k$ triple points and $C \cap Q$. Therefore $\alpha(d)$ is bijective for Y'' sufficiently general when k is even and $k \geq 4$.

If k is odd and $k \geq 3$ we will proceed similarly. We choose 3 involutives lines belonging to the involutive ruling of Q and distinct from the $2k+1$ involutive lines in Y', say ℓ_1, ℓ_2, and ℓ_3. On the line the ℓ_1 (resp. the line ℓ_2) we place k triple points and choose the corresponding degenerate involutive conics of Y'' in such way that each of them intersects the lines ℓ_2 (resp. ℓ_1) and ℓ_3 in one point. We put 2 simple points on ℓ_1 with the corresponding involutive lines intersecting ℓ_2; and $k+2$ simple points on ℓ_3. We set A equal to the union of these $2k$ degenerate conics, and $k+4$ involutive lines. If the triple points and simple points in ℓ_3 are in general position then the support of the $2k$ triple points and the $k+2$ simple points outside of ℓ_3 coming from the involutive lines intersecting ℓ_3, will belong to $3k+2$ distinct lines of the non-involutive ruling of Q. Let B be a union of $\frac{(k-3)(3k+2)}{2}$ involutive lines such that $B \cap Q$ is contained in $k-3$ lines of the involutive ruling and each of these contains $3k+2$ points. If we set $Y'' = A \cup B$ then the same argument used for k even implies that there is no curve C

of bidegree $(3k+1, 3k+1)$ in Q containing schematically $Y'' \cap Q$. We conclude that $\alpha(3k+1)$ is bijective for $k \geq 3$. $\square$

6.10. **Synthesis.** We can summarize what we have proved so far in the following theorem.

Theorem 8. *If $d \neq 4$ then statement (H_d) holds true.*

6.11. **Proof of Theorem 7.** Set r_0 equal to $\lfloor \frac{1}{d+1}\binom{d+3}{3} \rfloor$. If $r \leq r_0$ and $d \neq 4$ then Theorem 8 guarantees the existence of a scheme Y which is the union of r_0 pairwise disjoint involutive lines such that the restriction morphism $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d)) \to H^0(Y, \mathcal{O}_Y(d))$ is surjective. After taking out $r_0 - r$ lines from Y we obtain a scheme Σ for which the analogous morphism is still surjective.

If $r > r_0$ and $d \neq 4$ then Theorem 8 gives a scheme Y equal to the union of r_0 pairwise disjoint involutives lines and $q = (d+1)\left(\frac{1}{d+1}\binom{d+3}{3} - r\right)$ contained in another involutive line disjoint from all the previous ones such that $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d)) \to H^0(Y, \mathcal{O}_Y(d))$ is a bijection. We take Σ as the union of lines in Y together with the line determined by the q points and $r - r_0 - 1$ involutive lines disjoint from the other ones, to obtain a scheme for which the restriction morphism in degree d is injective.

If $d = 4$ then to conclude the proof of Theorem 7 it remains to check that the restriction maps is surjective for $r < 7$, and injective for $r > 7$. It is sufficient to check surjectiveness for $r = 6$ (the remaining cases can be obtained from this one by taking lines out), and injectiveness for $r = 8$ (where the remaining cases can be obtained from this one by adding lines). Both verifications have been done by computer. $\square$

References

1. A. Dragt, D. Abell, *Symplectic maps and computation of orbits in particle accelerators.* Fields Inst. Commun. **10** (1995), 59–85.
2. E. Andersén, L. Lempert, *On the group of holomorphic automorphisms of* $\mathbb{C}^n$. Invent. Math. **110**, (1992), no. 2, 371–388.
3. A. Cayley, *A Memoir on Quartic Surfaces.* Proc. London Math. Soc. S1-3 no. 1, 19.
4. E. Carlini, M.V. Catalisano, A. Geramita, *S*ubspace arrangements, configurations of linear spaces and the quadrics containing them. J. Algebra **362** (2012), 70–83.
5. I. Dolgachev, *Classical algebraic geometry: a modern view.* to be published by the Cambride Univ. Press
6. F. Forstnerič, *Actions of* $(\mathbb{R}, +)$ *and* $(\mathbb{C}, +)$ *on complex manifolds.* Math. Z. **223**, (1996), 123–153.
7. P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Reprint of the 1978 original. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1994. xiv+813 pp.
8. R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977. xvi+496 pp.
9. R. Hartshorne, A. Hirschowitz, *Droites en position générale dans l'espace projectif.* Lecture Notes in Math., **961**, Springer, Berlin, 1982.
10. R. Hartshorne, A. Hirschowitz, *Courbes rationnelles et droites en position générale.* Ann. Inst. Fourier (Grenoble) **35** (1985), no. 4, 39–58.
11. J.A. Todd, *Configurations defined by six lines in space of three dimensions.* Proc. Camb. Philos. Soc. **29**, 52-68 (1933).

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