ON THE DIMENSION OF RESONANCE AND CHARACTERISTIC VARIETIES

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Abstract. We study codimension one foliations on complex projective spaces $\mathbb{P}^n$ associated to nontrivial resonance varieties of (complex projective) hyperplane arrangements. Using methods from theory of foliations we obtain certain upper bounds on the dimensions of those varieties as functions of n. Equivalently this gives upper bounds on the number of completely reducible fibers of pencils of hypersurfaces of $\mathbb{P}^n$. We obtain similar bounds for the dimensions of the characteristic varieties of the arrangement complements.

1. Introduction

1.1. **Resonance varieties.** Let $\mathcal{A} = \{H_1, \ldots, H_m\}$ be an arrangement of linear hyperplanes in $\mathbb{C}^{n+1}$, $|\mathcal{A}|$ its support, $M = \mathbb{C}^{n+1} \setminus |\mathcal{A}|$ its complement and $\mathcal{E} = \bigwedge(e_1, \ldots, e_m)$ the exterior algebra over $\mathbb{C}$ generated by degree-one elements e_i corresponding to the hyperplanes $H_i \in \mathcal{A}$. Define $\partial : \mathcal{E} \to \mathcal{E}$ of degree -1 as a $\mathbb{C}$-linear map subject to

$$\partial(e_{i_1} \cdots e_{i_p}) = \sum_{j=1}^{p} (-1)^{j-1} e_{i_1} \cdots \widehat{e_{i_p}} \cdots e_{i_p} .$$

If $\mathcal{I} \subset E$ is the homogeneous ideal generated by

$$\{\partial(e_{i_1} \cdots e_{i_p}) \,|\, H_{i_1}, \ldots, H_{i_p} \text{ is a minimal dependent subset of } \mathcal{A}\}$$

then the *Orlik-Solomon algebra* $A = A(\mathcal{A}) = \mathcal{E}/\mathcal{I}$ is a graded algebra. Notice that $\mathcal{I}$ does not contain elements of degree 0 or 1 and therefore $A^0 = \mathbb{C}$ and $A^1 = \mathbb{C}^m$. By the celebrated theorem by Arnold-Breiskorn-Orlik-Solomon (for instance, see [14]) A is naturally isomorphic to $H^*(M;\mathbb{C})$. The Orlik-Solomon algebra $A(\mathcal{A})$ is determined already by the combinatorics of $\mathcal{A}$, i.e., by its intersection lattice. In this paper we consider mainly *essential arrangements*, i.e., those with the intersection of all their hyperplanes equal to 0. For them the rank of an arrangement, that is the rank of the intersection lattice, equals $n + 1$.

Each $a \in A^1$ induces a cochain complex (A, a):

$$(A, a) : \quad 0 \to A^0 \to A^1 \to A^2 \to \cdots \to A^k \to A^{k+1} \to 0$$

where the differential is defined by the multiplication by a. The degree l *resonance variety* $\mathcal{R}^l(\mathcal{A})$ is defined as

$$\mathcal{R}^l(\mathcal{A}) = \{[a] \in \mathbb{P}(A^1) \cong \mathbb{P}^{m-1} \,|\, H^l(A, a) \neq 0\} .$$

In this paper we focus on the first resonance variety $\mathcal{R}^1(\mathcal{A})$. If $\mathcal{A}' \subset \mathcal{A}$ is a subarrangement then by definition $\mathcal{R}^1(\mathcal{A}') \subset \mathcal{R}^1(\mathcal{A})$. The support of an irreducible

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component Σ of $\mathcal{R}^1(\mathcal{A})$ is the smallest subarrangement $\mathcal{A}' \subset \mathcal{A}$ such that $\Sigma \subset \mathcal{R}^1(\mathcal{A}')$. The irreducible components with support equal to the whole $\mathcal{A}$ are called global components.

1.2. **Dimension of the first resonance varieties.** For our purposes it is convenient to projectivize linear arrangements, i.e., to deal with arrangements of hyperplanes in the projective space $\mathbb{P}^n$. We still call it essential if its linear cone is such. More explicitly this means that the rank n level of the intersection poset of the projective arrangement consists of more than one point of $\mathbb{P}^n$. In this case we say that the rank of the arrangement is n. For the Orlik-Solomon algebra $A(\mathcal{A})$ of a projective arrangement $\mathcal{A}$ and the resonance varieties we still use those of the linear cone of $\mathcal{A}$.

Our main interest lies in the dimensions of the first resonance varieties of these arrangements, very much in the spirit of [10, Problem 5.5] and [9, Problem 5.17]. Our main result can be stated as follows.

Theorem 1. *If $\mathcal{A}$ is an arrangement of hyperplanes in $\mathbb{P}^n$ and $\Sigma \subset \mathbb{P}(A^1)$ is an irreducible component of $\mathcal{R}^1(\mathcal{A})$ then the following assertions hold:*

(1) If $\dim \Sigma > 3$ then the rank of the support of Σ is one;
(2) If $\dim \Sigma > 1$ then the rank of the support of Σ is at most two;
(3) If $\dim \Sigma > 0$ then the rank of the support of Σ is at most four.

The result (1) has been proved in [11] for nets and in [6] for multinets with all multiplicities of lines equal 1 (see 2.3). Roughly speaking Theorem 1 says that the non-triviality of resonance varieties is a low-dimensional phenomenon.

The starting point of our study is the recent description in [6] of irreducible components of $\mathcal{R}^1(\mathcal{A})$ in terms of pencils with the union of completely reducible fibers equal to $\mathcal{A}$. We recall this description in Section 2.

1.3. **Completely reducible fibers.** It turns out that the description mentioned in the previous paragraph implies that Theorem 1 is equivalent to the following result.

Theorem 2. *If $\mathcal{P}$ is a pencil of hypersurfaces on $\mathbb{P}^n$ with irreducible generic fiber and k is the number of completely reducible fibers of $\mathcal{P}$ then the following assertions hold*

(1) If $k > 5$ then $\mathcal{P}$ is a pencil of hyperplanes;
(2) If $k > 3$ then $\mathcal{P}$ is the linear pull-back of a pencil on $\mathbb{P}^2$;
(3) If $k > 2$ then $\mathcal{P}$ is the linear pull-back of a pencil on $\mathbb{P}^4$.

Despite its evident classical taste the result seems to be new, i.e., we have not found it in the classical literature. Of course this is not a guarantee of originality and the possibility that a similar statement had been made by the end of the nineteenth century is not completely excluded.

The proof of Theorem 2 is based on the projective geometry of codimension one foliations on projective spaces reflected by some properties of theirs Gauss maps.

1.4. **Characteristic varieties.** Much of the interest in the resonance varieties comes from the fact that they are strictly related to the support loci for the cohomology of local systems on the complement of the arrangement – the so called *characteristic varieties.* If for every $\rho \in \text{Hom}\,(\pi_1(M), \mathbb{C}^*)$ we write $\mathcal{L}_\rho$ for the associated rank one local system then the characteristic varieties $\mathcal{V}^l(M)$ are defined as follows

$$\mathcal{V}^l(M) = \left\{\rho \in \text{Hom}(\pi_1(M, \mathbb{C}^*) \,|\, \text{H}^l(M, \mathcal{L}_\rho) \neq 0\right\}.$$

The main result of [3] implies that the projectivization of the tangent cone of $\mathcal{V}^l(M)$ at the trivial representation is isomorphic to $\mathcal{R}^l(\mathcal{A})$.

Nevertheless there exist components of characteristic varieties that do not contain the trivial representation, see [17].

In a recent preprint Dimca [4] clarified the link between the positive dimensional irreducible components of $\mathcal{V}^1(M)$ and pencils of hypersurfaces. Combining his results with our methods we obtain an analogue of Theorem 1 for characteristic varieties.

Theorem 3. *If $\mathcal{A}$ is an arrangement of hyperplanes in $\mathbb{P}^n$ and $\Sigma \subset \mathcal{V}^1(M)$ is an irreducible component then the following assertions hold:*

(1) If $\dim \Sigma > 4$ then the rank of the support of Σ is one;
(2) If $\dim \Sigma > 2$ then the rank of the support of Σ is at most two;
(3) If $\dim \Sigma > 1$ then the rank of the support of Σ is at most four;
(4) If $\dim \Sigma > 0$ then the rank of the support of Σ is at most six.

Notice that the difference with Theorem 1 in the dimensions of Σ is due to the projectivization in that theorem. The definition of the support of Σ can be found in Section 5.

We do not know if the bounds given in the Theorems 1, 2 and 3 are sharp in all the cases. In Section 6.1 we collect some old and new examples of positive dimensional resonance varieties.

We also do not know if there are restrictions on the rank of the support of the zero-dimensional characteristic varieties.

2. Foliations and the structure of $\mathcal{R}^1(\mathcal{A})$

No new results are presented in this Section. Its purpose is to recall some basic definitions and the structure of resonant arrangements.

As we mentioned in Introduction the Orlik-Solomon algebra A is graded isomorphic to the DeRham cohomology of the complement $\mathbb{C}^{n+1} \setminus |\mathcal{A}^c|$ where $\mathcal{A}^c$ is the linear cone of a projective arrangement $\mathcal{A}$. The generator e_i of A can be identified with the logarithmic 1-form $\omega_i = d\log\alpha_i$ where α_i is linear polynomial cutting out H_i. Under this identification the multiplication corresponds to the wedge product and ∂ corresponds to the interior product with the radial (or Euler) vector field $R = \sum_{i=0}^n x_i \frac{\partial}{\partial x_i}$.

2.1. **Foliations.** In this paper we will adopt a utilitarian definition for codimension one singular foliations on $\mathbb{P}^n$, from now on just foliation on $\mathbb{P}^n$. A foliation $\mathcal{F}$ on $\mathbb{P}^n$ will be an equivalence class of homogeneous rational differential 1-forms on $\mathbb{C}^{n+1}$ under the equivalence relation

$$\omega \sim \omega' \text{ if and only if there exists } h \in \mathbb{C}(x_0, \ldots, x_n) \setminus 0 \text{ for which } \omega = h\omega',$$

such that $i_R\omega = 0$ and $\omega \wedge d\omega = 0$ for every representative ω. Of course, to ensure the validity of the two conditions for every representative it is sufficient to check it just for one of them.

Among the representatives of $\mathcal{F}$ there are privileged ones - the homogeneous polynomial 1-forms with singular, i.e. vanishing, set of codimension at least two. Any two such forms that are equivalent differ by a non-zero multiplicative constant. If such a form has coefficients of degree $d+1$ then we say that $\mathcal{F}$ is a degree d foliation. The shift in the degree is motivated by the geometric interpretation of the degree. It is the number of tangencies between $\mathcal{F}$ and a generic line in $\mathbb{P}^n$.

Outside the singular set the well-known Frobenius Theorem ensures the existence of local submersions with connected level sets whose tangent space at a point is the kernel of a defining 1-form at this point. These level sets are the local leaves of $\mathcal{F}$. The leaves are obtained by patching together level sets of distinct submersions that have non-empty intersection. Although the data is algebraic the leaves, in general, have a transcendental nature.

Lemma 2.1. *If $[\omega] \in \mathcal{R}^k(\mathcal{A})$ then ω defines a codimension one foliation $\mathcal{F}_\omega$ on $\mathbb{P}^n$.*

Proof. Since ω is homogenous, rational, and $d\omega = 0$ we have just to check that $i_R\omega = 0$, i.e., $\partial\omega = 0$. This is well-known and follows directly from the condition on ω (see [18], Proposition 2.1). □

2.2. **The structure of the first resonance variety.** In the result below we collect some information of what is currently known about the structure of the irreducible components of $\mathcal{R}^1(\mathcal{A})$.

Theorem 2.1. *The irreducible components of $\mathcal{R}^1(\mathcal{A})$ are pairwise disjoint positive dimensional projective subspaces of $\mathbb{P}(A^1)$. Moreover for any irreducible global component of $\mathcal{R}^1(\mathcal{A})$ of dimension d there exists a pencil of hypersurfaces with an irreducible generic member and with $d+2$ completely decomposable fibers whose disjoined union is $\mathcal{A}$.*

The linearity of the irreducible components of $\mathcal{R}^1(\mathcal{A})$ was proved in [3] answering a conjecture of M. Falk. The proof that these irreducible components are disjoint and positive-dimensional appeared in [11]. The relation of the dimension of the irreducible global components with the number of completely decomposable fibers of pencils of hypersurfaces first appeared in [11] for a special case and in [6] in general.

We remark that the result does not hold for the Orlik-Solomon algebra defined over a field of positive characteristic, cf. [5].

The irreducible components of $\mathcal{R}^1(\mathcal{A})$ are precisely the projectivization of maximal linear subspaces $E \subset A^1$ with dimension at least 2 and isotropic with respect to the product $A^1 \times A^1 \to A^2$, cf. [11, Corollaries 3.5, 3.7]. In particular, if E is one of these subspaces then all the homogeneous rational 1-forms, whose cohomology classes belong to E, are proportional over rational functions whence correspond to the same foliation. Moreover the foliation in question admits a rational first integral $F : \mathbb{P}^n \dashrightarrow \mathbb{P}^1$ of a rather special kind. If we write $F = \frac{A}{B}$, where A, B are relatively prime homogenous polynomials, then $sA + tB$ is irreducible for generic $[s:t] \in \mathbb{P}^1$ and the cardinality of the set

$$\left\{ [s:t] \in \mathbb{P}^1 \mid \text{ all irreducible components of } sA + tB \text{ have degree one } \right\}$$

is $\dim E + 1$.

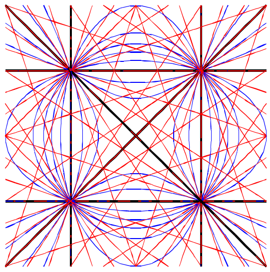


FIGURE 1. A_3 arrangement (in bold) and Bol's 5-web

Remark 2.1 (Webs associated to Resonant Arrangements)**.** As we have just explained to each arrangement $\mathcal{A}$ with $\mathcal{R}^1(\mathcal{A}) \neq \emptyset$ we can canonically associate a finite collection of foliations on $\mathbb{P}^n$ induced by the rational maps $F : \mathbb{P}^n \dashrightarrow \mathbb{P}^1$, one for each irreducible component of $\mathcal{R}^1(\mathcal{A})$. This finite collection forms a global web on $\mathbb{P}^n$. The field of web geometry has a venerable history and the present days are witnessing a lot of activity on webs and their abelian relations. From this viewpoint one of the key problems, at least according to Chern, is the classification of planar webs with the maximal number of abelian relations that are not algebrizable, cf. [2]. For a long time the only example that appeared in the literature was Bol's example. It consists of the 5-web formed by four pencils of lines through four generic points on $\mathbb{P}^2$ and a pencil of conics through these four points. It is a rather intriguing fact that the 5-web, canonically associated to the Coxeter arrangement of type A_3, is precisely Bol's web. It corresponds to four resonance components supported on pencils of lines and one global component. It would be rather interesting to pursue the determination of the rank of the webs associated to resonant line arrangements in $\mathbb{P}^2$.

2.3. **The multinet property.** When $\mathcal{A}$ is an arrangement in $\mathbb{P}^2$ a more precise description of the rational maps induced by the irreducible components of $\mathcal{R}^1(\mathcal{A})$ is given in [6]. The description is based on the combinatorics of lines and points. There it is shown that the existence of a global irreducible component $\Sigma \subset \mathcal{R}^1(\mathcal{A})$ of dimension d is equivalent to the existence of a partition

$$\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \cdots \cup \mathcal{A}_{d+2}$$

in, so called, classes and a multiplicity function $m : \mathcal{A} \to \mathbb{Z}_{>0}$ satisfying

(a) $\sum_{H \in \mathcal{A}_i} m(H)$ is independent of $i = 1, \ldots, d+2$;

(b) If p is the point of intersection of two lines from two different classes then the sum
$$n(p) = \sum_{H \in \mathcal{A}_i, p \in H} m(H)$$
is independent of $i = 1, \ldots, d+2$.

Then the following can be additionally achieved.

(c) $\gcd_{H \in \mathcal{A}} m(H) = 1$;

(d) The partition into classes is the most refined for a fixed set $\mathcal{X}$ of all the intersection points of lines from different classes.

The collection of lines and points $(\mathcal{A}, \mathcal{X})$ satisfying conditions (a)-(d) is called a *(d+2)-multinet.* If $n(p) = 1$ for all $p \in \mathcal{X}$ (whence $m(H) = 1$ for all $H \in \mathcal{A}$) then it is a *(d+2)-net.*

Intersecting an arrangement in $\mathbb{P}^n$ with a general position plane $\mathbb{P}^2$ one readily sees that the similar equivalence holds for arrangements of hyperplanes in $\mathbb{P}^n$.

3. The Gauss map of foliations

3.1. **Gauss map.** Let ω be a homogeneous polynomial differential 1-form on $\mathbb{C}^{n+1}$ such that $i_R\omega = 0$ and $\omega \wedge d\omega = 0$. Let $\mathcal{F}$ be the foliation defined by ω on $\mathbb{P}^n$. The Gauss map of $\mathcal{F}$ is the rational map

$$\begin{array}{rcl}\mathcal{G}_\omega = G_{\mathcal{F}} : \mathbb{P}^n & \dashrightarrow & (\mathbb{P}^n)^\vee \\ p & \mapsto & T_p\mathcal{F} \,.\end{array}$$

that takes every point $p \in \mathbb{P}^n \setminus \mathrm{sing}(\mathcal{F})$ to the hyperplane tangent to $\mathcal{F}$ at p. Under a suitable identification of $\mathbb{P}^n$ with $(\mathbb{P}^n)^\vee$ the Gauss map $\mathcal{G}_\omega$ is nothing more than the rational map defined in homogeneous coordinates by the coefficients of ω.

We say that a foliation $\mathcal{F}$ has degenerate Gauss map when $\mathcal{G}_{\mathcal{F}}$ is not dominant, i.e., its image is not dense in $(\mathbb{P}^n)^\vee$. On $\mathbb{P}^2$ a foliation with degenerate Gauss map has to be a pencil of lines. Indeed the restriction of the Gauss map to a leaf of the foliation coincides with the Gauss map of the leaf and the only (germs of) curves on $\mathbb{P}^2$ with degenerate Gauss map are (germs of) lines. Thus all the leaves of a foliation with degenerate Gauss map are open subsets of lines. Now it is clear that these lines should have just one intersection point whence the foliation has to be a pencil of lines.

On $\mathbb{P}^3$ the situation is more subtle and a complete classification can be found in [1]. Some results toward the classification of foliations with degenerate Gauss map on $\mathbb{P}^4$ have been recently obtained by T. Fassarela [7].

Proposition 3.1. *Let $\mathcal{A}$ be an essential arrangement of hyperplanes in $\mathbb{P}^n$. If Σ is a global irreducible component of $\mathcal{R}^1(\mathcal{A})$ then the Gauss map of the associated foliation is non-degenerate.*

Proof. We use induction on n. Suppose $n = 2$. The only way to have a pencil of lines as the foliation associated to Σ is for $\mathcal{A}$ to be a pencil of lines itself. But a pencil of lines is not essential in $\mathbb{P}^2$. Thus the Gauss map is not degenerate in this case.

Suppose that the result holds for essential arrangements in $\mathbb{P}^n$ and let $\mathcal{A}$ be an essential arrangement in $\mathbb{P}^{n+1}$ with Σ a global irreducible component of $\mathcal{R}^1(\mathcal{A})$ of dimension d. Let $\mathcal{A} = \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_{d+2}$ be the corresponding partition of $\mathcal{A}$ and $m : \mathcal{A} \to \mathbb{Z}_{\geq 0}$ the multiplicity function , cf. 2.3. The foliation associated to Σ can be thus defined by the 1-form

$$\omega = FdG - GdF \ ,$$

where

$$F = \prod_{H \in \mathcal{A}_1} {\alpha_H}^{m(H)} \quad \text{and} \quad G = \prod_{H \in \mathcal{A}_2} {\alpha_H}^{m(H)} \,.$$

and α_H is a linear form on $\mathbb{C}^{n+1}$ whose kernel is H.

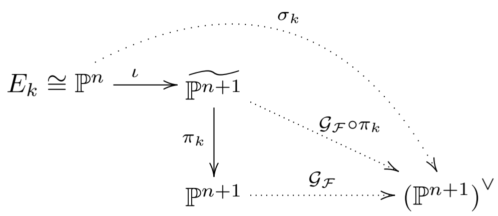
Using the definitions of essential arrangements and multinets one can find two distinct points $p_1, p_2 \in \mathbb{P}^{n+1}$ such that

(1) For $k = 1, 2$ the subarrangements
$$\mathcal{B}_k = \bigcup_{p_k \in H, H \in \mathcal{A}} H$$
have rank n;

(2) For $k = 1, 2$ the point p_k belongs to $\bigcap_i |\mathcal{A}_i|$.

Let $\pi_k : \widetilde{\mathbb{P}^{n+1}} \to \mathbb{P}^{n+1}$ be the blow-up of $\mathbb{P}^{n+1}$ at p_k. The restriction to exceptional divisor $E_k \cong \mathbb{P}^n$ of the strict transforms of the hyperplanes in $\mathcal{B}_k$ induces a non-degenerate arrangement of hyperplanes in E_k which we will still denote by $\mathcal{B}_k$.

If $\iota : E_k \to \widetilde{\mathbb{P}^{n+1}}$ is the natural inclusion then we claim that the closure of the image of the rational map $\sigma_k = \mathcal{G}_\mathcal{F} \circ \pi_k \circ \iota$ (see the diagram) is the hyperplane $H^{(k)}$ in $(\mathbb{P}^{n+1})^\vee$ dual to p_k.



Indeed the arrangement $\mathcal{B}_k \subset E_k$ admits a partition
$$\mathcal{B}_k = \bigcup_{i=1}^{d+2} \mathcal{B}_k \cap \mathcal{A}_i$$
and a function $m_k = m_{|\mathcal{B}_k}$ satisfying the multinet properties. Notice that due to choice of the point p_k we have $\mathcal{B}_k \cap \mathcal{A}_i \neq \emptyset$ for every $i = 1, \ldots, d+2$. Therefore the irreducible component $\Sigma \subset \mathcal{R}^1(\mathcal{A})$ induces an irreducible component $\Sigma_k \subset \mathcal{R}^1(\mathcal{B}_k)$ of the same dimension.

On the one hand the foliation on $\mathbb{P}^n$ induced by Σ_k can be defined by the 1-form $F_k dG_k - G_k dF_k$ with
$$F_k = \prod_{H \in \mathcal{A}_1 \cap \mathcal{B}_k} \alpha_H^{m(H)} \quad \text{and} \quad G_k = \prod_{H \in \mathcal{A}_2 \cap \mathcal{B}_k} \alpha_H^{m(H)}.$$
On the other hand the first non-zero jet of $FdG - GdF$ at p_k is $F_k dG_k - G_k dF_k$ viewing now α_H as linear forms on $\mathbb{C}^{n+2}$. More precisely let us assume that $p_k = [0 : \ldots : 0 : 1]$ and write $F = F_k F'_k$, $G = G_k G'_k$. Then the map $\mathcal{G}_\mathcal{F}$ (after division by $F'_k(p_k) G'_k(p_k)$) can be written in the homogeneous coordinates $[x_0 : x_1 : \ldots : x_n : x_{n+1}]$ of $\mathbb{P}^{n+1}$ as
$$\begin{aligned} \mathcal{G}_\mathcal{F} &= \left[F_k \frac{\partial G_k}{\partial x_0} - G_k \frac{\partial F_k}{\partial x_0} + b_0 : \cdots : F_k \frac{\partial G_k}{\partial x_n} - G_k \frac{\partial F_k}{\partial x_n} + b_n : b_{n+1} \right] \\ &= \left[F_k dG_k - G_k dF_k + \sum_{i=0}^{n+1} b_i dx_i \right], \end{aligned}$$
where $b_i \in \mathfrak{m}^{2 \deg F}$ for $i = 0, \ldots, n, n+1$, $\mathfrak{m}$ being the maximal ideal of $\mathbb{C}[x_0, \ldots, x_n]$ supported at $0 \in \mathbb{C}^{n+1}$.

If we consider now $(x_0 : \ldots : x_n) \in \mathbb{P}^n$ as a homogenous system of coordinates on the exceptional divisor E_k then in these coordiantes

$$\sigma_k = \left[F_k \frac{\partial G_k}{\partial x_0} - G_k \frac{\partial F_k}{\partial x_0} : \cdots : F_k \frac{\partial G_k}{\partial x_n} - G_k \frac{\partial F_k}{\partial x_n} \right] = [F_k dG_k - G_k dF_k]$$

since the coefficients of $F_k dG_k - G_k dF_k$ lie in $\mathfrak{m}^{2 \deg F - 1}$. Thus σ_k can be identified with the Gauss map of the foliation $\mathcal{F}_k$ on $\mathbb{P}^n$ induced by Σ_k composed with ϕ, where ϕ is the isomorphism of $(\mathbb{P}^n)^\vee$ with $H^{(k)}$ defined by the coordinates.

Using the inductive hypothesis we have now that the closure of the image of $\mathcal{G}_\mathcal{F}$ contains at least two distinct hyperplanes. Since $\mathbb{P}^{n+1}$ is irreducible the closure of the image of $\mathcal{G}_\mathcal{F}$ must be $(\mathbb{P}^{n+1})^\vee$. This completes the proof. □

3.2. **The parabolic divisor.** We will need some simple properties of the parabolic divisor of $\mathcal{F}$.

Definition 3.1. Let $\mathcal{F}$ be a foliation on $\mathbb{P}^n$ with non-degenerate Gauss map. If $\omega = \sum_{i=0}^n a_i dx_i$ is a homogeneous polynomial 1-form without codimension one zeroes defining $\mathcal{F}$ then $P_\mathcal{F}$, the *parabolic divisor* of $\mathcal{F}$, is the divisor in $\mathbb{P}^n$ defined by the vanishing of the polynomial

$$D = \det \left(\frac{\partial a_i}{\partial x_j} \right)$$

(with multiplicities taken into account).

Proposition 3.2. *If $\mathcal{F}$ is a foliation on $\mathbb{P}^n$ with non-degenerate Gauss map and H is a hyperplane invariant with respect to $\mathcal{F}$ then*

$$(n-1)H \le P_\mathcal{F},$$

i.e., α_H^{n-1} divides D.

Proof. Without loss of generality we can assume that $\alpha_H = x_0$. If $\omega = \sum_{i=0}^n a_i dx_i$ is a homogenous polynomial 1-form defining $\mathcal{F}$ then the $\mathcal{F}$-invariance of H means that all the coefficients of $\omega \wedge dx_0$ are divisible by x_0. Thus $x_0 | a_i$ for every $i = 1, \ldots, n$. In particular except for the entries of the first collum and the first row all the elements of the matrix $\left(\frac{\partial a_i}{\partial x_j} \right)$ are divisible by x_0. Using the Laplacian expansion of D with respect to the first column it is then clear that D is divisible by x_0^{n-1}. □

It is immediate from the definition of the parabolic divisor that

$$\deg(D) = (n+1)\deg(\mathcal{F}).$$

This simple observation together with Proposition 3.2 allows us to exhibit an upper bound on the number of hyperplanes invariant with respect to a foliation with non-degenerate Gauss map.

Corollary 3.1. *If $\mathcal{F}$ is a degree d foliation on $\mathbb{P}^n$ with non-degenerate Gauss map then the number of invariant hyperplanes is at most*

$$\left(\frac{n+1}{n-1} \right) \cdot \deg(\mathcal{F}).$$

Example 3.1. Here is an example showing that the bound in Corollary 3.1 is sharp. Let $\mathcal{F}$ be a foliation on $\mathbb{P}^n$, $n \geq 2$, induced by a logarithmic 1-form

$$\sum_{i=0}^{n} \lambda_i \frac{dx_i}{x_i}$$

where $\sum \lambda_i = 0$ and no λ_i is equal to zero. (This form represents a degree one generic element from the Orlik-Solomon algebra of the coordinate arrangement). Then $\mathcal{F}$ has degree $(n-1)$ and its Gauss map is non-degenerate. Also $\mathcal{F}$ leaves invariant all the $n+1$ hyperplanes of the arrangement.

For $n = 2, 3$ there are examples of degree d, $d > n-1$, foliations $\mathcal{F}$ on $\mathbb{P}^n$ with exactly $\frac{n+1}{n-1}\deg(\mathcal{F})$ invariant hyperplanes. On $\mathbb{P}^2$ we are aware of three sporadic examples (Hesse pencil, Hilbert modular foliation [13], and a degree 7 foliation leaving invariant the extended Hessian arrangement of all the reflection hyperplanes of the reflection group of order 1296 - see [14], p. 227) and one infinite family [15], [6, Example 4.6], consisting of degree d, $d \geq 2$, foliations leaving invariant the arrangement

$$xyz(x^{d-1} - y^{d-1})(x^{d-1} - z^{d-1})(z^{d-1} - y^{d-1}).$$

On $\mathbb{P}^3$ we are aware of just one example with $d > n-1$ attaining the bound, see 6.2 below.

4. Resonance versus pencils: proofs of Theorems 1 and 2

It follows from Theorem 2.1 that Theorem 2 is equivalent to Theorem 1. We will prove Theorem 2.

Let $\mathcal{P}$ be a pencil of hypersurfaces on $\mathbb{P}^n$ with generators F and G and k completely reducible fibers. If $k \leq 2$ there is nothing to prove. So we will assume that $k \geq 3$. Let $\omega_0 = FdG - GdF$ be a representative of the associated foliation $\mathcal{F}$. It is a classical (and elementary) result of Darboux that the divisor of zeros of ω_0 is equal to

$$\sum (m(A) - 1)A\,,$$

where A runs over all irreducible components of elements of $\mathcal{P}$ and $m(A)$ denotes the multiplicity of A in the pencil. In particular, if $\tilde{Q} \in \mathbb{C}[x_0, \ldots, x_n]$ denotes the product of the completely reducible fibers and Q denotes the reduced polynomial with the same zero set then the 1-form

$$\omega = \frac{Q}{\tilde{Q}}\omega_0$$

is an homogeneous polynomial 1-form defining $\mathcal{F}$. Notice that

$$\deg(\mathcal{F}) \leq 2\deg(F) - 2 - \deg\left(\frac{\tilde{Q}}{Q}\right).$$

In the rest of the proof we will assume that the pencil $\mathcal{P}$ has full rank, i.e., the arrangement $Q = 0$ is essential. Proposition 3.1 implies that the Gauss map of $\mathcal{F}$ is non-degenerate. It follows from Proposition 3.2 that $\bar{Q}^{n-1}$ divides

$$\left(\frac{\tilde{Q}}{Q}\right)^{n-1} \det\left(\frac{\partial a_i}{\partial x_j}\right),$$

where $\omega = \sum a_i dx_i$. Since $\deg(\tilde{Q}) = k\deg(F)$ we obtain that

$$\begin{aligned} (n-1)k\deg(F) &\leq (n+1)\left(2\deg(F) - 2 - \deg\left(\frac{\tilde{Q}}{Q}\right)\right) + (n-1)\deg\left(\frac{\tilde{Q}}{Q}\right) \\ &= (n+1)\left(2\deg(F)-2\right) - 2\deg\left(\frac{\tilde{Q}}{Q}\right). \end{aligned}$$

Therefore

$$k \leq \frac{n+1}{n-1}\left(2 - \frac{2}{\deg(F)}\right) - \frac{2}{(n-1)\deg(F)}\deg\left(\frac{\tilde{Q}}{Q}\right) < 2\frac{n+1}{n-1}.$$

In particular

$$\begin{aligned} n \geq 5 &\quad \text{implies} \quad k \leq 2, \\ n \geq 3 &\quad \text{implies} \quad k \leq 3, \\ n \geq 2 &\quad \text{implies} \quad k \leq 5. \end{aligned}$$

Theorem 2 follows. □

5. Characteristic varieties versus pencils: proof of Theorem 3

If $\mathcal{A}$ is an arrangement on $\mathbb{P}^n$, $M = \mathbb{P}^n \setminus |\mathcal{A}|$ its complement and $\mathcal{A}' \subset \mathcal{A}$ is a subarrangement with complement M' then the inclusion of M into M' induces an inclusion of $\mathrm{Hom}(\pi_1(M'), \mathbb{C}^*)$ into $\mathrm{Hom}(\pi_1(M), \mathbb{C}^*)$ and also of $\mathcal{V}^1(M')$ into $\mathcal{V}^1(M)$. As in the case of resonance varieties, the support of an irreducible component Σ of $\mathcal{V}^1(M)$ is the smallest subarrangement $\mathcal{A}'$ such that $\Sigma \subset \mathcal{V}^1(\mathcal{A}')$.

If $\Sigma \subset \mathcal{V}^1(M)$ is an irreducible component of dimension $d > 0$ containing $1 \in \mathrm{Hom}(\pi_1(M), \mathbb{C}^*)$ then, as it was already pointed out in the introduction, the projectivization of its tangent space is a irreducible component of the resonance variety. In particular the description of the resonance varieties carried out in Section 2 applies.

The description of the translated components of $\mathcal{V}^1(M)$, i.e. the irreducible components not containing 1, has some extra ingredients. We invite the reader to consult [4] for a more extensive description. Here we will recall just what is strictly necessary for our purposes.

If Σ is a translated component of dimension at least two then after translating it to 1 one obtains an irreducible component of $\mathcal{V}^1(M)$ through 1.

If Σ is a translated component of dimension one then there exists a pencil of hypersurfaces with generic irreducible fiber and exactly two completely reducible fibers with support contained in the support of Σ. The support of Σ is the union of the hyperplanes that appear as components of elements of the pencil. Moreover there is at least one extra fiber such that its components are either hyperplanes in the support of Σ or non reduced hypersurfaces.

5.1. A multinet-like property.

Lemma 5.1. *Let $\mathcal{P}$ be a pencil of hypersurfaces on $\mathbb{P}^n$ with irreducible generic fiber generated by two completely reducible fibers F and G. If there exists a third fiber* *that is a product of linear forms and non-reduced polynomials then*

$$\sum_{\alpha_H | F, p \in H} m(H) = \sum_{\alpha_H | G, p \in H} m(H)$$

for every p *in the base locus of* $\mathcal{P}$.

Proof. Let $\omega = FdG - GdF$ and p be a point in the base locus of the pencil. Choose affine coordinates $(x_1, \ldots, x_n)$ where p is the origin and write $F = F_1 \cdot F_2$ and $G = G_1 \cdot G_2$ where $F_2, G_2 \notin \mathfrak{m}$ and all the irreducible components of F_1 and G_1 are in $\mathfrak{m}$, $\mathfrak{m}$ being the maximal ideal $(x_1, \ldots, x_n)$.

Put now $R = x_1 \frac{\partial}{\partial x_1} + \cdots + x_n \frac{\partial}{\partial x_n}$. Notice that unlike in the previous sections R is now the radial vector field in $\mathbb{C}^n$. Then the Leibniz formula implies that

$$i_R \omega = F_2 G_2 i_R \left(F_1 dG_1 - G_1 dF_1\right) + F_1 G_1 i_R \left(F_2 dG_2 - G_2 dF_2\right).$$

To prove the lemma it suffices to show that $i_R(F_1 dG_1 - G_1 dF_1) = 0$.

Suppose this is not true. Then we have $i_R(F_1 dG_1 - G_1 dF_1) = cF_1 G_1$ for a $c \in \mathbb{C}^*$ and $i_R \omega = F_1 G_1 g$ where g is a polynomial in $x_1, \ldots, x_n$ such that $g(0) \neq 0$. Our hypothesis implies that there exists a hyperplane or a non reduced hypersurface passing through 0. In the latter case, there is an irreducible polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ such that f^m divides, say $K = F - G$ for some $m > 1$ whence f divides the coefficients of ω. Thus f divides the polynomial $i_R \omega$ whence it also divides $F_1 G_1$ which is a contradiction.

In the former case, there is a linear form $\alpha \in \mathbb{C}[x_1, \ldots, x_n]$ that divides K. Thus α divides $K_0 = F_2(p) F_1 - G_2(p) G_1$ whence α divides also the polynomial $i_R(F_1 dG_1 - G_1 dF_1) = cF_1 G_1$. This is again a contradiction. □

Let $\mathcal{A}$ be the arrangement formed by the hyperplanes that appear as irreducible components of elements of a pencil $\mathcal{P}$ as in Lemma 5.1. Combining Lemma 5.1 with (the proof of) Proposition 3.1 we obtain the following Corollary.

Corollary 5.1. *If* $\mathcal{A}$ *has full rank then the Gauss map of the foliation induced by* $\mathcal{P}$ *is non-degenerate.*

With this result at hand we will mimic below the proof of Theorem 1 to obtain Theorem 3.

5.2. **Proof of Theorem 3.** From the discussion at the beginning of this Section it suffices to show that $n < 7$ if there exists a pencil $\mathcal{P}$ of hypersurfaces on $\mathbb{P}^n$ with completely reducible generators F and G inducing a full rank arrangement and at least one extra fiber, say $K = F - G$, that can be written as

$$K = \tilde{U} \cdot \tilde{V}$$

where $\tilde{U} \in \mathbb{C}[x_0, \ldots, x_n]$ is a product of linear forms and $\tilde{V} \in \mathbb{C}[x_0, \ldots, x_n]$ is a product of non-trivial powers of irreducible polynomials of degree at least two. We will denote by U, V the reduced polynomial with the same zero set of $\tilde{U}, \tilde{V}$. We point out that

$$2 \deg \left(\frac{\tilde{V}}{V} \right) \geq \deg(\bar{V})\,. \tag{1}$$

If $\tilde{Q} \in \mathbb{C}[x_0, \ldots, x_n]$ denotes the product FG and Q denotes the reduced polynomial with the same zero set then the 1-form

$$\omega = \frac{U}{\tilde{U}} \frac{V}{\bar{V}} \frac{Q}{\tilde{Q}} \omega_0$$

is a homogeneous polynomial 1-form defining $\mathcal{F}$. In particular

$$\deg(\mathcal{F}) \leq \deg(\tilde{Q}) - 2 - \deg\left(\frac{\tilde{U}\tilde{V}\tilde{Q}}{UVQ}\right).$$

Corollary 5.1 implies that the Gauss map of $\mathcal{F}$ is non-degenerate. It follows from Proposition 3.2 that $(\tilde{U}\tilde{Q})^{n-1}$ divides

$$\left(\frac{\tilde{U}\tilde{Q}}{UQ}\right)^{n-1} \det\left(\frac{\partial a_i}{\partial x_j}\right),$$

where $\omega = \sum a_i dx_i$. We obtain that

$$\begin{aligned}(n-1)\deg(\tilde{U}\tilde{Q})) &\leq (n+1)\left(\deg(\tilde{Q}) - 2 - \deg\left(\frac{\tilde{U}\tilde{V}\tilde{Q}}{UVQ}\right)\right) + (n-1)\deg\left(\frac{\tilde{U}\tilde{Q}}{UQ}\right) \\ &= (n+1)\left(\deg(\tilde{Q}) - 2\right) - 2\deg\left(\frac{\tilde{U}\tilde{Q}}{UQ}\right) - (n+1)\deg\left(\frac{\tilde{V}}{V}\right).\end{aligned}$$

Therefore, if we suppose that $n \geq 3$, delete the term $-2\deg\left(\frac{\tilde{U}\tilde{Q}}{UQ}\right)$ and use (1) then

$$\begin{aligned}(n-1)\deg(\tilde{Q}) &< (n+1)\left(\deg(\tilde{Q}) - 2\right) - \frac{n+1}{2}\deg\left(\tilde{V}\right) - (n-1)\deg(\tilde{U}) \\ &\leq (n+1)\left(\deg(\tilde{Q}) - 2\right) - \frac{n+1}{2}\deg\left(\tilde{U}\tilde{V}\right) < \frac{3(n+1)}{4}\deg\left(\bar{Q}\right).\end{aligned}$$

In particular $n < 7$. Theorem 3 follows. □

6. Some Examples

6.1. **Pencils on $\mathbb{P}^2$.** The inequalities of Theorem 2 allow pencils on $\mathbb{P}^2$ with five completely reducible fibers. However no such pencils (of full rank) are known. This existence problem is not even settled for the case of nets, cf. [19, Problem 2]. The smallest possible example would be a (5,7)-net in $\mathbb{P}^2$ realizing an orthogonal triple of Latin squares of order 7.

Concerning pencils with four completely reducible fibers just one example is known - the Hesse pencil based on the (4,3)-net. This pencil can be succinctly describe as the pencil generated by a smooth cubic and its Hessian.

Concerning pencils on $\mathbb{P}^2$ with three completely reducible fibers, plenty of them are known, including families with analytic moduli. The existence of such families with analytic moduli can be inferred from the realization result for nets presented in [19, Theorem 4.4]. For an explicit example one can consider hyperplane sections of the examples presented in 6.2.

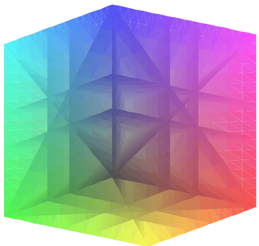


FIGURE 2. Arrangement of 12 hyperplanes on $\mathbb{P}^3$ associated to D_4

6.2. **Pencils on $\mathbb{P}^3$.** Let d be a positive integer and consider the pencil on $\mathbb{P}^3$ generated by

$$\begin{aligned} F_d &= \left(x_0^d - x_1^d\right)\left(x_2^d - x_3^d\right) \\ G_d &= \left(x_0^d - x_2^d\right)\left(x_1^d - x_3^d\right) . \end{aligned}$$

Then

$$F_d - G_d = \left(x_0^d - x_3^d\right)\left(x_2^d - x_1^d\right) .$$

The arrangements $\mathcal{A}_d$ corresponding to these pencils have many interesting properties. Each $\mathcal{A}_d$ consists of $6d$ hyperplanes that are the reflecting hyperplanes for the monomial group $G(d,d,4)$ generated by complex reflections (e.g., see [14]). For $d = 1, 2$ the group is the Coxeter group of type A_3 and D_4 respectively. For all $d \geq 2$ the arrangements $\mathcal{A}_d$ are essential and carrying each a global irreducible resonance component of dimension one.

Moreover the combinatorics of these arrangements can be called a (3,d)-net in $\mathbb{P}^3$ (for $d \geq 2$) if one substitutes in the definition of a net in $\mathbb{P}^2$ planes for lines and lines for points. This implies that the intersection of $\mathcal{A}_d$ with a generic plane gives a net in $\mathbb{P}^2$. On the other hand, the intersection with a special plane (say, the one defined by $x_3 = 0$) gives the family of multinets mentioned at the end of Section 3 whence representing these multinets as limits of nets.

Let us look closer at the combinatorics of $\mathcal{A}_d$. For each ζ such that $\zeta^d = 1$ denote by $H_{i,j}(\zeta)$ the hyperplane defined by $x_i = \zeta x_j$ $(1 \leq i < j \leq 4)$. Then identify the collection of hyperplanes corresponding to the linear divisors of F_d, G_d. and $F_d - G_d$ via

$$a_\zeta = H_{1,2}(\zeta^{-1}) = H_{1,3}(\zeta) = H_{2,3}(\zeta),$$

and

$$b_\zeta = H_{3,4}(\zeta) = H_{2,4}(\zeta) = H_{1,4}(\zeta).$$

It is straightforward to check that the Latin square of $\mathcal{A}_d$ becomes after the identification the multiplication table of the dihedral group D_d with a_ζ forming the cyclic subgroup of order d and $\{b_\zeta\}$ being the complementary set of involutions. Intersecting $\mathcal{A}_d$ with $d \geq 3$ with a general plane we obtain a series of 3-nets in $\mathbb{P}^2$ realizing non-commutative groups (cf. [19]). In particular these nets are not algebrizable. We remark that another non-algebraizable example of a 3-net in $\mathbb{P}^2$ has been found by J. Stipin in [16]. He has exhibited a (3,5)-net that does not realize $\mathbb{Z}_5$.

Finally (as we learned from [16]) the general fibers of the above pencil for $d = 2$ were studied by R. M. Mathews in [12] under the name of *desmic* surfaces. Thus for arbitrary d these fibers can be considered as generalizations of desmic surfaces.

The foliations $\mathcal{F}_d$ induced by $\omega_d = F_d dG_d - G_d dF_d$ have degree $4d - 2$. Thus the bound of Corollary 3.1 is attained by $\mathcal{F}_2$ and no other foliation in the family.

6.3. **Pencils on $\mathbb{P}^4$.** We do not know any example of a pencil on $\mathbb{P}^4$ with three completely reducible fibers that is not a linear pullback of a pencil from a smaller dimension. One can deduce from careful reading of the proof of Theorem 2 that the degree of such a pencil must be at least 10.

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