Invariant Hypersurfaces for Positive Characteristic Vector Fields

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Abstract

We show that a generic vector field on an affine space of positive characteristic
admits an invariant algebraic hypersurface. This is in sharp contrast with the char-
acteristic zero case where Jouanolou's Theorem says that a *generic* vector field on
the complex plane does not admit any invariant algebraic curve.

Key words: positive characterisitic vector fields, invariant hypersurfaces

1 Introduction

Jouanolou, in his celebrated Lecture Notes [3], proved that a generic vector
field of degree greater than one on $\mathbb{P}^2_{\mathbb{C}}$ does not admit any invariant algebraic
curve. Here, by generic we mean outside an enumerable union of algebraic
varieties. In this paper we investigate what happens if we change the field of
complex numbers to a field of positive characteristic.

It turns out that the situation is completely different, and we prove that
outside an algebraic variety in the space of affine vector fields of a fixed degree
a vector field **does** admit an invariant algebraic hypersurface. More precisely,
we prove the following result.

Theorem 1 *Let X be a vector field on $\mathbb{A}^n_k$, where k is a field of positive
characteristic. If the divergent of X is different from zero, then X admits an
invariant reduced algebraic hypersurface. If $\mathrm{div}(X) = 0$ then the irreducible
polynomials cutting out invariant hypersurfaces appear as factors of a polyno-
mial F completely determined by X.*

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Our methods are quite elementary, and we start by investigating collections of n vector fields on the n-dimensional affine space over any field, and their dependency locus. We give conditions for the dependency locus to be invariant by every vector field in the collection. These conditions turn out to be necessary and sufficient conditions in the two-dimensional characteristic zero case, see proposition 5. In positive characteristic such conditions imply Theorem 1.

Despite its simplicity the theorem and its proof illustrate some particular features of vector fields in positive characteristic.

2 Preliminaries

In this section we define the basic vocabulary that will be used in the rest of the paper. We try to keep the language as simple as possible.

2.1 Derivations and vector fields

Denote by R the ring $k[x_1, ..., x_n]$, and by $\Lambda(\mathbb{A}^n)$ the graded R–module of differential forms.

Definition 1 A k-*derivation* X of R is a k-linear transformation of R in itself that satisfies Leibniz's rule, i.e. $X(ab) = aX(b) + bX(a)$ for arbitrary $a, b \in R$.

A derivation X can be written as $X = \sum_{i=1}^{n} X(x_i)\frac{\partial}{\partial x_i}$, and understood as a vector field on $\mathbb{A}_k^n$.

Definition 2 The *inner product* of X and a p-form ω is the $(p-1)$-form $i_X\omega$ defined as
$$i_X\omega(v_1, ..., v_{p-1}) = \omega(X, v_1, ..., v_{p-1}) \ .$$
Note that i_X is an antiderivation of degree -1 of $\Lambda(\mathbb{A}^n)$.

Sometimes to simplify the notation we are going to denote $i_{X_1} i_{X_2} \ldots i_{X_n}\omega$ by $i_{X_1, X_2, \ldots, X_n}\omega$.

Definition 3 Given a vector field X on $\mathbb{A}_k^n$, its *Lie derivative* L_X is the derivation of degree 0 of $\Lambda(\mathbb{A}^n)$ defined by

$$L_X = i_X d + d i_X$$

The proof of the next proposition can be found in [2], pages 93 and 94.

Proposition 1 *If X and Y are two vector fields on $\mathbb{A}^n$, then*

$$[L_X, i_Y] = i_{[X,Y]}$$

$$[L_X, L_Y] = L_{[X,Y]}$$

Definition 4 We say that the hypersurface given by the reduced equation $(F = 0)$ is invariant by X if $\frac{X(F)}{F}$ is a polynomial. In case that $X(F) = 0$ we say that F is a first integral or a non-trivial constant of derivation of X.

2.2 Derivations in characteristic p

The derivations in positive characteristic have very particular properties when compared with the derivations in characteristic zero. Some of these special properties can be seen in the next two results, and these will be essential for the proof of Theorem 1.

Proposition 2 *Let X be a derivation over a field of characteristic $p > 0$. Then X^p is a derivation.*

Proof: It is sufficient to verify that X^p satisfies Leibniz's rule. In fact,

$$X^p(fg) = \sum_{i=1}^{p} \binom{p}{i} X^{p-i}(f) X^i(g) = X^p(f)g + X^p(g)f.$$

■

Theorem 2 *Let X be a derivation of R, where k is a field of characteristic $p > 0$. Then X admits a non-trivial constant of derivation if and only if $X \wedge X^p \wedge \cdots \wedge X^{p^{n-1}} = 0$.*

Proof: See Lecture III on [4](more precisely pages 56—57) or [1]. ■

3 Invariant hypersurfaces in $\mathbb{A}_k^n$

In this section we define the notion of a polynomially involutive family of vector fields and show how it can be used to guarantee the existence of invariant algebraic hypersurfaces for vector fields in such a family.

3.1 Dependency locus of vector fields

Definition 5 Let $X_1, \dots, X_n$ be vector fields on $\mathbb{A}_k^n$. Their *dependency locus* is the hypersurface cut out by $\mathrm{Dep}(X_1, \dots, X_n)$, where :

$$\mathrm{Dep}(X_1, \dots, X_n) = i_{X_1} \cdots i_{X_n} dx_1 \wedge \cdots \wedge dx_n$$

Proposition 3 *If $X_1, \dots, X_n$ are generically independent vector fields in $\mathbb{A}_k^n$ then there exist polynomials $p_{ij}^{(k)}$ and a non-negative integer m such that*

$$[X_i, X_j] = \sum_{k=1}^{n} \frac{p_{ij}^{(k)}}{\mathrm{Dep}(X_1, \dots, X_n)^m} X_k$$

Proof: In the principal open set $\{\mathbb{A}^n \setminus (\mathrm{Dep}(X_1, \dots, X_n) = 0)\}$ the vector fields are independent, whence the lemma follows. $\blacksquare$

Lemma 1 (Fundamental Lemma) *Let $X_1, .., X_n$ be vector fields on $\mathbb{A}_k^n$ and $a_{ij}^{(k)}$ be rational functions such that $[X_i, X_j] = \sum_{k=1}^{n} a_{ij}^{(k)} X_k$. Denoting $\mathrm{Dep}(X_1, \dots, X_n)$ by F, then*

$$i_{X_k} \Omega \wedge dF = \left[\left((-1)^{k+1} div(X_k) + \sum_{j=k+1}^{n} a_{kj}^{(j)} + \sum_{i=1}^{k-1} (-1)^{i-k+1} a_{ik}^{(i)} \right) F \right] \Omega \ .$$

where $\Omega = dx_1 \wedge \cdots \wedge dx_n$.

Proof: The proof of the lemma follows from a few manipulations with the formulas given in Proposition 1. From the definition of the Lie derivative, we can see that

$$\begin{aligned}
d\mathrm{Dep}(X_1, \cdots, X_n) &= d i_{X_1} \cdots i_{X_n} \Omega = (L_{X_1} - i_{X_1} d) i_{X_2} \cdots i_{X_n} \Omega \\
&= \sum_{i=1}^{n} (-1)^{i+1} i_{X_1} \cdots i_{X_{i-1}} L_{X_i} i_{X_{i+1}} \cdots i_{X_n} \Omega \ .
\end{aligned}$$

We can write the last expression on the formula above as

$$\sum_{i=1}^{n}(-1)^{i+1}i_{X_1,\dots,X_{i-1}}L_{X_i}(i_{X_{i+1},\dots,X_n}\Omega)$$

$$=\sum_{i=1}^{n}(-1)^{i+1}\left(div(X_i)\beta_i+\sum_{j=i+1}^{n}i_{X_1,\dots,X_{i-1},X_{i+1},\dots,X_{j-1},[X_i,X_j],X_{j+1},\dots,X_n}\right)\Omega$$

$$=\sum_{i=1}^{n}(-1)^{i+1}\left(div(X_i)\beta_i+\sum_{j=i+1}^{n}(-1)^{i-j+1}a_{ij}^{(i)}\beta_j+a_{ij}^{(j)}\beta_i\right),$$

where $\beta_i=i_{X_1}\cdots i_{X_{i-1}}i_{X_{i+1}}\cdots i_{X_n}\Omega$. Since

$$i_{X_k}\Omega\wedge\beta_l=\delta_{kl}(i_{X_1}\cdots i_{X_n}\Omega)\Omega,$$

we obtain that $i_{X_k}\Omega\wedge di_{X_1}\cdots i_{X_n}\Omega$ is equal to

$$\left[\left((-1)^{k+1}div(X_k)+\sum_{j=k+1}^{n}a_{kj}^{(j)}+\sum_{i=1}^{k-1}(-1)^{i-k+1}a_{ik}^{(i)}\right)i_{X_1}\cdots i_{X_n}\Omega\right]\Omega,$$

and the lemma is proved. ∎

3.2 Polynomially involutive vector fields

Definition 6 A collection of vector fields $X_1,\dots,X_n$ of $\mathbb{A}_k^n$ is *polynomially involutive* if there exist polynomials $p_{ij}^{(k)}$ such that

$$[X_i,X_j]=\sum_{k=1}^{n}p_{ij}^{(k)}X_k.$$

Proposition 4 *Let k be a field and $X_1,\dots,X_n$ a collection of vector fields on $\mathbb{A}_k^n$. Suppose that $\{X_i\}_{i=1}^{n}$ is a polynomially involutive system of vector fields. If the dependency locus is not a constant of derivation then it is invariant by X_j for each $j=1,\dots,n$.*

Proof: Let $F:=\mathrm{Dep}(X_1,\cdots,X_n)$. By the fundamental lemma,

$$X_k(F)=\frac{i_{X_k}\Omega\wedge dF}{\Omega}=\left((-1)^{k+1}div(X_k)+\sum_{j=k+1}^{n}a_{kj}^{(j)}+\sum_{i=1}^{k-1}(-1)^{i-k+1}a_{ik}^{(i)}\right)F.$$

Since $\{X_i\}_{i=1}^{n}$ is a polynomially involutive system of vector fields, one can see that

$$\frac{X_k(F)}{F}$$

is a polynomial. Therefore, if dF is different from zero, the dependency locus is invariant by X_j. ∎

In general the converse of the Proposition above does not hold, as the next example shows.

Example 1 Consider the vector fields X, Y, Z on $\mathbb{A}^3_\mathbb{C}$ given by

$$X = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} \ , \ \ Y = x\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} \ \text{e} \ \ Z = \frac{\partial}{\partial x} \ .$$

Then $\text{Dep}(X, Y, Z) = z^2$ and $[X, Z] = \frac{\partial}{\partial y} = \frac{Y - xZ}{z}$. Therefore the vector fields X, Y, Z are not polynomially involutive, but the dependency locus is invariant by all of them.

Although in the two-dimensional case polynomially involutiviness completely caracterizes the invariance of the dependency locus for pairs of vector fields.

Proposition 5 *Let X and Y be vector fields on $\mathbb{A}^2_k$ such that $\text{Dep}(X, Y)$ is not a constant of derivation. Then X and Y are polinomially involutive if, and only if, $\text{Dep}(X, Y)$ is invariant by both X and Y.*

Proof: By proposition 4, we have just to prove that the invariance of $\text{Dep}(X, Y)$ by both X and Y implies that X and Y are polynomially involutive. We know that $[X, Y] = \frac{p}{\text{Dep}(X,Y)^m}X + \frac{q}{\text{Dep}(X,Y)^m}Y$. By the fundamental lemma,

$$X(\text{Dep}(X, Y)) = (div(X) + \frac{q}{\text{Dep}(X, Y)^m})\text{Dep}(X, Y) \ ,$$

and from our hypotheses we can deduce that $\frac{q}{\text{Dep}(X,Y)^m}$ is a polynomial. Mutatis mutandis, we can conclude that $\frac{p}{\text{Dep}(X,Y)^m}$ is also a polynomial. Hence X and Y are polynomially involutive. ∎

4   Proof of Theorem 1

If k is a field of positive characteristic $p > 0$ and X is a vector field on $\mathbb{A}^n_k$, it is fairly simple to decide whether or not X has an invariant hypersurface. This simplicity is in sharp contrast with the characteristic zero case, where decidability is not known.

The fact is that in positive characteristic we have a polynomially involutive system of vector fields canonically associated to X. When X is a vector field on $\mathbb{A}^n_k$ then the polynomially involutive system is

$$X, X^p, \dots, X^{p^{n-1}}.$$

In fact the former system is commutative. By Theorem 2, if

$$\mathrm{Dep}(X,\dots,X^{p^{n-1}}) = 0$$

then X admits a first integral and, in particular, an invariant hypersurface. If $\mathrm{div}(X) \neq 0$ and

$$\mathrm{Dep}(X,\dots,X^{p^{n-1}}) \neq 0$$

then by the Proposition 4, X admits an invariant hypersurface.

When $div(X) = 0$, if there exists an invariant hypersurface then its reduced equation will divide $\mathrm{Dep}(X,\dots,X^{p^{n-1}})$. In fact if F is an invariant algebraic hypersurface then F divides $X(F)$, and consequently, F also divides $X^k(F)$, for any positive integer k. This is sufficient to guarantee that F cut out the dependency locus of $X, \dots, X^{p-1}$. ∎

Example 2 In general, when $div(X) = 0$, we can't guarantee the existence of an invariant hypersurface. For example, if $X = y^3 \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}$ and we are in characteristic two, then $X^2 = xy^2 \frac{\partial}{\partial x} + y^3 \frac{\partial}{\partial y}$ and $\mathrm{Dep}(X, X^2) = (y^3 + xy)^2$. Therefore the only possible invariant curves are y and $y^2 + x$, which are not invariant as one can promptly verify. Hence X does not admit any invariant curve.

Corollary 1 *Let X be a vector field on $\mathbb{A}^2_k$, where k is a field of characteristic $p > 0$. If the degree of X is less than $p - 1$ then X admits an invariant curve.*

Proof: By Theorem 1 we can suppose that $div(X) = 0$. Then the 1-form $\omega = i_X dx_1 \wedge dx_2$ is closed, and its coefficients have degree smaller than $p - 1$. In this case, the closedness is sufficient to guarantee that $\omega = df$, for some $f \in R$. ∎

Example 3 Over the complex numbers, Jouanolou [3] showed that $X = (1 - xy^d)\frac{\partial}{\partial x} + (x^d - y^{d+1})\frac{\partial}{\partial y}$ does not have any algebraic invariant curve for $d \geq 2$. In characteristic two, for example, if d is odd then $x^{2d+1}y^{d-1} + x^d y^d + x^{d-1} + y^{2d+1}$ is invariant, and if d is even X has a first integral of the form $y^{d+1}x + x^{d+1} + y$. Observe that for $d = 2$ the first integral is Klein's quartic, a curve of genus 3 that has 168 automorphisms. It is interesting to observe that, in characteristic two Jouanolou's example has many more automorphisms than in characteristic zero, where it has 42.

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