

# ON THE DENSITY OF ALGEBRAIC FOLIATIONS WITHOUT ALGEBRAIC INVARIANT SETS

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ABSTRACT. Let  $X$  be a smooth complex projective variety of dimension greater than or equal to 2,  $\mathcal{L}$  an ample line bundle and  $k \gg 0$  an integer. We prove that a generic global section of the twisted tangent sheaf  $\Theta_X \otimes \mathcal{L}^{\otimes k}$  gives rise to a foliation of  $X$  without any proper algebraic invariant subvarieties of nonzero dimension. As a corollary we obtain a dynamical characterization of ampleness for line bundles over smooth projective surfaces.

## 1. INTRODUCTION

The study of holomorphic foliations over projective varieties can be traced back to the work of G. Darboux and H. Poincaré in the 19th century. In the papers [6] and [19] they studied differential equations over the complex projective plane and posed several questions concerning projective algebraic curves invariant under holomorphic foliations, many of which are still actively pursued.

In the late 1970s, J. P. Jouanolou reworked and extended the work of Darboux [6] in the framework of modern algebraic geometry. One of the key results of Jouanolou's celebrated monograph [10, théorème 1.1, p. 158] states that a very generic holomorphic foliation of the projective plane, of degree at least 2, does not have any invariant algebraic curves. Recall that a property  $P$  holds for a *very generic point* of a variety  $V$  if the set of points on which it fails is contained in a countable union of hypersurfaces of  $V$ . Jouanolou's theorem has been extended in various ways; see [11], [12], [13], [14], [16] and [20].

In this paper we prove a generalization of Jouanolou's result for one dimensional foliations over any smooth projective variety. Our result is related to a problem posed by V. I. Arnold in [2, §10, pp. 6-7]. It also leads to a simpler proof of [13, Corollary B] and of [14, Theorem 2, p. 533], from which we draw our general strategy. Throughout the paper  $X$  denotes a smooth complex projective variety of dimension  $d \geq 2$ .

**Theorem 1.1.** *Let  $\mathcal{L}$  be an ample line bundle. Then, for  $k \gg 0$ , let  $f$  be a very generic global section of the twisted tangent sheaf  $\Theta_X \otimes \mathcal{L}^{\otimes k}$ . The foliation of  $X$  determined by  $f$  has no proper invariant algebraic subvarieties of nonzero dimension.*

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It should be noted that a similar result does not hold for foliations of codimension 1 when the underlying variety has dimension greater than 2. This follows from the fact that the space of codimension one foliations of any degree over  $\mathbb{P}^n$  has a logarithmic component, in the sense of [5, Theorem 2, p. 580], whenever  $n \geq 3$ . The result also fails for varieties over fields of positive characteristic [18].

On the other hand, it is not difficult to extend Theorem 1.1 to fields of  $m$ -vectors, or Pfaff equations as they are sometimes called, as we show in section 7. As an application of Theorem 1.1, we prove in section 8, the following *dynamical* characterization of ampleness when  $X$  is a surface.

**Theorem 1.2.** *A line bundle  $\mathcal{L}$  on a smooth projective surface  $X$  is ample if, and only if,  $\mathcal{L}^2 > 0$  and there exists a positive integer  $k$  such that the generic section of  $\Theta_X \otimes \mathcal{L}^{\otimes k}$  induces a foliation of  $X$  without invariant algebraic curves.*

We finish the introduction with a sketch of the proof of Theorem 1.1. Given an ample line bundle  $\mathcal{L}$  of  $X$  and an integer  $k \gg 0$  (which depends on  $\mathcal{L}$ ), let  $\Sigma = \mathbb{P}(\mathrm{H}^0(X, \Theta_X \otimes \mathcal{L}^{\otimes k}))$ . Define two subsets of  $\Sigma \times X$ , by  $\mathcal{Y} = \{([f], x) : [f] \text{ is singular at } x\}$ , and

$$\mathcal{S}_\chi = \{([f], x) : x \text{ is in a subscheme, invariant under } f, \text{ of Hilbert polynomial } \chi\}.$$

Let  $p_1 : \Sigma \times X \rightarrow \Sigma$  be the projection on the first coordinate. Since  $\mathcal{S}_\chi$  is a closed set by Proposition 2.1, it follows that  $p_1(\mathcal{S}_\chi)$  is a closed subset of  $\Sigma$ . Suppose, by contradiction, that  $p_1(\mathcal{S}_\chi) = \Sigma$ . Thus, every  $[f] \in \Sigma$  admits an invariant algebraic subvariety with Hilbert polynomial  $\chi$ . However, by Proposition 5.3, every variety invariant under  $f$  must contain a singularity of  $f$ . Therefore,

$$p_1(\mathcal{S}_\chi \cap \mathcal{Y}) = \Sigma, \quad \text{and} \quad \dim(\mathcal{S}_\chi \cap \mathcal{Y}) \geq \dim(\Sigma).$$

On the other hand, by Proposition 2.4,  $\mathcal{Y}$  is a closed irreducible subset of  $\Sigma \times X$  whose dimension is equal to  $\dim(\Sigma)$ ; so that  $\mathcal{Y} = \mathcal{S}_\chi \cap \mathcal{Y}$ . In particular, we have shown that, given  $[f] \in \Sigma$  and a singularity  $x$  of  $[f]$ , there exists a subvariety with Hilbert polynomial  $\chi$  that is invariant under  $[f]$  and contains  $x$ , which is in direct contradiction with Proposition 4.1. Therefore,  $p_1(\mathcal{S}_\chi)$  is a proper closed subset of  $\Sigma$ . Since there are only countably many Hilbert polynomials, the proof of Theorem 1.1 is complete.

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## 2. PRELIMINARIES

**2.1. Fields of  $m$ -vectors.** Throughout the paper  $X$  denotes a smooth complex projective variety of dimension  $d \geq 2$ . Let  $\Theta_X$  be the tangent sheaf of  $X$  and let  $\mathcal{L}$  be a line bundle over  $X$ . A *field of  $m$ -vectors* of  $X$  is an  $\mathcal{O}_X$ -homomorphism  $f : \Omega_X^m \rightarrow \mathcal{L}$ .

A field of 1-vectors  $f$  determines a *singular foliation of dimension one*  $\mathcal{F}$  of  $X$ . The same foliation is also completely defined by the kernel of  $f$ . In fact there exists a coherent sheaf  $N\mathcal{F}^*$  and an injective morphism  $\alpha : N\mathcal{F}^* \rightarrow \Omega_X^1$  such that the kernel of  $f$  is generated, as a sheaf, by the image of  $\alpha$ . Throughout the paper, such an  $\mathcal{F}$  is simply called a *foliation* of  $X$ . The bundle  $\mathcal{L}$  is sometimes called the *cotangent bundle* of the foliation  $\mathcal{F}$  and  $N\mathcal{F}^*$  is its *conormal sheaf*. The dual of  $N\mathcal{F}^*$  is the *normal sheaf* of  $\mathcal{F}$  and will be denoted by  $N\mathcal{F}$ .

Still keeping to the case when  $f$  is a foliation we have an exact sequence

$$0 \rightarrow N\mathcal{F}^* \rightarrow \Omega_X^1 \rightarrow \mathcal{L}.$$

Moreover, if  $f$  is surjective outside a set of codimension two, then we have the following *adjunction formula*

$$(2.1) \quad K_X \cong \det(N\mathcal{F}^*) \otimes \mathcal{L}.$$

The field of  $m$ -vectors  $f : \Omega_X^m \rightarrow \mathcal{L}$  can also be defined by

- (1) a global section of  $\bigwedge^m \Theta_X \otimes \mathcal{L}$ ; or
- (2) the  $\mathcal{O}_X$ -homomorphism  $f^\vee : \mathcal{L}^\vee \rightarrow \bigwedge^m \Theta_X$ ;

where  $\mathcal{L}^\vee = \mathcal{H}om(\mathcal{L}, \mathcal{O}_X)$ . We swap between these definitions, whenever needed, without further comment. Moreover, we do not always distinguish between a foliation  $\mathcal{F}$  and the map or section  $f$  that is used to define it.

A *singularity* of the field of  $m$ -vectors  $f$  is a point  $x \in X$  such that  $f$  is not surjective at  $x$ . The set of all singularities of  $f$  is denoted by  $\text{Sing}(f)$ .

**2.2. Invariant Subschemes.** A subscheme  $Y$  of  $X$  is *invariant* under  $f$  if there exists a map  $\Omega_Y^m \rightarrow \mathcal{L}|_Y$  such that the diagram

$$\begin{array}{ccc} \Omega_X^m|_Y & \xrightarrow{f|_Y} & \mathcal{L}|_Y \\ \downarrow & \nearrow \text{dotted} & \\ \Omega_Y^m & & \end{array}$$

is commutative. In particular, if  $Y$  is an irreducible subvariety and  $\dim(Y) < m$  then  $\Omega_Y^m = 0$  over the generic point and, consequently,  $Y \subseteq \text{Sing}(f)$  since  $\text{Sing}(f)$  is closed.

Let  $\mathcal{L}$  be a line bundle,  $m$  be a positive integer and  $\Sigma$  denote the projective space  $\mathbb{P}(\mathcal{H}^0(X, \bigwedge^m \Theta_X \otimes \mathcal{L}))$ . For  $\chi \in \mathbb{Q}[t]$  define  $\mathcal{S}$ , a subset of  $\Sigma \times X$ , by

$$\mathcal{S} = \{([f], x) : x \text{ is in a subscheme, invariant under } f, \text{ of Hilbert polynomial } \chi\}.$$

We will write  $\mathcal{S}_\chi$  if we need to call attention to the corresponding Hilbert polynomial. The main result of this subsection is

**Proposition 2.1.**  $\mathcal{S}_\chi$  is a closed subset of  $\Sigma \times X$ .

Before proving Proposition 2.1 we need to settle some notation and establish some technical lemmata.

Denote by  $\mathbb{T}$  the trivial bundle with fibre  $H^0(X, \wedge^m \Theta_X \otimes \mathcal{L})$ . There exists a map of vector bundles  $u : \mathbb{T} \rightarrow \wedge^m T_X \otimes \mathcal{L}$  which takes  $(x, \theta) \in \mathbb{T}$  to the  $m$ -vector  $\theta(x) \in \wedge^m T_x X \otimes \mathcal{L}$ .

If  $\pi : \mathbb{P}\mathbb{T} \rightarrow X$  is the standard projection, there exists a diagram

$$(2.2) \quad \begin{array}{ccc} \pi^*(\mathbb{T}) & \xrightarrow{\pi^*(u)} & \pi^*(\wedge^m T_X \otimes \mathcal{L}) \\ j \uparrow & \nearrow v & \\ \mathcal{O}_{\mathbb{T}}(-1) & & \end{array}$$

Now  $v$  gives rise to a map

$$(2.3) \quad \Omega_{\Sigma \times X / \Sigma}^m \rightarrow \mathcal{O}_{\mathbb{T}}(1) \otimes \pi^*(\mathcal{L}),$$

which plays the role of a *universal field of  $m$ -vectors* over  $X$ . Note that  $\mathbb{P}(\mathbb{T}) = X \times \Sigma$ . Let  $S$  be a scheme and consider the diagram

$$\begin{array}{ccc} \mathcal{X} = \Sigma \times X \times S & \xrightarrow{q_1} & \Sigma \times S \\ \downarrow q_3 & & \downarrow \\ \Sigma \times X & \longrightarrow & \Sigma \end{array}$$

where  $q_1$  and  $q_3$  are the canonical projections. Then it follows from (2.3) by base change that

$$g : \Omega_{\mathcal{X}/\Sigma \times S}^m \rightarrow q_3^*(\mathcal{O}_{\mathbb{T}}(1) \otimes \pi^*(\mathcal{L})).$$

On the other hand, let  $V \subset X \times S$  be a flat family over  $S$ . The pull-back  $\tilde{V} \subset \mathcal{X}$  of  $V$  under the canonical projection  $q_2 : \mathcal{X} \rightarrow X \times S$  is a flat family over  $T = \Sigma \times S$ . Moreover, for a given  $t = ([f], s) \in \Sigma \times S$  the scheme  $V_s$  is invariant under the field of  $m$ -vectors  $f : \Omega_X^1 \rightarrow \mathcal{L}$  if and only if the map  $\theta$  defined by

$$(2.4) \quad \begin{array}{ccc} 0 & & \\ \downarrow & & \\ \mathcal{K} & \searrow \theta & \\ \downarrow & & \\ (\Omega_{\mathcal{X}/T}^m)_{|\tilde{V}} & \xrightarrow{g|_{\tilde{V}}} & q_3^*(\mathcal{O}_{\mathbb{T}}(1) \otimes \pi^*(\mathcal{L}))_{|\tilde{V}} \\ \downarrow & & \\ \Omega_{\tilde{V}/T}^m & & \\ \downarrow & & \\ 0 & & \end{array}$$

is zero at  $t$ . We want to show that the set

$$\mathcal{Z}_V = \{([f], s) \in T : V_s \text{ is invariant under } f\} = \{t \in T : \theta_t = 0\}$$

is closed in  $T = \Sigma \times S$ . But first we need a technical lemma.

**Lemma 2.2.** *Let  $p : \mathcal{X} \rightarrow T$  be a projective morphism. Assume that  $\mathcal{F}$  is a  $p$ -flat coherent  $\mathcal{O}_{\mathcal{X}}$ -module such that  $R^1 p_* \mathcal{F} = 0$ . If  $\mathcal{G}$  is a quasi-coherent  $\mathcal{O}_T$ -module and  $\sigma : p^* \mathcal{G} \rightarrow \mathcal{F}$  is a homomorphism of  $\mathcal{O}_{\mathcal{X}}$ -modules, then the set  $\{t \in T : \sigma_t = 0\}$  is closed in  $T$ .*

*Proof.* The result follows immediately from [1, Proposition 2.3, p. 16] if we prove that  $p_* \mathcal{F}$  is locally free, and that its formation commutes with base change.

However,  $p$  is projective and  $\mathcal{F}$  is  $\mathcal{O}_{\mathcal{X}}$ -coherent, so that  $p_*(\mathcal{F})$  is  $\mathcal{O}_T$ -coherent by [9, Theorem 8.8, p. 252]. Since  $\mathcal{F}$  is  $p$ -flat and  $R^1 p_* \mathcal{F} = 0$  it follows from [9, Theorem 12.11, p. 290] that  $p_* \mathcal{F}$  is locally free and that its formation commutes with base change.  $\square$

**Lemma 2.3.** *Let  $S$  be a scheme and let  $V \subset X \times S$  be a flat family over  $S$ . The set  $\mathcal{Z}_V$  is closed in  $T = \Sigma \times S$ .*

*Proof.* Let  $\mathcal{M}$  be a very ample sheaf over  $\tilde{V}$ , the pullback of  $V$  under  $q_2 : \mathcal{X} \rightarrow T$ . Given an integer  $r \gg 0$  it follows by Serre's Theorem that, for some positive integer  $N$ , there exists a surjective map  $\alpha : \mathcal{O}_{\tilde{V}}^{\oplus N} \rightarrow \mathcal{K} \otimes \mathcal{M}^{\otimes r}$ . Denote by  $\sigma$  the composition

$$\begin{array}{c} \mathcal{O}_{\tilde{V}}^{\oplus N} \xrightarrow{\alpha} \mathcal{K} \otimes \mathcal{M}^{\otimes r} \longrightarrow (\Omega_{\mathcal{X}/T}^m)_{|\tilde{V}} \otimes \mathcal{M}^{\otimes r} \longrightarrow q_3^*(\mathcal{O}_T(1) \otimes \mathcal{L})_{|\tilde{V}} \otimes \mathcal{M}^{\otimes r}. \\ \sigma \qquad \qquad \qquad \theta \end{array}$$

Since  $\alpha$  is surjective,  $\theta_t = 0$ , for some  $t \in T$  if and only if  $\sigma_t = 0$ . But this implies that  $\mathcal{Z}_V = \{t \in T : \sigma_t = 0\}$ . We must show that this set is closed in  $T$ . In order to do this we apply Lemma 2.2 with

$$\mathcal{F} = q_3^*(\mathcal{O}_T(1) \otimes \mathcal{L})_{|\tilde{V}} \otimes \mathcal{M}^{\otimes r} \quad \text{and} \quad \mathcal{G} = \mathcal{O}_T^{\oplus N}.$$

Denoting by  $p$  the composition of the embedding of  $\tilde{V}$  in  $\mathcal{X}$  with the projection  $q_1$ , we have that  $p^* \mathcal{G} = \mathcal{O}_{\tilde{V}}^{\oplus N}$ , and  $\sigma$  is a map  $p^* \mathcal{G} \rightarrow \mathcal{F}$ . Moreover,  $R^i p_* \mathcal{F} = 0$  for  $i > 0$  by [9, Theorem III.8.8, p. 252]. The result follows from Lemma 2.2.  $\square$

We are now ready to establish Proposition 2.1.

*Proof of Proposition 2.1.* Denote by  $\mathbf{Hilb}_{\chi}(X)$  the Hilbert scheme of  $X$  with respect to the Hilbert polynomial  $\chi$  and recall that  $\mathbf{Hilb}_{\chi}(X)$  is a projective scheme. Let

$$\begin{array}{ccc} \Sigma \times X \times \mathbf{Hilb}_{\chi}(X) & \xrightarrow{q_1} & \Sigma \times \mathbf{Hilb}_{\chi}(X) \\ \downarrow q_2 & \searrow q_3 & \\ X \times \mathbf{Hilb}_{\chi}(X) & & \Sigma \times X \end{array}$$

be the canonical projections, and let  $\mathcal{C}$  be the universal family in  $X \times \mathbf{Hilb}_{\chi}(X)$ . Denote by  $\mathcal{C}_s$  the closed subscheme of  $X$  that corresponds to  $s \in \mathbf{Hilb}_{\chi}(X)$ . Then, it follows from Lemma 2.3 that

$$\mathcal{Z}_{\mathcal{C}} = \{([f], s) : \mathcal{C}_s \text{ is invariant under } f\}$$

is a closed subset of  $\Sigma \times \mathbf{Hilb}_{\chi}(X)$ . Thus,

$$\mathcal{S} = q_3(q_1^{-1}(\mathcal{Z}_{\mathcal{C}}) \cap q_2^{-1}(\mathcal{C}))$$

is closed in  $\Sigma \times X$ , as we wished to prove.  $\square$

**2.3. Irreducibility of the Universal Singular Set.** Denote by  $[f]$  the class of  $f \in H^0(X, \bigwedge^m \Theta_X \otimes \mathcal{L})$  in  $\Sigma = \mathbb{P}H^0(X, \bigwedge^m \Theta_X \otimes \mathcal{L})$ , and define a subset of  $\Sigma \times X$  by

$$\mathcal{Y} = \{([f], x) : [f] \in \Sigma \text{ and } x \in \text{Sing}(f)\}.$$

**Proposition 2.4.** *If  $\bigwedge^m \Theta_X \otimes \mathcal{L}$  is generated by global sections and*

$$h^0(X, \bigwedge^m \Theta_X \otimes \mathcal{L}) > \text{rk}(\bigwedge^m \Theta_X)$$

*then  $\mathcal{Y}$  is an irreducible subvariety of  $\Sigma \times X$  of dimension*

$$d + \dim(\Sigma) - \binom{d}{m}.$$

*Proof.* It is clear that  $\mathcal{Y}$  is a closed set, we must show that it is irreducible.

As in subsection 2.2, denote by  $\mathbb{T}$  the trivial bundle with fiber  $H^0(X, \bigwedge^m \Theta_X \otimes \mathcal{L})$  and by  $\pi : \mathbb{P}\mathbb{T} \rightarrow X$  the standard projection. Since  $\bigwedge^m \Theta_X \otimes \mathcal{L}$  is generated by global sections and  $h^0(X, \bigwedge^m \Theta_X \otimes \mathcal{L}) > \text{rk}(\bigwedge^m \Theta_X)$ , there exists a surjective map of vector bundles with nontrivial kernel  $u : \mathbb{T} \rightarrow \bigwedge^m \Theta_X \otimes \mathcal{L}$  which takes  $(x, \theta) \in \mathbb{T}$  to the vector  $\theta(x) \in T_x X$ . Moreover, since  $u$  is surjective,  $\ker(u)$  is also a vector bundle, and we have an exact sequence

$$\begin{array}{ccccc} \pi^*(\ker u) & \longrightarrow & \pi^*(\mathbb{T}) & \xrightarrow{\pi^*(u)} & \pi^*(\bigwedge^m \Theta_X \otimes \mathcal{L}) \\ & & \uparrow j & & \\ & & \mathcal{O}_{\mathbb{T}}(-1) & & \end{array}$$

Now  $([f], x) \in \mathcal{Y}$  if and only if  $\pi^*(u)j$  is zero when restricted to the fibre over  $([f], x)$ . Note that the zero scheme of  $\pi^*(u)j$ , which is  $\mathcal{Y}$ , is isomorphic to  $\mathbb{P}(\ker(u))$ . But  $\mathbb{P}(\ker(u))$  is irreducible of dimension

$$\dim(X) + h^0(X, \bigwedge^m \Theta_X \otimes \mathcal{L}) - \text{rk}(\bigwedge^m \Theta_X) - 1 = d + \dim(\Sigma) - \binom{d}{m}$$

so the same holds for  $\mathcal{Y}$ . □

The above result for  $X = \mathbb{P}^n$  and  $m = 1$  appeared in [8, Lemma 1.1].

**Corollary 2.5.** *Let  $2 \leq m \leq d - 2$  be an integer and let  $f$  be a generic section of  $\bigwedge^m \Theta_X \otimes \mathcal{L}$ . If  $\bigwedge^m \Theta_X \otimes \mathcal{L}$  is generated by global sections then:*

- (1)  $\text{Sing}(f) = \emptyset$ ;
- (2) *any subscheme of  $Y$  invariant under  $m$  must have dimension at least  $m$ .*

*Proof.* Let  $p : \Sigma \times X \rightarrow \Sigma$  be the projection on the first component of the product. Since  $2 \leq m \leq d - 2$ , it follows from Proposition 2.4 that  $\dim(\mathcal{Y}) < \dim(\Sigma)$ . Hence,  $p(\mathcal{Y}) \subsetneq \Sigma$ . But, by the definition of  $\mathcal{Y}$  we have that every  $f \in \Sigma \setminus p(\mathcal{Y}) \neq \emptyset$  has an empty singular set. Now (2) follows from (1) and the fact that every closed subscheme of  $X$ , invariant under a field of  $m$ -vectors and of dimension smaller than  $m$  must be contained in  $\text{Sing}(f)$ . □

## 3. LOCAL ANALYSIS ON THE SINGULAR SET

Let  $\mathcal{F}$  be a foliation of  $X$  defined by the  $\mathcal{O}_X$ -homomorphism  $f : \Omega_X^1 \rightarrow \mathcal{L}$ , where  $\mathcal{L}$  is a line bundle over  $X$ . Denote by  $(X, x)$  the germ of  $X$  at  $x$ . If  $x$  is a singularity of  $\mathcal{F}$  then, taking a local system of coordinates  $\alpha : (\mathbb{C}^d, 0) \rightarrow (X, x)$ , which maps 0 to  $x$ , we have the following commutative diagram:

$$\begin{array}{ccc} \Omega_{X,x}^1 & \xrightarrow{f} & \mathcal{L}_x \\ \downarrow \alpha^* & & \downarrow \alpha^* \\ \Omega_{\mathbb{C}^d,0}^1 & \longrightarrow & \mathcal{O}_{\mathbb{C}^d,0} \end{array}$$

In this way we can identify the foliation  $\mathcal{F}$  on a neighbourhood of  $x$  with a germ of section of  $\text{Hom}(\Omega_{\mathbb{C}^d}^1, \mathcal{O}_{\mathbb{C}^d})$ ; that is, with a germ of holomorphic vector field  $Z$  defined on a neighbourhood of 0 in  $\mathbb{C}^d$ .

The *algebraic multiplicity* of  $Z$  at 0, or equivalently of  $\mathcal{F}$  at  $x$ , denoted by  $m = m(Z, 0) = m(\mathcal{F}, x)$ , is the total degree of the first nonzero jet of  $Z$ . In other words,

$$(3.1) \quad Z = \sum_{i=m}^{+\infty} Z_i,$$

where  $Z_i$  is a homogeneous vector field of degree  $i$ , and  $Z_m \neq 0$ . We say that  $0 \in \mathbb{C}^d$  is a *dicritical* singularity of  $Z$  if  $m \geq 0$  and  $Z_m$  is a multiple of the radial vector field  $R = \sum_{i=1}^d x_i \partial / \partial x_i$ .

Twisting the Euler sequence [9, Example 8.20.1, p. 182] with  $\mathcal{O}_{\mathbb{P}^{d-1}}(m-1)$ , we obtain

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^{d-1}}(m-1) \rightarrow \mathcal{O}_{\mathbb{P}^{d-1}}(m)^{d+1} \rightarrow \Theta_{\mathbb{P}^{d-1}}(m-1) \rightarrow 0,$$

Thus, if  $Z_m$  is not a multiple of  $R$ , then it induces a holomorphic foliation  $\mathcal{F}_x$  of  $\mathbb{P}^{d-1}$  with cotangent bundle given by  $\mathcal{O}_{\mathbb{P}^{d-1}}(m-1)$ . Let  $\phi$  be the blowup of  $\mathbb{C}^d$  at 0. A simple computation shows that  $\mathcal{F}_x$  coincides with the restriction to  $\phi^{-1}(0) \cong \mathbb{P}^{d-1}$  of  $\phi^*(Z)$ , the pullback of the (local) foliation induced by  $Z$  under the blowup at 0. Note that the singular set of  $\phi^*(Z)$  may have codimension 2 on the underlying variety, while its restriction to the exceptional divisor has codimension one.

**Lemma 3.1.** *Let  $\mathcal{F}$  be a foliation of  $X$  and let  $x \in X$  be a nondicritical singularity of  $\mathcal{F}$ . Denote by  $\phi$  the blowup of  $X$  at  $x$ . If  $(W, x)$  is an irreducible germ of subvariety invariant under  $\mathcal{F}$ , then the restriction of  $\widetilde{W}$ , the strict transform of  $W$  under  $\phi$ , to the exceptional divisor  $\phi^{-1}(x)$  is invariant under  $\mathcal{F}_x$ .*

*Proof.* By hypothesis  $x$  is a nondicritical singularity and therefore the exceptional divisor  $E = \phi^{-1}(x)$  is invariant under  $\mathcal{F}_x$ . Since  $\widetilde{W}$  is invariant under  $\mathcal{F}_x$ , so is its intersection with  $E$ , and the lemma follows.  $\square$

Let  $\Xi_r$  be the coherent subsheaf of  $\Theta_X$  generated by the sections of  $\Theta_X$  whose germs at  $x$  have algebraic multiplicity at least  $r$ . Denote by  $\mathfrak{m}_x$  the sheaf of sections of  $\mathcal{O}_X$  vanishing at  $x$ .

**Lemma 3.2.** *Let  $x \in X$  and let  $\mathcal{G}$  be a holomorphic foliation of  $\mathbb{P}^{d-1}$  of degree  $r$ , i.e.,  $\mathcal{G}$  is induced by a global section of  $\Theta_{\mathbb{P}^{d-1}}(r-1)$ . If  $H^1(X, \Xi_r \otimes \mathfrak{m}_x \otimes \mathcal{L}) = 0$  then there exists a section of  $\Theta_X \otimes \mathcal{L}$  which induces a foliation  $\mathcal{F}$  of  $X$  such that:*

- (1)  $\mathcal{F}$  has a singularity at  $x$ , and
- (2)  $\mathcal{F}_x = \mathcal{G}$ .

*Proof.* Since the germ of a section of  $\Xi_r$  at  $x$  is given by equation (3.1) with  $m = r$ , we have an exact sequence

$$0 \rightarrow \Xi_r \otimes \mathfrak{m}_x \rightarrow \Xi_r \rightarrow H^0(\mathbb{P}^{d-1}, \mathcal{O}_{\mathbb{P}^{d-1}}(r))^d \rightarrow 0,$$

where the rightmost sheaf is the skyscraper sheaf supported at  $x$  whose stalk is the  $\mathbb{C}$ -vector space  $H^0(\mathbb{P}^{d-1}, \mathcal{O}_{\mathbb{P}^{d-1}}(r))^d$ .

Since  $H^1(X, \Xi_r \otimes \mathfrak{m}_x \otimes \mathcal{L}) = 0$ , if  $Z \in H^0(\mathbb{P}^{d-1}, \mathcal{O}_{\mathbb{P}^{d-1}}(r))^d$  is a homogeneous vector field on  $\mathbb{C}^d$  inducing  $\mathcal{G}$  then there exists  $f \in H^0(X, \Xi_r \otimes \mathcal{L}) \subset H^0(X, \Theta_X \otimes \mathcal{L})$  inducing a foliation  $\mathcal{F}$  of  $X$  with the required properties.  $\square$

Before stating the next lemma we need to recall some terminology from the local theory of holomorphic foliations. Assume that the germ of vector field  $Z$  has an isolated singularity at the origin. Let  $\mu_1, \dots, \mu_d$  be the eigenvalues of the 1-jet of  $Z$  at 0. Then,  $Z$  is in the *Poincaré domain* if the convex hull of  $\mu_1, \dots, \mu_d$  in  $\mathbb{C}$  does not contain the origin. The singularity 0 of  $Z$  is *hyperbolic* if

$$\mu_i/\mu_j \in \mathbb{C} \setminus \mathbb{R}$$

for every  $1 \leq i < j \leq d$ ; and it is *resonant* if there exist non-negative integers  $m_1, \dots, m_d$  such that

$$\sum_{i=1}^d m_i \mu_i = \mu_j \quad \text{and} \quad \sum_{i=1}^d m_i \geq 2,$$

for some  $1 \leq j \leq d$ . A germ of a vector field will be called *adequate* if it belongs to the Poincaré domain, and if its singularity at 0 is nonresonant and hyperbolic. Let  $M_d(\mathbb{C})$  be the space of  $d \times d$  matrices with coefficients in  $\mathbb{C}$ , and let  $A \in M_d(\mathbb{C})$ . If the linear vector field

$$Z = Z_A = (x_1, \dots, x_d) \cdot A \cdot \left( \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d} \right)^T,$$

is adequate, then we say that the matrix  $A$  is adequate. As the next lemma shows, if we choose a matrix  $B$  that is sufficiently general, then  $I + \epsilon B$  is adequate for small choices of  $\epsilon$ .

**Lemma 3.3.** *For most diagonal matrices  $B \in M_d(\mathbb{C})$  there exists  $\kappa > 0$  such that the matrix  $I + \epsilon B$  is adequate for all  $\epsilon \in (0, \kappa) \subset \mathbb{R}$ .*

*Proof.* Let  $\beta_1, \dots, \beta_d$  be the eigenvalues of  $B$ . Most diagonal matrices  $B \in M_d(\mathbb{C})$  satisfy

- (1)  $1, \beta_1, \dots, \beta_d$  are linearly independent over  $\mathbb{Q}$ ,
- (2)  $B$  is hyperbolic; that is  $\beta_i/\beta_j \in \mathbb{C} \setminus \mathbb{R}$ , for  $1 \leq i < j \leq d$ .

We claim that for every  $B$  satisfying the assumptions (1) and (2) there exists a positive real number  $\kappa$  such that  $I + \epsilon B$  is adequate for every  $\epsilon \in (0, \kappa)$ .

By assumption (1), the matrix  $sI + tB$  is nonresonant for every  $[s : t] \in \mathbb{P}_{\mathbb{R}}^1$ . Since  $\mathbb{R}$  is a closed subset of  $\mathbb{C}$  in the real Zariski topology, it follows from (2) that

$$V = \{[s : t] \in \mathbb{P}_{\mathbb{R}}^1 : sI + tB \text{ is not hyperbolic}\}$$



is a proper closed subset of  $\mathbb{P}_{\mathbb{R}}^1$  in the real Zariski topology. Therefore,  $V$  is finite. Moreover,  $[1 : 0] \in V$ . Hence, there exists  $\kappa_1 > 0$  such that  $I + \epsilon B$  is nonresonant and hyperbolic for every  $\epsilon \in (0, \kappa_1)$ .

On the other hand,

$$U = \{A \in M_d(\mathbb{C}) : A \text{ is in the Poincaré domain}\},$$

is an open subset of  $M_d(\mathbb{C})$ . Since the identity matrix belongs to  $U$ , there exists  $\kappa_2 > 0$  such that  $I + \epsilon B$  is in the Poincaré domain for every  $\epsilon \in (0, \kappa_2)$ . Now take  $\kappa = \min\{\kappa_1, \kappa_2\}$ .  $\square$

A germ of irreducible analytic curve  $\Gamma$  at 0 is called a *separatrix* of a vector field germ  $Z$  if  $\Gamma$  is invariant under  $Z$ . In particular,  $\Gamma \setminus \{0\}$  is a leaf of the local foliation induced by  $Z$ . The following lemma is well-known to the experts, we include it here only for the sake of completeness.

**Lemma 3.4.** *Let  $Z$  be a germ of vector field in a neighbourhood of  $0 \in \mathbb{C}^d$  with an isolated singularity at 0. If  $Z$  is adequate then it has exactly  $d$  separatrices at 0. Moreover, the topological closure of a leaf sufficiently close to 0 contains at least one of the separatrices.*

*Proof.* Since  $Z$  is adequate, it follows, by the Poincaré-Dulac Theorem [3, p. 183], that there exists an analytic change of coordinates which carries  $Z$  to its linear part. The remainder of the argument is a straightforward generalization to higher dimensions of the two dimensional case dealt with in [22, Corollary 3.18].  $\square$

#### 4. EXISTENCE OF SINGULARITIES WITHOUT ALGEBRAIC SEPARATRICES

Let  $x$  and  $y$  be two distinct points of  $X$ . Denote by  $\Theta_{X,x,y}$  the coherent subsheaf of  $\Theta_X$  generated by the sections vanishing at  $x$  and  $y$  and by  $\mathfrak{m}_x$  the sheaf of ideals vanishing at  $x$ . Identifying  $\text{Hom}(T_x X, T_x X)$  with the skyscraper sheaf which has this vector space as its stalk at  $x$ , we have an exact sequence,

$$(4.1) \quad 0 \rightarrow \Theta_{X,x,y} \otimes \mathfrak{m}_x \otimes \mathfrak{m}_y \rightarrow \Theta_{X,x,y} \rightarrow \text{Hom}(T_x X, T_x X) \oplus \text{Hom}(T_y X, T_y X) \rightarrow 0.$$

Observe that the image of the last map in this sequence is obtained by taking the linear part of the local vector fields at the singular points  $x$  and  $y$ .

**Proposition 4.1.** *Let  $\chi \in \mathbb{Q}[t]$  and  $\mathcal{L}$  be a line bundle. Suppose that*

$$H^1(X, \Theta_{X,x,y} \otimes \mathfrak{m}_x \otimes \mathfrak{m}_y \otimes \mathcal{L}) = 0.$$

*Then, there exists  $f \in \Theta_X \otimes \mathcal{L}$  such that the corresponding foliation  $\mathcal{F}$  of  $X$  has a singularity at  $x$  which is not contained in any algebraic curve with Hilbert polynomial  $\chi$  that is invariant under  $f$ .*

*Proof.* We begin by fixing some notation. If  $x \in X$ , then  $\text{Hom}(T_x X, T_x X) \cong M_d(\mathbb{C})$ . Choose  $y \neq x$  in  $X$ , and  $A \in \text{Hom}(T_y X, T_y X)$  such that the corresponding matrix is adequate; see section 3 for the definition. Let

$$\lambda : M_d(\mathbb{C}) \rightarrow \text{Hom}(T_x X, T_x X) \oplus \text{Hom}(T_y X, T_y X)$$

be given by  $\lambda(B) = (B + \text{Id}, A)$  where  $B \in \text{Hom}(T_x X, T_x X)$ . By hypothesis  $H^1(X, \Theta_{x,y} \otimes \mathfrak{m}_x \otimes \mathfrak{m}_y \otimes \mathcal{L}) = 0$ . Thus, it follows from the long exact sequence derived from (4.1) that  $\lambda$  lifts to a morphism  $\Lambda : M_d(\mathbb{C}) \rightarrow H^0(X, \Theta_X \otimes \mathcal{L})$ .

Let  $\phi : \bar{X} \rightarrow X$  be the blowup of  $X$  at  $x$ , and extend it to a morphism

$$\Phi : \Sigma \times \bar{X} \rightarrow \Sigma \times X,$$

by  $\Phi(\theta, z) = (\theta, \phi(z))$ . Denote by  $\overline{\mathcal{S}}$  the Zariski closure of  $\Phi^{-1}(\mathcal{S}_\chi \setminus (\Sigma \times \{x\}))$  in  $\Sigma \times \overline{X}$ . On the other hand, given a matrix  $B \in M_d(\mathbb{C})$ , let

$$\mathcal{X}_\epsilon = \mathcal{S}_\chi \cap ([\Lambda(\epsilon B)] \times X).$$

Since  $\mathcal{S}_\chi$  is closed by Lemma 2.3 it follows that  $\mathcal{X}_\epsilon$  is also closed. Denote by  $\overline{\mathcal{X}_\epsilon}$  the Zariski closure of  $\Phi^{-1}(\mathcal{X}_\epsilon \setminus \{(\Lambda(\epsilon B), x)\})$ . We identify the set

$$D_\epsilon = \Phi^{-1}(\Lambda(\epsilon B) \times \{x\}) \cap \mathcal{X}_\epsilon,$$

with a subset of the exceptional divisor  $E = \phi^{-1}(x) \cong \mathbb{P}^{d-1}$ . Thus, every point of  $D_\epsilon$  corresponds to a separatrix of  $\Lambda(\epsilon B)$  passing through  $x$ , and contained in the support of subschemes with Hilbert polynomial  $\chi$  that are left invariant by  $\Lambda(\epsilon B)$ .

Let  $\Sigma_x \subset \Sigma$  be the subset of foliations with a singularity at  $x$ . Suppose now that, for every foliation  $[\theta] \in \Sigma_x = \mathbb{P}(\mathbb{H}^0(X, \Theta_{X,x} \otimes \mathcal{L}))$  there exists a one dimensional subscheme, with Hilbert polynomial  $\chi$ , that is invariant under  $\theta$  and contains  $x$ . We will aim at a contradiction.

We show first that  $D_0 = E$ . Note that the singularity of  $\Lambda(0)$  at  $x$  has the identity as its linear part. Thus, the origin is a nonresonant singularity of  $\Lambda(0)$  that belongs to the Poincaré domain. Hence, by the Poincaré-Dulac Theorem [3, p. 183],  $\Lambda(0)$  is locally conjugated to the foliation induced by the radial vector field. Therefore, for every line  $\ell$  through the origin of the tangent space of  $X$  at  $x$ , the foliation  $\Lambda(0)$  has a smooth separatrix through  $x$  whose tangent is  $\ell$ .

Suppose, by contradiction, that  $D_0$  is a proper closed subset of  $E$ . Fix a basis  $\beta$  of  $T_x X$  such that the  $d$  points of  $E$  determined by this basis are disjoint from  $D_0$ . Once we have fixed a basis we have a natural identification of  $\text{Hom}(T_x X, T_x X)$  with  $M_d(\mathbb{C})$ . By Lemma 3.3 there exists a diagonal matrix  $B \in \text{Hom}(T_x X, T_x X)$  and a positive real number  $\kappa$  such that  $\Lambda(\epsilon B)$  is adequate at  $x$  for all  $0 < \epsilon < \kappa$ .

Let  $\mathcal{F}_\epsilon$  be the foliation associated to  $\Lambda(\epsilon B)$ . Thus,  $\mathcal{F}_\epsilon$  has an isolated singularity at  $x$ . Moreover, this singularity is adequate for all  $\epsilon \in (0, \kappa)$ . Hence, by Lemma 3.4,  $\mathcal{F}_\epsilon$  has precisely  $d$  separatrices at  $x$  whose tangents at  $x$  correspond to the vectors of  $\beta$ . Thus, the points of  $D_\epsilon$  correspond to the one dimensional subschemes with Hilbert polynomial  $\chi$  that are invariant under  $\mathcal{F}_\epsilon$  and tangent to some vector of  $\beta$  at  $x$ . Note that, by hypothesis,  $D_\epsilon \neq \emptyset$ . Since  $\beta$  is finite, there exists  $p \in D_\epsilon$  and an infinite sequence  $\epsilon_k$  such that

$$\lim_{k \rightarrow \infty} \epsilon_k = 0 \quad \text{and} \quad (\Lambda(\epsilon_k B), p) \in \overline{\mathcal{S}}.$$

But  $\overline{\mathcal{S}}$  is a closed set, so that

$$\lim_{k \rightarrow \infty} (\Lambda(\epsilon_k B), p) = (\Lambda(0), p) \in \overline{\mathcal{S}}.$$

In particular, there exists a one dimensional subscheme with Hilbert polynomial  $\chi$  that is invariant under  $\mathcal{F}_\epsilon$  and tangent to some vector of  $\beta$  at  $x$ . This contradicts the choice of  $\beta$  and shows that  $D_0 = E$ .

But if  $D_0 = E$ , then  $\mathcal{X}_0$  is a Zariski closed set which contains a full analytic neighbourhood of  $[\Lambda(0)] \times \{x\}$ . Thus  $\mathcal{X}_0 = [\Lambda(0)] \times X$ . In particular, every leaf of  $\mathcal{F}_0$  is algebraic. However, it follows from the hypotheses on  $A$  and from Lemma 3.4 that the foliation  $\Lambda(0)$  has an isolated singularity at  $y$ , analytically conjugated to the foliation of  $\mathbb{C}^d$  determined by the linear vector field induced by  $A$ . Moreover, this foliation has exactly  $d$  separatrices, and every leaf of  $\Lambda(0)$  that is sufficiently close to  $y$  contains at least one of the separatrices on its closure. Therefore, the leaves that are sufficiently close to  $y$ , and that are not contained in one of the  $d$

separatrices, must be nonalgebraic. But this contradicts the fact that all leaves are algebraic, thus proving the proposition.  $\square$

## 5. SINGULARITIES MEET INVARIANT SUBSCHEMES

**Lemma 5.1.** *Let  $\mathcal{F}$  be a foliation of  $X$ . If  $W$  is an irreducible closed subvariety of  $X$  invariant under  $\mathcal{F}$  then the singular locus of  $W$  is invariant under  $\mathcal{F}$ . In particular, if the singularities of  $W$  are isolated then*

$$\text{Sing}(W) \subset \text{Sing}(\mathcal{F}).$$

*Proof.* We may assume, without loss of generality, that  $W$  is a subvariety of  $X$  whose singular locus is not contained in  $\text{Sing}(\mathcal{F})$ . Thus, in order to prove the global statement it is enough to show that  $\text{Sing}(W)$  is invariant under  $\mathcal{F}$  in the neighbourhood of every point which is not a singularity of  $\mathcal{F}$ .

Let  $p \in \text{Sing}(W) \setminus \text{Sing}(\mathcal{F})$ , and let  $U$  be a neighbourhood of  $p$  over which  $\mathcal{F}|_U$  is described by a nowhere vanishing holomorphic vector field  $Z$ . Let  $V \subset U$  be a neighbourhood of  $p$  where the local flow of  $Z$  is defined. In other words, there exists a holomorphic map  $\Phi : (\mathbb{C}, 0) \times V \rightarrow U$  such that for every  $t$ ,  $\Phi(t, \cdot)$  is biholomorphic onto its image and

$$(5.1) \quad \frac{d}{dt} \Phi(t, z) = Z(\Phi(t, z)) \quad \text{and} \quad \Phi(0, z) = z.$$

Since  $W$  is invariant under  $Z$  it follows that  $\Phi(t, W \cap V) \subset W \cap U$ , for every  $t \in (\mathbb{C}, 0)$ . Moreover,  $\Phi(t, \cdot)$  is a biholomorphism, so it must preserve the singular set of  $W$ . In other words,

$$\Phi(t, \text{Sing}(W) \cap V) \subset \text{Sing}(W) \cap U,$$

for every  $t \in (\mathbb{C}, 0)$ . This proves that  $\text{Sing}(W)$  is invariant under  $\mathcal{F}$ .

When  $p$  is an isolated singularity of  $W$  we have that  $\Phi(t, p) = p$ , for every  $t \in (\mathbb{C}, 0)$ . Together with (5.1) this implies that  $Z(p) = 0$ , which shows that every isolated singularity of  $W$  must be a singularity of  $\mathcal{F}$ .  $\square$

**Lemma 5.2.** *Let  $\mathcal{G}$  be a foliation of  $X$  such that the determinant of its normal sheaf is ample. If an irreducible smooth closed algebraic subvariety  $V$  of  $X$  is invariant under  $\mathcal{G}$  then  $\text{Sing}(\mathcal{G}) \cap V \neq \emptyset$ .*

*Proof.* Suppose, by contradiction, that  $\mathcal{G}$  does not have singularities on  $V$ . Then, by [21, Theorem 6.4, p. 195], any polynomial of degree  $n = \dim(V)$  on the Chern classes of the normal sheaf  $\mathcal{N}_{\mathcal{G}}$  must vanish when restricted to  $V$ . In particular,

$$c_1(\mathcal{N}_{\mathcal{G}})^n \cdot V = \det(\mathcal{N}_{\mathcal{G}})^n \cdot V = 0,$$

which contradicts the ampleness of  $\det(\mathcal{N}_{\mathcal{G}})$ .  $\square$

**Proposition 5.3.** *Let  $\theta \in H^0(X, \Theta_X \otimes \mathcal{L})$  for some line bundle  $\mathcal{L}$  such that  $K_X^\vee \otimes \mathcal{L}$  is ample. If  $\text{Sing}(\theta)$  has codimension at least 2 and  $W$  is a closed subscheme of  $X$  invariant under  $\theta$  then  $W \cap \text{Sing}(\theta) \neq \emptyset$ .*

*Proof.* By the adjunction formula (2.1) we have that

$$\det(N_{\mathcal{F}}) = K_X^\vee \otimes \mathcal{L}.$$

Thus  $\det(N_{\mathcal{F}})$  is an ample line bundle. If  $W$  is contained in the singular set of  $\theta$ , then there is nothing to do. Otherwise,  $W$  is invariant under  $\mathcal{F}$ , the foliation of

$X$  induced by  $\theta$ . But this implies that each irreducible component of the reduced scheme  $W_{\text{red}}$  is also invariant under  $\mathcal{F}$ . Therefore, we may assume that  $W$  is an irreducible closed subvariety of  $X$ .

We proceed by induction on  $\dim(W)$ . If  $\dim(W) = 0$  then  $W$  is an invariant point, hence a singularity of  $\mathcal{F}$ . Suppose that the result holds for all invariant closed subvarieties of dimension less than  $r$ . If  $W$  is an invariant subvariety of dimension  $r$ , then either  $W$  is smooth, or it has a singularity set  $V$  of dimension smaller than  $r$ . In the first case, the result follows from Lemma 5.2. If  $W$  is singular, then its singularity set  $V$  is invariant under  $\mathcal{F}$  by Lemma 5.1. Thus,

$$\emptyset \neq V \cap \text{Sing}(\mathcal{F}) \subseteq W \cap \text{Sing}(\mathcal{F}),$$

by the induction hypothesis; and the proposition is proved.  $\square$

## 6. PROOF OF MAIN THEOREM

The following notation will hold throughout this section. Let  $\mathcal{L}$  be an ample line bundle and choose  $k > 0$  such that

- (1)  $\Theta_X \otimes \mathcal{L}^{\otimes k}$  is generated by its global sections;
- (2)  $K_X^\vee \otimes \mathcal{L}^{\otimes k}$  is ample;
- (3)  $H^1(X, \Theta_{X,x,y} \otimes \mathfrak{m}_x \otimes \mathfrak{m}_y \otimes \mathcal{L}^{\otimes k}) = 0$  for some distinct points  $x, y \in X$  (see section 4 for the definition of  $\Theta_{x,y}$ );
- (4)  $H^1(X, \Xi_2 \otimes \mathfrak{m}_z \otimes \mathcal{L}^{\otimes k}) = 0$  for some point  $z \in X$  (see section 3 for the definition of  $\Xi_r$ ).

Note that it follows from Serre's Vanishing Theorem that all the conditions above are automatically satisfied for  $k \gg 0$ . Note also that (1) together with (3) imply  $h^0(X, \Theta_X \otimes \mathcal{L}^{\otimes k}) > \text{rk}(\Theta_X)$ ; and that for  $(X, \mathcal{L}) = (\mathbb{P}^d, \mathcal{O}_{\mathbb{P}^d}(1))$  we can choose  $k = 1$ .

Write  $\Sigma = \mathbb{P}(H^0(X, \Theta_X \otimes \mathcal{L}^{\otimes k}))$ . The class of  $f \in H^0(X, \Theta_X \otimes \mathcal{L}^{\otimes k})$  in  $\Sigma$  will be denoted by  $[f]$ . Before we proceed, recall from sections 2.2 and 2.3 that for a given  $\chi \in \mathbb{Q}[t]$  we have two subsets of  $\Sigma \times X$ , namely

$$\mathfrak{S} = \{([f], x) : x \text{ is in a subscheme, invariant under } f, \text{ of Hilbert polynomial } \chi\}.$$

and

$$\mathfrak{Y} = \{([f], x) : [f] \in \Sigma \text{ and } x \in \text{Sing}(f)\}.$$

We will write  $\mathfrak{S}_\chi$  if we need to call attention to the corresponding Hilbert polynomial. Let  $p$  be the restriction to the subvariety  $\mathfrak{S}$  of the projection of  $\Sigma \times X$  on the first coordinate.

**Lemma 6.1.** *Let  $U \subset \Sigma$  be an open Zariski subset. Suppose that*

- (1)  $p^{-1}([f]) \cap \text{Sing}(f) \neq \emptyset$  for every  $[f] \in U$  for which  $p^{-1}([f]) \neq \emptyset$ , and that
- (2) there exists a foliation  $\mathcal{F}$  of  $X$  with cotangent bundle  $\mathcal{L}^{\otimes k}$  and a singularity  $x$  of  $\mathcal{F}$  which is not contained in any positive dimensional closed subscheme of  $X$  with Hilbert polynomial  $\chi$  invariant under  $\mathcal{F}$ .

*Then  $p(\mathfrak{S})$  is a proper closed subset of  $\Sigma$ .*

*Proof.* Since  $p$  is a proper map, it is enough to show that  $p(\mathfrak{S}) \neq \Sigma$ . We will assume, by contradiction, that  $p(\mathfrak{S}) = \Sigma$ . Thus, for every  $[f] \in \Sigma$  the subvariety  $p^{-1}([f])$  is invariant under  $f$ . Moreover, (1) implies that  $p(\mathfrak{S} \cap \mathfrak{Y}) \supseteq U$  and therefore

$$p(\mathfrak{S} \cap \mathfrak{Y}) = \Sigma.$$

Since  $\dim(\mathcal{Y}) = \dim(\Sigma)$ , it follows that  $\dim(\mathcal{S} \cap \mathcal{Y}) = \dim(\mathcal{Y})$ . However,  $\mathcal{Y}$  is irreducible by Proposition 2.4, therefore  $\mathcal{S} \cap \mathcal{Y} = \mathcal{Y}$ . This means that given  $f \in \Sigma$  and  $x \in \text{Sing}(f)$ , there exists a closed subscheme  $\mathcal{C}_x$  of  $X$ , with Hilbert polynomial  $\chi$  such that  $x \in \mathcal{C}_x$ , which contradicts (2).  $\square$

We are ready to prove Theorem 1.1

*Proof of Theorem 1.1* For every  $(d, r) \in \mathbb{N}^2$  with  $d \geq 2$  and  $1 \leq r \leq d - 1$ , consider the following statement:

$A(d, r)$ : if  $X$  is a  $d$ -dimensional smooth projective variety and  $\mathcal{L}$  is an ample line bundle then a very generic section of  $\Theta_X \otimes \mathcal{L}^{\otimes k}$  does not have any invariant closed  $r$ -dimensional subschemes.

Note, first of all, that since  $\Sigma$  is irreducible, and since there are only countably many Hilbert polynomials, then  $A(d, r)$  follows if we prove that

$$p(\mathcal{S}_\chi) \neq \Sigma = \mathbb{P}(\mathbb{H}^0(X, \Theta_X \otimes \mathcal{L}^{\otimes k})),$$

for all  $\chi \in \mathbb{Q}[t]$  of degree  $r$ . Thus we may assume, from now on, that  $\chi \in \mathbb{Q}[t]$  is such a polynomial.

We begin by proving that  $A(d, 1)$  holds for all  $d \geq 2$ . Suppose that  $\chi$  has degree one. It follows from Proposition 5.3 that

$$p^{-1}([f]) \cap \text{Sing}([f]) \neq \emptyset$$

for every  $[f] \in U \subset \Sigma$ , where  $U$  is the set of foliations whose singular sets have codimension at least 2. Thus, by Proposition 4.1 and Lemma 6.1,  $p(\mathcal{S}_\chi)$  is a proper closed subset of  $\Sigma$ . As we noted above this is enough to prove  $A(d, 1)$ .

In order to prove the theorem by induction it is enough to show that  $A(d, r)$  follows from  $A(d - 1, r - 1)$ , for every  $r \geq 2$ . But, just as in the case  $r = 1$  dealt with above,  $A(d, r)$  follows from Proposition 5.3 and Lemma 6.1 if we prove the following statement

there exists a foliation  $\mathcal{F}$  of  $X$ , with cotangent bundle  $\mathcal{L}^{\otimes k}$ , and a singularity  $x$  of  $\mathcal{F}$  which is not contained in any closed  $r$ -dimensional subscheme of  $X$  invariant under  $\mathcal{F}$ .

The case  $d = 2$  is covered by  $A(d, 1)$ . We show that for  $d \geq 3$ , the statement follows from  $A(d - 1, r - 1)$  applied to  $X = \mathbb{P}^{d-1}$ . If  $f$  is a very generic section of  $\Theta_{\mathbb{P}^{d-1}}(k)$  then  $A(d - 1, r - 1)$  implies that the only proper closed subschemes invariant under  $f$  are its singularities. By Lemma 3.2 there exists a foliation  $\mathcal{F}$  of  $X$ , singular at  $x$ , and such that, restricting the pullback of  $\mathcal{F}$  under the blowup at  $x$  to the exceptional divisor, we get the foliation induced by  $f$ .

If there exists a germ of  $r$ -dimensional subvariety at  $x$  invariant under  $\mathcal{F}$  then, by Lemma 3.1,  $f$  admits a proper invariant algebraic set of dimension  $r - 1$ . But this is impossible by the choice of  $f$ . Therefore  $x$  is a singularity of  $\mathcal{F}$  which is not contained in any invariant  $r$ -dimensional germ of subvariety. We have proved the statement above, and the proof of the theorem is complete.

REMARK. In the above proof, the integer  $k$  depends on the variety  $X$  and on the line bundle  $\mathcal{L}$ . However, the choice of  $k$  does not interfere with the inductive step since we use that  $A(d - 1, r - 1)$  holds only for  $\mathbb{P}^{d-1}$ . Moreover, the induction argument

also shows that, for the projective space  $\mathbb{P}^{d-1}$ , the result is true for  $\mathcal{L} = \mathcal{O}_{\mathbb{P}^{d-1}}(1)$  and for every  $k \geq 1$ .

## 7. DENSITY OF PFAFF EQUATIONS WITHOUT ALGEBRAIC SOLUTIONS

Let  $k \gg 0$  and  $m$  be positive integers. As in section 2,  $\mathcal{L}$  denotes an ample line bundle, and  $\Sigma$  the projective space  $\mathbb{P}(H^0(X, \bigwedge^m \Theta_X \otimes \mathcal{L}^{\otimes k}))$ . For  $i = 1, \dots, m$ , let  $k_i \gg 0$ , be positive integers which add up to  $k$ , and write

$$\Psi : \bigoplus_{i=1}^m H^0(X, \Theta_X \otimes \mathcal{L}^{\otimes k_i}) \rightarrow H^0(X, \bigwedge^m \Theta_X \otimes \mathcal{L}^{\otimes k})$$

for the natural map.

**Lemma 7.1.** *For  $i = 1, \dots, m$ , let  $k_i \gg 0$ , be positive integers which add up to  $k$ . If  $f$  is a generic element of  $\bigoplus_{i=1}^m H^0(X, \Theta_X \otimes \mathcal{L}^{\otimes k_i})$  then*

$$\dim \text{Sing}(\Psi(f)) = m - 1.$$

*Proof.* For  $m = 1$  the result follows from Proposition 2.4. Suppose that the result holds for  $m - 1$  and let  $g = (g_1, \dots, g_{m-1})$  be an element of  $\bigoplus_{i=1}^{m-1} H^0(X, \Theta_X \otimes \mathcal{L}^{\otimes k_i})$  such that

$$\dim \text{Sing}(\Psi(g)) = m - 2.$$

Denote by  $U$  the complement of  $\text{Sing}(\Psi(g))$  in  $X$ . Consider the trivial bundle  $\mathbb{T}_U$  over  $U$  with fibre  $H^0(U, \Theta_U \otimes \mathcal{L}|_U^{\otimes k_m})$ . Since the codimension of  $\text{Sing}(\Psi(g))$  is at least 2, it follows that  $H^0(X, \Theta_X \otimes \mathcal{L}^{\otimes k_m}) \cong H^0(U, \Theta_U \otimes \mathcal{L}|_U^{\otimes k_m})$ . Thus,  $\mathbb{T}_U$  is the restriction to  $U$  of the bundle  $\mathbb{T}$  defined in section 2. Once again we have a map of vector bundles  $u : \mathbb{T}_U \rightarrow \bigwedge^m T_U \otimes \mathcal{L}|_U^{\otimes k}$ , of constant rank, which takes  $(x, \theta) \in \mathbb{T}_U$  to the  $m$ -vector  $(g \wedge \theta)(x) \in \bigwedge^m T_x U$ . Hence,  $\ker(u)$  has dimension

$$\dim X + h^0(U, \Theta_U \otimes \mathcal{L}|_U^{\otimes k}) - \text{rank}(\text{Im}(u)).$$

But,  $\text{rank}(\text{Im}(u)) = \dim X - (m - 1)$ , so that  $\dim \ker(u) = h^0(U, \Theta_U \otimes \mathcal{L}|_U^{\otimes k}) + (m - 1)$ . Thus for a generic  $\theta$ ,

$$\dim \text{Sing}(\Psi(g \wedge \theta)) = m - 1.$$

□

We may now prove the main result of this section.

**Proposition 7.2.** *Let  $\mathcal{L}$  be an ample line bundle. Suppose that  $k \gg 0$  is an integer and that  $f$  is a very generic section of  $\bigwedge^m \Theta_X \otimes \mathcal{L}^{\otimes k}$ .*

- (1) *If  $1 \leq m \leq d - 1$ , then  $f$  has no proper invariant algebraic subvarieties of nonzero dimension.*
- (2) *If  $2 \leq m \leq d - 2$ , then  $f$  has no singular points.*

*Proof.* If  $m = 1$  then the theorem has already been proved, so we may assume that  $m > 1$ .

Since  $\mathbb{Q}[t]$  is a countable set, it is enough to prove that, for a given  $\chi \in \mathbb{Q}[t]$ , the generic field of  $m$ -vectors does not have any invariant subvariety of Hilbert polynomial  $\chi$ .

But the set of  $m$ -vectors which do not admit an invariant closed subvariety of Hilbert polynomial  $\chi$  is open in  $\Sigma$  by Proposition 2.1. Thus, the result follows if

we prove that this open set is nonempty. If  $m > \deg(\chi)$  this is a consequence of Corollary 2.5. So we may assume that  $m \leq \deg(\chi)$ .

For  $1 \leq i \leq m$  choose integers  $k_i \gg 0$  which add up to  $k$ . It follows from Theorem 1.1 that there exist sections  $g_i$  of  $\Theta_X \otimes \mathcal{L}^{\otimes k_i}$  which do not have any proper invariant closed algebraic subvarieties apart from their singularities. Write  $g = (g_1, \dots, g_m)$ . By Lemma 7.1 the singularity set of  $\Psi(g)$  has dimension  $m - 1 < d$ . But, if a proper closed subvariety  $Y$  of  $X$ , invariant under  $\Psi(g)$ , goes through a non-singular point of  $\Psi(g)$  then it must be invariant under each  $g_i$ . Thus  $\dim Y = 0$  and the proof of (1) is complete. (2) follows from Corollary 2.5.  $\square$

## 8. DYNAMICAL CHARACTERIZATION OF AMPLENESS

We will use some results of the birational theory of foliations. For more information on the subject see [15, 17], and specially the last three chapters of [4]. Throughout this section  $S$  denotes a smooth complex projective surface and  $\mathcal{F}$  a foliation  $f : \Omega_S^1 \rightarrow \mathcal{L}$  over  $S$ .

Let  $C$  be a curve on  $S$  and  $p$  a point of  $C$ . Denote by  $\mathcal{O}_p$  the local algebra of  $S$  at  $p$ . If the curve has local equation  $f = 0$  and the foliation is described by a vector field  $v$  in a neighbourhood of  $p$ , let

$$\text{tang}(\mathcal{F}, C, p) = \dim_{\mathbb{C}} \frac{\mathcal{O}_p}{(f, v(f))}.$$

Note that this number is 0 except at the finite number of points where  $f$  is not transverse to  $C$ . Define the *tangency number* between  $\mathcal{F}$  and  $C$  by

$$\text{tang}(\mathcal{F}, C) = \sum_{p \in C} \text{tang}(\mathcal{F}, C, p).$$

**Lemma 8.1.** *If  $\mathcal{F}$  is a foliation without algebraic invariant curves, then  $\mathcal{L} \cdot C \geq 0$  for every irreducible curve  $C$ .*

*Proof.* By Miyaoka's Theorem [4, Theorem 7.1, p. 89]  $\mathcal{L}$  is pseudo-effective. Thus, by [7, Theorem 1.12, p. 108],  $\mathcal{L}$  can be decomposed in the form  $P + N$ , where  $P$  is a semi-positive  $\mathbb{Q}$ -divisor and  $P \cdot C = 0$  for each irreducible component  $C$  of the support of  $N$ . The result follows if we show that the support of  $N$  is empty.

Assume, by contradiction, that the support of  $N$  is nonempty. Then, by [7, Theorem 1.12(c), p. 108] it contains an irreducible component  $E$  such that  $\mathcal{L} \cdot E < 0$  and  $E^2 < 0$ . Since  $\mathcal{F}$  does not have any invariant algebraic curves,  $E$  cannot be invariant under  $\mathcal{F}$ . Then, by [4, Proposition 2.2, p.23],

$$\mathcal{L} \cdot E = \text{tang}(\mathcal{F}, E) - E^2,$$

so that  $\mathcal{L} \cdot E > 0$ , contradicting the choice of  $E$ .  $\square$

*Proof of Theorem 1.2* Let  $\mathcal{F}$  be a foliation of  $S$  with no algebraic invariant curves and suppose that  $\mathcal{L} \cdot C = 0$  for some irreducible curve  $C$  on  $S$ . Since  $\mathcal{L}^2 > 0$ , by hypothesis, it follows from the Hodge Index Theorem [9, Theorem 1.9, p. 364] that  $C^2 < 0$ . However, [4, Proposition 2.2, p.23] implies that  $C^2 = \text{tang}(\mathcal{F}, C) \geq 0$ , and we conclude that there exists no such  $C$ . Hence, it follows from Lemma 8.1 that  $\mathcal{L} \cdot C > 0$  for every irreducible curve  $C$ . Thus  $\mathcal{L}$  is ample by the Nakai-Moishezon criterion [9, Theorem 1.10, p. 365]. The converse is a straightforward consequence of the main theorem.  $\square$

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