ON THE HOLONOMY GROUP OF ALGEBRAIC CURVES INVARIANT BY HOLOMORPHIC FOLIATIONS

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To César Camacho, on his 60^{th} birthday

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Abstract. We prove a realization result for the linear holonomy group of algebraic curves invariant by one-dimensional foliations of projective varieties. In the case of projective surfaces we also treat the prescription of higher order jets of the holonomy group. Applications are given including the construction of stably minimal foliations of arbitrary projective varieties.

1. Introduction and Statement of Results

The dynamical behavior of a foliation can be studied in a neighborhood of a leaf by the holonomy representation of its fundamental group. This representation is constructed as follows. We take a point p on the leaf L and a germ of transversal Σ at p. Lifting closed paths γ starting at p along the leaves of the foliation induces germs of diffeomorphisms $h_\gamma : (\Sigma, p) \to (\Sigma, p)$ which do not depend on the homotopy class of the path. The *holonomy representation* of $\pi_1(L, p)$ is the morphism defined by

$$\begin{aligned} \mathrm{Hol}(L) : \pi_1(L,p) &\to \mathrm{Diff}(\Sigma, p) \\ \gamma &\mapsto h_\gamma \,, \end{aligned}$$

and the *holonomy group* of the foliation along L is the image of this map (which will be confounded with the representation itself). Different points in the leaf and different sections give rise to representations conjugated by germs of holomorphic diffeomorphisms.

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This paper deals with one-dimensional holomorphic foliations of projective varieties and the holonomy groups of leaves whose closure are algebraic. First we will focus on the first order approximation of the holonomy representation, namely, the linear representation obtained from above by taking the derivative of the maps at p:

$$\begin{aligned} \mathrm{Hol}_{lin}(L) : \pi_1(L,p) &\to GL(T_p\Sigma) \\ \gamma &\mapsto Dh_\gamma(p)\,. \end{aligned}$$

The image of this map is the *linear holonomy group* of the foliation along L.

We are concerned here with realizing subgroups of $GL(T_p\Sigma)$ as linear holonomy groups of leaves which are smooth, projective curves (perhaps deprived of a finite number of points).

Theorem 1.1. *Let M be a smooth projective variety of dimension $m+1$, $C \subset M$ a smooth algebraic curve, $q \in C$ a point of C, $\mathcal{P} = \{p_1, \ldots, p_k\}$ a set of points of $C \setminus \{q\}$ and Σ a germ of transversal to C at q. Given a homomorphism $\phi : \pi_1(C \setminus \mathcal{P}, q) \to GL(T_p\Sigma)$ there exists a holomorphic foliation $\mathcal{F}$ of M satisfying the following properties:*

i. *C is invariant by $\mathcal{F}$, $q \notin \mathrm{Sing}(\mathcal{F})$ and $\mathcal{P} \subset \mathrm{Sing}(\mathcal{F})$.*
ii. *the linear holonomy group of $\mathcal{F}$ along $C \setminus \mathrm{Sing}(\mathcal{F})$ is $\phi \circ i_*$, where $i_* : \pi_1(C \setminus \mathrm{Sing}(\mathcal{F}), q) \to \pi_1(C \setminus \mathcal{P}, q)$ is the natural homomorphism.*

The singularities in $\mathcal{S}ing(\mathcal{F}) \setminus \mathcal{P}$ give no contribution to the linear holonomy group; for that reason, we call them *apparent singularities* and we will simply say that *the linear holonomy group of $\mathcal{F}$ along $C \setminus \mathcal{P}$ is ϕ*. There are situations where the presence of apparent singularities cannot be avoided. For instance when $\mathcal{P}$ is empty and C has non-zero self-intersection the Index Theorem, see [3], implies $\mathrm{Sing}(\mathcal{F}) \cap C \neq \emptyset$.

It is worthwhile mentioning the classical Riemann-Hilbert problem. One has to construct a rank m meromorphic linear differential equation over $\overline{\mathbb{C}}$ such that the induced foliation of $M = \overline{\mathbb{C}} \times \mathbb{C}^m$ has holonomy group along the leaf $C = \overline{\mathbb{C}} \times \{0\}$ given a priori by ϕ. $\mathrm{Sing}(\mathcal{F})$ is demanded to be exactly $\mathcal{P}$ and its elements should be as simple as possible (fuchsian singularities); no apparent singularities are allowed.

A particular case of Theorem 1.1 was proved in [12] for plane foliations, including an estimate of its degree as a function of the degree of the curve; this means that the cotangent bundle of the foliation can be controlled. In our more general context, we no longer are able to have such a refinement. Although here we are able to prescribe the linear holonomy also for germs of holomorphic families of representations close to the identity at $\mathcal{P}$. See Section 4 for a precise statement.

One of the motivations of this study is to present tools to produce examples of foliations with prescribed dynamical properties. In particular, we may use Theorem 1.1 to find stably minimal foliations on projective varieties following the ideas from [10]. We will denote by Θ_M the tangent sheaf of M and by $\mathrm{Fol}(M, \mathcal{L})$ the space of foliations of the projective variety M with cotangent bundle isomorphic to $\mathcal{L}$, i.e., $\mathrm{Fol}(M, \mathcal{L}) = \mathbb{P}\mathrm{H}^0(M, \Theta_M \otimes \mathcal{L}))$.

A holomorphic foliation of a projective variety is a *minimal foliation* if every leaf is dense. We will say that $\mathcal{F} \in \mathrm{Fol}(M, \mathcal{L})$ is a *stably minimal foliation* if there exists an open set $U \subset \mathrm{Fol}(M, \mathcal{L})$ such that every foliation of U is minimal.

As an application of Theorem 1.1 we obtain

Theorem 1.2. *Let $\mathcal{L}$ be an ample line-bundle on a projective variety V. Then for $n >> 0$ the set of stably minimal foliations with cotangent bundle isomorphic to $\mathcal{L}^{\otimes n}$ is non-empty.*

As in the approach of [10], we obtain stably minimal foliation perturbing foliations with invariant algebraic curves (projective lines in [10]).

We are also interested in the higher order jets of the holonomy group of an algebraic leaf. So let k be a positive integer and H_k be the subgroup of $\mathrm{Diff}(\Sigma, p)$ formed by the elements which are tangent to the identity up to order k. Since H_k is a normal subgroup the cokernel of the inclusion of H_k in $\mathrm{Diff}(\Sigma, p)$ is the *group of k-jets of* $\mathrm{Diff}(\Sigma, p)$. This group will denoted by $J^k\mathrm{Diff}(\Sigma, p)$. Observe that $\mathrm{J}^1\mathrm{Diff}(\mathbb{C}^m, 0)$ is isomorphic to $\mathrm{GL}(m, \mathbb{C})$ and for $k \geq 2$ we have the exact sequence

$$\mathrm{Id} \to \frac{H^{k-1}}{H^k} \to \mathrm{J^k Diff}(\Sigma, p) \to \mathrm{J^{k-1} Diff}(\Sigma, p) \to \mathrm{Id}$$

If L is a leaf of a foliation $\mathcal{F}$ and Σ is a transversal to $\mathcal{F}$ passing through $p \in L$ then the *k-jet of the holonomy representation* will be the morphism

$$\mathrm{Hol}_k(L) : \pi_1(L, p) \to \mathrm{J^k Diff}(\Sigma, p)\,,$$

obtained by composing the usual holonomy representation with the natural quotient map

$$\mathrm{Diff}(\Sigma, p) \to \mathrm{J^k Diff}(\Sigma, p)\,.$$

As before we are concerned here with realizing subgroups of $\mathrm{J^k Diff}(\Sigma, p)$ as k-jets of the holonomy groups of leaves which are smooth projective curves (perhaps deprived of a finite number of points).

Theorem 1.3. *Let M be a smooth projective surface, $C \subset M$ a smooth algebraic curve, $q \in C$ a point of C, $\mathcal{P} = \{p_1, \ldots, p_k\}$ a set of points of $C \setminus \{q\}$ and Σ a germ of transversal to C at q. Given a homomorphism $\phi : \pi_1(C \setminus \mathcal{P}, q) \to J^k\mathrm{Diff}(\Sigma, p)$ there exists a holomorphic foliation $\mathcal{F}$ of M satisfying the following properties:*

i. *C is invariant by $\mathcal{F}$, $q \notin \mathrm{Sing}(\mathcal{F})$ and $\mathcal{P} \subset \mathrm{Sing}(\mathcal{F})$.*
ii. *the k-jet of the holonomy group of $\mathcal{F}$ along $C \setminus \mathrm{Sing}(\mathcal{F})$ is $\phi \circ i_*$, where $i_* : \pi_1(C \setminus \mathrm{Sing}(\mathcal{F}), q) \to \pi_1(C \setminus \mathcal{P}, q)$ is the natural homomorphism.*

If M is a projective variety and $C \subset M$ is a smooth projective curve then we will denote by $\Theta_{M,C}$ the sheaf of vector fields on M tangent to C and by $\mathrm{Fol}_C(M, \mathcal{L})$ the space of foliations of M leaving the curve C invariant and with cotangent bundle isomorphic to $\mathcal{L}$, i.e., $\mathrm{Fol}_C(M, \mathcal{L}) = \mathbb{P}\mathrm{H}^0(M, \Theta_{M,C} \otimes \mathcal{L})$. We also define $\mathrm{Fol}^{\mathrm{sol}}_C(M, \mathcal{L})$ as *the subset of foliations in $\mathrm{Fol}_C(M, \mathcal{L})$ whose holonomy group along C is solvable*; it is postulated that if C is contained in the singular set of a foliation $\mathcal{F}$ then its holonomy group is trivial.

As an application of Theorem 1.3 we obtain

Proposition 1.4. *Let $\mathcal{L}$ be an ample line-bundle and $n \gg 0$ then $\mathrm{Fol}^{\mathrm{sol}}_C(M, \mathcal{L}^{\otimes n})$ is a closed subset of $\mathrm{Fol}_C(M, \mathcal{L}^{\otimes n})$ with empty interior.*

2. The Linear Holonomy Group

Let M be a projective variety of dimension $m + 1 \in \mathbb{N}$ and $C \subset M$ a smooth, compact curve. Let also Θ_M denote the $\mathcal{O}_m$ sheaf of holomorphic vector fields on M and $\Theta_{M,C}$ its subsheaf formed by the germs of vector fields which leave C invariant. The sheaf $\Theta_{M,C}$ fits into the exact sequence

$$0 \to \Theta_{M,C} \to \Theta_M \to N_C \to 0\,,$$

where N_C denotes the normal bundle of C on M.

If $\mathcal{I}_C$ is the ideal sheaf defining C then we have the short exact sequence

$$0 \to \Theta_{M,C} \otimes \mathcal{I}_C \to \Theta_{M,C} \to \mathcal{N}^{(1)}_{M,C} \to 0\,, \tag{2.1}$$

where $\mathcal{N}^{(1)}_{M,C}$ is the quotient sheaf $\frac{\Theta_{M,C}}{\Theta_{M,C} \otimes \mathcal{I}_C}$. Observe that $\mathcal{N}^{(1)}_{M,C}$ is supported on C and naturally carries a structure of a sheaf of $\mathcal{O}_C$-modules.

A section of $\mathcal{N}^{(1)}_{M,C}$ is described in local coordinates as follows. First we cover a neighborhood of C in M by open sets $\{U_\alpha\}$, where the coordinates $(x_\alpha, y^{(\alpha)}) = (x_\alpha, y^{(\alpha)}_1, \ldots, y^{(\alpha)}_m)$ are choose in such a way that the points in $U_\alpha \cap C$ satisfy $y^{(\alpha)}_1 = \ldots = y^{(\alpha)}_m = 0$. In each U_α we have a vector field

$$X_\alpha(x_\alpha, y^{(\alpha)}_1, \ldots, y^{(\alpha)}_m) = A_\alpha(x_\alpha)\frac{\partial}{\partial x_\alpha} + \sum_{j,k=1}^{n} B^{(\alpha)}_{jk}(x_\alpha) y^{(\alpha)}_j \frac{\partial}{y^{(\alpha)}_k}\,,$$

where A_α, $B^{(\alpha)}_j$ are holomorphic functions, $1 \le j \le m$. Furthermore, if $U_\alpha \cap U_\beta \neq \emptyset$ and $(x_\alpha, y^{(\alpha)}) = \phi_{\alpha\beta}(x_\beta, y^{(\beta)})$, then the obvious truncation of

$\phi^*_{\alpha\beta} X_\alpha$ is equal to X_β. This truncation simply replaces

$$\phi_{\alpha\beta}(x_\beta, y^{(\beta)}) = (f_{\alpha\beta}(x_\beta) + \dots, g_{\alpha\beta}(x_\beta) \cdot y^{(\beta)} + \dots)$$

by

$$\psi_{\alpha\beta}(x_\beta, y^{(\beta)}) = (f_{\alpha\beta}(x_\beta), g_{\alpha\beta}(x_\beta) \cdot y^{(\beta)}) \, ,$$

where $\dots$ means higher order terms in $y^{(\beta)}$ and $g_{\alpha\beta}(x_\beta)$ is a $n \times n$ matrix. We have that $\psi^*_{\alpha\beta} X_\alpha = X_\beta$ whenever $U_\alpha \cap U_\beta \neq \emptyset$.

Clearly we may use the collection $\{\phi_{\alpha\beta}(x_\beta)\}$ as the transition functions, and consequently $\{X_\alpha\}$ defines a section of $\mathcal{N}^{(1)}_{N_C,C}$, where N_C denotes the total space of the normal vector bundle of C on M, C denotes the zero section and

$$\mathcal{N}^{(1)}_{N_C,C} = \frac{\Theta_{N_C,C}}{\Theta_{N_C,C} \otimes I_C} \, .$$

In other terms $\{X_\alpha\}$ defines a foliation of N_C, which leaves C, the zero section, invariant and is transversal to the fibers except when $A_\alpha(x_\alpha) = 0$.

From the previous discussion we obtain the following.

Lemma 2.1. *As sheaves of $\mathcal{O}_C$-modules $\mathcal{N}^{(1)}_{M,C}$ and $\mathcal{N}^{(1)}_{N_C,C}$ are isomorphic.*

If $\mathcal{F}$ is foliation of M with cotangent bundle isomorphic to $\mathcal{L}$ and is tangent to C then $\mathcal{F}$ is defined by an element $\sigma_\mathcal{F}$ of $\mathrm{H}^0(M, \Theta_{M,C} \otimes \mathcal{L})$. The sequence (2.1) induces a map

$$\xi_{M,\mathcal{L}} : \mathrm{H}^0(M, \Theta_{M,C} \otimes \mathcal{L}) \to \mathrm{H}^0(C, \mathcal{N}^{(1)}_{M,C} \otimes \mathcal{L}) \, , \tag{2.2}$$

The proof of the following Proposition is standard.

Proposition 2.2. *If $\sigma_\mathcal{F} \in \mathrm{H}^0(M, \Theta_{M,C} \otimes \mathcal{L})$ defines a foliation $\mathcal{F}$ such that $\mathrm{Sing}(\mathcal{F}) \cap C$ is a proper subset of C then the linear holonomy group of $\mathcal{F}$ along C is completely determined by the image of $\sigma_\mathcal{F}$ under the map $\xi_{M,\mathcal{L}}$.*

Lemma 2.3. *Let M be a smooth projective variety of dimension $m+1$, $C \subset M$ a smooth curve and $\mathcal{L}$ an ample line-bundle on M. If $n \gg 0$ then*

i. $\dim_\mathbb{C} \mathrm{H}^0(M, \Theta_{M,C} \otimes \mathcal{L}^{\otimes n}) \gg 0$;
ii. *the map $\xi_{M,\mathcal{L}^{\otimes n}}$ (from (2.2)) is surjective.*

Proof. It is enough to choose $n \gg 0$ to have $\mathrm{H}^1(M, \Theta_{M,C} \otimes \mathcal{I}_C \otimes \mathcal{L}^{\otimes n}) = 0$. Since $\Theta_{M,C}$ is a coherent sheaf, this is granted by Serre's Vanishing Theorem(see [7], Theorem 5.2, pg. 228). ☐

3. Prescribing the Linear Holonomy Group

The proof of Theorem 1.1 can be divided in three steps. The first two steps are known; they are sketched here for the reader's convenience.

Step 1: We start constructing a rank m vector-bundle E over C and a holomorphic foliation $\mathcal{G}$ of E satisfying the following properties

i. the zero section of E, identified with C, is invariant by $\mathcal{G}$;
ii. the singular set of $\mathcal{G}$ is contained in the fibers of E over $\mathcal{P}$;
iii. the holonomy group of $C \setminus \mathcal{P}$ is conjugated to

$$\phi : \pi_1(C \setminus \mathcal{P}, q) \to GL(T_q\Sigma).$$

Let $\tilde{C}$ be the universal covering of $C \setminus \mathcal{P}$, G the group of its covering transformations and

$$\alpha : \pi_1(C \setminus \mathcal{P}, q) \to G$$

the canonical isomorphism. The representations ϕ and α induce an action, $\alpha \times \phi$, on the trivial vector bundle over $\tilde{C}$ defined as follows

$$\begin{array}{rcl} \alpha \times \phi : \pi_1(C \setminus \mathcal{P}, q) \times \tilde{C} \times T_q\Sigma & \to & \tilde{C} \times T_q\Sigma \\ (g, (x, v)) & \mapsto & (\alpha(g)(x), \phi(g) \cdot v) \, . \end{array}$$

The natural projection $\tilde{C} \times T_q\Sigma \to T_q\Sigma$ induces a non-singular 1–dimensional foliation $\tilde{\mathcal{G}}_0$ of $\tilde{C} \times T_q\Sigma$ which is invariant by the action $\alpha \times \phi$. Therefore the quotient of $\tilde{C} \times T_q\Sigma$ by $\alpha \times \phi$ is a vector bundle E_0 over $C \setminus \mathcal{P}$ carrying a non-singular foliation $\mathcal{G}_0$. By construction the zero section of E_0 is invariant by $\mathcal{G}_0$ and has holonomy group given by ϕ.

Let $\mathbb{D} \subset C$ be a small disc centered at $p \in \mathcal{P}$, $\gamma \subset \mathbb{D}$ be small loop around p and $\tilde{p} \in \gamma$; let us fix some path α joining q to $\tilde{p}$ and define $\gamma_p = \alpha * \gamma * \alpha^{-1}$. Let also A be a $n \times n$ complex matrix such that $exp(A) = \phi(\gamma_p)$. If $(x, (y_1, \ldots, y_m)^T) = Y$ are coordinates for $\mathbb{C} \times \mathbb{C}^m$ then the linear differential equation $xdY - A \cdot Y dx = 0$ defines a 1–dimensional foliation over $\mathbb{C} \times \mathbb{C}^m$ which leaves the horizontal axis $\{y_1 = \ldots = y_m = 0\}$ invariant, has a unique singularity at 0 and the local holonomy map along γ is $exp(A)$. This foliation is holomorphically conjugated over $\mathbb{D} \setminus \{p\} \times \mathbb{C}^m$ to $\mathcal{G}$, by a vector bundle automorphism. Therefore one can extend E_0 to a holomorphic vector bundle E over C and $\mathcal{G}_0$ to a foliation $\mathcal{G}$ of E . By construction $\mathcal{G}$ satisfies i, ii and iii.

Step 2: We show know how to pull-back $\mathcal{G}$ to a foliation of N_C.

Let $\mathcal{L}'$ be a convenient power of an ample line bundle over C such that both $E \otimes \mathcal{L}'$ and $N_C \otimes \mathcal{L}'$ are generated by global holomorphic sections (see [7], Thm. 5.17). Let us fix $q \in C$; there exist global sections $e_1, \ldots, e_m$ and $n_1, \ldots, n_m$ such that $\{e_1(q), \ldots, e_m(q)\}$ is a basis for the fiber E_q and $\{n_1(q), \ldots, n_m(q)\}$ is a basis for the fiber N_q. If l is a section of $\mathcal{L}$ not vanishing at q then $\{\frac{e_1}{l}, \ldots, \frac{e_m}{l}\}$ generates E and $\{\frac{n_1}{l}, \ldots, \frac{n_m}{l}\}$ generates

N_C over a Zariski open subset U of C. The map

$$\begin{array}{rcl} T_{|U} : N_{C|U} & \rightarrow & E_{|U} \\ \sum a_j \dfrac{e_j}{l} & \mapsto & \sum a_j \dfrac{n_j}{l} \end{array}$$

defines an isomorphism of vector bundles which extends to a bimeromorphic map $T : N_C \dashrightarrow E$.

The pull-back of $\mathcal{G}$ by T is a holomorphic foliation $\mathcal{H}$ of N_C whose holonomy group along $C \setminus \mathcal{P}$ is ϕ.

Step 3: Finally we construct the foliation with the desired properties in M.

If $\sigma_{\mathcal{H}} \in H^0(N_C, \Theta_{N_C,C} \otimes T^*\mathcal{H})$ defines $\mathcal{H}$, we set $\theta = \xi_{N_C,T^*\mathcal{H}}(\sigma_{\mathcal{H}})$ (see (2.2)).

To conclude the proof of the Theorem we will take an ample line bundle $\mathcal{L}$ on M and show that, for some $n \in \mathbb{N}$, there exists $\sigma \in H^0(M, \Theta_{M,C} \otimes \mathcal{L}^{\otimes n})$ and $f \in H^0(C, (\mathcal{L}^{\otimes n} \otimes T\mathcal{H})_{|C})$ such that $\xi_{M,\mathcal{L}^{\otimes n}}(\sigma) = f \otimes \theta$

Since $\deg(\mathcal{L}_{|C}) > 0$, an easy application of Riemann–Roch's Theorem allows us to choose a non-zero section $f \in H^0(C, (\mathcal{L}^{\otimes n} \otimes T\mathcal{H})_{|C})$ for some $n \gg 0$. It follows that $f \otimes \theta \in H^0(C, \mathcal{N}^{(1)}_{M,C} \otimes (\mathcal{L}^{\otimes n})_{|C})$.

From Lemma 2.3 we obtain, for $n \gg 0$, a section $\sigma \in H^0(M, \Theta_{M,C} \otimes \mathcal{L}^{\otimes n})$ such that $\xi_{M,\mathcal{L}^{\otimes n}}(\sigma) = f \otimes \theta$. The foliation of M induced by σ has the desired properties. □

4. Prescribing Families of Linear Holonomy Groups

Recall that $\gamma_p = \alpha * \gamma * \alpha^{-1}$ where $\gamma \subset C$ is an arbitrarily small loop around p, $\tilde{p} \in \gamma$ and α is an arbitrary path joining q to $\tilde{p}$.

Definition 4.1. Let $\{\phi_s : \pi_1(C \setminus \mathcal{P}, q)\}_{s \in (\mathbb{C}^l,0)} \rightarrow GL(T_q\Sigma)\}$ be a germ of holomorphic family of homomorphisms parameterized by $(\mathbb{C}^l, 0)$. We will say that $\{\phi_s\}$ is *close to the identity at* $\mathcal{P}$ if $\phi_0(\gamma_p) = \mathrm{Id}$ for every $p \in \mathcal{P}$.

As stated in the introduction Theorem 1.1 has a version for families of representations close to the identity at $\mathcal{P}$.

Theorem 4.2. *Let M be a smooth projective variety of dimension $m+1$, $C \subset M$ a smooth algebraic curve, $q \in C$ a point of C, $\mathcal{P} = \{p_1, \ldots, p_k\}$ a set of points of $C \setminus \{q\}$ and Σ a germ of transversal to C at q. Given a germ of holomorphic family of homomorphisms*

$$\{\phi_s : \pi_1(C \setminus \mathcal{P}, q) \rightarrow GL(T_q\Sigma)\}_{s \in (\mathbb{C}^l,0)}$$

close to the identity at $\mathcal{P}$ there exists a germ of families of holomorphic foliations $\{\mathcal{F}_s\}_{s \in (\mathbb{C}^l,0)}$ of M satisfying the following properties:

i. *for all* $s \in (\mathbb{C}^l, 0)$, C *is invariant by* $\mathcal{F}_s$, $q \notin \mathrm{Sing}(\mathcal{F}_s)$ *and* $\mathcal{P} \subset \mathrm{Sing}(\mathcal{F}_s)$.
ii. *the linear holonomy group of* $\mathcal{F}_s$ *along* $C \backslash \mathcal{P}$ *is equal to* $\phi_s \circ i_{s*}$, *where* $i_{s*} : \pi_1(C \backslash \mathrm{Sing}(\mathcal{F}_s), q) \to \pi_1(C \backslash \mathcal{P}, q)$ *is the natural homomorphism.*
iii. *if* $s, s' \in (\mathbb{C}^l, 0)$ *then the cotangent bundles of* $\mathcal{F}_s$ *and* $\mathcal{F}_{s'}$ *are isomorphic.*

Proof. The proof is essentially the same as the proof of Theorem 1.1. In the first step the construction can repeated as it is for the family $\{\phi_s\}_{s \in (\mathbb{C}^l, 0)}$ since the $\{\phi_s\}_{s \in (\mathbb{C}^l, 0)}$ is close to the identity at $\mathcal{P}$ and therefore we can take an inverse branch of the exponential containing $\phi_s(\gamma_p)$ for every $p \in \mathcal{P}$ and every $s \in (\mathbb{C}^l, 0)$.

The families of vector bundles and foliations obtained in Step 1 can be interpreted as a foliation of a vector bundle over M defined over $\mathrm{Spec}(\mathcal{O})$, where $\mathcal{O}$ is the ring of germ of holomorphic functions on $(\mathbb{C}^l, 0)$. Since $\mathcal{O}$ is a noetherian ring and Serre's Vanishing Theorem of Serre is still true on this more general context then Theorem 4.2 follows. □

5. Hyperbolic Singularities

We say that a singularity is *hyperbolic* if all its eigenvalues are distinct non-zero complex numbers and the quotient of any two of them is not a real number.

The following Proposition appears in [6, Thm. 6.7] when $\dim_{\mathbb{C}} V = 2$.

Proposition 5.1. *Let* $\mathcal{L}$ *be an ample line-bundle on a projective variety* V. *If* $n \gg 0$, *there exists an open and dense subset of* $\mathbb{P}(\mathrm{H}^0(V, \Theta_V \otimes \mathcal{L}^{\otimes n}))$ *such that the induced foliations have only hyperbolic singularities.*

Let $\mathcal{L}$ be a line bundle on V, $\Sigma = \mathbb{P}(\mathrm{H}^0(V, \Theta_V \otimes \mathcal{L}^{\otimes n}))$ and define the subset S of $\Sigma \times V$ by

$$S = \{(\mathcal{F}, x) \in \Sigma \times (V) \ : \ x \in \mathrm{Sing}(\mathcal{F})\} .$$

We will denote by π the standard projection of S to Σ.

The proof of the proposition above is based in the next Lemma, which adapts an argument from [4].

Lemma 5.2. *If* $\Theta_V \otimes \mathcal{L}$ *is generated by global sections then* S *is a smooth subvariety of* $\Sigma \times V$. *Moreover* $\dim_{\mathbb{C}} S = \dim_{\mathbb{C}} \Sigma$.

Proof. Denote by $\mathbb{T}$ the trivial bundle over V with fiber $\mathrm{H}^0(V, \Theta_V \otimes \mathcal{L})$ and by $\pi : \mathbb{P}(\mathbb{T}) \to V$ the standard projection. There exists a surjective map of vector bundles $u : \mathbb{T} \to T_V \otimes \mathcal{L}$ which takes $(x, \theta) \in \mathbb{T}$ to the vector $\theta(x) \in T_x V$. Since $\Theta_V \otimes \mathcal{L}$ is generated by global sections it follows that u

is surjective, therefore $ker(u)$ is also a vector bundle. Since $S = \mathbb{P}(\ker u)$, the assertions about S are proved.

□

Proof of proposition 5.1. The crucial point is the following *prescription property*: let $p \in V$ be chosen; for $n \gg 0$, there exists a foliation in $\mathrm{H}^0(M, \Theta_V \otimes \mathcal{L}^{\otimes n})$ which has a hyperbolic singularity at p. In fact, let us consider the short exact sequence

$$0 \to \Theta_{V,p} \otimes \mathcal{L}^{\otimes n} \otimes m_p \to \Theta_{V,p} \otimes \mathcal{L}^{\otimes n} \to \frac{\Theta_{V,p} \otimes \mathcal{L}^{\otimes n}}{\Theta_{V,p} \otimes \mathcal{L}^{\otimes n} \otimes m_p} \to 0\,,$$

where m_p is the maximal ideal associated to p and $\Theta_{V,p}$ is the subsheaf of Θ_V whose local sections vanish at p. If $n \gg 0$ then $\mathrm{H}^1(M, \Theta_{V,p} \otimes \mathcal{L}^{\otimes n} \otimes m_p) = 0$, so that we have a surjetive map

$$\mathrm{H}^0(V, \Theta_{V,p} \otimes \mathcal{L}^{\otimes n}) \to \mathrm{H}^0\left(V, \frac{\Theta_{V,p} \otimes \mathcal{L}^{\otimes n}}{\Theta_{V,p} \otimes \mathcal{L}^{\otimes n} \otimes m_p}\right).$$

It follows that we can prescribe any linear part for the singularity at p, in particular to get a hyperbolic one.

Let us first guarantee (always by increasing $n \in \mathbb{N}$ if necessary) that foliations in some Zariski open subset of $\mathrm{H}^0(V, \Theta_V \otimes \mathcal{L}^{\otimes n})$ have singularities with Milnor number one whose eigenvalues are distinct. In fact, the subset of points $(\mathcal{F}, x) \in S$ such that there are repeated eigenvalues of the singularity x of the foliation $\mathcal{F}$ is an analytic subvariety S_0; the prescription property implies that it is a strict subvariety. For the same reason, $S_1 = \{(\mathcal{F}, x) \in S;\ \pi|_S \text{ is not a submersion at } (\mathcal{F}, \mathrm{x})\}$ is also an analytic subvariety strictly contained in S. It follows that $\pi(S_0 \cup S_1)$ is also a strict analytic subvariety of Σ; let U_0 be its complement. Finally, we consider the Zariski open subset U_1 of Σ such that $\pi : \pi^{-1}(U_1) \to U_1$ is a finite to one map. It follows that the function defined in $S' = \pi^{-1}(U_0 \cap U_1))$ that assigns to $(\mathcal{F}, x) \in S'$ the Milnor number of $x \in V$ as a singularity of $\mathcal{F}$ is locally constant, therefore constant since S' is connected. We conclude that the Milnor number of any singularity of a foliation in $U_0 \cap U_1$ has to be one, because of the prescription property.

Inside S' we define the analytic subvariety $\widehat{S}$ of foliations such that some quotient of eigenvalues of the singularities belongs to $\mathbb{R}$. Once more the prescription property implies that this subvariety is strict. The statement of the Proposition is therefore true for $U = (U_0 \cap U_1) \setminus \pi(\widehat{S})$. □

The arguments can be easily adapted if we want to get hyperbolic singularities along some invariant, smooth, algebraic subvariety H; we replace $\mathbb{P}(\mathrm{H}^0(V, \Theta_V \otimes \mathcal{L}^{\otimes n}))$ by $\mathbb{P}(\mathrm{H}^0(V, \Theta_{V,H} \otimes \mathcal{L}^{\otimes n}))$, where $\Theta_{V,H}$ is the sheaf of germs of holomorphic vector fields in V which leave H invariant.

6. Prescribing Minimality

This section is devoted to proving Theorem 1.2. As already noted we adapt an argument from [10], the *induction trick*. The proof consists in checking the steps of that argument in our context. Stably minimal foliations will appear close to special foliations, which correspond in [10] to the construction of foliations of $\mathbb{P}^n_{\mathbb{C}}$ possessing a flag of invariant projective spaces and a convenient holonomy. The next lemma allow us to apply a similar construction in projective varieties.

Lemma 6.1. *Let V be a $(m+1)$–dimensional projective variety and $\mathcal{L}$ a ample line-bundle. Then if $n \gg 0$ there exists a germ of family of holomorphic foliations $\{\mathcal{F}_t\}_{t\in(\mathbb{C}^l,0)}$ with cotangent bundle $\mathcal{L}^{\otimes n}$ satisfying the following properties:*

1. *$\mathcal{F}_0$ is tangent to smooth varieties $C = V_1 \subset V_2 \subset \cdots \subset V_j \subset \cdots \subset V_n = V$, with $\dim_{\mathbb{C}} V_j = j$ and V_i is ample in V_{i+1}.*
2. *the generators of the linear holonomy group of $\mathcal{F}_0$ along $C \setminus \mathrm{Sing}(\mathcal{F}_0)$ are equal to Id;*
3. *any subgroup G of $GL(\mathbb{C}^m)$ generated by two elements on an open set W with $Id \in W$ is contained in the linear holonomy group of $\mathcal{F}_t$ for some $t \in \mathbb{C}$ small.*

Proof. Suppose we have subvarieties of the form $V_{i+1} \subset V_{i+2} \subset \ldots \subset V_{m+1} = V$. Take V_i to be a generic hyperplane section of V_{i+1}. By Bertini's and Lefschetz' Theorems V_i is smooth and connected.

The Lemma follows from Theorem 4.2.

□

We will need also the following

Lemma 6.2. *Let V be a projective variety, H a hypersurface on V and $\mathcal{L}$ an ample line-bundle on V. If $n \gg 0$ then every foliation on H with cotangent bundle given by $\mathcal{L}^{\otimes n}_{|H}$ extends to V with $\mathcal{L}^{\otimes n}$ as its cotangent bundle.*

Proof. If we denote by $\Theta_{V,H}$ the sheaf of germ of vector fields leaving H invariant then $\Theta_{V,H}$ fits in the exact sequence,

$$0 \to \Theta_{V,H} \otimes \mathcal{I}_H \to \Theta_{V,H} \to \Theta_H \to 0\,.$$

Therefore if $n \gg 0$ then by Serre's vanishing theorem we obtain that $\mathrm{H}^1(V, \Theta_{V,H} \otimes I_H \otimes \mathcal{L}^{\otimes n}) = 0$ and the surjectivity of the restriction map

$$\mathrm{H}^0(V, \Theta_{V,H} \otimes \mathcal{L}^{\otimes n}) \to \mathrm{H}^0(H, \Theta_H \otimes \mathcal{L}^{\otimes n})\,.$$

□

Theorem 1.3 follows from the next Proposition.

Proposition 6.3. *Let V, $\mathcal{L}$ and $\mathcal{F}_t$ be as in Lemma 6.1. Then there exist stably minimal foliations arbitrarily close to $\mathcal{F}_0$.*

Proof. We assume in the sequel that whenever necessary the foliations are taken in $\mathbb{P}\mathrm{H}^0(V, TM \otimes \mathcal{L}^{\otimes n})$ for $n >> 0$.

The proof goes by induction in the dimension of V. First of all let $\dim_{\mathbb{C}} V = 2$ and C be an ample curve of V.

(1) given $\mathcal{F}_0$, there exists a nearby foliation $\mathcal{H}$ whose singularities along C (which remains invariant) are hyperbolic (by Proposition 4.1). In particular, the singular set of $\mathcal{H}$ is finite; all leaves, except a finite number of algebraic leaves, have to intersect a transverse section L to C since the complement of C is a Stein space (see [5], Thm. 6.5) . Note that this is a stable property, that is, it persists for a neighborhood of $\mathcal{H}$.

    When we perturb $\mathcal{H}$ to some $\mathcal{H}'$, the generators of the holonomy group of $\mathcal{H}$ change to the generators of a pseudo-group of diffeomorphisms acting on L; we still call this the *pseudo-group of holonomy of $\mathcal{H}'$ associated to C*.

(2) we approximate $\mathcal{H}$ by $\mathcal{H}'$ which has no algebraic invariant curves (by Theorem 1 from [4]). It follows that all leaves of $\mathcal{H}'$ have to intersect L and the generators of the pseudo-group of holonomy of $\mathcal{H}'$ associated to C have no common fixed point (such a property is obviously a stable one). According to Corollary 5.2 from [10], this pseudo-group acts minimally on L. Therefore $\mathcal{H}'$ and its neighbors are minimal foliations.

Now we consider a foliation $\mathcal{F}_0$ as in the statement of Lemma 6.1. We assume by induction that its restriction to V_m can be approximated by a stably minimal foliation $\mathcal{H}_0$ in V_m.

(1) Lemma 6.2 implies that $\mathcal{H}_0$ extends to V as a foliation close to $\mathcal{F}_0$. By its turn, this foliation can be approximated by some $\mathcal{H}_1$ all of whose singularities along V_m (which remains invariant) are hyperbolic, because of Proposition 4.1. In particular, the restriction of $\mathcal{H}_1$ to V_m is still stably minimal. Let L be a transverse section to C. All leaves of $\mathcal{H}$ which intersect a neighborhood of V_m intersect L as well, except perhaps the separatrices of the singularities on V_m which are transversal to this variety (this is a stable property, by Lemma 9.5 of [10]). Since V_m is ample in V, we may say (again by [5], Thm. 6.5): all leaves of $\mathcal{H}_1$ except for a finite number of algebraic leaves intersect L(the separatrices we mentioned previously are contained in algebraic curves, since the singular set is finite).

(2) we approximate $\mathcal{H}_1$ by $\mathcal{H}_2$ which has no algebraic leaves ([4], Theorem 1); therefore, all leaves of $\mathcal{H}_2$ intersect L. This is still a stable property.
(3) a further convenient approximation $\mathcal{H}_3$ of $\mathcal{H}_2$ guarantees that its pseudo-group of holonomy associated to C acts minimally in L ([10], Corollary 5.2), so that $\mathcal{H}_3$ is a minimal foliation. This last approximation may be done because of condition 3 of Lemma 6.1; in particular, minimality of $\mathcal{H}_3$ is a stable property.

□

7. Higher Order Jets of the Holonomy Group

As before let $\mathcal{F}$ be a holomorphic foliation by curves of a projective variety M, $\dim_{\mathbb{C}} M = m+1$, and $C \subset M$ a smooth, compact curve. If $\mathcal{I}_C$ is the ideal sheaf defining C then we have the short exact sequence

$$0 \to \Theta_{M,C} \otimes {\mathcal{I}_C}^k \to \Theta_{M,C} \to \mathcal{N}^{(k)}_{M,C} \to 0\,, \tag{7.1}$$

where $\mathcal{N}^{(k)}_{M,C}$ is the quotient sheaf $\frac{\Theta_{M,C}}{\Theta_{M,C}\otimes \mathcal{I}_C^k}$.

Remark 7.1. If $k \geq 2$ then, in contrast with the case $k=1$, the sheaf $\mathcal{N}^{(k)}_{M,C}$, although geometrically supported on C, **does not** carry a structure of a sheaf of $\mathcal{O}_C$-modules compatible with the exact sequence (7.1). The compatibility depends on the sheaf $\mathcal{N}^{(k)}_{M,C}$ to be seen as a sheaf of $\mathcal{O}_M/I_C^k\mathcal{O}_M$-modules. This will be one of the sources of difficulties in this more general setting.

If $\mathcal{F}$ is foliation of M with cotangent bundle isomorphic to $\mathcal{L}$ and is tangent to C then $\mathcal{F}$ is defined by an element $\sigma_{\mathcal{F}}$ of $\mathrm{H}^0(M, \Theta_{M,C}\otimes\mathcal{L})$. The sequence (7.1) induces a map

$$\xi^{(k)}_{M,\mathcal{L}} : \mathrm{H}^0(M, \Theta_{M,C}\otimes\mathcal{L}) \to \mathrm{H}^0(C^{(k)}, \mathcal{N}^{(k)}_{M,C}\otimes\mathcal{L})\,, \tag{7.2}$$

where $C^{(k)}$ is the non-reduced subscheme of M associated to the sheaf of ideals I^k_C. Despite the differences with the linear case the Proposition 2.2 and the lemma 2.3 admit straightforward generalizations as stated below.

Proposition 7.2. *If $\sigma_{\mathcal{F}} \in \mathrm{H}^0(M, \Theta_{M,C}\otimes\mathcal{L})$ defines a foliation $\mathcal{F}$ such that $\mathrm{Sing}(\mathcal{F}) \cap C$ is a proper subset of C then the k-jet of the holonomy representation of $\mathcal{F}$ along C is completely determined by the image of $\sigma_{\mathcal{F}}$ under the map $\xi^{(k)}_{M,\mathcal{L}}$.*

Lemma 7.3. *Let M be a smooth projective variety of dimension $m+1$, $C \subset M$ a smooth curve and $\mathcal{L}$ an ample line-bundle on M. If $n \gg 0$ then*

i. $\dim_{\mathbb{C}}\mathrm{H}^0(M, \Theta_{M,C}\otimes\mathcal{L}^{\otimes n}) \gg 0$;

ii. *the map* $\xi^{(k)}_{M,\mathcal{L}^{\otimes n}}$ *(from (7.2)) is surjective.*

8. Prescribing the k-jets

The proof of Theorem 1.3 will follow the same strategy of the proof of Theorem 1.1. The only obstruction to generalize Theorem 1.3 to arbitrary projective varieties is in Step 2. The argument there is based on Grauert's Tubular Neighborhood Theorem valid only for surfaces. The general case will be treated elsewhere.

On the case of k-jets an extra step will be needed. This will be the

Step 0: Let

$$\phi : \pi_1(C \setminus \mathcal{P}, q) \to J^k\mathrm{Diff}(\Sigma, p)$$

be a homomorphism. We will construct a homomorphism

$$\widetilde{\phi} : \pi_1(C \setminus \widetilde{\mathcal{P}}, q) \to \mathrm{Diff}(\Sigma, p)\,,$$

where $\widetilde{\mathcal{P}}$ is the union of $\mathcal{P}$ with an extra point of C distinct from q, such that the following diagram

$$\begin{array}{ccc} \pi_1(C \setminus \widetilde{\mathcal{P}}, q) & \xrightarrow{\ \widetilde{\phi}\ } & \mathrm{Diff}(\Sigma, p) \\ \downarrow{\scriptstyle i_*} & & \downarrow \\ \pi_1(C \setminus \mathcal{P}, q) & \xrightarrow{\ \phi\ } & J^k\mathrm{Diff}(\Sigma, q) \end{array}$$

commutes. Of course the vertical arrows correspond to the natural maps.

Let $\gamma_1, \ldots, \gamma_l$ be the generators of $\pi_1(C \setminus \mathcal{P}, q)$. We can choose a point p in $C \setminus \mathcal{P}$ such that after setting $\widetilde{\mathcal{P}}$ as $\mathcal{P} \cup \{p\}$ then $\pi_1(C \setminus \widetilde{\mathcal{P}}, q)$ is a free group with generators $\gamma_1, \ldots, \gamma_l$.

Once we have chosen a system of coordinates for Σ we can define an injective map $\tau_k : J^k\mathrm{Diff}(\Sigma, p) \to \mathrm{Diff}(\Sigma, p)$ by setting the terms of order greater than k as 0. The map τ_k is a homomorphism only for $k = 1$, although, since $\pi_1(C \setminus \widetilde{\mathcal{P}}, q)$ is the free group generated by $\gamma_1, \ldots, \gamma_l$ we can set $\widetilde{\phi}([\gamma_i]) = \tau_k(\phi([\gamma_i])$ for $i = 1 \ldots l$, and this will extend as a homomorphism.

Step 1: An analogous construction to the one made in Step 1 of the proof of Theorem 1.1 will assure the existence of a complex surface U containing C and a holomorphic foliation $\mathcal{G}$ of U satisfying the following properties:

i. the curve C is invariant by $\mathcal{G}$;
ii. the singular set of $\mathcal{G}$ is contained in C and contains $\widetilde{\mathcal{P}}$;
iii. the holonomy group of $C \setminus \widetilde{\mathcal{P}}$ is conjugated to

$$\widetilde{\phi} : \pi_1(C \setminus \widetilde{\mathcal{P}}, q) \to \mathrm{Diff}(\Sigma, p).$$

Away from the set $\widetilde{\mathcal{P}}$ the construction is the essentially the same as before. The local construction near the points of $\widetilde{\mathcal{P}}$ is more subtle and requires the use of a realization theorem by Pérez-Marco and Yoccoz, see [11]. Since a completely similar construction has been carried out by Ilyashenko in [9] we invite the reader to consult these works to provide the details for the construction of $\mathcal{G}$.

Blowing-up points of $C \subset U$ and enlarging the singular set of $\mathcal{G}$ we can assume that

iv. $\deg(N_{C/U}) \ll 0$;

where $N_{C/U}$ denotes the normal bundle of C in U. It follows from Riemann-Roch's Theorem that there exists a non identically zero section $s \in \mathrm{H}^0(C, N^*_{C/U} \otimes N_{C/M})$.

Thus we will make these assumptions, preparing the ground for the next step where we will use Grauert's Theorem on holomorphic tubular neighborhoods, see [2, Theorem 4.4, pg. 68].

Step 2: Let $s \in \mathrm{H}^0(C, N^*_{C/U} \otimes N_{C/M})$ be the section as above. Let $\pi : M_0 \to M$ be the blow-up M at the points of $(s)_0$ the divisor of zeros of s, a subset of C. Therefore, as $\mathcal{O}_C$-sheafs, we have that

$$N_{C/M_0} \cong N_{C/M} \otimes \mathcal{O}_C(-(s)_0)\,.$$

From the choice of s we conclude that the restrictions of N_{C/M_0} and $N_{C/U}$ to C are isomorphic invertible sheaves.

By [iv] we can use Grauert's Theorem to assure the existence of a biholomorphism between small neighborhoods of C in U and in M_0. Therefore the foliation $\mathcal{G}$ pulls-back to a foliation $\mathcal{H}$, defined in a small neighborhood $U_0 \subset M_0$ of C.

Note that the singularities of $\mathcal{H}$ on $C \setminus \widetilde{\mathcal{P}}$ which appears along the above construction have the identity as its local holonomy map. Thus, we can say that the holonomy group of $\mathcal{H}$ along C is given by $\widetilde{\phi}$.

Step 3: We will now we construct the foliation with the desired properties in M_0.

If $\sigma_{\mathcal{H}} \in H^0(U_0, \Theta_{U_0,C} \otimes T^*\mathcal{H})$ defines $\mathcal{H}$, we define θ as the image of $\sigma_{|H}$ under the natural map 7.2, i.e., $\sigma = \xi^{(k)}_{U_0,T^*\mathcal{H}}(\sigma_{\mathcal{H}})$.

If $\mathcal{L}$ is an ample line-bundle on M_0 then $\mathcal{L}$ is still ample when restricted to $C^{(k)}$. Thus follows from Serre's Vanishing Theorem that for $n \in \mathbb{N}$ large enough there exists $f \in H^0(C^{(k)}, (\mathcal{L}^{\otimes n} \otimes T\mathcal{H})_{|C^{(k)}})$ not vanishing identically when restricted to C. It follows that

$$f \otimes \theta \in H^0(C^{(k)}, (\mathcal{N}^{(k)}_{M_0,C^{(k)}} \otimes \mathcal{L}^{\otimes n})_{|C^{(k)}})\,.$$

From Lemma 7.3, since $n \gg 0$, we obtain a section $\sigma \in H^0(M_0, \Theta_{M_0,C} \otimes \mathcal{L}^{\otimes n})$ such that $\xi^{(k)}_{M_0,\mathcal{L}^{\otimes n}}(\sigma) = f \otimes \theta$. The foliation $\mathcal{F}_0$ of M_0 induced by σ has the desired properties, except for being a foliation of M_0 instead of M. The Theorem follows taking $\mathcal{F} = \pi_* \mathcal{F}_0$. □

9. Openness of non-solvable dynamics

9.1. **Solvable subgroups of** $\mathrm{Diff}(\mathbb{C}, 0)$**.** If G is a group then we say that G is *solvable* if there exists a positive integer l such that the l^{th}-derived group $D^l G$ is trivial, where $D^l G$ is defined by the following recurrence relations

(1) $D^0 G = G$;
(2) $D^{l+1} G = [D^l G, D^l G]$, where $[D^l G, D^l G]$ is the group generated by the commutators of $D^l G$.

We will say that G has *length* l when l is the smallest positive integer such that $D^l G$ is trivial. When G is a subgroup of $\mathrm{Diff}(\mathbb{C}, 0)$ it is well-know that G is solvable, if and only if, G is meta-abelian, i.e., G has length at most two. An immediate consequence of this fact is the following

Lemma 9.1. *If G is a subgroup of $\mathrm{Diff}(\mathbb{C}, 0)$ then G is solvable if, and only if, for every $k \in \mathbb{N}$ the group $\mathrm{J}^k G$ has length at most 2.*

Proposition 9.2. *Let $\mathcal{L}$ be a line bundle on a projective surface S and $C \subset S$ be a smooth projective curve. Then the subset $\mathrm{Fol}^{\mathrm{sol}}_C(S, \mathcal{L})$ of $\mathrm{Fol}_C(S, \mathcal{L})$ formed by the foliations such that the holonomy group of C is solvable is a closed subset.*

Proof. Given a foliation $\mathcal{F}$ on the surface S such that $C \subsetneq Sing(\mathcal{F})$, let $\Gamma = \pi_1(C \setminus Sing(\mathcal{F}) \cap C, q)$. If $U \subset Fol_C(S, \mathcal{L})$ is a small neighborhood of $\mathcal{F}$, then for every $k \in \mathbb{N}$ we obtain $j_k : U \to \mathrm{Hom}(\Gamma, J^k \mathrm{Diff}(\mathbb{C}, 0))$ such that

(1) $j_k(\mathcal{F}) = \mathrm{Hol}_k(C, \mathcal{F})$;
(2) $j_k(\mathcal{G})$ is the k^{th}-jet of the subgroup of $\mathrm{Hol}(C, \mathcal{G})$ generated by using the holonomy maps associated to Γ.

Let us remark that j_k is a holomorphic map in the sense that $H^k_\gamma(\mathcal{G}) = j_k(\mathcal{G})(\gamma)$ is holomorphic for each $\gamma \in \Gamma$.

If $\mathcal{F} \in U$ has non-solvable holonomy group along C, then by Lemma 9.1 there exists $k_0 \in \mathbb{N}$ and $\gamma_0 \in D^2\Gamma$ such that $H^{k_0}_{\gamma_0}(\mathcal{F}) \neq \mathrm{Id}$. It follows that $H^{k_0}_{\gamma_0}(\mathcal{G}) \neq \mathrm{Id}$ for every $\mathcal{G}$ in a neighborhood V of $\mathcal{F}$, that is, every $\mathcal{G} \in V$ has non-solvable holonomy group along C. □

Let $\Delta_1 \subset \mathrm{Fol}_C(S, \mathcal{L})$ be the Zariski open, connected subset of foliations with singularities along C which have Milnor number equal to 1 (see also Section 5).

Proposition 9.3. $\Delta_1 \cap \mathrm{Fol}_C^{sol}(S,\mathcal{L})$ *is an analytic subset of* Δ_1.

Proof. Let $\mathcal{F} \in \Delta_1 \cap \mathrm{Fol}_C(S,\mathcal{L})$; since the statement is of local nature, it is enough to prove it in a neighborhood U of $\mathcal{F}$ as above. We have then that

$$\mathrm{Fol}_C^{sol}(S,\mathcal{L}) \cap U = \bigcap_{\gamma \in D^2\Gamma, k \in \mathbb{N}} (H_\gamma^k)^{-1}(Id)$$

so that $\mathrm{Fol}_C^{sol}(S,\mathcal{L}) \cap U$ is an analytic subset of U. □

Remark 9.4. If we denote by Δ_2 the set of foliations with exactly one singularity of C with Poincaré-Hopf index two we can repeat the arguments above to prove that $\Delta_2 \cap \mathrm{Fol}_C^{\mathrm{sol}}(M,\mathcal{L})$ is a closed analytic subset. We can also proceed to obtain a stratification of $\mathrm{Fol}_C(M,\mathcal{L})$ in quasi-projective varieties such that the set of foliations with solvable holonomy group along C in each of these strata is a closed analytic subvariety.

It would be interesting to know if $\mathrm{Fol}_C^{\mathrm{sol}}(M,\mathcal{L})$ has a finite number of irreducible components.

9.2. **Proof of Proposition 1.4.** If $\mathcal{L}$ is an ample line-bundle and $n \gg 0$ then by Theorem 1.3 we can construct a foliation in $\mathrm{Fol}_C(S,\mathcal{L}^{\otimes n})$ such that the k-jet, $k \geq 3$, of the holonomy group along C has length at least 3. By Lemma 9.1 this foliation has a non-solvable holonomy group along C. The result follows from Proposition 9.2 and Proposition 9.3. □

Remark 9.5. To generalize Proposition 9.2 to arbitrary projective varieties is sufficient to extend Lemma 9.1 to the case of subgroups of $\mathrm{Diff}(\mathbb{C}^m,0)$. Once this is done Proposition 1.4 follows easily from the existence of finitely generated non-solvable subgroups of $\mathrm{GL}(m,\mathbb{C})$, $m \geq 2$, and Theorem 1.1.

The case of subgroups of $\mathrm{Diff}(\mathbb{C}^2,0)$ has been treated in [1].

Proposition 9.6. *Let G be a finitely generated subgroup of* $\mathrm{Diff}(\mathbb{C}^2,0)$. *Then G is a solvable if, and only if, G has length at most* 7. *In particular G is solvable if, and only if, for every $k \in \mathbb{N}$ the group J^kG has length at most* 7.

A statement similar to Proposition 9.6 should hold for $\mathrm{Diff}(\mathbb{C}^m,0)$, $m > 2$ and Propositions 1.4 and 9.2 should hold for arbitrary projective varieties.

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