FOLIATIONS WITH TRIVIAL CANONICAL BUNDLE ON FANO 3-FOLDS

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Abstract. We classify the irreducible components of the space of foliations on Fano 3-folds with rank one Picard group. As a corollary we obtain a classification of holomorphic Poisson structures on the same class of 3-folds.

1. Introduction

Let X be a projective manifold and N be a line bundle on it. A holomorphic 1-form with coefficients in L defines a codimension one foliation $\mathcal{F}$ if and only if it satisfies the Frobenius integrability condition $\omega \wedge d\omega = 0$ in $H^0(X, \Omega^3_X \otimes N^{\otimes 2})$. If this is the case and ω has zeros of codimension at least two then N is called the normal bundle of $\mathcal{F}$. For a fixed line-bundle N on a fixed projective manifold X, it is natural to study the irreducible components of the variety

$$\mathrm{Fol}(X, N) = \left\{ [\omega] \in \mathbb{P}H^0(X, \Omega^1_X \otimes N) \;\middle|\; \omega \wedge d\omega = 0; \operatorname{codim} \operatorname{sing}(\omega) \geq 2 \right\}$$

which we call the space of codimension one foliations on X with normal bundle N.

If X has dimension two then the integrability condition is automatically satisfied and the space of foliations with a given normal bundle N is either empty or has only one irreducible component which as open subset of the projective space $\mathbb{P}H^0(X, \Omega^1_X \otimes N)$. The discussion from now one will focus on projective manifolds of dimension at least three. When $X = \mathbb{P}^n$ the normal bundle of a codimension one foliation $\mathcal{F}$ is $\mathcal{O}_{\mathbb{P}^n}(d+2)$ where d is the degree of the foliation defined as the number of tangencies $\mathcal{F}$ with a general line. The irreducible components of $\mathrm{Fol}(\mathbb{P}^n, d) = \mathrm{Fol}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d+2))$ for $d = 0$ and $d = 1$ are described by Jouanolou in [18] using elementary methods. In the celebrated work [6], Cerveau and Lins Neto give a complete description of the irreducible components of $\mathrm{Fol}(\mathbb{P}^n, 2) = \mathrm{Fol}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(4))$, $n \geq 3$. The methods are considerably more involved and rely on the study of the Gauss map of the foliations, Dulac's classification of centers of degree 2 polynomial planar vector fields [12], and computer-assisted calculations.

The canonical bundle of a foliation $\mathcal{F}$ with normal bundle N can be defined as $K\mathcal{F} = KX \otimes N^*$, where KX is the canonical bundle of the ambient manifold. Notice that in $\mathbb{P}^n$ the foliations with trivial canonical bundle are precisely those of degree $n-1$. In the particular case of $\mathbb{P}^3$, Cerveau-Lins Neto classification is the classification of the irreducible components of the space of foliations on $\mathbb{P}^3$ with trivial canonical bundle. The main purpose of this paper is to extend this classification to the other Fano 3-folds (3-folds with ample anticanonical bundle) having Picard group isomorphic to $\mathbb{Z}$. Our main result and Cerveau-Lins Neto

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classification of foliations of degree two on $\mathbb{P}^3$ are summarized in Table 1. For more precise statements see Theorems 5.2, 6.1, 7.1, and 8.1. Our results also give a classification of holomorphic Poisson structures on Fano 3-folds with rank one Picard group, see Section 9.

Manifold	**Irreducible component**	dim
Projective space $\mathbb{P}^3$	$\mathrm{Rat}(1,3)$	21
	$\mathrm{Rat}(2,2)$	16
	$\mathrm{Log}(1,1,1,1)$	14
	$\mathrm{Log}(1,1,2)$	17
	$\mathrm{LPB}(2)$	17
	Aff	13
Hyperquadric Q^3	$\mathrm{Rat}(1,2)$	17
	$\mathrm{Log}(1,1,1)$	14
	Aff	8
Hypersurface of degree 6 in $\mathbb{P}(1,1,1,2,3)$	$\mathrm{Rat}(1,1)$	2
Hypersurface of degree 4 in $\mathbb{P}(1,1,1,1,2)$	$\mathrm{Rat}(1,1)$	4
Cubic in $\mathbb{P}^4$	$\mathrm{Rat}(1,1)$	6
Intersection of quadrics in $\mathbb{P}^5$	$\mathrm{Rat}(1,1)$	8
X_5	$\mathrm{Rat}(1,1)$	10
	Aff	1
Mukai-Umemura 3-fold	Aff	1

Table 1. Irreducible components of the space of foliations with $K\mathcal{F} = 0$ on Fano 3-folds with rank one Picard group.

Our main technical tool is the following result obtained by combining Theorem 3.5 and Theorem 3.8 of [23].

Theorem 1.1. *Let (X, H) be a polarized complex projective manifold and $\mathcal{F}$ be a codimension one foliation on X with numerically trivial canonical bundle and semi-stable tangent sheaf. Suppose $c_1(TX)^2 \cdot H^{n-2} > 0$. Then at least one of the following statements holds true:*

1. *$T\mathcal{F}$ is stable and $\mathcal{F}$ is a rationally connected foliation, i.e., the general leaf of $\mathcal{F}$ is a rationally connected algebraic variety;*
2. *$T\mathcal{F}$ is strictly semi-stable and there is a rationally connected foliation $\mathcal{H}$ tangent to $\mathcal{F}$ and with $K\mathcal{H} \cdot H^{n-1} = 0$; or*
3. *$\mathcal{F}$ is defined by a closed rational 1-form with coefficients in a flat line-bundle and without divisorial components in its zero set.*

Indeed we will show that when $\mathrm{Pic}(X) = \mathbb{Z}$, statement (1) implies statement (3). This will be achieved through a study of fibers of rational maps $F : X \dashrightarrow \mathbb{P}^1$, which seen to have some independent interest.

1.1. **Number of reducible fibers of first integrals.** Let $\mathcal{F}$ be a codimension one foliation on a projective manifold X defined by the levels of a rational map $F : X \dashrightarrow C$ from X to some algebraic curve C. If we further assume that F has irreducible general fiber (what can always be done after replacing F by its Stein factorization) and, following [29], define its base number as

$$r(\mathcal{F}) = r(F) = \sum_{x \in C} \left(\#\{ \text{ irreducible components of } F^{-1}(x)\} - 1 \right) .$$

then we obtain a rather strong bound on $r(\mathcal{F})$ under the additional assumption that $T\mathcal{F}$ is stable/semi-stable and has zero/positive first Chern class.

Theorem 1. *Let $\mathcal{F}$ be such a codimension one foliation on a polarized projective manifold (X, H) of dimension at least three. If $T\mathcal{F}$ is H-semi-stable and $K\mathcal{F} \cdot H^{n-1} < 0$, or $T\mathcal{F}$ is H-stable and $K\mathcal{F} \cdot H^{n-1} = 0$, then*

$$r(\mathcal{F}) \leq \operatorname{rank} NS(X) - 1 \,,$$

where $NS(X)$ is the Neron-Severi group of X. In particular, if $X = \mathbb{P}^n$, $n \geq 3$, then $r(\mathcal{F}) = 0$.

Combining this result with a classical Theorem by Halphen about pencils on projective spaces (which we generalize to simply connected projective manifolds in Theorem 3.3) we are able to control the first integrals of (semi)-stable foliations on Fano manifolds with rank one Picard group having (negative) zero canonical bundle.

1.2. **Plan of the paper.** In Section 2 we have collected basic results about foliations that will be used in the sequel. Section 3 studies the relationship between the existence of invariant hypersurfaces and the semi-stability of the tangent sheaf. Besides the proof of Theorem 1, it contains a generalization of a classical result of Halphen, and the classification of foliations with $K\mathcal{F} < 0$ on Fano 3-folds with rank one Picard group (Proposition 3.7). Section 4 gives a rough classification (Theorem 4.1) of foliations with trivial canonical bundle on Fano 3-folds with rank one Picard group. In Section 5 we give a complete classification of foliations with $K\mathcal{F} = 0$ on three-dimensional quadrics, Theorem 5.2. In Section 6 we recall the statement of Cerveau-Lins Neto classification (Theorem 6.1), give a classification of the foliations on $\mathbb{P}^n$ of degree one and arbitrary codimension (Theorem 6.2), and show how to deduce the Cerveau-Lins Neto for $n > 3$ from the classification for $n = 3$ using the classification of foliations of degree one. Sections 7 and 8 deals with cases of index two (Theorem 7.1) and one (Theorem 8.1), respectively. And finally in Section 9 we spell out the classification of holomorphic Poisson structures on Fano 3-folds with rank one Picard group in Theorem 9.1.

2. Basic concepts

2.1. **Foliations as subsheaves of the tangent and cotangent bundles.** A foliation $\mathcal{F}$ on a complex manifold is determined by a coherent subsheaf $T\mathcal{F}$ of the tangent sheaf TX of X which

(1) is closed under the Lie bracket (involutive), and
(2) the inclusion $T\mathcal{F} \to TX$ has torsion free cokernel.

The locus of points where $TX/T\mathcal{F}$ is not locally free is called the singular locus of $\mathcal{F}$, denoted here by $\operatorname{sing}(\mathcal{F})$. Condition (2) implies, in particular, that the codimension of $\operatorname{sing}(\mathcal{F})$ is at least two. The dimension of $\mathcal{F}$, $\dim \mathcal{F}$ for short, is by definition the generic rank of $T\mathcal{F}$. The codimension of $\mathcal{F}$, $\operatorname{codim} \mathcal{F}$, is defined as the integer $\dim X - \dim \mathcal{F}$.

The dual of $T\mathcal{F}$ is the cotangent sheaf of $\mathcal{F}$ and will be denoted by $T^*\mathcal{F}$. The determinant of $T^*\mathcal{F}$, i.e. $(\wedge^{\dim \mathcal{F}} T^*\mathcal{F})^{**}$ is what we will call the canonical bundle of $\mathcal{F}$ and will be denoted by $K\mathcal{F}$.

There is a dual point of view where $\mathcal{F}$ is determined by a subsheaf $N^*\mathcal{F}$ of the cotangent sheaf $\Omega^1_X = T^*X$ of X. The involutiveness asked for in condition (1) above is replace by integrability: $dN^*\mathcal{F} \subset N^*\mathcal{F} \wedge \Omega^1_X$ where d is the exterior derivative. Condition (2) is unchanged: $\Omega^1_X / N^*\mathcal{F}$ is torsion free. The normal bundle of $\mathcal{F}$ is defined as the dual of $N^*\mathcal{F}$. Over the smooth locus $X - \mathrm{sing}(\mathcal{F})$ we have the following exact sequence

$$0 \to T\mathcal{F} \to TX \to N\mathcal{F} \to 0\,,$$

but this is not valid over the singular locus. Anyway, as the singular set has codimension at least two we obtain the adjunction formula

$$KX = K\mathcal{F} \otimes \det N^*\mathcal{F}$$

valid in the Picard group of X.

2.2. **Foliations as q-forms and spaces of foliations.** If $\mathcal{F}$ is a codimension q foliation on a complex variety X then the q-th wedge product of the inclusion

$$N^*\mathcal{F} \longrightarrow \Omega^1_X$$

determines a differential q-form ω with coefficients in the line bundle $\det N\mathcal{F} = (\wedge^q N\mathcal{F})^{**}$ having the following properties:

- **Local decomposability**: the germ of ω at the general point of X decomposes as the product of q germs of holomorphic 1-forms
$$\omega = \omega_1 \wedge \cdots \wedge \omega_q .$$
- **Integrability**: the decomposition of ω at the general point of X satisfies Frobenius integrability condition
$$d\omega_i \wedge \omega = 0 \quad \text{for every } i = 1, \ldots, q\,.$$

The tangent bundle of $\mathcal{F}$ can be recovered as the kernel of the morphism

$$TX \to \Omega^{q-1}_X \otimes \det N\mathcal{F}$$

defined by contraction with ω.

Reciprocally, if $\omega \in H^0(X, \Omega^q \otimes N)$ is a twisted q-form with coefficients in a line bundle N which is locally decomposable and integrable then the kernel of ω has generic rank $\dim X - q$, and it is the tangent bundle of a holomorphic foliation $\mathcal{F}$. Moreover, if the zero set of ω has codimension at least two then $N = \det N\mathcal{F}$.

Example 2.1 (Foliations on $\mathbb{P}^n$ and homogeneous forms)**.** Let $\mathcal{F}$ be a codimension q-foliation on $\mathbb{P}^n$ given by $\omega \in H^0(\mathbb{P}^n, \Omega^q_{\mathbb{P}^n} \otimes N)$. If $i : \mathbb{P}^q \to \mathbb{P}^n$ is a general linear immersion then $i^*\omega \in H^0(\mathbb{P}^q, \Omega^q_{\mathbb{P}^q} \otimes N)$ is a section of a line bundle, and its zero divisor reflects the tangencies between $\mathcal{F}$ and $i(\mathbb{P}^q)$. The degree of $\mathcal{F}$ is, by definition, the degree of such tangency divisor. It is commonly denoted by $\deg(\mathcal{F})$. Since $\Omega^q_{\mathbb{P}^q} \otimes N = \mathcal{O}_{\mathbb{P}^q}(\deg(N) - q - 1)$, it follows that $N = \mathcal{O}_{\mathbb{P}^n}(\deg(\mathcal{F}) + q + 1)$.

The Euler sequence implies that a section ω of $\Omega^q_{\mathbb{P}^n}(\deg(\mathcal{F}) + q + 1)$ can be thought as a polynomial q-form on $\mathbb{C}^{n+1}$ with homogeneous coefficients of degree $\deg(\mathcal{F}) + 1$, which we will still denote by ω, satisfying (*) $i_R \omega = 0$ where $R = x_0 \frac{\partial}{\partial x_0} + \cdots + x_n \frac{\partial}{\partial x_n}$ is the radial vector field. Thus the study of foliations of degree d on $\mathbb{P}^n$ reduces to the study of locally decomposable, integrable homogeneous q-forms of degree $d+1$ on $\mathbb{C}^{n+1}$ satisfying the relation (*).

2.3. **Harder-Narasimhan filtration.** Let $\mathcal{E}$ be a torsion free coherent sheaf on a n-dimensional smooth projective variety X polarized by the ample line bundle H. The slope of $\mathcal{E}$ (more precisely the H-slope of $\mathcal{E}$) is defined as the quotient

$$\mu(\mathcal{E}) = \frac{c_1(\mathcal{E}) \cdot H^{n-1}}{\mathrm{rank}(\mathcal{E})} \, .$$

If the slope of every proper subsheaf $\mathcal{E}'$ of $\mathcal{E}$ satisfies $\mu(\mathcal{E}') < \mu(\mathcal{E})$ (respectively $\mu(\mathcal{E}') \le \mu(\mathcal{E})$) then $\mathcal{E}$ is called stable (respectively semi-stable). A sheaf which is semi-stable but not stable is said to be strictly semi-stable.

There exists a unique filtration of $\mathcal{E}$ by torsion free subsheaves

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_r = \mathcal{E}$$

such that $\mathcal{G}_i := \mathcal{E}_i/\mathcal{E}_{i-1}$ is semi-stable, and $\mu(\mathcal{G}_1) > \mu(\mathcal{G}_2) > \ldots > \mu(\mathcal{G}_r)$. This filtration is called the Harder–Narasimhan filtration of $\mathcal{E}$. Of course $\mathcal{E}$ is semi-stable if and only if $r = 1$. Usually one writes $\mu_{max}(\mathcal{E}) = \mu(\mathcal{G}_1)$ and $\mu_{min}(\mathcal{E}) = \mu(\mathcal{G}_r)$.

We will say that a foliation F is stable/semi-stable/strictcly semi-stable when its tangent sheaf $T\mathcal{F}$ is stable/semi-stable/strictcly semi-stable . When $\mathcal{E}$ is the tangent sheaf of a foliation $\mathcal{F}$, the proof of [20, Chapter 9, Lemma 9.1.3.1] (see also [23, Proposition 2.1]) implies the following result.

Proposition 2.2. *Let $\mathcal{F}$ be a foliation on a polarized smooth projective variety (X, H) satisfying $\mu(T\mathcal{F}) \ge 0$. If $\mathcal{F}$ is not semi-stable then the maximal destabilizing subsheaf of $T\mathcal{F}$ is involutive. Thus there exists a semi-stable foliation $\mathcal{G}$ tangent to $\mathcal{F}$ and satisfying $\mu(T\mathcal{G}) > \mu(T\mathcal{F})$.*

Example 2.3. If $\mathcal{F}$ is a foliation of $\mathbb{P}^n$ then the slope of $T\mathcal{F}$ is

$$\mu(T\mathcal{F}) = \frac{\dim(\mathcal{F}) - \deg(\mathcal{F})}{\dim(\mathcal{F})} \, .$$

Therefore $T\mathcal{F}$ is semi-stable if and only if for every distribution $\mathcal{D}$ tangent to $\mathcal{F}$ we have $\frac{\deg(\mathcal{D})}{\dim(\mathcal{D})} \ge \frac{\deg(\mathcal{F})}{\dim(\mathcal{F})}$. Of course, $T\mathcal{F}$ is stable if and only if the strict inequality holds for every proper distribution $\mathcal{D}$.

If $\mathcal{F}$ is unstable and $\deg(\mathcal{F}) \le \dim(\mathcal{F})$ then there exists a foliation $\mathcal{G}$ contained in $\mathcal{F}$ satisfying

$$\frac{\deg(\mathcal{G})}{\dim(\mathcal{G})} < \frac{\deg(\mathcal{F})}{\dim(\mathcal{F})} \, .$$

2.4. **Miyaoka-Bogomolov-McQuillan Theorem.** We recall that an algebraic variety Y is rationally connected if through any two points $x, y \in Y$ there exists a rational curve C in Y containing x and y. Foliations with all leaves algebraic and with rationally connected general leaf will be called rationally connected foliations. Beware that there exists rationally connected foliation with some leaves non rationally connected, see for instance [23, §2.3]. A fundamental result in the study of holomorphic foliations is the Miyaoka-Bogomolov-McQuillan Theorem see [24, Theorem 8.5], [20, Chapter 9], [3].

Theorem 2.4. *Let $\mathcal{F}$ be a semi-stable foliation on a n-dimensional polarized projective variety (X, H). If $K\mathcal{F} \cdot H^{n-1} < 0$ then $\mathcal{F}$ is a rationally connected foliation.*

2.5. **Closed 1-forms without divisorial components in theirs zero sets.** Let X be a simply-connected projective manifold of dimension at least three. If $D = \sum \lambda_i H_i$ is a $\mathbb{C}$-divisor on X with zero (complex) first Chern class then there exists a unique closed rational 1-form $\eta = \eta_D$ on X with simple poles and residue equal to $\sum \lambda_i H_i$. The associated foliation has normal bundle equal to $\mathcal{O}_X((\eta)_\infty - (\eta)_0)$. If we multiply η by the defining equations of the (reduced) hypersurfaces H_i then we obtain a section of $H^0(X, \Omega^1_X \otimes \mathcal{O}_X(\sum H_i))$. If X has Picard group $\mathbb{Z}$ (generated by $\mathcal{O}_X(1)$) and the divisors H_i have degree d_i then we get a rational

$$\Phi : \Sigma \times \left(\prod_{i-1}^{k} \mathbb{P}H^0(X, \mathcal{O}_X(d_i)) \right) \dashrightarrow \mathbb{P}H^0\big(X, \Omega^1_X(\sum d_i)\big)$$

$$((\lambda_1 : \cdots : \lambda_k), f_1, \ldots, f_k) \mapsto \left(\prod_{i=1}^{k} f_i \right) \left(\sum_{i=1}^{k} \lambda_i \frac{df_i}{f_i} \right) ,$$

where $\Sigma \subset \mathbb{P}^{k-1}$ is the hyperplane $\{\sum \lambda_i d_i = 0\}$. If the domain is not empty then the closure of the image Φ will be denoted by $\mathrm{Log}(d_1, \ldots, d_k)$ when $k \geq 3$, and by $\mathrm{Rat}(d_1, d_2)$ when $k = 2$. Under mild assumptions Calvo-Andrade [4] proved that these subvarieties are irreducible components of the space of foliations on X with normal bundle $\mathcal{O}_X(\sum d_i)$. They are usually called the logarithmic components (when $k \geq 3$) or the rational components (when $k = 2$) of the space of foliations. Notice that in general the image of Φ is not closed: the confluence of hypersurfaces may give rise to indeterminacy points of Φ, and at the closure of its image one may find foliations defined by closed rational 1-forms with poles of higher order. Nevertheless, it can be verified that for any 1-form $[\omega] \in \mathbb{P}H^0(X, \Omega^1_X(\sum d_i))$ in the closure of the image of Φ there exists a section f of $H^0(X, \mathcal{O}_X(\sum d_i))$ such that $f^{-1}\omega$ is a closed rational 1-form.

Even if the assumptions of Calvo-Andrade's result does not hold, a straightforward adaptation of the proof of [6, Lemma 8] shows that any foliation with normal bundle $\mathcal{O}_X(d)$ defined by a closed rational 1-form without divisorial components in its zero set lies in a variety of the form $\mathrm{Log}(d_1, \ldots, d_k)$ with $\sum d_i = d$.

Lemma 2.5. *Let X be a projective manifold with $H^0(X, \Omega^1_X) = 0$. Let $\mathcal{F}$ be codimension one foliation on X defined by a closed rational 1-form ω with zero set of codimension at least two and polar divisor $(\omega)_\infty = \sum_{i=1}^k r_i D_i$. Then there exists a holomorphic family of foliations $\mathcal{F}_t$, $t \in (\mathbb{C}, 0)$, such that*

(1) $\mathcal{F}_0 = \mathcal{F}$;
(2) $N\mathcal{F}_t = N\mathcal{F} = \mathcal{O}_X(\sum_{i=1}^k r_i D_i)$ *for every* $t \in \mathbb{C}$; *and*
(3) $\mathcal{F}_t$ *is defined by a logarithmic* 1*-form for every* $t \neq 0$.

3. First integrals of (semi)-stable foliations

Miyaoka-Bogomolov-McQuillan Theorem (Theorem 2.4) tells us that semi-stable foliations with negative canonical bundle have algebraic leaves and that the general one is rationally connected. The goal of this section is to complement this result for codimension one foliations by giving more information about the first integral. We also deal with stable foliations with numerically trivial canonical bundle having rational first integrals, and the results here presented will play an important role in proof of the classification of codimension one foliations with $K\mathcal{F} = 0$ on Fano 3-folds with rank one Picard group.

3.1. **Invariant hypersurfaces and subfoliations.** Let $\mathcal{F}$ be a foliation of codimension q on a compact Kähler manifold X. Let $\mathrm{Div}(\mathcal{F}) \subset \mathrm{Div}(X)$ be the subgroup of the group of divisors of X generated by irreducible hypersurfaces invariant by $\mathcal{F}$. The arguments used in [15] to prove Jouanolou's theorem lead us to the following result.

Lemma 3.1. *Suppose the dimension of $\mathcal{F}$ is greater than or equal to two. If $D \in \mathrm{Div}(\mathcal{F})$ satisfies $c_1(D) = m \cdot c_1(N\mathcal{F})$ for a suitable $m \in \mathbb{Z}$ then at least one of the following assertions holds true:*

(a) *the integer m is non-zero and $\mathcal{F}$ is, after a ramified abelian covering of degree m and a bimeromorphic morphism, defined by a meromorphic closed q-form with coefficients in a flat line bundle; or*

(b) *the integer m is zero and $\mathcal{F}$ is tangent to a codimension one logarithmic foliation with poles at the support of D and integral residues; or*

(c) *there exists a foliation $\mathcal{G}$ of codimension $q+1$ tangent to $\mathcal{F}$ with normal sheaf satisfying*

$$\det N\mathcal{G} = \det N\mathcal{F} \otimes \mathcal{O}_X(-\Delta)$$

for some effective divisor $\Delta \geq 0$.

Proof. Let $N = \det N\mathcal{F}$ and $\omega \in H^0(X, \Omega^q_X \otimes N)$ be a twisted q-form defining $\mathcal{F}$. Write D as $\sum \lambda_\alpha H_\alpha$ with $\lambda_\alpha \in \mathbb{Z}$.

Our hypothesis ensure the existence of an open covering of $\mathcal{U} = \{U_i\}$ where

$$H_\alpha \cap U_i = \{h^{(i)}_\alpha = 0\} \quad \text{and} \quad \sum \lambda_\alpha \left(\frac{dh^{(i)}_\alpha}{h^{(i)}_\alpha} - \frac{dh^{(j)}_\alpha}{h^{(j)}_\alpha} \right) = m \frac{dg_{ij}}{g_{ij}}$$

where $\{g_{ij}\} \in H^1(\mathcal{U}, \mathcal{O}^*_X)$ is a cocycle defining N, i.e. ω is defined by a collection of q-forms $\{\omega_i \in \Omega^q_X(U_i)\}$ which satisfies $\omega_i = g_{ij}\omega_j$.

On U_i, set $\eta_i = \sum \lambda_\alpha \frac{dh^{(i)}_\alpha}{h^{(i)}_\alpha}$ and define

$$\theta_i = \eta_i \wedge \omega_i + m \cdot d\omega_i \, .$$

As the hypersurfaces H_α are invariant by $\mathcal{F}$, θ_i is a holomorphic $(q+1)$-form. It is also clear that θ_i is locally decomposable and integrable. Moreover, on $U_i \cap U_j$ we have the identity

$$\theta_i = \left(\eta_j - m \frac{dg_{ij}}{g_{ij}} \right) \wedge g_{ij}\omega_j + m \cdot d(g_{ij}\omega_j) = g_{ij}\theta_j \, .$$

Hence the collection $\{\theta_i\}$ defines a holomorphic section θ of $\Omega^{q+1}_X \otimes N$. If this section is non-zero then it defines a foliation $\mathcal{G}$ with $\det N\mathcal{G} = \det N\mathcal{F} \otimes \mathcal{O}_X(-(\theta)_0)$. We are in case (c).

Suppose now that θ is identically zero. If $m = 0$ then $\eta_i = \eta_j$ on $U_i \cap U_j$ and we can patch then together to obtain a logarithmic 1-form η with poles at the support of D. Clearly we are in case (b).

If $m \neq 0$ then on U_i the (multi-valued) meromorphic q-form

$$\Theta_i = \exp\left(\int \frac{1}{m} \eta_i \right) \omega_i = \left(\prod h_{\alpha_i}^{\lambda_\alpha/m} \right) \omega_i$$

is closed. Moreover, if $U_i \cap U_j \neq \emptyset$ then $\Theta_i = \mu_{ij}\Theta_j$ for suitable $\mu_{ij} \in \mathbb{C}^*$. It is a simple matter to see that we are in case (a). □

3.2. **Number of reducible fibers of first integrals.** Let $\mathcal{F}$ be a codimension one foliation on a polarized projective manifold (X, H) having a rational first integral. Stein's factorization ensures the existence of a rational first integral $F : X \dashrightarrow C$ with irreducible general fiber. We are interested in bounding the number of non-irreducible fibers of f. More precisely we want to bound the number

$$r(\mathcal{F}) = r(F) = \sum_{x \in C} \Big(\#\{ \text{ irreducible components of } F^{-1}(x) \} - 1 \Big) \ ,$$

where we do not count the multiplicity of the irreducible components of $F^{-1}(x)$. This problem, for rational functions $F : X \dashrightarrow \mathbb{P}^1$ has been investigated by A. Vistoli and others. In [29] he obtains a bound in function of the rank of the Neron-Severi group of X and what he calls the base number of F. In particular, when X is $\mathbb{P}^n$, he proves that $r(F) \le \deg(F)^2 - 1$ where $\deg(F)$ is the degree of a general fiber of F. Our result below gives much stronger bounds for the first integrals obtained through Theorem 2.4 when $\dim X \ge 3$.

Theorem 3.2. *Suppose the dimension X is at least three. If $\mathcal{F}$ is semi-stable and $c_1(T\mathcal{F}) \cdot H^{n-1} > 0$, or $\mathcal{F}$ is stable and $c_1(T\mathcal{F}) \cdot H^{n-1} = 0$ then*

$$r(\mathcal{F}) \le \operatorname{rank} NS(X) - 1 \, ,$$

where $NS(X)$ is the Neron-Severi group of X. In particular, if $X = \mathbb{P}^n$, $n \ge 3$, then $r(\mathcal{F}) = 0$.

Proof. Let $x_1, \ldots, x_k$ be the points of C for which $F^{-1}(x)$ is non-irreducible, and let $n_1, \ldots, n_k$ be the number of irreducible components of $F^{-1}(x_i)$. Choose $n_i - 1$ irreducible components in each of the non-irreducible fibers and denote them by $F_1, \ldots, F_{r(\mathcal{F})}$. If $r(\mathcal{F}) \ge \operatorname{rank} NS(X)$ then an irreducible fiber F_0 is numerically equivalent to a $\mathbb{Q}$-divisor supported on $F_1 \cup \cdots \cup F_{r(\mathcal{F})}$. Thus there exists a nonzero $D \in \operatorname{Div}(\mathcal{F})$ with zero Chern class and supported on $F_0 \cup \cdots \cup F_{r(\mathcal{F})}$.

Lemma 3.1 implies that either there exists a codimension two foliation $\mathcal{G}$ contained in $\mathcal{F}$ with $\det N\mathcal{G} = N\mathcal{F} \otimes \mathcal{O}_X(-\Delta)$, for some $\Delta \ge 0$; or $\mathcal{F}$ is defined by a logarithmic 1-form η with poles in D. We will now analyze these two possibilities.

If there exists $\mathcal{G}$ as above and $c_1(T\mathcal{F}) \cdot H^{n-1} < 0$ then

$$0 < c_1(T\mathcal{F}) \cdot H^{n-1} = (c_1(T\mathcal{G}) - \Delta) \cdot H^{n-1} \le c_1(T\mathcal{G}) \cdot H^{n-1} \, ,$$

which implies $\mu(T\mathcal{G}) > \mu(T\mathcal{F})$ contradicting the semi-stability of $\mathcal{F}$. Similarly, when $c_1(T\mathcal{F}) \cdot H^{n-1} = 0$ we deduce $\mu(T\mathcal{G}) \ge \mu(T\mathcal{F}) = 0$ contradicting the stability of $\mathcal{F}$.

Suppose now that $\mathcal{F}$ is defined by η. As the general fiber of F is irreducible, there exists a 1-form η' on C such that $\eta = F^*\eta'$. Consequently the polar set of η is set-theoretically equal to a union of fibers of F. This contradicts the choice of $F_1, \ldots, F_{r(\mathcal{F})}$, and concludes the proof. □

3.3. **Multiple fibers of rational maps to $\mathbb{P}^1$.** A classical result of Halphen [16, Chapitre 1] says that a rational map $F : \mathbb{P}^n \dashrightarrow \mathbb{P}^1$ with irreducible general fiber has at most two multiple fibers. In this section we follow closely the exposition of Lins Neto [22] to establish the following generalization.

Theorem 3.3. *Let X be a simply-connected compact Kähler manifold and $F : X \dashrightarrow \mathbb{P}^1$ be meromorphic map. If the general fiber of F is irreducible then F has at most two multiple fibers.*

We will say that a line bundle $\mathcal{L}$ is primitive if its Chern class $c_1(\mathcal{L}) \in H^2(X,\mathbb{Z})$ generates a maximal rank 1 submodule of $H^2(X,\mathbb{Z})$. To adapt Lins Neto's proof of Halphen's Theorem to other manifolds we will need the following lemma.

Lemma 3.4. *Let X be a simply-connected compact complex manifold. If $\mathcal{L} \in \mathrm{Pic}(X)$ is a primitive line bundle on X then the total space of $\mathcal{L}$ minus its zero section is simply-connected.*

Proof. Let E be the total space of $\mathcal{L}$ minus its zero section. As E is a $\mathbb{C}^*$-bundle, we can use Gysin sequence

$$H^1(X,\mathbb{Z}) \to H^1(E,\mathbb{Z}) \to H^0(X,\mathbb{Z}) \stackrel{\wedge c_1(\mathcal{L})}{\longrightarrow} H^2(X,\mathbb{Z})$$

to deduce that the fundamental group of E is torsion. If E is not simply-connected then its universal covering is a $\mathbb{C}^*$-bundle over X, and the associated line bundle divides $\mathcal{L}$. This contradicts the primitiveness of $\mathcal{L}$. □

Proof of Theorem 3.3. Let $\mathcal{L}$ be a primitive line bundle and k a positive integer such that $\mathcal{L}^{\otimes k} = F^*\mathcal{O}_{\mathbb{P}^1}(1)$. If E is the total space of the $\mathbb{C}^*$-bundle defined by $\mathcal{L}^*$ then sections of $\mathcal{L}$ and its positive powers naturally define holomorphic functions on E. Moreover, if $f \in H^0(X,\mathcal{L}^{\otimes k})$ then the element of $H^0(E,\mathcal{O}_E)$ determined by f, which we still denote by f, is homogeneous of degree k with respect to $\mathbb{C}^*$-action on E given by fiberwise multiplication. In particular, if R is the vector field on E with flow defining this $\mathbb{C}^*$-action then we have the Euler identity $i_R df = kf$ on E.

Now suppose $F : X \dashrightarrow \mathbb{P}^1$ has three multiple fibers, of multiplicity p,q,r. Assume that they are over the points $[0:1],[1:0],[1:-1]$. Thus we can write $F = f^p/g^q$ with

$$f^p + g^q + h^r = 0, \tag{1}$$

and $f^p, g^q, h^r \in H^0(X,\mathcal{L}^{\otimes k})$. If we interpret f,g,h now as functions on E then taking the differential of the relation (1) we get

$$pf^{p-1}df + qg^{q-1}dg + rh^{r-1}dh = 0\,.$$

Taking the wedge product first with df and then with dg, we deduce the following equalities between holomorphic 2-forms

$$\frac{df\wedge dg}{h^{r-1}} = \frac{dg\wedge dh}{f^{p-1}} = \frac{df\wedge dh}{g^{q-1}}$$

where we have deliberately omitted irrelevant constants. If we contract these identities with R we get

$$\omega = \frac{\frac{k}{p}fdg - \frac{k}{q}gdf}{h^{r-1}} = \frac{\frac{k}{q}gdh - \frac{k}{r}hdg}{f^{p-1}} = \frac{\frac{k}{p}fdh - \frac{k}{r}hdf}{g^{q-1}}$$

and ω can be interpreted as holomorphic section of $\Omega^1_X \otimes \mathcal{L}^a$ where

$$\frac{a}{k} = \frac{k}{p} + \frac{k}{q} - \frac{(r-1)k}{r} = \frac{k}{q} + \frac{k}{r} - \frac{(p-1)k}{p} = \frac{k}{p} + \frac{k}{r} - \frac{(q-1)k}{q}\,.$$

Since X is Kähler and simply-connected, $H^0(X,\Omega^1_X \otimes \mathcal{L}^{\otimes b}) = 0$ for any $b \leq 0$. Thus $a > 0$, and from this inequality we deduce that

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1 + a > 1\,.$$

This implies that the triple (p,q,r), after reordering, must be one of the following: $(2,2,m),(2,3,3),(2,3,4)$, or $(2,3,5)$.

If (p,q,r) belongs to this list then $\mathbb{C}^2\setminus\{0\}$ is the universal covering of the surface $S_{p,q,r}=\{(x,y,z)\in\mathbb{C}^3\setminus\{0\}|x^p+y^q+z^r=0\}$. Moreover, the entries of the covering map $p=(F,G,H):\mathbb{C}^2\setminus\{0\}\to S_{p,q,r}$ are homogeneous polynomials in two variables satisfying $F^p+G^q+H^r=0$, see [22, Introduction].

Recall that E is simply-connected according to Lemma 3.4. Since the indeterminacy set of F has codimension two, the manifold $E\setminus\{f=g=h=0\}$ is also simply-connected. Therefore we can lift the map

$$\begin{aligned}\varphi : E\setminus\{f=g=h=0\} &\longrightarrow S_{p,q,r}\\ x &\longmapsto (f,g,h)\,.\end{aligned}$$

through the covering map p to a map $\tilde\varphi : E\setminus\{f=g=h=0\}\to\mathbb{C}^2\setminus\{0\}$. The particular form of the covering map described above implies that $\tilde\varphi$ sends fibers of the $\mathbb{C}^*$-bundle E to lines through origin of $\mathbb{C}^2$, and therefore it descends to a rational map $G:X\dashrightarrow\mathbb{P}^1$ which fits into the diagram below.

$$\begin{tikzcd} & \mathbb{P}^1 \arrow[d] \\ X \arrow[ur, dashed, "G"] \arrow[r, dashed, "F"'] & \mathbb{P}^1 \end{tikzcd}$$

Since the vertical arrow is not invertible, the general fiber of F is not irreducible. With this contradiction we conclude the proof. ☐

3.4. **Codimension one stable foliations with first integrals.** Having Theorem 3.3 at hand we are able to give precisions about the structure of the first integrals of semi-stable foliations of codimension one having negative canonical bundle on projective manifolds with rank one Picard group.

Proposition 3.5. *Let X be a projective manifold with $\mathrm{Pic}(X)=\mathbb{Z}$ and $\mathcal{F}$ be a codimension one foliation on X. Suppose*

(a) *$\mathcal{F}$ is semi-stable and $K\mathcal{F}<0$, or*
(b) *$\mathcal{F}$ is stable, has a rational first integral, and $K\mathcal{F}=0$.*

Then $\mathcal{F}$ admits a rational first integral of the form $(f^p:g^q):X\dashrightarrow\mathbb{P}^1$ where p,q are relatively prime positive integers; and f,g are sections of line bundles $\mathcal{L}_1,\mathcal{L}_2$ which satisfy

$$\mathcal{L}_1^{\otimes p}=\mathcal{L}_2^{\otimes q}\quad \text{and}\quad N\mathcal{F}=\mathcal{L}_1\otimes\mathcal{L}_2.$$

In particular $\mathcal{F}$ is defined by a logarithmic 1-form without divisorial components in its zero set.

Proof. Let $F:X\dashrightarrow\mathbb{P}^1$ be a rational first integral for $\mathcal{F}$ with irreducible general fiber. Notice that the target has to be $\mathbb{P}^1$ since $\mathrm{Pic}(X)=\mathbb{Z}$. Theorem 3.2 implies that every fiber of F is irreducible, and Theorem 3.3 tells us that there are at most two non-reduced fibers. Assume that they are over $0,\infty\in\mathbb{P}^1$ and write $F^{-1}(0)=pH_0$, $F^{-1}(\infty)=qH_\infty$ where H_0 and H_∞ are reduced and irreducible hypersurfaces. If we take the logarithmic 1-form on $\mathbb{P}^1$ given in homogeneous coordinates by $dx/x-dy/y$ and we pull-back it by F then the resulting logarithmic 1-form, which defines $\mathcal{F}$, has polar divisor equal to H_0+H_∞ and empty zero divisor. Therefore $N\mathcal{F}=\mathcal{O}_X(H_0+H_\infty)$ and the rational function F can be written as $(f^p:g^q)$ with $f\in H^0(X,\mathcal{O}_X(H_0))$, $g\in H^0(X,\mathcal{O}_X(H_\infty))$. The proposition follows. ☐

Corollary 3.6. *Let $\mathcal{F}$ be a semi-stable codimension one foliation on $\mathbb{P}^n$, $n \geq 3$. If* $\deg(\mathcal{F}) < n-1$ *then $\mathcal{F}$ admits a rational first integral of form $(F^p : G^q)$ where F and G are homogeneous polynomials and p, q are relatively prime positive integers such that* $p\deg(F) = q\deg(G)$ *and* $\deg(\mathcal{F}) = \deg(F) + \deg(G) - 2$.

3.5. **Very negative foliations on Fano manifolds with rank one Picard group.** A projective manifold X is Fano if its anticanonical bundle $-KX$ is ample. Let H be an ample generator of the Picard group of a Fano manifold with $\rho(X) = 1$ ($\rho(X)$ is the rank of the Picard group of X). The **index** of X, denoted by $i(X)$, is defined through the relation $-KX = i(X)H$. The index of a Fano manifold of dimension n is bounded by $n+1$ and the extremal cases are $\mathbb{P}^n$ ($i(X) = n+1$) and hyperquadrics $Q^n \subset \mathbb{P}^{n+1}$ ($i(X) = n$), see [19].

A codimension one foliation of degree one on $\mathbb{P}^n$ has canonical bundle $K\mathcal{F}$ equal to $\mathcal{O}_{\mathbb{P}^n}(2-n)$, see Example 2.1. Our next result can be thought as a generalization of Jouanolou's classification of codimension one foliations of degree one on $\mathbb{P}^n$ [18, Chapter I, Proposition 3.5.1] to arbitrary Fano manifolds with $\rho(X) = 1$.

Proposition 3.7. *Let X be a Fano manifold of dimension $n \geq 3$ and Picard number $\rho(X) = 1$. Let H be an ample generator of* $\mathrm{Pic}(X)$. *If $\mathcal{F}$ is a codimension one foliation on X with $K\mathcal{F} = (2-n)H$ then $\mathcal{F}$ is a foliation of degree one on $\mathbb{P}^n$, or $\mathcal{F}$ is the restriction of a pencil of hyperplanes on $\mathbb{P}^{n+1}$ to a hyperquadric Q^n.*

Proof. Assume first that $\mathcal{F}$ is semi-stable. Theorem 2.4 implies $\mathcal{F}$ has a rational first integral. Proposition 3.5 implies $N\mathcal{F} \geq 2H$. Since $KX = K\mathcal{F} - N\mathcal{F}$, it follows that $KX \leq -nH$. Therefore $KX = -(n+1)H$, $N\mathcal{F} = 3H$ and $X = \mathbb{P}^n$, or $KX = -nH$, $N\mathcal{F} = 2H$ and $X = Q^n$. Proposition 3.5 implies $\mathcal{F}$ is a pencil of quadrics with a non-reduced member in the first case, and a pencil of hyperplane sections of Q^n in the second case.

Suppose now that $\mathcal{F}$ is not semi-stable and let $\mathcal{G}$ be its maximal destabilizing foliation. Therefore

$$-K\mathcal{G} = c_1(T\mathcal{G}) > \frac{-K\mathcal{F}}{\dim(\mathcal{F})} \cdot \dim(\mathcal{G}) \geq (\dim(\mathcal{G}) - 1)H\,.$$

and, consequently, $-K\mathcal{G} \geq \dim(\mathcal{G})H$ and we can produce a non-zero section of $\wedge^{\dim(\mathcal{G})} TX \otimes \mathcal{O}_X(-\dim(\mathcal{G})H)$. It follows from [1, Theorem 1.2] that $X = \mathbb{P}^n$ and $\mathcal{G}$ is a foliation of degree zero on $\mathbb{P}^n$. These have been classified in [7, Théorème 3.8]: a codimension q foliation of degree zero on $\mathbb{P}^n$ is defined by a linear projection from $\mathbb{P}^n$ to $\mathbb{P}^q$. It follows that $\mathcal{F}$ is the linear pull-back of a foliation of degree one on $\mathbb{P}^{n-\dim(\mathcal{G})}$. □

In [2] codimension one foliations with $K\mathcal{F} = (2-n)H$ are called codimension one del Pezzo foliations.

4. Rough structure

The goal of this section is to prove the following result.

Theorem 4.1. *Let X be a Fano 3-fold with* $\mathrm{Pic}(X) = \mathbb{Z}$, *and let $\mathcal{F}$ be a codimension one foliation on X with trivial canonical bundle. If $\mathcal{F}$ is not semi-stable then $X = \mathbb{P}^3$ and $\mathcal{F}$ is the linear pull-back of a degree two foliation on $\mathbb{P}^2$. If $\mathcal{F}$ is semi-stable then at least one of the following assertions holds true:*

(1) *$T\mathcal{F} = \mathcal{O}_X \oplus \mathcal{O}_X$ and $\mathcal{F}$ is induced by an algebraic action;*

(2) *there exists an algebraic action of* $\mathbb{C}$ *or* $\mathbb{C}^*$ *with non-isolated fixed points is tangent to* $\mathcal{F}$*;*
(3) $\mathcal{F}$ *is given by a closed rational* 1*-form without divisorial components in its zero set.*

4.1. **Division Lemma.** To prove Theorem 4.1 we will use the following lemma.

Lemma 4.2. *Let* X *be a projective* 3*-fold,* $\mathcal{G}$ *be a one-dimensional foliation on* X *with isolated singularities, and* $\mathcal{F}$ *a codimension one foliation containing* $\mathcal{G}$*. If* $H^1(X, KX \otimes K\mathcal{G}^{\otimes -2} \otimes N\mathcal{F}) = 0$ *then* $T\mathcal{F} \cong T\mathcal{G} \oplus T\mathcal{H}$ *for a suitable one-dimensional foliation* $\mathcal{H}$*.*

Proof. Let $v \in H^0(X, TX \otimes K\mathcal{G})$ be a twisted vector field defining $\mathcal{G}$. By hypothesis v has isolated zeros. Therefore (see for instance [13, Exercise 17.20]) contraction of differential forms with v defines a resolution of the singular scheme $\mathrm{sing}(\mathcal{G})$ of $\mathcal{G}$:

$$0 \to \Omega^3_X \to \Omega^2_X \otimes K\mathcal{G} \xrightarrow{\Phi} \Omega^1_X \otimes K\mathcal{G}^{\otimes 2} \to K\mathcal{G}^{\otimes 3} \to \mathcal{O}_{\mathrm{sing}(\mathcal{G})} \to 0\,.$$

After tensoring by $N\mathcal{F} \otimes K_{\mathcal{G}}^{\otimes -2}$, we obtain from the exact sequence above the following exact sequences

$$0 \to Im\Phi \otimes K\mathcal{G}^{\otimes -2} \otimes N\mathcal{F} \to \Omega^1_X \otimes N\mathcal{F} \to K\mathcal{G} \otimes N\mathcal{F}\,,$$

and

$$0 \to \Omega^3_X \otimes K\mathcal{G}^{\otimes -2} \otimes N\mathcal{F} \to \Omega^2 \otimes N\mathcal{F} \otimes K\mathcal{G}^{-1} \to Im\Phi \otimes K\mathcal{G}^{\otimes -2} \otimes N\mathcal{F} \to 0\,.$$

If $\omega \in H^0(X, \Omega^1_X \otimes N\mathcal{F})$ defines $\mathcal{F}$ then, since $\mathcal{F}$ contains $\mathcal{G}$, ω belongs to the kernel of

$$H^0(X, \Omega^1_X \otimes N\mathcal{F}) \to H^0(X, K\mathcal{G} \otimes N\mathcal{F})\,.$$

The first sequence tells us that we can lift ω to $H^0(X, Im\Phi \otimes K\mathcal{G}^{\otimes -2} \otimes N\mathcal{F})$. The second exact sequence, together with our cohomological hypothesis, ensures the existence of $\theta \in H^0(X, \Omega^2_X \otimes N\mathcal{F} \otimes K\mathcal{G}^{-1})$ such that $\omega = i_v\theta$. The twisted 2-form θ defines the sought foliation $\mathcal{H}$. □

4.2. **Automorphisms of a foliation.** Let $\mathcal{F}$ be a codimension one foliation on a projective manifold X. The automorphism group of $\mathcal{F}$, $\mathrm{Aut}(\mathcal{F})$, is the subgroup of $\mathrm{Aut}(X)$ formed by automorphisms of X which send $\mathcal{F}$ to itself. It is a closed subgroup of $\mathrm{Aut}(X)$, and therefore the connected component of the identity is a finite dimensional connected Lie group. We will denote by $\mathfrak{aut}(\mathcal{F})$ its Lie algebra, which can be identified with a subalgebra of $\mathfrak{aut}(X) = H^0(X, TX)$. If $\mathcal{F}$ is defined by $\omega \in H^0(X, \Omega^1_X \otimes N\mathcal{F})$ then we define the $\mathfrak{fix}(\mathcal{F})$ as the subalgebra of $\mathfrak{aut}(\mathcal{F})$ annihilating ω, i.e.

$$\mathfrak{fix}(\mathcal{F}) = \{v \in \mathfrak{aut}(\mathcal{F}) \,|\, i_v\omega = 0\}\,.$$

Notice that $\mathfrak{fix}(\mathcal{F})$ is nothing more than $H^0(X, T\mathcal{F})$. We also point out that $\mathfrak{fix}(\mathcal{F})$ is an ideal of $\mathfrak{aut}(\mathcal{F})$, and that subgroup $\mathrm{Fix}(\mathcal{F}) \subset \mathrm{Aut}(\mathcal{F})$ generated by $\mathfrak{fix}(\mathcal{F})$ is not necessarily closed.

Lemma 4.3. *The following assertions hold true:*

(1) *If* $\mathfrak{fix}(\mathcal{F}) = \mathfrak{aut}(\mathcal{F}) \neq 0$ *then there exists a non-trivial algebraic action with general orbit tangent to* $\mathcal{F}$*.*
(2) *If* $\mathfrak{fix}(\mathcal{F}) \neq \mathfrak{aut}(\mathcal{F})$ *then* $\mathcal{F}$ *is generated by a closed rational* 1*-form without divisorial components in its zero set.*

Proof. The connected component of the identity of $\mathrm{Aut}(\mathcal{F})$ is closed. If $\mathfrak{fix}(\mathcal{F}) = \mathfrak{aut}(\mathcal{F})$ then $\mathrm{Fix}(\mathcal{F})$ is also closed and therefore correspond to an algebraic subgroup of $\mathrm{Aut}(X)$. Item (1) follows. To prove Item (2), let v be a vector field in $\mathfrak{aut}(\mathcal{F}) - \mathfrak{fix}(\mathcal{F})$. If $\omega \in H^0(X, \Omega^1_X \otimes N\mathcal{F})$ is a twisted 1-form defining $\mathcal{F}$ then [26, Corollary 2] implies $(i_v\omega)^{-1}\omega$ is a closed meromorphic 1-form. Since the singular set of ω has codimension at least two, the same holds true for the zero set of $(i_v\omega)^{-1}\omega$. □

4.3. **Proof of Theorem 4.1.** If $T\mathcal{F}$ is not semi-stable then Proposition 2.2 implies the existence of a foliation by curves $\mathcal{G}$ tangent to $\mathcal{F}$ and with $\mu(T\mathcal{G}) > 0$. According to Wahl's Theorem [30], X is isomorphic to $\mathbb{P}^3$ and $T\mathcal{G} = \mathcal{O}_{\mathbb{P}^3}(1)$. Thus $\mathcal{G}$ is a foliation of degree zero and, consequently, its leaves are the lines through a point $p \in \mathbb{P}^3$. It follows that $\mathcal{F}$ is a pullback of foliation on $\mathbb{P}^2$ of degree two under the linear projection $\pi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ determined by $\mathcal{G}$.

Suppose now that $T\mathcal{F}$ is stable. If $\mathcal{F}$ is a foliation by algebraic leaves then Proposition 3.5 implies that also in this case $\mathcal{F}$ is defined by a logarithmic 1-form without codimension one components in its zero set. Since Fano manifolds are simply-connected [10, Corollary 4.29], every flat line-bundle on X is trivial. Theorem 1.1 implies that $\mathcal{F}$ is given by a closed rational 1-form without divisorial components in its zero set.

Finally, we will deal with the case where $T\mathcal{F}$ is strictly semi-stable. Now we have a foliation by curves $\mathcal{G}$ tangent to $\mathcal{F}$ with $T\mathcal{G} = \mathcal{O}_X$. In other words, $\mathcal{G}$ is induced by a vector field $v \in H^0(X, TX)$ with zeros of codimension at least two. Notice that $\mathbb{C}v \subset \mathfrak{fix}(\mathcal{F})$.

Suppose $\mathfrak{fix}(\mathcal{F}) = \mathfrak{aut}(\mathcal{F})$. If $\mathfrak{fix}(\mathcal{F}) = \mathbb{C}v$ then we claim $\mathcal{G}$ is defined by an algebraic action of $\mathbb{C}$ or $\mathbb{C}^*$ with non-isolated fixed points. Indeed Lemma 4.3 implies $\mathcal{F}$ is tangent to an action of a one-dimensional Lie group. Moreover, if the action has only isolated fixed points then we can apply Lemma 4.2 to deduce that the tangent bundle of $\mathcal{F}$ is $\mathcal{O}_X \oplus \mathcal{O}_X$. Notice that the hypothesis of Lemma 4.2 are satisfied since $KX \otimes K\mathcal{G}^{\otimes -2} \otimes N\mathcal{F} = \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$ for Fano manifolds.

If we still assume $\mathfrak{fix}(\mathcal{F}) = \mathfrak{aut}(\mathcal{F})$ but now with $\dim \mathfrak{fix}(\mathcal{F}) > 1$ then, as v has no divisorial components in its zero set, any two elements in it will generate $T\mathcal{F}$. Thus $T\mathcal{F} = \mathcal{O}_X \oplus \mathcal{O}_X$ in this case and $\mathcal{F}$ is defined by an algebraic action since $\mathrm{Aut}(\mathcal{F})$ is closed.

Finally, if $\mathfrak{fix}(\mathcal{F}) \neq \mathfrak{aut}(\mathcal{F})$ then Lemma 4.3 implies $\mathcal{F}$ is given by a closed meromorphic 1-form with zero set of codimension at least two. □

5. Foliations on the 3-dimensional quadric

We will now classify the foliations with $K\mathcal{F} = 0$ on the 3-dimensional quadric. We start by presenting an example.

Example 5.1. Identify $\mathbb{P}^4$ with the set of 4 unordered points in $\mathbb{P}^1$. This identification gives a natural action of $PSL(2, \mathbb{C}) \simeq \mathrm{Aut}(\mathbb{P}^1)$ on $\mathbb{P}^4$. Let $p_0 \in \mathbb{P}^4$ be the point defined by the set $\{1, -1, i, -i\} \subset \mathbb{P}^1$. The closure of the $PSL(2, \mathbb{C})$-orbit of p_0 is a smooth quadric $Q^3 \subset \mathbb{P}^4$, see [25]. This quadric can be decomposed as the union of three orbits of $PSL(2, \mathbb{C})$: a closed orbit of dimension one isomorphic to a rational normal curve Γ_4 of degree 4 corresponding to points on $\mathbb{P}^1$ counted with multiplicity 4; an orbit S of dimension two corresponding to two distinct points on $\mathbb{P}^1$, one with multiplicity three and the other with multiplicity one(in more geometric terms this orbit is the tangent surface of Γ_4); and the open orbit of dimension three corresponding to 4 distinct points isomorphic to $\{1, -1, i, -i\}$ The affine subgroup $\mathrm{Aff}(\mathbb{C}) \subset \mathrm{Aut}(\mathbb{P}^1)$ acts on Q^3 fixing the point $p = 4\infty$, and defines on it a codimension one foliation $\mathcal{F}$ with trivial tangent bundle. Notice that the singular set of $\mathcal{F}$ has three irreducible components: Γ_4; a twisted cubic Γ_3 corresponding to points of the form $3p+\infty$; and a line corresponding to points of the form $p+3\infty$. Notice that the foliation $\mathcal{F}$ leaves invariant the surface S (which belongs to the linear system $|\mathcal{O}_{Q^3}(3)|$, see [5, §2.4]), and that the quadratic cone through p (which belongs to $|\mathcal{O}_{Q^3}(1)|$) is the unique hyperplane section invariant by $\mathcal{F}$. This is sufficient to show that $\mathcal{F}$ is not in $\mathrm{Rat}(1,2)$ nor in $\mathrm{Log}(1,1,1)$. Indeed, [8, §5.3.1] implies that the image of the rational parametrization of $\mathrm{Rat}(1,2)$ defined in §2.5 is closed. In particular, foliations in this component do not leave irreducible elements of $|\mathcal{O}_{Q^3}(3)|$, like S, invariant. The rational parametrization of $\mathrm{Log}(1,1,1)$ do not have closed image, but if an element is not on the image then the polar divisor of the corresponding closed rational 1-form η must $2H+H'$ or $3H$ where H, H' are distinct elements of $|\mathcal{O}_X(1)|$. According to the structure of closed rational 1-forms on projective manifolds [31, appendix to Chapter VII], in the first case η is proportional to $h^2h'\left(\frac{dh}{h} - \frac{dh'}{h'} + d(\frac{f}{h})\right)$, and in the second case η is proportional to $h^3 d\left(\frac{g}{h^2}\right)$, where $f, h, h' \in H^0(Q^3, \mathcal{O}_{Q^3}(1))$ and $g \in H^0(Q^3, \mathcal{O}_{Q^3}(2))$. In the former case, the general leaf is not algebraic while in the latter case the general leaf if an element of $|\mathcal{O}_{Q^3}(2)|$. In neither cases the foliation leaves an irreducible element of $|\mathcal{O}_{Q^3}(3)|$ invariant. We conclude that $\mathcal{F}$ does not belong to $\mathrm{Rat}(1,2)$ nor to $\mathrm{Log}(1,1,1)$.

Theorem 5.2. *The irreducible components of space of codimension one foliations with $K\mathcal{F} = 0$ on the hyperquadric Q^3 are* $\mathrm{Rat}(2,1)$, $\mathrm{Log}(1,1,1)$, *and* Aff *(the general element is conjugated to the foliation presented in Example 5.1).*

Theorem 5.2 follows from Theorem 4.1 combined with the next three propositions and Lemma 2.5.

Proposition 5.3. *Let $\mathcal{F}$ be a codimension one foliation on Q^3 with $K\mathcal{F} = 0$. If $\mathcal{F}$ is tangent to an algebraic $\mathbb{C}^*$-action with non-isolated fixed points then $\mathcal{F}$ is given by a closed rational 1-form without divisorial components in its zero set.*

Proof. We can assume that $Q^3 \subset \mathbb{P}^4$ is given by the equation $\{x_0^2 + x_1x_2 + x_3x_4 = 0\}$ and that $\mathbb{C}^* \subset \mathrm{Aut}(Q^3)$ is a subgroup of the form

$$\varphi_\lambda(x_0 : x_1 : x_2 : x_3 : x_4) = (x_0 : \lambda^a x_1 : \lambda^{-a} x_2 : \mu^b x_3 : \mu^{-b} x_4), ,$$

with $a, b \in \mathbb{N}$ relatively prime, since $\mathrm{Aut}(Q^3) = \mathbb{P}O(5, \mathbb{C})$ has rank two. If a and b are distinct non-zero natural numbers then the fixed points of the action are isolated. Thus we have to analyze only two cases: $(a, b) = (0, 1)$ and $(a, b) = (1, 1)$.

Let us start with the case $(a, b) = (0, 1)$. Consider the rational map

$$\Phi : \mathbb{P}^4 \dashrightarrow \mathbb{P}(1,1,1,2) \subset \mathbb{P}^6$$

$$(x_0 : x_1 : x_2 : x_3 : x_4) \mapsto (x_0^2 : x_0x_1 : x_0x_2 : x_1^2 : x_1x_2 : x_2^2 : x_3x_4)\,,$$

which identifies $\mathbb{P}(1,1,1,2)$ with a cone over the Veronese surface in $\mathbb{P}^5$. Notice that the quadric Q^3 is mapped to a hyperplane section of $\mathbb{P}(1,1,1,2)$ not passing through the vertex $(0 : 0 : 0 : 0 : 0 : 0 : 1)$, which is of course isomorphic to $\mathbb{P}^2$. We will denote by Φ_0 the induced rational map $\Phi_0 : Q^3 \dashrightarrow \mathbb{P}^2$. The general fiber of Φ_0 is an orbit of φ, and therefore the foliation $\mathcal{F}$ must be the pull-back of a foliation $\mathcal{H}$ on $\mathbb{P}^2$. Notice also that $\Phi_0^*\mathcal{O}_{\mathbb{P}^2}(1)$ is equal to $\mathcal{O}_{Q^3}(1)$. A simple computation shows that the critical set of Φ_0 has codimension greater than two. Thus $\mathcal{O}_{Q^3}(3) = N\mathcal{F} = \Phi_0^* N\mathcal{H}$. It follows that $N\mathcal{H} = \mathcal{O}_{\mathbb{P}^2}(3)$, i.e., $\mathcal{H}$ has degree one. Since every foliation of degree one on $\mathbb{P}^2$ is induced by a closed meromorphic 1-form with isolated singularities [18, Chapter 1, Section 2] the proposition follows in this case.

Suppose now that $(a,b) = 1$, and consider the rational map

$$\begin{aligned} \Phi : \mathbb{P}^4 &\dashrightarrow \mathbb{P}^4 \\ (x_0 : x_1 : x_2 : x_3 : x_4) &\mapsto (x_0^2 : x_1x_2 : x_1x_4 : x_2x_3 : x_3x_4)\,. \end{aligned}$$

Its image is contained in a cone over a smooth quadric surface in $\mathbb{P}^3$. The quadric Q^3 is mapped into a smooth hyperplane section of this cone which is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. If we denote by $\Phi_0 : Q^3 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ the induced rational map then $\Phi_0^*\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(c,d) = \mathcal{O}_{Q^3}(c+d)$. The only divisorial component of the critical set of Φ_0 is the intersection of the hyperplane $\{x_0 = 0\}$ with Q^3. The image of this critical set is a $(1,1)$ curve C in $\mathbb{P}^1 \times \mathbb{P}^1$. If $\mathcal{G}$ is a foliation on $\mathbb{P}^1 \times \mathbb{P}^1$ with normal bundle $\mathcal{N}\mathcal{G} = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(c,d)$ then

$$N\Phi_0^*\mathcal{G} = \begin{cases} \mathcal{O}_{Q^3}(c+d) & \text{if} \quad C \text{ is not } \mathcal{G}\text{-invariant} \\ \mathcal{O}_{Q^3}(c+d-1) & \text{if} \quad C \text{ is } \mathcal{G}\text{-invariant}\,. \end{cases}$$

Therefore if $\mathcal{F} = \Phi_0^*\mathcal{G}$ and $N\mathcal{F} = \mathcal{O}_{Q^3}(3)$ then $c = d = 2$ and C is $\mathcal{G}$-invariant. A foliation $\mathcal{G}$ on $\mathbb{P}^1 \times \mathbb{P}^1$ with $N\mathcal{G} = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2,2)$ is given by a closed rational 1-form $\omega = \pi_1^*\omega_1 + \pi_2^*\omega_2$ where $\pi_1, \pi_2 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ are the natural projections and the 1-forms ω_i have polar set of degree two. Since the $(1,1)$-curve C is $\mathcal{G}$-invariant, we must have $\omega_1 = -\omega_2 = dx_0/x_0 - dx_1/x_1$ in a suitable choice of coordinates where $C = \{x_0y_1 - y_0x_1 = 0\}$. Therefore

$$\omega = \frac{dx_0}{x_0} - \frac{dx_1}{x_1} - \frac{dy_0}{y_0} + \frac{dy_1}{y_1}\,.$$

Notice that ω is proportional to

$$\alpha = \left(\frac{d(x_0y_1 - y_0x_1)}{x_0y_1 - y_0x_1} - \frac{dx_0}{x_0} - \frac{dy_1}{y_1}\right).$$

and the pull-back of α under Φ_0 is closed 1-form without divisorial components in its zero set. $\square$

Proposition 5.4. *Let $\mathcal{F}$ be a codimension one foliation on Q^3 with $K\mathcal{F} = 0$. If $\mathcal{F}$ is tangent to an algebraic $\mathbb{C}$-action with non-isolated fixed points then $\mathcal{F}$ is given by a closed rational 1-form without divisorial components in its zero set.*

Proof. Let $\varphi : \mathbb{C} \times Q^3 \to Q^3$ be an algebraic $\mathbb{C}$-action. As such, it must be of the form $\varphi(t) = \exp(t\cdot n)$ where n is a nilpotent element of the Lie algebra $\mathfrak{aut}(Q^3) = \mathfrak{so}(5,\mathbb{C})$. In $\mathfrak{so}(5,\mathbb{C})$ there are exactly three $\mathrm{Aut}(Q^3) = \mathbb{P}O(5,\mathbb{C})$-conjugacy classes of non-zero nilpotent elements. The Jordan normal forms of the corresponding matrices in $End(\mathbb{C}^5)$ have: (1) only one Jordan block of order 5; (2) one Jordan block of order 3 and two trivial (order one) Jordan blocks ; or (3) two Jordan blocks of order 2 and one trivial Jordan block.

The action in case (1) has isolated fixed points and is excluded by hypothesis. To deal with case (2) we can assume that $n = x_1\frac{\partial}{\partial x_0} + x_2\frac{\partial}{\partial x_1}$ and that the quadric

Q^3 is $\{x_1^2 - 2x_0x_2 + x_3^2 + x_4^2 = 0\}$. The general fiber of the rational map

$$\Phi : \mathbb{P}^4 \dashrightarrow \mathbb{P}^6$$
$$(x_0 : x_1 : x_2 : x_3 : x_4) \mapsto (x_1^2 - 2x_0x_2 : x_2^2 : x_2x_3 : x_2x_4 : x_3^2 : x_3x_4 : x_4^2)$$

coincides with an orbit of φ, and sends $\mathbb{P}^4$ to a cone over the second Veronese embedding of $\mathbb{P}^2$. The image of the quadric Q^3 avoids the vertex of this cone and is isomorphic to $\mathbb{P}^2$. Moreover, the critical set of $\Phi_0 : Q \dashrightarrow \mathbb{P}^2$ (the restriction of Φ to Q) has no divisorial components. Therefore every foliation $\mathcal{F}$ on Q^3 tangent to φ is of the form $\Phi_0^*\mathcal{G}$ for some foliation on $\mathbb{P}^2$ and its normal bundle satisfies $N\mathcal{F} = \Phi_0^* N\mathcal{G}$. Since $\Phi_0^*\mathcal{O}_{\mathbb{P}^2}(1) = \mathcal{O}_Q(1)$, it follows that $\mathcal{F}$ is the pull-back of a foliation $\mathcal{G}$ on $\mathbb{P}^2$ of degree one and, as such, is given by a closed 1-form without zeros of codimension one [18, Chapter 1, Section 2].

Case (3) is very similar to case (2). Now the vector field n is of the form $x_1\frac{\partial}{\partial x_0} + x_3\frac{\partial}{\partial x_2}$, the quadric is $Q = \{x_0x_3 - x_1x_2 + x_4^2 = 0\}$ and the quotient map is

$$\Phi : \mathbb{P}^4 \dashrightarrow \mathbb{P}^6$$
$$(x_0 : x_1 : x_2 : x_3 : x_4) \mapsto (x_0x_3 - x_1x_2 : x_1^2 : x_1x_3 : x_1x_4 : x_3^2 : x_3x_4 : x_4^2)\,.$$

The restriction of Φ to Q has critical set of codimension at least two, and therefore the conclusion is the same: $\mathcal{F}$ is the pull-back under $\Phi_{|Q}$ of a foliation on $\mathbb{P}^2$ of degree one, and as such is defined by a closed rational 1-form with zeros of codimension at least two. □

Proposition 5.5. *Let $\mathcal{F}$ be a codimension one foliation on Q^3 with trivial canonical bundle. Suppose that $\mathcal{F}$ is induced by an algebraic action of a two dimensional Lie subgroup of* $\mathrm{Aut}(Q^3)$. *Then $\mathcal{F}$ is defined by a closed* 1*-form without zeros of codimension one, or $\mathcal{F}$ is conjugated to the foliation presented in Example 5.1.*

Proof. Let $G \subset \mathrm{Aut}(Q^3)$ be the subgroup defining $\mathcal{F}$, and $\mathfrak{g} \subset \mathfrak{so}(5,\mathbb{C})$ the corresponding Lie subalgebra. If G is abelian then it must be of the form $\mathbb{C}^* \times \mathbb{C}^*$, $\mathbb{C} \times \mathbb{C}^*$, or $\mathbb{C} \times \mathbb{C}$. In the first case every element in $\mathfrak{g}$, the Lie algebra of G, is a semi-simple element of $\mathfrak{so}(5,\mathbb{C})$. Since the rank of $\mathfrak{so}(5,\mathbb{C})$ is two, $\mathfrak{g}$ is a Cartan subalgebra of $\mathfrak{so}(5,\mathbb{C})$. Therefore, we can find $\mathbb{C}^* \subset G$ inducing an algebraic action with non-isolated fixed points tangent to $\mathcal{F}$. We can apply Proposition 5.3 to conclude that $\mathcal{F}$ is induced by a closed 1-form without codimension one zeros. In the two remaining cases, $\mathfrak{g}$ contains a nilpotent element n which defines an algebraic subalgebra $\mathbb{C} \subset G$. If the corresponding action has non-isolated fixed points then Proposition 5.4 implies $\mathcal{F}$ is defined by a closed rational 1-form without divisorial components in its zero set.

If the corresponding action has only isolated fixed points then we can assume that Q is defined by the quadratic form $q = x_2^2 - 2x_1x_3 + 2x_0x_4$ and that n, seen as an element of $\mathfrak{so}(q,\mathbb{C})$, has only one Jordan block of order 5. The centralizer $C(n)$ of n in $\mathfrak{so}(q,\mathbb{C})$ is thus formed by nilpotent matrices of the form

$$\begin{pmatrix} 0 & \alpha & 0 & \beta & 0 \\ 0 & 0 & \alpha & 0 & \beta \\ 0 & 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

In particular, since $\mathfrak{g} \subset C(n)$, $\mathfrak{g}$ contains another nilpotent element which defines a $\mathbb{C}$-action with non-isolated fixed points. Proposition 5.4 implies $\mathcal{F}$ is defined by a closed 1-form without codimension one zeros.

Suppose now that G is not abelian. Its Lie algebra $\mathfrak{g}$ is isomorphic to the affine Lie algebra $\mathbb{C}x \oplus \mathbb{C}y$ with the relation $[x, y] = y$. This relation implies that y is a nilpotent element of $\mathfrak{so}(5, \mathbb{C}) \subset \mathfrak{sl}(5, \mathbb{C})$. As before, using Proposition 5.4, we can reduce to the case where y is in Jordan normal form and has only one Jordan block of order 5. The elements $x \in \mathfrak{so}(q, \mathbb{C})$ satisfying $[x, y] = y$ are of the form

$$\begin{pmatrix} 2 & \alpha & 0 & \beta & 0 \\ 0 & 1 & \alpha & 0 & \beta \\ 0 & 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & -1 & \alpha \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix}.$$

After one last conjugation by an element of $SO(q, \mathbb{C})$ we can suppose that $\beta = 0$. We have just proved that up to automorphisms of Q^3 there is only one foliation defined by an algebraic action of an algebraic subgroup $G \subset \mathrm{Aut}(Q^3)$ which is not invariant an algebraic action of a one-dimensional Lie group with non-isolated fixed points. Therefore it must be the foliation described in Example 5.1. □

6. Foliations on projective spaces

Let us recall the main result of [6].

Theorem 6.1. *The irreducible components of the space of codimension one foliations of degree* 2 *on* $\mathbb{P}^n$*,* $n \geq 3$*, are* $\mathrm{Rat}(2, 2)$*,* $\mathrm{Rat}(1, 3)$*,* $\mathrm{Log}(1, 1, 1, 1)$*,* $\mathrm{Log}(2, 1, 1)$*,* $\mathrm{LPB}(2)$*, and* $Exc(2)$*. The general element of* $Exc(2)$ *is a linear pull-back from the foliation on* $\mathbb{P}\mathbb{C}_3[x]$ *induced by the natural action of the affine group.*

For a detailed description of the general element of $Exc(2)$ on $\mathbb{P}^3$ see [5].

Following essentially the same steps as used to proof Theorem 5.2 one can recover Theorem 6.1 for $n = 3$ without using Dulac's classification of quadratic centers [12] (see also [6, Theorem 7]) and bypassing the computer-assisted calculations used to prove [6, Theorem E']. Nevertheless, to establish analogues of Propositions 5.3, and 5.4 following the same strategy as above, one would be lead to a lengthy case-by-case analysis which we have chosen to not carry out here for details but which can be found in version 2 of [23] at the arXiv. Instead, we will present below a proof of Theorem 6.1 for $n > 3$ assuming that it holds true for $n = 3$, as it may serve as a model to extend the results of the previous section to higher dimensional hyperquadrics. We start with the classification of degree one foliations of arbitrary codimension on $\mathbb{P}^n$, a result of independent interest which is used in [2].

6.1. The space of foliations on $\mathbb{P}^n$ of degree one (arbitrary codimension).

We already recalled the classification of the foliations of degree zero in the proof of Proposition 3.7: a codimension q foliation of degree zero on $\mathbb{P}^n$ is defined by a linear projection from $\mathbb{P}^n$ to $\mathbb{P}^q$. The classification of foliations of degree one can be easily deduce from Medeiros classification of locally decomposable integrable homogeneous q-forms of degree one ([9, Theorem A]) as we show below.

Theorem 6.2. *If* $\mathcal{F}$ *be a foliation of degree* 1 *and codimension* q *on* $\mathbb{P}^n$ *then we are in one of following cases:*

(1) *$\mathcal{F}$ is defined a dominant rational map $\mathbb{P}^n \dashrightarrow \mathbb{P}(1^q, 2)$ with irreducible general fiber determined by q linear forms and one quadratic form; or*
(2) *$\mathcal{F}$ is the linear pull back of a foliation of induced by a global holomorphic vector field on $\mathbb{P}^{q+1}$.*

Proof. We start by recalling [9, Theorem A]: if ω is a locally decomposable integrable homogeneous q-form of degree 1 on $\mathbb{C}^{n+1}$ then

(a) there exist $q-1$ linear forms $L_1, \ldots, L_{q-1}$ and a quadratic form Q such that $\omega = dL_1 \wedge \cdots \wedge dL_{q-1} \wedge dQ$; or
(b) there exist a linear projection $\pi : \mathbb{C}^{n+1} \to \mathbb{C}^{q+1}$, and a locally decomposable integrable homogeneous q-form $\widetilde{\omega}$ of degree 1 on $\mathbb{C}^{q+1}$ such that $\omega = \pi^* \widetilde{\omega}$.

Let ω be a locally decomposable, integrable homogeneous q-form on $\mathbb{C}^{n+1}$ defining $\mathcal{F}$. Since $\mathcal{F}$ has degree 1, the degree of the coefficients of ω is 2. It is immediate from the definitions that the differential of a locally decomposable integrable q-form is also locally decomposable and integrable. Therefore we can apply [9, Theorem A] to $d\omega$. To recover information about ω we will use that $i_R \omega = 0$ implies $i_R d\omega = (q+2) \cdot \omega$.

If $d\omega$ is in case (a) then $d\omega$ is the pull-back of $dx_0 \wedge \cdots \wedge dx_q$ under the map

$$\mathbb{C}^{n+1} \ni (x_0, \ldots, x_n) \mapsto (L_1, \ldots, L_q, Q) \in \mathbb{C}^{q+1},$$

and $(q+2)\omega = i_R d\omega$ is the pull-back of $i_{R(1^q,2)} dy_0 \wedge \cdots \wedge dy_q$ where $R(1^q, 2) = y_0 \frac{\partial}{\partial y_0} + \cdots + y_q \frac{\partial}{\partial y_q} + 2 y_{q+1} \frac{\partial}{\partial y_{q+1}}$. We are clearly in case (a) of the statement with rational map from $\mathbb{P}^n \dashrightarrow \mathbb{P}(1^q, 2)$ described in homogeneous coordinates as above. It still remains to check that the general fiber is irreducible. As ω has zero set of codimension at least two, the same holds true for $d\omega$ and consequently the map considered does not ramify in codimension one. Since $\mathbb{P}^n$ is simply-connected, the irreducibility of the general fiber follows.

If $d\omega$ is in case (b) then, in suitable coordinates, $d\omega$ depends only on $q+2$ variables, say $x_0, \ldots, x_{q+1}$. Being a $(q+1)$-form with coefficients of degree 1, there exists a linear vector field X such that $d\omega = i_X dx_0 \wedge \cdots \wedge dx_{q+1}$. The result follows. □

Corollary 6.3. *The space of foliations of degree* 1 *and codimension q on $\mathbb{P}^n$ has two irreducible components.*

Proof that Theorem 6.1 for $n = 3$ implies Theorem 6.1 for $n > 3$. Notice that when $n > 3$, a foliation of degree two has negative canonical bundle. Thus, if $\mathcal{F}$ is semi-stable Proposition 3.5 implies that $\mathcal{F}$ is either a pencil of quadrics or a pencil of cubics having a hyperplane with multiplicity three as a member.

Suppose now that $\mathcal{F}$ is unstable and let $\mathcal{G}$ be its maximal destabilizing foliation. Recall from Example 2.3 that

$$\frac{\deg(\mathcal{G})}{\dim(\mathcal{G})} < \frac{\deg(\mathcal{F})}{\dim(\mathcal{F})} .$$

Therefore $\deg(\mathcal{G}) < 2$. If $\mathcal{G}$ has degree zero then $\mathcal{F}$ is a linear pull-back of a foliation of degree two on a lower-dimensional projective space and we can proceed inductively. Suppose now that the degree of $\mathcal{G}$ is one. The classification of foliations of degree one, Theorem 6.2, implies that the semi-stable foliations of degree one are either defined by a rational map to $\mathbb{P}(1^q, 2)$ or have dimension one. The maximal destabilizing foliation $\mathcal{G}$, which is semi-stable by definition, does not fit into the former case as we would have $1 < \deg(\mathcal{F})/\dim(\mathcal{F})$. Thus $\mathcal{G}$ must be defined by a rational map to $\mathbb{P}(1^q, 2)$. It is not hard to verify that in this case the foliation $\mathcal{F}$ must be in the component $\mathrm{Log}(1, 1, 2)$. □

7. Foliations on Fano 3-folds of index two

It remains to deal with foliations with $K\mathcal{F} = 0$ on Fano 3-folds of index one and two. Unlikely in the cases where the index is four (projective space) or three (quadric), these 3-folds have moduli. As will be seen below the space of foliations with $K\mathcal{F} = 0$ on them behaves rather uniformly with respect to the moduli, with only two exceptions. The exceptions are the quasi-homogeneous $PSL(2, \mathbb{C})$–manifolds of index one and two.

Let X be a Fano 3-fold with $\mathrm{Pic}(X) = \mathbb{Z}H$ and index $i(X) = 2$ which means, by definition, $-KX = 2H$. In this case the classification is very precise (see [17] and references therein) and says that X is isomorphic to a 3-fold fitting in one of the following classes:

(1) $H^3 = 1$. Hypersurface of degree 6 in $\mathbb{P}(1, 1, 1, 2, 3)$;
(2) $H^3 = 2$. Hypersurface of degree 4 in $\mathbb{P}(1, 1, 1, 1, 2)$;
(3) $H^3 = 3$. Cubic in $\mathbb{P}^4$;
(4) $H^3 = 4$. Intersection of two quadrics in $\mathbb{P}^5$;
(5) $H^3 = 5$. Intersection of the Grassmannian $\mathrm{Gr}(2, 5) \subset \mathbb{P}^9$ with a $\mathbb{P}^6$.

Although not evident from the description above, the 3-folds falling in class (5) are all isomorphic to a 3-fold $X_5 \subset \mathbb{P}^6$. In [25] X_5 is described as an equivariant compactification of $\mathrm{Aut}(\mathbb{P}^1)/\Gamma$ where Γ is the octahedral group. Explicitly, if we consider the point $p_0 \in \mathrm{Sym}^6\,\mathbb{P}^1$ defined by the polynomial $xy(x^4 - y^4)$ then X_5 is the closure of the $\mathrm{Aut}(\mathbb{P}^1)$-orbit of p_0 under the natural action.

Theorem 7.1. *Let X be a Fano 3-fold with $\mathrm{Pic}(X) = \mathbb{Z}H$ and index $i(X) = 2$. If $X \neq X_5$ then the space of codimension one foliations on X with trivial canonical bundle is irreducible. If $X = X_5$ then the space of codimension one foliations on X with trivial canonical bundle has two irreducible components.*

As we will see from its proof the result is much more precise as it describes quite precisely the irreducible components. We summarize the description in the Table below.

Manifold	**Irreducible component**	dim
Hypersurface of degree 6 in $\mathbb{P}(1, 1, 1, 2, 3)$	$\mathrm{Rat}(1, 1)$	2
Hypersurface of degree 4 in $\mathbb{P}(1, 1, 1, 1, 2)$	$\mathrm{Rat}(1, 1)$	4
Cubic in $\mathbb{P}^4$	$\mathrm{Rat}(1, 1)$	6
Intersection of quadrics in $\mathbb{P}^5$	$\mathrm{Rat}(1, 1)$	8
X_5	$\mathrm{Rat}(1, 1)$	10
	Aff	1

Lemma 7.2. *The dimension of $H^0(X_5, TX_5)$ is 3, and every $v \in H^0(X_5, TX_5)$ has isolated singularities.*

Proof. Let Σ be the variety of lines contained in X_5. According to [14], Σ is isomorphic to $\mathbb{P}^2$. The induced action of $\mathrm{Aut}(\mathbb{P}^1)$ on it has one closed orbit isomorphic to a conic $C \subset \mathbb{P}^2$, and one open orbit isomorphic to $\mathbb{P}^2 \setminus C$. It can be identified with the natural action of $\mathrm{Aut}(\mathbb{P}^1)$ in $\mathrm{Sym}^2\,\mathbb{P}^1 \simeq \mathbb{P}^2$. If an automorphism of X_5 induces the identity on Σ then it must be identity since through every point of every line contained in X_5 passes at least another line, loc. cit. Corollary 1.2. This suffices to show that $h^0(X_5, TX_5) = 3$.

Let now $v \in H^0(X_5, TX_5)$ be a non-zero vector field, and $H = \exp(\mathbb{C}v) \subset \mathrm{Aut}(X_5)$ be the one-parameter subgroup generated by it. The description of the induced action of $\mathrm{Aut}(X)$ on Σ implies that the induced action of H on Σ has isolated fixed points. Therefore, if the zero set of v has positive dimension then it must be contained in a finite union of lines. If we take ℓ as one of these lines then the action of H on Σ would fix all the lines intersecting ℓ. This contradicts the description of the induced action of $\mathrm{Aut}(X)$ on Σ. $\square$

Lemma 7.3. *Let $\mathbb{P} = \mathbb{P}(q_0, q_1, q_2, q_3, q_4)$ be a well-formed weighted projective space of dimension four with $q_0 \le q_1 \le q_2 \le q_3 \le q_4$, and $X \subset \mathbb{P}$ be a smooth hypersurface. If $\deg(X) \ge q_2 + q_3 + q_4$ then $h^0(X, TX) = 0$.*

Proof. Set $d = \deg(X)$ and $Q = \sum_{i=0}^4 q_i$. By [11, Theorem 3.3.4], $\Omega^3_X = \mathcal{O}_X(d-Q)$. Consequently $TX = \Omega^2_X \otimes \mathcal{O}_X(Q-d)$. From the long exact sequence associated to

$$0 \to \Omega^1_X \otimes N^*_X \otimes \mathcal{O}_X(Q-d) \to \Omega^2_{\mathbb{P}X}(Q-d) \to \Omega^2_X(Q-d) \to 0$$

we see that $h^0(X, TX) = 0$ when $h^0(X, \Omega^2_{\mathbb{P}X}(Q-d)) = h^1(X, \Omega^1_X(Q-2d)) = 0$.

To compute $h^1(X, \Omega^1_X(Q-2d))$, consider the conormal sequence of $X \subset \mathbb{P}$ tensored by $\mathcal{O}_X(Q-2d)$

$$0 \to N^*_X(Q-2d) \to \Omega^1_{\mathbb{P}X}(Q-2d) \to \Omega^1_X(Q-2d) \to 0\,.$$

On the one hand, as the intermediary cohomology of $\mathcal{O}_X(n)$ vanishes for every $n \in \mathbb{Z}$ [11, Theorem 3.2.4 (iii)], $h^2(X, N^*_X(Q-2d)) = h^2(X, \mathcal{O}_X(Q-3d)) = 0$. On the other hand $H^1(X, \Omega^1_{\mathbb{P}X}(Q-2d))$ can be computed with the exact sequence

$$0 \to \Omega^1_{\mathbb{P}}(Q-3d) \to \Omega^1_{\mathbb{P}}(Q-2d) \to \Omega^1_{\mathbb{P}|X}(Q-2d) \to 0\,.$$

Now [11, Theorem 2.3.2] tell us that $h^2(\mathbb{P}, \Omega^1_{\mathbb{P}}(n) = 0$ for every $n \in \mathbb{Z}$, and $h^1(\mathbb{P}, \Omega^1_{\mathbb{P}}(n)) = 0$ if and only if $n \neq 0$. But $d \ge q_2 + q_3 + q_4$, as we have assumed, implies $2d > Q$. Thus $h^1(X, \Omega^1_{\mathbb{P}X}(Q-2d)) = 0$ as wanted.

It remains to show that $h^0(X, \Omega^2_{\mathbb{P}|X}(Q-d)) = 0$. To do it, consider the exact sequence

$$0 \to \Omega^2_{\mathbb{P}}(Q-2d) \to \Omega^2_{\mathbb{P}}(Q-d) \to \Omega^2_{\mathbb{P}|X}(Q-d) \to 0\,.$$

The vanishing of $h^1(\mathbb{P}, \Omega^2_{\mathbb{P}}(Q-2d))$ is assured by [11, Theorem 2.3.4]. Finally, [11, Corollary 2.3.4] implies $h^0(\mathbb{P}, \Omega^2_{\mathbb{P}}(Q-d)) \neq 0$ if and only if

$$d < Q - q_0 - q_1\,.$$

The lemma follows. $\square$

Lemma 7.3 together with the classification of Fano 3-folds of index two imply the following corollary.

Corollary 7.4. *If X is a Fano 3-fold with $\rho(X) = 1$ and $i(X) = 2$ then $h^0(X, TX) \neq 0$ if and only if X is isomorphic to X_5.*

Proof of Theorem 7.1. Let X be a Fano 3-fold of index two with $\mathrm{Pic}(X) = \mathbb{Z}\cdot H$, where H is an ample divisor, and $\mathcal{F}$ a codimension one foliation on X with $K\mathcal{F} = 0$. If $H^3 \leq 4$ then Corollary 7.4 implies X has no vector fields. Therefore by Theorem 4.1 any foliation on X with $K\mathcal{F} = 0$ is given by a closed 1-form without codimension one zeros and with polar divisor linearly equivalent to $2H$. The result follows Lemma 2.5. Notice that the dimension of $H^0(X, \mathcal{O}_X(H))$ is equal to $H^3 + 2$, [21, Chapter V, Exercise 1.12.6].

Suppose now that $H^3 = 5$, i.e., $X = X_5$. Lemma 7.2 implies that every algebraic action of $\mathbb{C}$ or $\mathbb{C}^*$ has isolated fixed points. Theorem 4.1 tells us that a foliation on X_5 with trivial canonical bundle is either induced by an algebraic action of two dimensional Lie group or is given by a closed 1-form without codimension one zeros and with polar divisor linearly equivalent to $2H$. The Lie algebra of regular vector fields on X_5 is isomorphic to $\mathfrak{sl}(2,\mathbb{C})$ (Lemma 7.2) and the two dimension subalgebras are all $PSL(2,\mathbb{C})$-conjugated, and isomorphic to the affine Lie algebra $\mathfrak{aff}(\mathbb{C})$. Let $\mathcal{F}$ be a foliation of X_5 determined by any of the affine Lie algebras contained into $\mathfrak{sl}(2,\mathbb{C})$. The induced action of $\mathrm{Aff}(\mathbb{C}) \subset \mathrm{Aut}(X)$ on $\mathbb{P}H^0(X, \Omega^1_X(H))$ has only one fixed point, therefore $\mathrm{Aff}(\mathbb{C})$ is tangent to only one hyperplane section of $X_5 \subset \mathbb{P}^6$. It follows that $\mathcal{F}$ is not defined by a closed 1-form without codimension one zeros since in this case the action would have to preserve a pencil of hyperplane sections. As there is a smooth $\mathbb{P}^1$ of affine Lie subalgebras of $\mathfrak{sl}(2,\mathbb{C})$ we conclude that the space of foliations on X_5 with $K\mathcal{F} = 0$ has two disjoint irreducible components: one corresponding to foliations defined by closed 1-forms and the other defined by affine subalgebras of $\mathfrak{aut}(X_5)$. Notice that they are both smooth, with the second one corresponding to a closed orbit of $\mathrm{Aut}(X_5)$ in $\mathbb{P}H^0(X_5, \Omega^1_{X_5}(2H))$. □

8. Foliations on Fano 3-folds of index one

Most of the work for the classification of foliations with $K\mathcal{F} = 0$ on Fano 3-folds with $\mathrm{Pic}(X) = \mathbb{Z}$ and of index one has already been done by Jahnke and Radloff in [17]. In [17, Proposition 1.1] it is proved that $h^0(X, \Omega^1_X(1)) \neq 0$ implies that the genus of X, which by definition is $g(X) = h^0(X, -KX) + 2 = \frac{1}{2}KX^3 + 1$, is 10 or 12. This considerably reduces the amount of work to prove the final bit in the classification of foliations with $K\mathcal{F} = 0$ on Fano 3-folds with rank one Picard group.

Theorem 8.1. *If $\mathcal{F}$ is a codimension one foliation with trivial canonical bundle on a Fano 3-fold with $\mathrm{Pic}(X) = \mathbb{Z}$ and $i(X) = 1$ then X is the Mukai-Umemura 3-fold and $\mathcal{F}$ is induced by an algebraic action of the affine group.*

Recall that the Mukai-Umemura 3-fold is the quasi-homogeneous 3-fold obtained by the closure of the $\mathrm{Aut}(\mathbb{P}^1)$-orbit of the point of in $\mathrm{Sym}^{12}\,\mathbb{P}^1$ determined by the polynomial $xy(x^{10} + 11x^5y^5 + y^{10})$. It is an equivariant compactification of the quotient of $\mathrm{Aut}(\mathbb{P}^1)$ by the icosahedral group A_5, [25].

Proof. In [28] the Fano 3-folds of index one and $g \geq 7$ carrying vector fields are classified. There are two rigid examples (Mukai-Umemura 3-fold with $\mathrm{Aut}^0(X) = \mathbb{P}SL(2,\mathbb{C})$ and a 3-fold with $\mathrm{Aut}^0(X) = (\mathbb{C},+)$) and a one parameter family of examples with $\mathrm{Aut}^0(X) = (\mathbb{C}^*,\cdot)$. All the cases can be obtained from X_5, the Fano 3-fold of index two and degree 5, by means of a birational transformation defined by a linear system on X_5 of the form $|3H - 2Y|$ where Y is the closure of a $(\mathbb{C},+)$

or $(\mathbb{C}^*, \cdot)$-orbit in X_5. Thus Lemma 7.2 implies that the vector fields in X have, exactly as the vector fields in X_5, isolated fixed points.

Theorem 4.1 implies that any codimension one foliation on X with $K\mathcal{F} = 0$ must be induced by an algebraic group. It follows that X is the Mukai-Umemura 3-fold and that $\mathcal{F}$ is induced by an action of the affine group. $\square$

Remark 8.2. In the main result of [17] there is an imprecision. They claim that a general section of $H^0(X, \Omega^1_X(1))$ for a general deformation of the Mukai-Umemura 3-fold is integrable. This cannot happen since $h^0(X, \Omega^1_X(1)) = 3$ for any sufficiently small deformation of the Mukai-Umemura 3-fold ([17, Proposition 2.6]) and therefore the closedness of Frobenius integrability condition would imply that every element of $H^0(X, \Omega^1_X(1)) \simeq (\mathfrak{sl}_2)^*$ is integrable. Apparently, their mistake is at the proof of their Proposition 2.16. More specifically, at the determination of the integer a from the exact sequence $0 \to \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathcal{O}_{\mathbb{P}^1}(-a+1) \oplus \tau \to 0$, where τ is a torsion sheaf.

9. Holomorphic Poisson structures

A (non-trivial) holomorphic Poisson structure on projective manifold X is an element of $[\Pi] \in \mathbb{P}H^0(X, \bigwedge^2 TX)$ such that $[\Pi, \Pi] = 0$, where $[\cdot, \cdot]$ is the Schouten bracket, see [27]. In dimension three, a Poisson structure is equivalent to a pair $(\mathcal{F}, D)$ where $\mathcal{F}$ is a codimension one foliation with $K\mathcal{F} = \mathcal{O}_X(-D)$ and an effective divisor D. Our classifications of irreducible components of the space of foliations with $K\mathcal{F}$ very negative (Proposition 3.7) and with $K\mathcal{F} = 0$ on Fano 3-folds with rank one Picard group implies at once a description of the irreducible components of the space of Poisson structures

$$\mathrm{Poisson}(X) = \left\{ \Pi \in \mathbb{P}H^0(X, \bigwedge^2 TX) \,\middle|\, [\Pi, \Pi] = 0 \right\}$$

on these manifolds.

Theorem 9.1. *If X is a Fano 3-fold with rank one Picard group then* $\mathrm{Poisson}(X)$ *has 9 irreducible components when $X = \mathbb{P}^3$; 4 irreducible components when $X = Q^3$; 2 irreducible components when $X = X_5$; 1 irreducible component when X has index two and is distinct from X_5; 1 irreducible component when X is the Mukai-Umemura 3-fold; and is empty when X has index one and is not the Mukai-Umemura 3-fold.*

To wit, when $X = \mathbb{P}^3$ then besides the irreducible components of the space of foliations of degree two we have three extra irreducible components: one parametrized by the product of $\mathbb{P}H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))$ and $\mathrm{Rat}(1,1)$; one parametrized by the product of $\mathbb{P}H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$ and $\mathrm{Rat}(1,2)$; and one parametrized by the product of $\mathbb{P}H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$ and $\mathrm{Log}(1,1,1)$. When $X = Q^3$, we have just one extra component corresponding to the product of $\mathbb{P}H^0(Q^3, \mathcal{O}_{Q^3}(1))$ and $\mathrm{Rat}(1,1)$. For manifolds of index one or two there are no extra components.

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