GLOBAL STABILITY FOR HOLOMORPHIC FOLIATIONS ON KAEHLER MANIFOLDS

JORGE VITÓRIO PEREIRA

Instituto de Matemática Pura e Aplicada, IMPA, Estrada Dona Castorina, 110
Jardim Botânico, 22460-320 - Rio de Janeiro, RJ, Brasil. email : jvp@impa.br

Abstract. We prove the following theorem for Holomorphic Foliations on compact complex kaehler manifolds: if there exist a compact leaf with finite holonomy, then every leaf is compact with finite holonomy.

1. Introduction

The question of global stability is recurrent in the theory of foliations. The work of Ehresmann and Reeb establishes the so called global stability theorem, which says that if $\mathcal{F}$ is a transversely orientable codimension one foliation in a compact connected manifold M that has a compact leaf L with finite fundamental group, then every leaf of L is compact with finite holonomy group [**?**]. Counterexamples for codimension greater than one are known since the birth of the theorem. Here we want to abolish the hypothesis on the codimension for a special kind of foliation, namely holomorphic foliations in complex Kaehler manifolds. In other words we are going to prove the following :

Theorem 1. *Let $\mathcal{F}$ be a holomorphic foliation of codimension q in a compact complex Kaehler manifold. If $\mathcal{F}$ has a compact leaf with finite holonomy group then every leaf of $\mathcal{F}$ is compact with finite holonomy group.*

Another kind of stability problem was posed by Reeb and Haefliger. The question was the stability of compact foliations, that is, if a foliation has all leaves compact is the leaf space Hausdorff? Positive answers to this problem arose in the work of Epstein[Ep], Edwards-Millet-Sullivan[EMS], Holmann[Ho], etc. There are plenty situations where the leaf space is not Hausdorff. Sullivan found a example in the C^∞ case[Su], Thurston in the analytic case[Su] and Müller in the holomorphic case[Ho]. The examples of Sullivan and Thurston live in compact manifolds, and Müller's in a non-compact non-Kaehler manifold. As corollary of the theorem we reobtain Holmann's result and a special case of [EMS]'s outstanding Theorem.

Corollary 1 (EMS,Ho)**.** *Suppose M is a complex Kaehler manifold. If $\mathcal{F}$ is a compact foliation, i.e., every leaf is compact, then every leaf has finite holonomy group. Consequently, there is an upper bound on the volume of the leaves, and the leaf space is Hausdorff.*

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2. The Leaf Volume Function

Let $\mathcal{F}$ be a holomorphic foliation of a complex Kaehler manifold (M, ω). As in [Br] we define

$$\Omega = \{p \in M | \text{ the leaf } L_p \text{ through p is compact with finite holonomy}\}$$

By the local stability theorem of Reeb, see for example [?], Ω is an open set of M. Set, for every $p \in \Omega$, $n(p) \in \mathbb{N}$ to be the cardinality of the holonomy group of L_p. If d is the dimension of the leaves then we define volume function of $\mathcal{F}$:

$$T : \Omega \longrightarrow \mathbb{R}^+, \; T(p) = n(p) \int_{L_p} \omega^d$$

Lemma 1. *T is a continuous locally constant function in Ω.*

Proof: The continuity is obvious. We have to prove that T is locally constant. To do this we have just to observe that it is constant in the residual subset of Ω, formed by the union of leaves without holonomy, see [?] p. 96. By the Reeb local stability theorem there is a saturated neighborhood for each leaf in this set where all leaves are homologous. Then using the closedness of ω^d and Stokes Theorem we prove the lemma. ∎

Remark 1. In fact, the proof of this lemma is essentially contained in [?].

3. A Lemma about Diff$(\mathbb{C}^n, 0)$

In 1905 Burnside [?] proved that if G is a subgroup of $GL(n, F)$, where F is a field of characteristic zero, with exponent e, then G is finite with $cardinality(G) \leq e^{n^3}$. Recall that a group has exponent e if every element g belonging to the group is such that $g^e = 1$. From the generalization of this result by Herzog-Praeger [?] we obtain :

Lemma 2. *If G is a subgroup of $Diff(\mathbb{C}^n, 0)$ with exponent e then G is finite with $cardinality(G) \leq e^n$.*

Proof: If for each element of G we consider its derivative we obtain a subgroup of $GL(n, \mathbb{C})$ with exponent e. Thus we only have to prove that the normal subgroup G_0 of G, formed by its elements tangent to the identity is the trivial group.
Let $g \in G_0$, then $g^e = Id$. Defining $H(x) = \sum_{i=1}^{e} Dg(0)^{-i} g^i(x)$, we see that :

$$H \circ g(x) = Dg(0)Dg(0)^{-1} \sum_{i=1}^{e} Dg(0)^{-i} g^{i+1}(x) = Dg(0)H(x)$$

Hence g is conjugated to its linear part, and therefore g must be the identity. ∎

4. Proof of the Results

Let $\mathcal{F}$ be as in the theorem. Consider the connected component of Ω containing the leaf L that is compact and with finite holonomy, and call it Ω_L. The volume function T is constant in Ω_L by Lemma 1, so if $p \in \partial\Omega_L$ we have that the leaf

through p is approximated by leaves with uniformly bounded volume, so it has bounded volume and is compact(here we use the fact that the manifold is compact to achieve the compactness of the leaf). The holonomy group of L_p has finite exponent, because for any transversal Σ of L_p, $\Sigma \cap \Omega_L$ will be an open set such that every leaf of Ω_L cuts it in at most m points. Thus for every holomy germ h of L_p, $(h^{m!})_{|\Sigma\cap\Omega_L} = Id$. Analytic continuation implies that $h^{m!} = Id$. Using Lemma 2, we see that $\partial\Omega_L = \emptyset$, and prove the theorem.
The Corollary follows observing that the set of leaves without holonomy is residual and that we don't need the compactness of the manifold to assure that a limit leaf is compact. Then the holonomy group of each leaf is finite and by the results of Epstein [**?**] we get the consequences.

Remark 2. The same proof works in a more general context. We have just to suppose that our foliation is transversely quasi-analytic and that there is a closed form which is positive on the $(n-q)$-planes of the distribution associated to the foliation.

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