# A CHARACTERIZATION OF DIAGONAL POISSON STRUCTURES

RENAN LIMA AND JORGE VITÓRIO PEREIRA

Abstract. The degeneracy locus of a generically symplectic Poisson structure on a Fano manifold is always a singular hypersurface. We prove that there exists just one family of generically symplectic Poisson structures in Fano manifold with cyclic Picard group having a reduced simple normal crossing degeneracy locus.

## 1. Introduction

1.1. **Poisson structures.** Let $X$ be a complex manifold. A Poisson structure on $X$ is a bivector field $\Pi \in H^0(X, \bigwedge^2 TX)$ such that the Schouten bracket $[\Pi, \Pi] \in H^0(X, \bigwedge^3 TX)$ vanishes identically. The vanishing of the Schouten bracket implies that the image of the morphism

$$\begin{aligned} \Omega^1_X &\longrightarrow TX \\ \eta &\longmapsto i_\eta \Pi \end{aligned}$$

is an involutive subsheaf of $TX$, and the induced foliation is called the symplectic foliation of $\Pi$. The most basic invariant attached to $\Pi$ is its rank which is the generic rank of this involutive subsheaf of $X$. Thanks to the anti-symmetry of $\Pi$, its rank is an even integer $2r$ where $r$ is the largest integer such that $\Pi^r$ does not vanish identically. When $n$ is even and the rank is equal to $n$, we say that the Poisson structure is generically symplectic. In this case, $\{\Pi^n = 0\}$ defines a divisor which we call the degeneracy divisor of $\Pi$.

1.2. **Diagonal Poisson structures.** The simplest examples of Poisson structures are the Poisson structures on $\mathbb{C}^n$ defined by constant bivector fields

$$\Pi = \sum_{i<j} c_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \, ,$$

where $c_{ij}$ are complex constants. Although rather particular Poisson structures, a Theorem of Darboux tells us that any Poisson structure at the neighborhood of a point where its rank is maximal is locally analytically equivalent to a constant Poisson structure.

The constant Poisson structures are invariant by the action of $\mathbb{C}^n$ on itself by translations, and therefore give rise to Poisson structures on quotients of $\mathbb{C}^n$ by discrete subgroups of itself. In particular, they define Poisson structures on $(\mathbb{C}^*)^n$. These are defined by bivectors of the form

$$\Pi = \sum_{i<j} c_{ij} x_i x_j \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \, .$$

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These bivector fields extend to bivector fields on $\mathbb{P}^n$, and we will call any Poisson structure on $\mathbb{P}^n$ projectively equivalent to the resulting Poisson structure, a **diagonal Poisson structure**. For generic choices of constants $c_{ij}$, the rank of the Poisson is $n$, when $n$ is even, or $n-1$, when $n$ is odd. In the even case, the degeneracy divisor is a simple normal crossing divisor supported on the $n+1$ coordinate hyperplanes. Our first main result tells that the diagonal Poisson structures are the only Poisson structures on even dimensional Fano manifolds of dimension at least four with cyclic Picard group having these properties.

**Theorem 1.** *Let $X$ be an even dimensional Fano manifold of dimension at least four and with cyclic Picard group. Suppose that $\Pi$ is a generically symplectic Poisson structure on $X$ such that its degeneracy divisor is a reduced normal crossing divisor. Then $X$ is the projective space $\mathbb{P}^{2n}$ and $\Pi$ is a diagonal Poisson structure.*

The hypothesis on the dimension of $X$ is indeed necessary, as a Poisson structure on a smooth projective surface $S$ is nothing but a section of the anti-canonical bundle of $S$. Although $\mathbb{P}^2$ is the only Fano surface with cyclic Picard group, on it any degree 3 divisor appears as the degeneracy divisor of some Poisson structure.

One of the key ideas in the proof of Theorem 1 is to show that the symplectic foliations on the irreducible components of the degeneracy locus are defined by logarithmic 1-forms and then study isolated singularities of these. Under less restrictive assumptions we are able to show that the symplectic foliation on a reduced and irreducible of the degeneracy divisor of generically symplectic Poisson structure is defined by closed rational 1-form, see Theorem 3.3.

1.3. **Spaces of Poisson structures.** In dimension two the integrability condition for Poisson structures, $[\Pi,\Pi]=0$, is vacuous. Starting from dimension three they impose strong constraints of $\Pi$. The study of the space of Poisson structures on a given projective manifold $X$,

$$\mathrm{Poisson}(X)=\left\{\Pi\in\mathbb{P}H^0(X,\bigwedge^2 TX)\,\middle|\,[\Pi,\Pi]=0\right\},$$

is a challenging problem.

In dimension three, we already know something about these spaces when $X$ is Fano with cyclic Picard group. A Poisson structure $\Pi$ on a smooth projective 3-fold $X$, if not zero, has rank 2 and defines a codimension one foliation $\mathcal{F}$ on $X$. The anti-canonical bundle of $\mathcal{F}$ is effective with section vanishing on the divisorial components of the zero set of $\Pi$. Therefore the study of Poisson structures on 3-folds is equivalent to the study of codimension one foliations with effective anti-canonical bundle. In the case of $X=\mathbb{P}^3$ the description of the irreducible components of $\mathrm{Poisson}(X)$ has been carried out in [4]. An analogue description, when $X$ is any other Fano 3-fold with cyclic Picard group, is presented in [13].

In this paper we prove that the diagonal Poisson structures form irreducible components of the space of Poisson structures on $\mathbb{P}^n$ for any $n\geq 3$.

**Theorem 2.** *If $n\geq 3$ the Zariski closure of the set of diagonal Poisson structures in $PH^0(\mathbb{P}^n,\bigwedge^2 T\mathbb{P}^n)$ is an irreducible component of $\mathrm{Poisson}(\mathbb{P}^n)$.*

The proof of this result relies on some observations concerning the stability of the loci where the Poisson structures have rank zero in the even dimensional case; and on the stability of codimension one logarithmic foliations [3] in the odd dimensional case.

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## 2. Poisson structures

In this section we present the basic theory of Poisson structures on projective manifolds following [16], [10] and [7].

2.1. **Basic definitions.** Let $X$ be a complex manifold. A Poisson structure on $X$ is a 2-derivation $\Pi \in H^0(X, \wedge^2 TX)$ such that the Schouten Bracket $[\Pi, \Pi]$ vanishes identically. If we set $\{f, g\} := \Pi(df \wedge dg)$ then the vanishing of the Schouten bracket is equivalent to the Jacobi identity for the Poisson bracket $\{\cdot, \cdot\} : \wedge^2 \mathcal{O}_X \to \mathcal{O}_X$.

If $\Pi^k(p) \neq 0$ and $\Pi^{k+1}(p) = 0$, we say that $\Pi$ has rank $2k$ at $p$ and write $\operatorname{rank}_p \Pi = 2k$. The biggest $2k$ such that $\Pi^k \neq 0$ is the rank of the Poisson structure.

We denote by $\Pi^\sharp : \Omega^1_X \to TX$ the $\mathcal{O}_X$-linear anchor map defined by contraction of 1-forms with $\Pi$.

A germ of vector field $v \in (TX)_p$ is Hamiltonian with respect to $\Pi$ if $v = \Pi^\sharp(df)$ for some $f \in \mathcal{O}_{X,p}$. If $\Pi$ can be understood from the context we just say that $v$ is a Hamiltonian vector field. A germ $v \in (TX)_p$ is a Poisson vector field with respect to $\Pi$, if $[v, \Pi] = 0$. Notice that every Hamiltonian vector field is Poisson, but the converse does not hold. Poisson vector fields are infinitesimal symmetries of the Poisson structure and do not need to belong to the image of $\Pi^\sharp$.

All the above definitions can be made on the more general case of complex varieties, or even schemes/analytic spaces, cf. [16] and [10]. For instance if $X$ is variety (reduced but perhaps singular), we denote by $\mathfrak{X}^q_X$ the sheaf of holomorphic $q$-derivations of $X$, i.e. the sheaf $\mathfrak{X}^q_X = \operatorname{Hom}(\Omega^q_X, \mathcal{O}_X)$. A Poisson structure on $X$ is then a 2-derivation $\Pi \in H^0(X, \mathfrak{X}^2_X)$ such that the Schouten Bracket $[\Pi, \Pi]$ vanishes identically. The sheaf $\mathfrak{X}^q_X$ coincides with $\wedge^q TX$ over the smooth locus of $X$, but in general the natural inclusion $\wedge^q TX \to \mathfrak{X}^q_X$ is strict.

2.2. **Poisson subvarieties and degeneracy loci.** Let $Y \subseteq X$ be a subvariety with defining ideal sheaf $\mathcal{I}_Y$. We say that $Y$ is a Poisson subvariety if $v(\mathcal{I}_Y) \subseteq \mathcal{I}_Y$ for every Hamiltonian vector fields $v$. Equivalentely, the global section $\Pi_{|Y} \in H^0(Y, (\wedge^2 TX)_{|Y})$ lies in the image of the natural map $H^0(Y, \mathfrak{X}^2_Y) \to H^0(Y, (\wedge^2 TX)_{|Y})$.

More generally, we say that $\mathcal{I} \subseteq \mathcal{O}_X$ is a Poisson ideal if $v(\mathcal{I}) \subseteq \mathcal{I}$ for every Hamiltonian vector fields $v$. We recall that the intersection and the sum of two Poisson ideals are Poisson ideals, and that radical of a Poisson ideal is a Poisson ideal, see [16, Lemma 1.1].

The $2k^{th}$ degeneracy ideal $\mathcal{I}_{2k}$ is the image of the morphism

$$\Omega_X^{2k+2} \xrightarrow{\Pi^{k+1}} \mathcal{O}_X.$$

Jacobi's identity implies that $\mathcal{I}_{2k}$ is a Poisson ideal, and the subvariety (or rather subscheme) $\mathrm{D}_{2k}(\Pi)$ defined by $\mathcal{I}_{2k}$ is a Poisson subvariety, called the $2k^{th}$ degeneracy locus. Notice that the support of $\mathrm{D}_{2k}(\Pi)$ satisfies

$$|\,\mathrm{D}_{2k}(\Pi)| = \{p \in X; \mathrm{rank}_p\, \Pi \le 2k\}.$$

If $\Pi$ is a Poisson structure of rank $2k$ then we define the degeneracy divisor of $\Pi$, denoted by $\mathrm{D}(\Pi)$ , as the divisorial component of the $(2k-2)^{\mathrm{th}}$ degeneracy locus of $\Pi$. Notice that if $\Pi$ is a generically symplectic Poisson structure then $\mathrm{D}_{2k-2}(\Pi) = \mathrm{D}(\Pi)$, but for a general Poisson structure of rank $2k$ the degeneracy divisor $\mathrm{D}(\Pi)$ does not need to coincide with $\mathrm{D}_{2k-2}(\Pi)$, all we have is the inclusion $\mathrm{D}(\Pi) \subset \mathrm{D}_{2k-2}(\Pi)$.

2.3. **Symplectic foliation.** Suppose that $\Pi$ is a Poisson structure of rank $2k$ on $X$. As already mentioned in the introduction, the image of the anchor map is an involutive subsheaf of $TX$. On $U = X - \mathrm{D}_{2k-2}(\Pi)$, the complement of the degeneracy locus of $\Pi$, this image is locally free and has locally free cokernel. Thus it defines a smooth foliation $\mathcal{F}_{|U}$. The Poisson structure induces a symplectic structure on the leaves of $\mathcal{F}_{|U}$. This foliations extends to a singular foliation $\mathcal{F}$ on $X$ with tangent sheaf $T\mathcal{F}$ equal to the saturation of $\mathrm{Im}\,\Pi^\sharp$ in $TX$. The singular set of $\mathcal{F}$ consists of the points over which the cokernel $TX/T\mathcal{F}$ is not locally free and therefore has codimension at least two. Notice that the singular set of $\mathcal{F}$ can be strictly smaller then the singular set of $\Pi$.

If there are no divisorial components on $\mathrm{D}_{2k-2}(\Pi)$, i.e. $\mathrm{D}(\Pi) = 0$, then

$$\det T\mathcal{F} := (\bigwedge^{rk\Pi} T\mathcal{F})^{**} \simeq \mathcal{O}_X.$$

In other words, the anti-canonical bundle of $\mathcal{F}$ is trivial. Otherwise, when there are divisorial components in $\mathrm{D}_{2k-2}(\Pi)$ we get that the anti-canonical bundle of $\mathcal{F}$ is effective, i.e. $\det T\mathcal{F} \simeq \mathcal{O}_X(\mathrm{D}(\Pi))$.

The conormal sheaf of $\mathcal{F}$ is, by definition, the kernel of the dual of the inclusion $T\mathcal{F} \to TX$. Therefore, it fits in the exact sequence

$$0 \to N^*\mathcal{F} \longrightarrow \Omega^1_X \longrightarrow T\mathcal{F}^* \,.$$

The rightmost map is surjective away from the singular set of $\mathcal{F}$ and, since $\mathrm{sing}(\mathcal{F})$ has codimension at least two, we have

$$\det N^*\mathcal{F} = K_X \otimes \det T\mathcal{F}.$$

If we set $N = \det(N^*\mathcal{F})^*$ then it follows that $\mathcal{F}$ is defined by a section $\omega$ of $\Omega^q_X \otimes N$ where $q = \dim X - \dim \mathcal{F}$. To wit, $T\mathcal{F}$ is the kernel of the morphism

$$TX \longrightarrow \Omega^{q-1}_X \otimes N$$

defined by contraction with $\omega$.

If $\Pi$ defines a codimension one foliation on $X$, $\dim X = 2n+1$, then $N^*\mathcal{F} = K_X \otimes \mathcal{O}_X(\mathrm{D}(\Pi))$.

2.4. **Poisson connections.** If $X$ is a manifold endowed with a Poisson structure $\Pi$ then a Poisson connection on a line-bundle $\mathcal{L}$ over $X$ is a morphism of $\mathbb{C}$-sheaves

$$\nabla : \mathcal{L} \longrightarrow TX \otimes \mathcal{L}$$

satisfying

$$\nabla(f\sigma) = f\nabla(\sigma) + \Pi^\sharp(df) \otimes \sigma\,,$$

where $\sigma$ is any germ of section of $\mathcal{L}$ and $f$ is any germ of function.

If $H$ is a Poisson hypersurface then the associated line-bundle $\mathcal{L} = \mathcal{O}_X(H)$ carries a natural Poisson connection defined as follows. If $H$ is defined by $f = \{f_i = 0\}$, then we can define a connection locally by

$$\nabla s_i = -\Pi^\sharp(\frac{df_i}{f_i}) \otimes s_i\,.$$

This is the Polishchuk connection associated to $H$, see [16, Section 7].

2.5. **Poisson structures with simple normal crossing degeneracy divisor.** We close this section on the basic theory of holomorphic Poisson manifolds by recalling a result by Polishchuk [16, Corollary 10.7] which is the starting point of our proof of Theorem 1.

**Theorem 2.1.** *Let $(X, \Pi)$ be a generically symplectic Poisson manifold with $\dim X = 2n$ and $X$ projective such that the ideal $\mathcal{I}_{2n-2}$ is reduced and the variety $\mathrm{D}(\Pi) = V_{2n-2}$ is composed by smooth irreducible components in normal crossing position. If $H^{(m)}$ consists of the points of $X$ where exactly $m$ irreducible components of $\mathrm{D}_{2n-2}(\Pi)$ meets, then $\Pi$ has constant rank at each connected component of $H^{(m)}$. Furthermore, we have $2n - 2m \leq \mathrm{rank}\ \Pi_{|H^{(m)}} \leq 2n - m$.*

In particular, $\Pi$ induces a codimension one Poisson foliation on each component of $\mathrm{D}(\Pi)$ and the singular locus of the foliation is contained in the intersection of, at least, two $H_i$. Moreover, let $\Pi_1 = \Pi_{|H_1}$ be the Poisson structure in $H_1$ induced by $\Pi$, then $\mathrm{D}(\Pi_1) \subseteq (H_2 \cup \ldots \cup H_k) \cap H_1$. If $\mathrm{D}(\Pi_1)$ does not contain $H_i \cap H_1$ for some $i$, then $H_i \cap H_1$ is invariant by the Poisson foliation of $\Pi_1$.

## 3. The degeneracy divisor

In this section we will reduce the proof of Theorem 1 to the four dimensional case.

3.1. **Index of a Fano manifold.** Recall a manifold $X$ is said to be Fano if its anti-canonical bundle $KX^*$ is ample. Assume that $X$ is Fano and has cyclic Picard group, i.e. Pic $X = \mathbb{Z}$. If $H$ is an ample generator of the Picard group of $X$ then the degree of a line bundle $\mathcal{L}$ is defined by the relation

$$\mathcal{L} = \mathcal{O}_X(\deg(\mathcal{L})H)\,.$$

The **index** of $X$, denoted by $i(X)$, is the degree of the anti-canonical divisor, i.e. $i(X) = \deg(KX^*)$. It was proved by Kobayashi and Ochiai in [11] that the index of a Fano manifold of dimension $n$ is bounded by $n+1$. Moreover, the extremal cases are $\mathbb{P}^n$ ($i(X) = n+1$) and hyperquadrics $Q^n \subset \mathbb{P}^{n+1}$ ($i(X) = n$).

**Lemma 3.1.** *Let $X$ be a Fano manifold with* $\mathrm{Pic}\ X = \mathbb{Z}$ *and* $\dim X \geq 4$. *Let $Y$ be a smooth hypersurface such that* $\deg Y < i(X)$. *Then $Y$ is a Fano manifold with* $\mathrm{Pic}\, Y = \mathbb{Z}$.

*Proof.* Lefschetz theorem for Picard groups [12, Example 3.1.25] implies that the restriction morphism $\operatorname{Pic} X \to \operatorname{Pic} Y$ is an isomorphism. In particular, $\operatorname{Pic} Y = \mathbb{Z}$.

Adjunction formula gives

$$KY = KX_{|Y} \otimes \mathcal{O}_Y(-Y) = \mathcal{O}_Y((i(X) - \deg(Y))Y) ,$$

and our assumptions implies that $-KY$ is ample. □

3.2. **The symplectic foliation on the degeneracy divisor.** Let $H = \sum H_i$ be a simple normal crossing divisor on a manifold $X$. A meromorphic 1-form $\omega$ on $X$ is logarithmic with poles on $H$ if for any germ of local equation $h$ of $H$, the differentials form $h\omega$ and $hd\omega$ are holomorphic. The sheaf of logarithmic 1-forms with poles on $H$, $\Omega^1_X(\log H)$ is locally free and fits into the exact sequence

$$0 \to \Omega^1_X \longrightarrow \Omega^1_X(\log H) \longrightarrow \bigoplus_{i=1}^m \mathcal{O}_{H_i} \to 0,$$

where the arrow on the right is the residue map.

**Proposition 3.2.** *Let $X$ be a projective manifold of dimension $2n$ and $\Pi$ be a generically symplectic Poisson structure on it. If $\mathrm{D}(\Pi)$ is a simple normal crossing divisor then for every irreducible component $Y$ of $\mathrm{D}(\Pi)$ the symplectic foliation on $Y$ is defined by an element of $H^0(Y, \Omega^1_Y(\log E))$ with non-zero residues on every irreducible component of $E = (\mathrm{D}(\Pi) - Y)_{|Y} - \mathrm{D}(\Pi_{|Y})$.*

*Proof.* Observe that $(\Pi_{|Y})^{n-1}$ is a section of $\wedge^{2n-2} TY \simeq \Omega^1_Y \otimes KY^*$, and therefore gives rise to a twisted 1-form $\omega \in H^0(Y, \Omega^1_Y \otimes KY^*)$. Notice that $\omega$ vanishes on $\mathrm{D}(\Pi_{|Y})$ and that Theorem 2.1 implies that the support of $\mathrm{D}(\Pi_{|Y})$ is contained in the support of $(\mathrm{D}(\Pi) - Y) \cap Y$. Let $s \in H^0(Y, KY^*)$ be a section vanishing on $(\mathrm{D}(\Pi) - Y) \cap Y$. The quotient $\frac{\omega}{s}$ is a rational 1-form on $Y$ with simple poles on the irreducible components of $(\mathrm{D}(\Pi) - Y) \cap Y - \mathrm{D}(\Pi_{|Y})$. But since these irreducible components are Poisson subvarieties of $\Pi_{|Y}$, the symplectic foliation of $\sigma$ is tangent to them. It follows that $\frac{\omega}{s}$ is logarithmic with polar set equal to $E$. □

3.3. **The symplectic foliation on the degeneracy divisor revisited.** Proposition 3.2 is a particular case of the more general result below. Although not strictly necessary for the proof of Theorem 1 we believe that it has some independent interest.

**Theorem 3.3.** *Let $X$ be a projective manifold of dimension $2n$ and $\Pi$ be a generically symplectic Poisson structure on it. Let $Y$ be a reduced irreducible component of $\mathrm{D}(\Pi)$ and let $Y^* = Y - \operatorname{sing}(Y)$ be its smooth locus. Then the symplectic foliation $\mathcal{F}_Y$ on $Y$ has codimension one and its restriction to $Y^*$ is defined by a closed meromorphic 1-form $\omega$ on $Y^*$ without divisorial components in its zero set, and with polar divisor satisfying*

$$(\omega)_\infty = (\mathrm{D}(\Pi) - Y) \cap Y^* - \mathrm{D}(\Pi_{|Y^*}) .$$

*Proof.* Let $p \in Y^*$ be a smooth point of $Y$. At a sufficiently small neighborhood of $p$ we can choose local analytic coordinates $(x_1, \ldots, x_{2n-1}, y)$ such that $Y = \{y = 0\}$ and

$$\Pi = y \frac{\partial}{\partial y} \wedge \left( \sum_i y^i v_i \right) + \left( \sum y^i \sigma_i \right)$$

where $v_i$ are vector fields in $\{y = 0\}$, and $\sigma_i$ are a bivector fields in $\{y = 0\}$. Since $Y$ is a reduced irreducible component of $\mathrm{D}(\Pi)$ and

$$\Pi^n = y\frac{\partial}{\partial y} \wedge v_0 \wedge \sigma_0^{n-1} + y^2\Theta$$

for some holomorphic $2n$-vector field $\Theta$, it follows that $v_0 \wedge \sigma_0^{n-1} \neq 0$. But $\sigma_0^{n-1}$ defines the symplectic foliation $\mathcal{F}_Y$ on $Y$, hence $\mathcal{F}_Y$ has codimension one.

The integrability condition $[\Pi, \Pi]$ implies the identity $[v_0, \sigma_0] = 0$. Therefore, $v_0$ is an infinitesimal symmetry of $\mathcal{F}_Y$. Moreover, $v_0 \wedge \sigma_0^{n-1} \neq 0$ implies that $v_0$ is generically transverse to $\mathcal{F}_Y$. Hence, if $\eta$ is any 1-form defining $\mathcal{F}_Y$ at a neighborhood of $p$ then the 1-form $\frac{\eta}{\eta(v_0)}$ is closed, see for instance [15, Corollary 2]. Notice that $\frac{\eta}{\eta(v_0)}$ has no divisorial components in its zero set, and its polar divisor is equal to $\{\sigma_0^{n-1} \wedge v_0 = 0\} - \{\sigma_0^{n-1} = 0\}$. Since

$$\left\{ \left(y^{-1}\Pi^n\right)_{|\{y=0\}} = 0 \right\} = \{\sigma_0^{n-1} \wedge v_0 = 0\} \,,$$

we see that the description of the poles of local closed meromorphic 1-form $\frac{\eta}{\eta(v_0)}$ is in accordance with the description of the poles of the sought global closed meromorphic 1-form $\omega$.

Now take a covering of a neighborhood of $Y^*$ by sufficiently small open subsets $U_i$ of $X - \mathrm{sing}(Y)$. Let $f_i$ be local equations for $Y$, satisfying $f_i = f_{ij} f_j$ for some $f_{ij} \in \mathcal{O}^*(U_i \cap U_j)$. Then

$$\Pi^\sharp\left(\frac{df_i}{f_i}\right) - \Pi^\sharp\left(\frac{df_j}{f_j}\right) = \Pi^\sharp\left(\frac{df_{ij}}{f_{ij}}\right) .$$

The righthand side is a Hamiltonian vector field, and in particular it is tangent to $Y$ and its restriction to $Y$ is tangent to $\mathcal{F}_Y$. The summands in the lefthand side are Poisson vector fields, also tangent to $Y$, but their restrictions to $Y$ are equal to generically transverse infinitesimal symmetries of $\mathcal{F}_Y$. Therefore, if $\mathcal{F}_Y$ is defined by a collection of 1-forms $\omega_i \in \Omega^1_{Y^*}(U_i)$ then over the open sets $Y^* \cap U_i \cap U_j$ we have the equality

$$\frac{\omega_i}{i_\Pi(\frac{df_i}{f_i} \wedge \omega_i)} = \frac{\omega_j}{i_\Pi(\frac{df_j}{f_j} \wedge \omega_j)} .$$

As argued before, these 1-forms are closed and therefore patch together to give the sought closed meromorphic 1-form defining $\mathcal{F}_Y$ on $Y^*$. □

If a hypersurface $Y$ of a smooth manifold $X$ has singularities of codimension at least two in $Y$, i.e. $\dim \mathrm{sing}(Y) \leq \dim X - 3$, then it is well-known that the singularities are normal and holomorphic functions defined on the complement of the singular set extend to holomorphic functions on the whole hypersurface. Similarly, if the codimension in $Y$ of the singularities is at least three then an analogue result holds for holomorphic 1-forms defined on the complement of the singular set [8]. When extensions results are concerned, closed holomorphic 1-forms are closer to functions than 1-forms as the lemma below shows.

**Lemma 3.4.** *Let $X$ be a polydisc in $\mathbb{C}^n$ and $Y \subset X$ be a reduced hypersurface with* $\dim\, \mathrm{sing}(Y) \leq n - 3$. *If $\omega$ is closed 1-form in $Y^* = Y - \mathrm{sing}(Y)$ then there exists a holomorphic 1-form on $X$ such that $i^*\Omega = \omega$, where $i : Y^* \to X$ is the inclusion.*

*Proof.* Since $\omega$ is closed we can find a Leray open covering $\mathcal{U} = \{U_i\}$ of $X - \mathrm{sing}(Y)$ and holomorphic functions $f_i \in \mathcal{O}_X(U_i)$ such that

$$\omega_{|U_i} = df_{i|U_i \cap Y^*}\,.$$

If $Y = \{h = 0\}$, with $h$ reduced, then over $U_i \cap U_j$ we have that

$$f_i - f_j = \lambda_{ij} + h \cdot g_{ij}$$

where $g_{ij} \in H^1(\mathcal{U}, \mathcal{O}_U)$ and $\lambda_{ij} \in H^1(\mathcal{U}, \mathbb{C})$. Since $\dim \mathrm{sing}(Y) \leq n - 3$, the cohomology group $H^1(\mathcal{U}, \mathcal{O}_U)$ is trivial ([9, pg. 133]) and we can produce $g_i \in \mathcal{O}(U_i)$ such that $g_{ij} = g_i - g_j$ over $U_i \cap U_j$. Therefore $d(f_i - hg_i) = d(f_j - hg_j)$ over $U_i \cap U_j$. Therefore we have a closed 1-form $\tilde{\Omega}$ defined on $(\mathbb{C}^n, 0) - \mathrm{sing}(Y)$. Hartog's Theorem allow us to extend $\tilde{\Omega}$ to a 1-form $\Omega$ globally defined on $X$. □

**Corollary 3.5.** *Let $X$ be a projective manifold of dimension $2n$ and $\Pi$ be a generically symplectic Poisson structure on it. Assume that $\mathrm{D}(\Pi) = Y$ is reduced and irreducible. If $\dim \mathrm{sing}(Y) \leq \dim X - 3$ then for any resolution of singularities of $Y$, $\pi : Z \to Y$, the pull-back $\pi^*\mathcal{F}_Y$ of the symplectic foliation on $Y$ is defined by closed holomorphic 1-form.*

*Proof.* if $Y^* = Y - \mathrm{sing}(Y)$ then Theorem 3.3 implies that the restriction of $\mathcal{F}_Y$ to $Y^*$ is defined by a closed holomorphic 1-form. Lemma 3.4 implies the result. □

Corollary 3.5 can be applied to given information on the structure of the generically symplectic Poisson structures on $\mathbb{P}^n$ with reduced and irreducible degeneracy divisor studied in [10, Section 8]. Their degeneracy divisors are secant varieties of elliptic normal curves on $\mathbb{P}^n$. The singular loci of these varieties satisfy the hypothesis of Corollary 3.5 and therefore the symplectic foliation defined on them are given by closed holomorphic 1-forms. Moreover, since these varieties are birationally equivalent to projective bundles over symmetric powers of an elliptic curve their space of holomorphic 1-forms is one dimensional, with generator coming by pull-back under the natural morphism $\mathrm{Sym}^{n/2} E \to E$. Therefore all leaves of the symplectic foliation are algebraic.

3.4. **Irreducible components of the degeneracy divisor.** After the brief digression about the symplectic foliation on general reduced and irreducible components of the degeneracy divisor of generically symplectic Poisson structures we return to the proof of Theorem 1.

**Lemma 3.6.** *If $X$ is a projective manifold with $\mathrm{Pic}\, X = \mathbb{Z}$ then there is no foliation on $X$ with trivial normal bundle. Moreover, there is no smooth codimension one foliation $\mathcal{F}$ on $X$.*

*Proof.* If the normal bundle of $\mathcal{F}$ is trivial then $\mathcal{F}$ is defined by a non-zero section of $\Omega^1_X$. But according to the Hodge decomposition we have that $H^0(X, \Omega^1_X) \simeq H^1(X, \mathcal{O}_X)$, and as $\mathrm{Pic}\, X = \mathbb{Z}$, the latter group is zero. This proves the first part of the statement.

For the second part, notice that if $\mathcal{F}$ has no singularities then Baum-Bott formula implies that $c_1(N\mathcal{F})^{\dim X} = 0$. Since $\mathrm{Pic}\, X = \mathbb{Z}$, it follows that $N\mathcal{F} = \mathcal{O}_X$ and we can conclude as before. □

**Proposition 3.7.** *Let $X$ be a Fano manifold of dimension $2n \geq 4$ with cyclic Picard group. Let $\Pi \in H^0(X, \bigwedge^2 TX)$ be a generically symplectic Poisson structure.*

*Assume that* $\mathrm{D}(\Pi)$ *is a reduced simple normal crossing divisor. If* $Y$ *is an irreducible component of* $\mathrm{D}(\Pi)$ *and* $\Pi_{|Y}$ *is the induced Poisson structure on* $Y$ *then* $\mathrm{D}(\Pi_{|Y}) \subsetneq (\mathrm{D}(\Pi) - Y) \cap Y$. *Moreover, there exists another irreducible component* $Z$ *of* $\mathrm{D}(\Pi)$ *such that the induced Poisson structure on* $Y \cap Z$ *is generically symplectic.*

*Proof.* Theorem 2.1 implies $\mathrm{D}(\Pi_{|Y}) \subseteq (\mathrm{D}(\Pi) - Y) \cap Y$. We want to prove that the inclusion is strict. If the inclusion is not strict then Proposition 3.2 implies that $\mathcal{F}_Y$ is defined by a global holomorphic 1-form. But this contradicts Lemma 3.6, proving that the inclusion must be strict.

If we take an irreducible component $Z$ of $\mathrm{D}(\Pi)$ such that $Y \cap Z$ is not contained in $\mathrm{D}(\Pi_{|Y})$ then the induced Poisson structure on $Y \cap Z$ has rank $2n-2 = \dim(Y \cap Z)$ according to Theorem 2.1, and is therefore generically symplectic. □

**Proposition 3.8.** *Let* $X$ *be a Fano manifold of dimension* $2n \geq 4$ *with cyclic Picard group. Let* $\Pi \in H^0(X, \bigwedge^2 TX)$ *be a generically symplectic Poisson structure. If* $\mathrm{D}(\Pi)$ *is a reduced simple normal crossing divisor then* $\mathrm{D}(\Pi)$ *has at least three distinct irreducible components. Moreover, if* $\mathrm{D}(\Pi)$ *has exactly three irreducible components then for any irreducible component* $Y$ *of* $\mathrm{D}(\Pi)$, *the symplectic foliation* $\mathcal{F}_Y$ *on* $Y$ *has normal bundle equal to* $KY^*$ *and trivial cotangent bundle.*

*Proof.* Let $Y$ be an irreducible component of $\mathrm{D}(\Pi)$. The symplectic foliation on $Y$ is defined by a logarithmic 1-form $\omega$. The residue of $\omega$ is a $\mathbb{C}$-divisor with zero Chern class. Therefore the polar set of $\omega$ has at least two distinct irreducible components. As $\operatorname{Pic} Y = \mathbb{Z}$, Lemma 3.1, any other irreducible component $Z$ of the simple normal crossing divisor $\mathrm{D}(\Pi)$ intersects $Y$ along an smooth and irreducible hypersurface. As $(\omega)_\infty$ is contained in $(\mathrm{D}(\Pi) - Y) \cap Y)$ it follows that $\mathrm{D}(\Pi)$ must have at least two other irreducible components besides $Y$.

The symplectic foliation on $Y$ is defined by a logarithmic 1-form with polar set equal to $(\mathrm{D}(\Pi) - Y) \cap Y - \mathrm{D}(\Pi_{|Y})$, and normal bundle given by the associated line-bundle. If we have only three irreducible components in $\mathrm{D}(\Pi)$ then $\mathrm{D}(\Pi_{|Y})$ is the zero divisor as there are no holomorphic 1-forms on $Y$, and any logarithmic 1-form must have at least two irreducible components in its polar divisor by the residue theorem. Therefore $N\mathcal{F}_Y = KY^*$ and $K\mathcal{F}_Y = \mathcal{O}_Y$. □

**Corollary 3.9.** *If a Fano manifold of dimension* $2n \geq 4$ *with cyclic Picard group admits a generically symplectic Poisson structure with simple normal crossing degeneracy divisor then the index of* $X$ *is at least three.*

*Proof.* Since $KX^* = \mathcal{O}_X(\mathrm{D}(\Pi))$, the index of $X$ is the sum of the degrees of the irreducible components of $\mathrm{D}(\Pi)$. Proposition 3.8 implies $i(X) \geq 3$ as wanted. □

3.5. **Induction argument.** Recall from the Introduction that a diagonal Poisson structure on $\mathbb{P}^n$ is defined on a suitable affine chart by a bivector field of the form $\sum_{ij} \lambda_{ij} x_i x_j \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$.

**Proposition 3.10.** *Let* $\Pi$ *be a generically symplectic Poisson structure on* $\mathbb{P}^{2n}$. *If* $\mathrm{D}(\Pi)$ *is the union of* $2n+1$ *hyperplanes in general position then* $\Pi$ *is projectively equivalent to a diagonal Poisson structure.*

*Proof.* A Poisson structure $\Pi$ on $\mathbb{P}^{2n}$ is determined by a Poisson structure $\tilde{\Pi}$ on $\mathbb{C}^{2n+1}$ defined by a homogenous quadratic bivector field, see for instance [16, Theorem 12.1] and [1]. Since the hyperplanes in $\mathrm{D}(\Pi)$ are in general position, we can choose homogeneous coordinates such that $\mathrm{D}(\Pi) = \{x_0 \cdots x_{2n} = 0\}$.

If we write

$$\tilde{\Pi} = \sum_{i \le j; k<l} a_{ij}^{kl} x_i x_j \frac{\partial}{\partial x_k} \wedge \frac{\partial}{\partial x_l}$$

then the Hamiltonian vector field induced by $x_m$ is

$$\sum_{i \le j; m<l} a_{ij}^{ml} x_i x_j \frac{\partial}{\partial x_l} - \sum_{i \le j; k<m} a_{ij}^{km} x_i x_j \frac{\partial}{\partial x_k} .$$

For every $m'$, the hypersurface $\{x_{m'} = 0\}$ is invariant by this vector field. Thus, if $m < m'$ then $a_{ij}^{mm'} x_i x_j$ must be divisible by $x_{m'}$. Changing the roles of $m$ and $m'$, we conclude that $a_{ij}^{mm'} = 0$ unless $\{i,j\} = \{m,m'\}$. It follows that $\tilde{\Pi}$ is diagonal and so is $\Pi$. □

**Proposition 3.11.** *If Theorem 1 holds for Fano manifolds with cyclic Picard group of dimension $2n$, with $n \ge 2$, then it also holds for Fano manifolds with cyclic Picard group of dimension $2(n+1)$.*

*Proof.* Let $X$ be a Fano manifold with cyclic Picard group of dimension $2(n+1)$ with a generically symplectic Poisson structure $\Pi$ having a reduced simple normal crossing degeneracy divisor. Let $Y$ and $Z$ be irreducible components of $\mathrm{D}(\Pi)$ as Proposition 3.7. Then $W = Y \cap Z$ is a Fano manifold with cyclic Picard group of dimension $2n$ and the Poisson structure $\Pi_{|W}$ is also generically symplectic and has a reduced simple normal crossing divisor. By our assumption, $W = \mathbb{P}^{2n}$ and the adjunction formula says that $i(X) = \deg Y + \deg Z + i(W)$. Kobayashi-Ochiai Theorem implies that $\deg Y = \deg Z = 1$ and $i(X) = 2n+3$, i.e., $X = \mathbb{P}^{2n+2}$. Since the degeneracy divisor of $\Pi_{|W}$ is the union of $2n+1$ hyperplanes in general position, we conclude that the singular set of $(X, \Pi)$ consists of $2n+3$ hyperplanes in general position. □

## 4. Poisson structures on Fano 4-folds

Proposition 3.11 reduces the proof of Theorem 1 to the four dimensional case which will be carried out in this section.

4.1. **Existence of global vector field and constraints on the index.** We will prove below, Proposition 4.2, that to prove Theorem 1 in dimension four we can assume that $X$ is $\mathbb{P}^n$ or an hyperquadric.

**Lemma 4.1.** *Let $\Pi$ be a generically symplectic Poisson structure on a projective manifold $X$, with $\dim X \ge 4$. Let $Y$ and $Z$ be two distinct effective divisors which are linearly equivalent. If $Y$ and $Z$ are Poisson divisors then there exists a non-zero global vector field $v \in H^0(X, TX)$ which is a Hamiltonian vector field in $X \setminus (Y \cup Z)$.*

*Proof.* Since $\Pi$ is a generically symplectic Poisson structure, the anchor morphism $\Pi^\sharp : \Omega^1_X \to TX$ is injective.

Write $\mathcal{L} = \mathcal{O}_X(Y) = \mathcal{O}_X(Z)$. Consider the Polishchuk connections $\nabla_Y$ and $\nabla_Z$ associated to $Y$ and $Z$ respectively. Then $\nabla_Y - \nabla_Z : \mathcal{L} \to TX \otimes \mathcal{L}$ is a $\mathcal{O}_X$-linear map and so, it induces a global vector field $v \in H^0(X, TX)$.

If $Y$ and $Z$ are locally defined by $\{f = 0\}$ and $\{g = 0\}$, respectively, then $v = \Pi^\sharp(d(\log \frac{f}{g}))$. Since $Y$ and $Z$ are distinct, we have $d(\log \frac{f}{g}) \ne 0$ and, as $\Pi$ is generically symplectic, $v$ does not vanish identically. From the local expression, we see that $v$ is Hamiltonian vector field in $X \setminus (Y \cup Z)$. □

**Proposition 4.2.** *Let $X$ be a Fano manifold of dimension four with cyclic Picard group . If there exists a generically symplectic Poisson structure on $X$ with simple normal crossing divisor then the index of $X$ is at least four. In particular, $X$ is a four dimensional hyperquadric or $\mathbb{P}^4$.*

*Proof.* As we already know that $i(X) \geq 3$, see Corollary 3.9, we can assume $i(X) = 3$. Let $\Pi$ be a Poisson structure on $X$ satisfying the assumptions, and $Y$ be an irreducible component of $\mathrm{D}(\Pi)$. Notice that $\mathrm{D}(\Pi)$ has exactly three irreducible components (say $Y, Z$, and $W$), each one of them has degree one, and any two of them are linearly equivalent.

By Lemma 4.1, we have a global vector field $v \in H^0(X, TX)$ induced by $Z$ and $W$, which is tangent to $Y$. Wahl's theorem ([17]) ensures that $v$ does not vanish identically along $Y$, and therefore we have a non zero vector $v_Y \in H^0(Y, TY)$. Since $v$ is a Hamiltonian vector field in $X - (Z \cup W)$, the vector field $v_Y$ is tangent to the symplectic foliation $\mathcal{F}_Y$ on $Y$.

Adjunction formula implies that $Y$ is a Fano 3-fold of index two, and we have just proved that $Y$ carries a foliation $\mathcal{F}_Y$ with trivial canonical bundle (Proposition 3.8) which satisfies $h^0(Y, T\mathcal{F}_Y) > 0$. This suffices to characterize $Y$ and $\mathcal{F}_Y$. On the one hand, [13, Lemma 7.2 and Lemma 7.3] implies that $Y$ is isomorphic to $X_5$, the unique Fano 3-fold with Picard group generated by an element $H$ satisfying $H^3 = 5$. On the other hand, [13, Lemma 4.2 and Theorem 7.1] imply that the foliation $\mathcal{F}_Y$ has trivial tangent bundle and is induced by an action of $\mathrm{Aff}(\mathbb{C})$ in $X_5$.

One of the first steps of the proof of [13, Theorem 7.1] is to show that the foliation on $X_5$ induced by the action of $\mathrm{Aff}(\mathbb{C})$ is not defined by a logarithmic 1-form with poles on two hyperplanes sections. This contradicts Proposition 3.2 and shows that $i(X) \geq 4$. □

4.2. **Singularities of logarithmic foliations.** Let $X$ be a four dimensional Fano manifold with cyclic Picard group and let $\Pi$ be a Poisson structure on $X$ with simple normal crossing degeneracy divisor.

Fix an irreducible component $Y$ of $\mathrm{D}(\Pi)$. We want to analyze the singularities of the symplectic foliation $\mathcal{F}_Y$ on $Y$. Recal that $\mathcal{F}_Y$ is defined by a logarithmic 1-form $\omega \in H^0(Y, \Omega^1_Y(\log E))$ where $E = (\mathrm{D}(\Pi) - Y) \cap Y - \mathrm{D}(\Pi_{|Y})$. If $p$ is a point at the intersection of $m$ irreducible components of $E$ then

$$\omega = \sum_{i=1}^{m} \lambda_i \frac{dx_i}{x_i} + \beta$$

where $x_1, \ldots, x_m$ are defining functions for the irreducible components of $E$ through $p$, $\lambda_1, \ldots, \lambda_m$ are nonzero complex numbers, and $\beta$ is a holomorphic 1-form. It follows that $\mathcal{F}_Y$ does not have isolated singularities at a neighborhood of $E$. Instead the singular set of $E$ coincides with the singular of $\mathcal{F}_Y$ at a neighborhood of $E$. Since $E$ is an ample normal crossing divisor on $Y$ then components of singular set of $\mathcal{F}_Y$ disjoint from $E$ must be zero dimensional, for details see [6]. The argument above also shows that $\omega$, seen as a section of $\Omega^1_Y(\log E)$, does not have zeros at a neighborhood of $E$. In particular the number of isolated singularities of $\mathcal{F}_Y$, counted with multiplicities, is equal to the top Chern class of $\Omega^1_Y(\log E)$.

**Lemma 4.3.** *Let $\Pi$ be a generically symplectic Poisson structure on a Fano 4-fold $X$ with simple normal crossing degeneracy divisor. If $Y$ is an irreducible component* *of* $\mathrm{D}(\Pi)$ *and* $\mathcal{F}_Y$ *is the symplectic foliation on* $Y$ *then any isolated singularity* $p$ *of* $\mathcal{F}_Y$ *lies at the intersection of at least three distinct irreducible components of* $\mathrm{D}(\Pi)$.

*Proof.* If $p$ is a singular point of $\mathcal{F}_1$, then $\Pi(p) = 0$. Locally, this means that $\Pi \in \mathfrak{m}_p \otimes \wedge^2 TX$. In particular, $\Pi \wedge \Pi \in \mathfrak{m}_p^2 \otimes \wedge^4 TX$. Since $\mathrm{D}(\Pi)$ is normal crossing, we can find local coordinates $(x_1, x_2, x_3, x_4)$ in a neighborhood of $p = 0$, such that $\Pi \wedge \Pi = x_1 x_2 V$, where $V$ is a 4-derivation. Write $\Pi = \Pi_1 + \Pi_2 + \dots$ the Taylor series of $\Pi$. To prove the lemma, we just need to check that $\Pi_1 \wedge \Pi_1 = 0$. Since $\Pi_1$ is a linear Poisson structure, it can be reinterpreted as a Lie algebras on $(\mathbb{C}^4)^*$ (see [7], chapter 1). We have a complete classification of the Lie algebra structure in dimension four and a simple check of the table in [2, Lemma 3], shows that $\Pi_1 \wedge \Pi_1 \neq 0$ just in the cases $\mathfrak{aff}(\mathbb{C}) \times \mathfrak{aff}(\mathbb{C})$, $\mathfrak{g}_6$ and $\mathfrak{g}_8(\alpha)$.

The last two cases are excluded because for them we have $\Pi_1 \wedge \Pi_1 = x_4^2 \frac{\partial}{\partial x_1} \wedge \dots \wedge \frac{\partial}{\partial x_4}$ which is not coherent with the assumption that $\mathrm{D}(\Pi)$ is a simple normal crossing divisor. To exclude the first case, we use the Theorem of Dufour and Molinier [7, Chapter 4] which states that we can find coordinates $(y_1, \dots, y_4)$, $H_1 = \{y_1 = 0\}$ such that

$$\Pi = y_1 \frac{\partial}{\partial y_1} \wedge \frac{\partial}{\partial y_2} + y_3 \frac{\partial}{\partial y_3} \wedge \frac{\partial}{\partial y_4}.$$

In particular, the Poisson foliation induced on $H_1$ is regular at $p$. This proves the lemma. □

**Lemma 4.4.** *Let* $\omega$ *be a logarithmic* 1*-form with simple normal crossing polar divisor* $D$ *on the quadric* 3*-fold* $Q^3$. *If the degree of* $D$ *is at most three then* $\omega$ *admits an isolated singularity.*

*Proof.* First assume that $D$ has degree two, i.e. $D$ is the union of two hyperplane sections $H_1$ and $H_2$ intersecting $Q^3$ transversely. Notice that $h^0(Q^3, \Omega^1_{Q^3}(\log H_1 + H_2)) = 1$, and the foliation induced by any section of $\Omega^1_{Q^3}(\log H_1 + H_2)$ is the pencil of hyperplane sections generated by $H_1$ and $H_2$. Since the dual variety of $Q^3$ is also a quadric, in the pencil generated by $H_1$ and $H_2$ there are two elements which intersect $Q^3$ on a cone over a two-dimension smooth quadric $Q^2$. Therefore, any nonzero $\omega_0 \in H^0(Q^3, \Omega^1_{Q^3}(\log H_1 + H_2))$ has two isolated singularities (counted with multiplicities).

Suppose now that $D$ is the union of three hyperplane sections intersecting transversely, say $H_1, H_2$, and $H_3$. Through the inclusion $\Omega^1_{Q^3}(\log H_1 + H_2) \to \Omega^1_{Q^3}(\log H_1 + H_2 + H_3)$ we can interpret $\omega_0 \in H^0(Q^3, \Omega^1_{Q^3}(\log H_1 + H_2))$ as a section of $\Omega^1_{Q^3}(\log H_1 + H_2 + H_3)$, and as such it vanishes not only at the zeros of the corresponding rational 1-form but also at $H_3$. Nevertheless, sufficiently small general perturbations of $\omega_0$ inside $H^0(Q^3, \Omega^1_{Q^3}(\log H_1 + H_2 + H_3))$ will still have isolated singularities near the original isolated singularity of $\omega_0$. This suffices to show that $c_3(\Omega^1_{Q^3}(\log H_1 + H_2 + H_3)) \geq c_3(\Omega^1_{Q^3}(\log H_1 + H_2)) \geq 2$.

The remaining case, $D$ is the union of a smooth hyperplane section and a smooth hypersurface of degree two intersecting transversely, can be dealt with similarly. Alternatively, a straightforward computation shows that $c_3(\Omega^1_X(\log D)) = 8$. □

**Remark 4.5.** The constraints on the degree of $D$ and on the dimension of hyperquadric $Q$ are not really necessary. The *continuity* argument used above can be pushed to prove that any logarithmic 1-form on $Q^n$, $n \geq 3$, with simple normal crossing polar divisor has isolated singularities.

4.3. **Proof of Theorem 1.** The result below together with Propositions 3.10 and 3.11 clearly imply Theorem 1.

**Proposition 4.6.** *Let $X$ be a projective 4-fold with cyclic Picard group. If there exists a generically symplectic Poisson structure on $X$ with simple normal crossing degeneracy divisor then $X$ is $\mathbb{P}^4$ and $\mathrm{D}(\Pi)$ is the union of five hyperplanes in general position.*

*Proof.* Proposition 4.2 implies $X$ is a four dimension hyperquadric $Q^4$ or $\mathbb{P}^4$. Aiming at a contradiction let us assume $X = Q^4$. Since the index of $Q$ is 4, we have that $\mathrm{D}(\Pi)$ has three or four irreducible components with degrees summing up to 4. Let $Y$ be an irreducible component of degree one. Thus $Y = Q^3$ is a three dimensional hyperquadric. The symplectic foliation on $Q^3$ is defined by logarithmic 1-form with at least two hypersurfaces in its polar set. According to Lemma 4.3 and Lemma 4.4, $\mathcal{F}_Y$ has isolated singularities which lies at the intersection of at least three distinct irreducible components of $\mathrm{D}(\Pi)$. Thus, it must lie also on the polar divisor of $\omega$. But, as already pointed out $\mathcal{F}_Y$ does not have isolated singularities at a neighborhood of the polar divisor of $\omega$. This contradiction proves that $X = \mathbb{P}^4$.

Assume now that $X = \mathbb{P}^4$. If the conclusion does not hold then $\mathrm{D}(\Pi)$ contains a hyperquadric $Q^3$ or a cubic hypersurface $C^3$. When $\mathrm{D}(\Pi)$ contains a quadric then the argument of the previous paragraph leads to a contradiction. If instead $\mathrm{D}(\Pi)$ contains a cubic then the symplectic foliation on $\mathcal{F}_C$ would be a pencil of smooth hyperplane sections intersecting transversely, and the singular members of this pencil (which exist because the dual of a smooth cubic is an hypersurface) would give isolated singularities for $\mathcal{F}_C$ disjoint of $\mathrm{D}(\Pi)$. In both cases we arrive at contradictions which imply that $D(\Pi)$ is a union of five hyperplanes as claimed. □

## 5. Stability of diagonal Poisson structures

5.1. **Curl operator.** Let us fix a neighborhood $U$ of $0 \in \mathbb{C}^n$ and a nowhere vanishing $n$-form $\Omega$ on $U$, e.g. $\Omega = dx_1 \wedge \ldots \wedge dx_n$. For every $p = 0, 1, \ldots, n$, the map

$$\Omega : \bigwedge^p TU \to \Omega_U^{n-p}$$

defined by $\Omega(A) = i_A(\Omega)$, is an $\mathcal{O}_U$-linear isomorphism from the space $\wedge^p TU$ to $\Omega^{n-p}_{\mathbb{C}^n}$. The inverse map will be denoted as $\Omega^{-1} : \Omega_U^{n-p} \to \bigwedge^p TU$.

The linear operator defined by the composition $\Omega^{-1} \circ d \circ \Omega$ is called the curl operator and it is denoted by $D_\Omega$.

If $\Pi$ is a Poisson structure on $U$ then the vector field $D_\Omega \Pi$ is called the curl vector field (with respect to $\Omega$) of $\Pi$. According to [7, Lemma 2.6.9] the identity

$$[D_\Omega \Pi, \Pi] = 0$$

holds true, i.e. $D_\Omega \Pi$ is a Poisson vector field for $\Pi$.

**Lemma 5.1.** *Assume $n = 2m \geq 4$. Let $\Pi = \sum_{i<j} \lambda_{ij} x_i x_j \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$ be a general diagonal Poisson structure on $U$, i.e. the complex numbers $\lambda_{ij}$ are general. Let $T$ be an irreducible complex variety containing a point $t_0$ and let $\Pi_t$, $t \in T$, be a holomorphic family of Poisson structures on $U$ such that $\Pi_{t_0} = \Pi$. Then, after restricting $U$ and $T$, there exists $\gamma : T \to U$ such that $\gamma(t)$ is the unique point in $U$ where $\Pi_t$ has rank zero. Moreover, the vanishing order of $(\Pi_t)^m$ at $\gamma(t)$ is at least $n = 2m$.*

*Proof.* Let $\Omega = dx_1 \wedge \cdots \wedge dx_n$ be the standard volume form on $\mathbb{C}^n$. The curl of $\Pi$ with respect to $\Omega$ is

$$D_\Omega \Pi = \sum_i \mu_i x_i \frac{\partial}{\partial x_i}, \qquad \text{where} \qquad \mu_i = \sum_{i<j} \lambda_{ij} - \sum_{i>j} \lambda_{ji} \,.$$

For a general choice of $\lambda_{ij}$ the origin is the unique singularity of $D_\Omega \Pi$. This singularity is simple in the sense that the ideal generated by the coefficients of $D_\Omega \Pi$ coincides with the maximal ideal of $\mathcal{O}_{U,0}$. Therefore for sufficiently small $t$, $D_\Omega \Pi_t$ has a unique simple singularity $\gamma(t)$ close to the origin. Implicit function theorem implies that $\gamma : T \to U$ is a holomorphic function.

Notice that $\sum_i \mu_i = 0$ and that the linear map

$$\sum_{ij} \lambda_{ij} x_i x_j \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \longmapsto \sum_i x_i \mu_i$$

has rank $n-1$. In particular, if the complex numbers $\lambda_{ij}$ are sufficiently general then $\sum_i c_i \mu_i = 0$ with $c_i \in \mathbb{Z}$ implies that $c_1 = \cdots = c_n$. For $t$ sufficiently general, the eigenvalues of the linear part of $D_\Omega \Pi_t$ at $\gamma(t)$ will have the same property. According to [14], we have a formal change of coordinates centered at $\gamma(t)$, which transforms $D_\Omega \Pi_t$ into a vector field

$$v_t = D_\Omega \Pi_t = \sum_i A_{i,t}(x_1 \cdots x_n) x_i \frac{\partial}{\partial x_i} \ ,$$

where $A_{i,t}$ are germs holomorphic functions in $(\mathbb{C}, 0)$ satisfying $A_{i,t}(0) = \mu_{i,t} \neq 0$.

Since $v_t$ is Poisson vector field for $\Pi_t$ we have that $[v_t, \Pi_t] = 0$. If we write $\Pi_t = \sum \pi_{ij,t} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$ then we obtain

$$\sum_{i,j} \pi_{ij,t}(0)(\mu_{i,t} + \mu_{j,t}) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} = 0 \,.$$

Therefore for every $i, j$ the identity $\pi_{ij,t}(0) = 0$ holds true since $\mu_{i,t}$ and $\mu_{j,t}$ are $\mathbb{Z}$-linearly independent. Thus $\Pi_t$ vanishes at zero. Similarly, looking at the linear part of $\Pi$ at zero we obtain that

$$(\mu_{i,t} + \mu_{j,t} - \mu_{k,t}) \frac{\partial \pi_{ij,t}}{\partial x_k}(0) = 0 \text{ for every } i,\ j \text{ and } k.$$

Again by the $\mathbb{Z}$-linear independence of $\mu_{i,t}, \mu_{j,t}$ and $\mu_{k,t}$, we deduce that the linear part of $\Pi_t$ also vanishes at zero. Therefore $(\Pi_t)^m$ vanishes at zero with order greater than or equal to $2m$. □

**Lemma 5.2.** *Let $H$ be a reduced hypersurface in $\mathbb{P}^k$ of degree $k+1$. Suppose that $H$ has $k+1$ singular points in general position. If the algebraic multiplicity of each of these $k+1$ points is $k$ then $H$ is the union of $k+1$ hyperplanes in general position. in general position.*

*Proof.* Let $p_0, \ldots, p_k$ be the $k+1$ points of $H$ with algebraic multiplicity $k$. We can assume that $p_0 = [1 : 0 : \ldots : 0], p_1 = [0 : 1 : \ldots : 0], \ldots, p_k = [0 : \ldots : 0 : 1]$. Let $f \in \mathbb{C}[x_0, \ldots, x_n]$ be a homogenous polynomial of degree $k+1$ cutting out $H$. Since $H$ has algebraic multiplicity $k$ at $p_i$, it follows that the polynomials $\frac{\partial^2 f}{\partial x_i^2}$ vanish identically for every $i \in \{0, \ldots, k\}$. In other words, every monomial contributing to the Taylor expansion of $f$ at $0 \in \mathbb{C}^{k+1}$ is square-free. But there is only one square-free monomial of degree $k+1$, $x_0 \cdots x_{k+1}$. The lemma follows. □

5.2. **Proof of Theorem 2.** Let us recall the statement of Theorem 2.

**Theorem 5.3.** *If we take sufficiently small deformations of a generic diagonal Poisson structure in $\mathbb{P}^n$ then the resulting Poisson structures are still diagonal Poisson structures.*

*Proof.* Assume first that $n = 2k+1$ is odd. If $\Pi$ is a generic Poisson structure then it has rank $2k$. The symplectic foliation is nothing but the logarithmic foliation defined by

$$\omega = \left(\prod_{i=0}^{2k+1} x_i\right)\left(\sum_{i=0}^{2k+1} \lambda_i \frac{dx_i}{x_i}\right) \in H^0(\mathbb{P}^{2k+1}, \Omega^1_{\mathbb{P}^{2k+1}}(2k+2))$$

where $\lambda_i \in \mathbb{C}$ satisfy $\sum_{i=0}^{2k+1} \lambda_i = 0$. Moreover, any choice of complex numbers $\lambda_i$ summing up to zero, defines a codimension one logarithmic foliation $\mathcal{F}_\omega$ which is the symplectic foliation of a diagonal Poisson structure. To prove this notice that the tangent sheaf of the foliation is trivial and its space of global sections is vector space of dimension $2k$ of commuting vector fields tangent to the hypersurface $\{x_0 \cdots x_{2k+1} = 0\}$. Any element of $\bigwedge^2 H^0(\mathbb{P}^{2k+1}, T\mathcal{F}_\omega)$ having rank $2k$ defines the sought Poisson structure. The stability of a general diagonal Poisson structure on $\mathbb{P}^{2k+1}$ follows from the corresponding result for the stability of codimension one logarithmic foliations with poles on $2k+2$ hyperplanes, see the main result of [3] or [5, Example 6.2].

Assume now that $n = 2k$ is even. If $\Pi$ is a generic diagonal Poisson structure in $\mathbb{P}^{2k}$ then it is generically symplectic and there are $2k+1$ points where $\Pi$ vanishes. Lemma 5.1 implies that any small deformation $\Pi_\varepsilon$ of $\Pi$ will still have $2k+1$ points where $\Pi_\varepsilon$ vanishes and $\Pi_\varepsilon^k$ has vanishing order $2k$ at each of these points. Consequently, the degeneracy divisor $\mathrm{D}(\Pi_\varepsilon)$ has $2k+1$ points of multiplicity $2k$ and Lemma 5.2 implies that it must be the union of $2k+1$ hyperplanes in general position. Theorem 2 follows from Proposition 3.10. □

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*E-mail address*: renan@impa.br

IMPA, Estrada Dona Castorina, 110, 22460-320, Rio de Janeiro, RJ, Brasil

*E-mail address*: jvp@impa.br

IMPA, Estrada Dona Castorina, 110, 22460-320, Rio de Janeiro, RJ, Brasil