ALGEBRAIC REDUCTION THEOREM FOR COMPLEX CODIMENSION ONE SINGULAR FOLIATIONS

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Abstract. Let M be a compact complex manifold equipped with $n = \dim(M)$ meromorphic vector fields that are independant at a generic point. The main theorem is the following. If M is not bimeromorphic to an algebraic manifold, then any codimension one complex foliation $\mathcal{F}$ with a codimension ≥ 2 singular set is the meromorphic pull-back of an algebraic foliation on a lower dimensional algebraic manifold, or $\mathcal{F}$ is transversely projective outside a proper analytic subset. The two ingredients of the proof are the Algebraic Reduction Theorem for the complex manifold M and an algebraic version of Lie's first Theorem which is due to J. Tits.

1. Introduction

Let M be a compact connected complex manifold of dimension $n \geq 2$. A (codimension 1 singular holomorphic) foliation $\mathcal{F}$ on M will be given by a covering of M by open subsets $(U_j)_{j\in J}$ and a collection of integrable holomorphic 1-forms ω_j on U_j, $\omega_j \wedge d\omega_j = 0$, having codimension ≥ 2 zero-set such that, on each non empty intersection $U_j \cap U_k$, we have

$$(*) \quad \omega_j = g_{jk} \cdot \omega_k, \quad \text{with} \quad g_{jk} \in \mathcal{O}^*(U_j \cap U_k).$$

Let $\text{Sing}(\omega_j) = \{p \in U_j \ ; \ \omega_j(p) = 0\}$. Condition $(*)$ implies that $\text{Sing}(\mathcal{F}) := \cup_{j\in J}\text{Sing}(\omega_j)$ is a codimension ≥ 2 analytic subset of M. If ω is an integrable meromorphic 1-form on M, $\omega \wedge d\omega = 0$, then we can associate to ω a foliation $\mathcal{F}_\omega$ as above. Indeed, at the neighborhood of any point $p \in M$, one can write $\omega = f \cdot \tilde{\omega}$ with f meromorphic, sharing the same divisor with ω; therefore, $\tilde{\omega}$ is holomorphic with codimension ≥ 2 zero-set and defines $\mathcal{F}_\omega$ on the neighborhood of p.

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The manifold M is called *pseudo-parallelizable*, if there exist n meromorphic vector fields $X_1, \ldots, X_n$ on M that are independent at a generic point. On such a manifold, differential calculus can be done likely as on an algebraic manifold and a foliation $\mathcal{F}$ is always defined by a global meromorphic 1-form ω (satisfying $\omega \wedge d\omega = 0$). Indeed, given a meromorphic vector field on M which is not identically tangent to $\mathcal{F}$, then there exists a unique meromorphic 1-form ω defining $\mathcal{F}$ and satisfying $\omega(X) \equiv 1$. We will denote $\mathcal{F} = \mathcal{F}_\omega$.

The notion of pseudo-parallelizable manifolds is invariant by bimeromorphic transformations; more generally, if $f : \tilde{M} \dashrightarrow M$ is meromorphic and generically étale, and if M is pseudo-parallelizable, then $\tilde{M}$ is also. Besides algebraic manifolds, one can find complex tori, Hopf manifolds, Iwasawa threefolds and homogeneous spaces among examples of such manifolds. Also, manifolds constructed in [6, 7] are pseudo-parallelizable but not of the previous type. Of course, even among surfaces, there are manifolds which are not pseudo-parallelizable.

We say that $\mathcal{F}_\omega$ is *transversely projective* if there exist meromorphic 1-forms $\omega_0 = \omega$, ω_1 and ω_2 on M satisfying

$$\begin{cases} d\omega_0 &= \omega_0 \wedge \omega_1 \\ d\omega_1 &= \omega_0 \wedge \omega_2 \\ d\omega_2 &= \omega_1 \wedge \omega_2 \end{cases} \tag{1}$$

This means that, outside the polar and singular set of the ω_i's, the foliation $\mathcal{F}$ is (regular and) transversely projective in the classical sense (see [3]) and this projective structure has "reasonable singularities". See [8, 10] for basic properties and examples of transversely projective foliations in the meromorphic singular sense above. When $\omega_2 = 0$ (i.e. $d\omega_1 = 0$) or $\omega_1 = 0$ (i.e. $d\omega_0 = 0$), we respectively say that $\mathcal{F}_\omega$ is actually *transversely affine* or *euclidian*. For instance, the foliation defined on $\mathbb{CP}(2)$ by the closed 1-form $\omega_0 = dx + \frac{dy}{y}$ is transversely euclidian in the main affine chart $(x, y) \in \mathbb{C}^2$ with poles along the line at infinity; the other euclidian structure given on $\mathbb{C}^2$ by $\tilde{\omega}_0 = e^x dx + dy$ does not extend meromorphically on $\mathbb{CP}(2)$. One can find in [10] examples of germs of foliations admitting a unique projective structure outside an analytic set, but that cannot extend meromorphically along this set. We do not know if such phenomena can happen for foliations on compact manifold.

Now, denote by $a(M)$ the algebraic dimension of M, that is the transcendence degree over $\mathbb{C}$ of the field $\mathcal{M}(M)$ of meromorphic functions on M. The algebraic Reduction Theorem (see [11] or section 2) provides a meromorphic map $f : M \dashrightarrow N$ onto a projective manifold N of dimension $a(M)$ such that $\mathcal{M}(M)$ identifies with $f^*\mathcal{M}(N)$. In fact, the fibers of f are the maximal subvarieties on which every meromorphic function on M is constant. Of course, the map f is unique up to birational modifications of N. We will denote by $\text{red} : M \dashrightarrow \text{red}(M)$ this map. There exist pseudo-parallelizable manifolds M of arbitrary dimension $n \geq 2$ with arbitrary algebraic dimension $0 \leq a(M) \leq n$.

When M is not algebraic (up to a bimeromorphism), i.e. $a(M) < n$, our main result is the following "Foliated Reduction Theorem".

Theorem 1.1. *Let $\mathcal{F}$ be a complex codimension one singular foliation on a pseudo-parallelizable compact complex manifold M. Then*

- *$\mathcal{F}$ is the pull-back by the reduction map $M \dashrightarrow \text{red}(M)$ of an algebraic codimension one foliation $\underline{\mathcal{F}}$ defined on $\text{red}(M)$,*
- *or $\mathcal{F}$ is transversely projective.*

More precisely, we are in the former case when the fibers of the map $M \dashrightarrow \text{red}(M)$ are contained in the leaves of $\mathcal{F}$. In the case $a(M) = 0$ (i.e. $\mathcal{M}(M) = \mathbb{C}$) or $a(M) = 1$, we have no alternative (if $\mathcal{F}$ is the pull-back of a foliation by points on a curve, then it is automatically transversely euclidean)

Corollary 1.2. *Let $\mathcal{F}$ and M be as above and assume $a(M) = 0$ or 1. Then $\mathcal{F}$ is transversely projective.*

When M is simply connected and $a(M) = 0$, it follows that $\mathcal{F}$ necessarily admits an invariant hypersurface, that is the singular set of the projective structure. Indeed, if the projective structure were not singular, the development map of the structure would provide a non constant meromorphic function on M, thus contradicting $a(M) = 0$.

In fact, our proof gives a more precise statement in the case the fibres of the map $M \dashrightarrow \text{red}(M)$ have dimension 1 or 2

Theorem 1.3. *Let $\mathcal{F}$ be a complex codimension one singular foliation on a pseudo-parallelizable compact complex manifold M and assume that $a(M) = n - 1$ or $n - 2$. Then*

- *$\mathcal{F}$ is the pull-back by the reduction map $M \dashrightarrow \text{red}(M)$ of an algebraic codimension one foliation $\underline{\mathcal{F}}$ defined on $\text{red}(M)$,*
- *or $\mathcal{F}$ is transversely affine.*

In the case M is a surface, we deduce

Corollary 1.4. *Let $\mathcal{F}$ be a singular foliation on a non algebraic pseudo-parallelizable compact surface S. Then $\mathcal{F}$ is transversely affine.*

Here, we use Chow-Kodaira Theorem ([11], p.249) that a surface having algebraic dimension 2 is actually projective.

In the case of a threefold, we resume

Corollary 1.5. *Let M be a 3-dimensional pseudo-parallelizable complex manifold and let $\mathcal{F}$ be a foliation of M. We have the following possibilities:*

(1) $a(M) = 3$ *and $\mathcal{F}$ is bimeromorphically equivalent to an algebraic foliation of an algebraic manifold;*
(2) $a(M) = 2$ *and $\mathcal{F}$ is the meromorphic pull-back of a foliation on an algebraic surface, or $\mathcal{F}$ is transversely affine;*
(3) $a(M) = 1$ *and $\mathcal{F}$ is transversely affine;*
(4) $a(M) = 0$ *and $\mathcal{F}$ is transversely projective.*

We now give example illustrating that the affine cases (2) and (3) and the projective case (4) do occur in the statement above.

Example 1.6. A complex torus $\mathbb{T}^n = \mathbb{C}^n/\Lambda$, Λ a co-compact lattice, can have arbitrary algebraic dimension $0 \leq a \leq n$ and carries the n independant 1-forms $dz_1, \ldots, dz_n$. The quotient by the involution $\sigma : z \mapsto -z$, $z = (z_1, \ldots, z_n)$ is a variety with conic singularities that can be desingularized after one blowing-up. The resulting smooth manifold M is pseudo-parallelizable as soon as $a > 0$, with algebraic dimension a. Indeed, the algebraic reduction of $\mathbb{T}^n$ is an algebraic torus $\mathbb{T}^a$ (see [2]) on which the involution σ is well defined; after chosing a meromorphic function $f \in \mathcal{M}(\mathbb{T}^a) = \mathcal{M}(\mathbb{T}^n)$ satisfying $f \circ \sigma = -f$, the meromorphic 1-forms $f \cdot dz_i$, $i = 1, \ldots, n$, are σ invariant and provide a pseudo-parallelism on the quotient manifold M. The linear foliation $\mathcal{F}$ defined by a generic linear combination of the 1-forms $f \cdot dz_i$ is transversely euclidean on $\mathbb{T}^n$ and transversely affine (and not better) on M.

Example 1.7. Consider the quotient $M := \Gamma \diagdown SL(2, \mathbb{C})$ by a co-compact lattice $\Gamma \subset SL(2, \mathbb{C})$. The left-invariant 1-forms define a parallelism on M. Following [5], there is no non constant meromorphic function on M (i.e. the algebraic dimension of M is $a(M) = 0$). Now, it is classical that the foliation $\mathcal{F}$ defined by a left-invariant 1-form (some of them are integrable) is transversely projective and not better since its monodromy is given by Γ.

One of the ingredients for the proof of Theorem 1.1 is the following algebraic version, due to J. Tits [9], of Lie's first Theorem

Lemma 1.8. *Let $\mathcal{L}$ be a finite dimensional Lie algebra over a field $\mathbb{K}$ of characteristic* 0*. If $\mathcal{L}$ has a codimension one Lie subalgebra $\mathcal{L}'$, then there exists a non trivial morphism $\phi : \mathcal{L} \to sl(2, \mathbb{K})$ such that the kernel of ϕ is contained in $\mathcal{L}'$.*

We end the paper by proving the following proposition generalizing some of the results obtained by É. Ghys in [4] for the foliations on complex tori.

Proposition 1.9. *Let $\mathcal{F}$ be a foliation on a compact manifold M and assume that there exist $n = \dim(M)$ independent closed meromorphic* 1*-forms on M. Then we have the following alternative:*

- *$\mathcal{F}$ is the pull-back of a foliation $\underline{\mathcal{F}}$ on red(M) via the algebraic reduction map $M \dashrightarrow$ red(M),*
- *or $\mathcal{F}$ is transversely euclidean, i.e. defined by a closed meromorphic* 1*-form.*

We would like to thank Étienne Ghys who helped us to improve the preliminary version [1] of our results. He brought reference [9] to our knowledge and provided an argument to avoid Godbillon-Vey sequences in order to conclude the proof of our Reduction Theorem. In particular, Theorem 1.3 was proved in [1] only for three-folds in the case $a(M) = n - 2$.

A natural question raised by the present work is: *can we avoid with "pseudo-parallelizable" assumption in our Reduction Theorem ?* In fact, even in the algebraic case, all known singular foliations on $\mathbb{CP}^n$, $n \geq 3$, are transversely projective, or pull-back by a rational map $\mathbb{CP}^n \dashrightarrow \mathbb{CP}^2$ of a foliation on the plane. *Is there a general principle ?* In a forthcoming paper, we further investigate these questions using Godbillon-Vey sequences. Among other results, we prove that our Reduction Theorem still holds whenever there exists at least one meromorphic vector field generically transversal to $\mathcal{F}$ on M.

2. Algebraic Reduction Theorem [11]

We now state the Algebraic Reduction Theorem. Let M be a compact connected complex manifold and consider the field $\mathcal{M}(M)$ of meromorphic functions on M. The *algebraic dimension*

$$a(M) \in \{0, 1, \ldots, \dim(M)\}$$

of M is the transcendence degree of $\mathcal{M}(M)$ over $\mathbb{C}$, i.e. the maximal number of elements $f_1, \ldots, f_a \in \mathcal{M}(M)$ satisfying

$$df_1 \wedge \cdots \wedge df_a \neq 0.$$

Recall that $a(M) = \dim(M)$ if, and only if, M is bimeromorphically equivalent to an algebraic manifold. If not, we have (see [11])

Theorem 2.1 (Algebraic Reduction). *Let M be a compact connected complex manifold of algebraic dimension $n = a(M)$. There exist*

1. *a bimeromorphic modification $\Psi : \tilde{M} \to M$,*
2. *an holomorphic projection $\pi : \tilde{M} \to N$ with connected fibers onto an n-dimensional algebraic manifold N*

such that $\Psi^*\mathcal{M}(M) = \pi^*\mathcal{M}(N)$.

We will denote by red the meromorphic map $\pi \circ \Psi^{-1}$. When $a(M) < \dim(M)$, i.e. M is not bimeromorphic to an algebraic manifold $\tilde{M}$, M is naturally equipped with the canonical codimension $a(M)$ fibration $\mathcal{G}$ induced by fibers of red.

The space $\mathcal{X}(M)$ of meromorphic vector fields over M acts by derivation on $\mathcal{M}(M)$ and, in this sense, preserves the fibration $\mathcal{G}$. Precisely, given any $X \in \mathcal{X}(M)$, the local flow of X sends fibers to fibers at the neighborhood of any point $p \in M$ where X and $\mathcal{G}$ are regular. In other words, any vector field X on M is a lifting of some vector field Y on the reduction $N = \text{red}(M)$. This can be seen also directly from the fact that a derivation on $\mathcal{M}(M) := \text{red}^*\mathcal{M}(N)$ is actually a derivation on $\mathcal{M}(N)$. The kernel $\mathcal{X}_0(M) = \{X \in \mathcal{X}(M) | X(f) = 0, \forall f \in \mathcal{M}(M)\}$ coincides with the subspace of those vector fields that are tangent to the fibration $\mathcal{G}$. The space $\mathcal{X}(M)$ is a Lie algebra over $\mathbb{C}$, having infinite dimension as soon as $a(M) \neq 0$, and $\mathcal{X}_0(M)$ is an ideal: $[\mathcal{X}_0(M), \mathcal{X}(M)] \subset \mathcal{X}_0(M)$. Observe that $\mathcal{X}_0(M)$ is also a Lie algebra over the field $\mathcal{M}(M)$, having dimension $\leq \dim(M) - a(M)$. We take care that the space of meromorphic vector fields $\mathcal{X}(F)$ on a given fiber F can actually be much bigger than the restriction $\mathcal{X}_0(M)|_F$: except in the case $a(M) = 0$, some of the fibers could carry non constant meromorphic functions (even, all fibers could be algebraic, like in Iwasawa three-fold).

Given a foliation $\mathcal{F}$ on M, we will distinguish between the case where $\mathcal{F}$ is tangent to the fibration $\mathcal{G}$ and the case where they are transversal at a generic point. The latter case will be studied in Section 3. The former case is completely understood by means of

Lemma 2.2. *Let $\mathcal{F}$ be a foliation on a complex manifold M. Let $\pi : M \to N$ be a surjective holomorphic map whose fibers are connected and tangent to $\mathcal{F}$, that is, contained in the leaves of $\mathcal{F}$. Then, $\mathcal{F}$ is the pull-back by π of a foliation $\tilde{\mathcal{F}}$ on N.*

Proof. In a small connected neighborhood $U \subset M$ of a generic point $p \in M$, the foliation $\mathcal{F}$ is regular, defined by a local submersion $f : U \to \mathbb{C}$. Since f is contant along the fibers of π in U, we can factorize $f = \tilde{f} \circ \pi$ for an holomorphic function $\tilde{f} : \pi(U) \to \mathbb{C}$. In particular, the function $\tilde{f}$ defines a codimension one singular foliation $\tilde{\mathcal{F}}$ on the open set $\pi(U)$. Of course, $\tilde{\mathcal{F}}$ does not depend on the choice of f. Moreover, since $f = \tilde{f} \circ \pi$, the function f extends to the whole tube $T := \pi^{-1}(\pi(U))$. By connectivity of U and the fibers of π, the tube T is connected and the foliation $\mathcal{F}$ is actually defined by f on the whole of T, coinciding with $\pi^*(\tilde{\mathcal{F}})$ on T. In this way, we can define a foliation $\tilde{\mathcal{F}}$ on $N \setminus S$, where $S = \{p \in N \ ; \ \pi^{-1}(p) \subset \mathrm{Sing}(\mathcal{F})\}$ such that $\mathcal{F} = \pi^*(\tilde{\mathcal{F}})$. We note that S has codimension ≥ 2 in N; therefore, $\tilde{\mathcal{F}}$ extends on N by Levi's Extension Theorem. □

Corollary 2.3. *Let $\mathcal{F}$ be a foliation on a compact manifold M. If the fibers of the algebraic reduction red : $M \dashrightarrow$ red(M) are tangent to $\mathcal{F}$, then $\mathcal{F}$ is actually the pull-back of an algebraic foliation $\tilde{\mathcal{F}}$ on red(M).*

In particular, even if M was not pseudo-parallelizable, $\mathcal{F}$ is *a posteriori* defined by a global meromorphic 1-form, namely the pull-back of any rational 1-form defining $\tilde{\mathcal{F}}$ on the algebraic manifold $\mathrm{red}(M)$.

3. Proof of the main result

3.1. **Foliated Algebraic Reduction: the case $a(M)=0$.** Recall first the classical

Lemma 3.1 (Lie). *Let $\mathcal{L}$ be a (finite dimensional) transitive Lie algebra of holomorphic vector fields defined on some neighborhood of $0\in\mathbb{C}$. Then, after a change of local coordinate, we are in one of the following three cases:*

1. $\mathcal{L}=\mathbb{C}\cdot\partial_z$;
2. $\mathcal{L}=\mathbb{C}\cdot\partial_z+\mathbb{C}\cdot z\partial_z$;
3. $\mathcal{L}=\mathbb{C}\cdot\partial_z+\mathbb{C}\cdot z\partial_z+\mathbb{C}\cdot z^2\partial_z$.

In particular, $\mathcal{L}$ is a representation of a subalgebra of $sl(2,\mathbb{C})$.

We need a technical Lemma. Given a vector field X on a manifold M, we denote by L_X the Lie derivative on differential k-forms. Notice that, when X is a meromorphic vector field on a compact manifold M, then L_X is trivial on the 0-forms $\mathcal{M}(M)$

$$L_X f=0,\quad \forall f\in\mathcal{M}(M)$$

if, and only if, the vector field X is actually tangent to the fibers of the algebraic reduction $\mathrm{red}: M\to\mathrm{red}(M)$ (see Theorem 2.1).

Lemma 3.2. *Let M be a compact manifold, ω be a meromorphic 1-form on M and X be a meromorphic vector field satisfying $\omega(X)=1$ and $L_X\mathcal{M}(M)=0$. Then, for all meromorphic vector field Y on M, we have*

$$L_X^{(i)}\omega(Y)=(-1)^i\omega(L_X^iY)$$

where $L_XY=[X,Y]$.

Proof. Since $\omega(X)=1$, we have

$$L_X\omega=d(\omega(X))+d\omega(X,.)=d\omega(X,.)\,.$$

Therefore, for any vector field Y, we have

$$L_X\omega(Y)=d\omega(X,Y)=L_X(\omega(Y))-L_Y(\omega(X))-\omega([X,Y])\,.$$

By assumption, we have that $L_X(\omega(Y))=L_X(\text{function})=0$ and $L_Y(\omega(X))=L_Y(\text{constant})=0$. Thus we conclude that

$$L_X\omega(Y)=-\omega([X,Y])\,.$$

The proof immediately follows by induction on i. □

Lemma 3.3. *Let $\mathcal{F}$ be the foliation defined by a meromorphic* 1*-form* ω *on a manifold* M *and* X *be a meromorphic vector field satisfying* $\omega(X) = 1$ *and* $L_X^{(3)}\omega = 0$*. Then, the meromorphic triple* $(\omega, L_X\omega, L_X^{(2)}\omega)$ *satisfies (1): the foliation* $\mathcal{F}$ *defined by* $\omega_0 = \omega$ *is transversely projective.*

Of course, when $L_X^{(2)}\omega = 0$ (respectively $L_X^{(1)}\omega = 0$) the foliation $\mathcal{F}$ is actually transversely affine (resp. euclidean).

Proof. First of all, the integrability condition of ω is equivalent to

$$\omega \wedge d\omega = 0 \quad \Leftrightarrow \quad d\omega = \omega \wedge L_X\omega. \tag{2}$$

Indeed, from $L_X\omega = d(\omega(X)) + d\omega(X, .) = d\omega(X, .)$, we derive

$$0 = \omega \wedge d\omega(X, ., .) = \omega(X) \cdot d\omega - \omega \wedge (d\omega(X, .)) = d\omega - \omega \wedge L_X\omega$$

(the converse is obvious). We thus obtain the first condition

$$d\omega_0 = \omega_0 \wedge \omega_1.$$

Now, applying Lie derivative to this equality yields

$$d(L_X\omega_0) = L_X(d\omega_0) = L_X\omega_0 \wedge \omega_1 + \omega_0 \wedge L_X\omega_1,$$

i.e. $d\omega_1 = \omega_0 \wedge \omega_2$ since $\omega_1 \wedge \omega_1 = 0$. Applying a last time the Lie derivative, we finally get

$$d\omega_2 = \omega_1 \wedge \omega_2 + \omega_0 \wedge \omega_3$$

by assumption, $\omega_3 = L_X^{(3)}\omega = 0$ and we get the projective relations (1). □

Let M be a pseudo-parallelizable compact manifold with no non constant meromorphic function. Therefore, the Lie algebra $\mathcal{L}$ of meromorphic vector fields on M has dimension $n = \dim(M)$. If $\mathcal{F}$ is a foliation on M, then the Lie algebra $\mathcal{L}'$ of those vector fields tangent to $\mathcal{F}$ has dimension $n-1$. Following Lemma 1.8, there exists a morphism $\phi : \mathcal{L} \to sl(2, \mathbb{C})$ such that $\ker(\phi) \subset \mathcal{L}' \subset \mathcal{L}$. Discussing on the codimension of $\ker(\phi)$, we construct a meromorphic vector field X satisfying $\omega(X) = 1$ (in particular $X \in \mathcal{L} \setminus \mathcal{L}'$) such that $L_X^{(3)}\omega = 0$.

3.1.1. *First Case:* $\ker(\phi)$ *has codimension* 1. Therefore, $\mathcal{L}' = \ker(\phi)$. In particular, $\mathcal{L}'$ is an ideal of $\mathcal{L}$: $[\mathcal{L}, \mathcal{L}'] \subset \mathcal{L}'$. Let X be any meromorphic vector field satisfying $\omega(X) = 1$. For every $Y \in \mathcal{L}$, we can write

$$Y = c \cdot X + Y' \,,$$

where $c \in \mathbb{C}$ and $Y' \in \mathcal{L}'$. Thus

$$\omega([X, Y]) = \omega([X, Y']) = 0 \quad \forall Y \in \mathcal{L},$$

allowing us to conclude that $L_X \omega = 0$ (see Lemma 3.2). Finally, $\omega_0 = \omega$ is closed.

3.1.2. *Second Case:* $\operatorname{codim} \ker(\phi) = 2$. One can choose a basis X_1, X_2 of $\mathcal{L}/\ker(\phi)$ such that $\omega(X_1) = 1$, X_2 is a basis for $\mathcal{L}'/\ker(\phi)$ and

$$\text{either } [X_1, X_2] = X_1, \quad \text{or } [X_1, X_2] = -X_2.$$

Indeed, after composing ϕ by an automorphism of $sl(2, \mathbb{C})$, the dimension two subalgebra $\phi(\mathcal{L})$ identifies with the Lie algebra generated by

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and $\phi(\mathcal{L}')$ is the one dimensional subalgebra generated by A or B. Then, just choose X_1 and X_2 so that correspondingly $(\phi(X_1), \phi(X_2)) = (B, A)$ or (A, B) and normalize $\omega_0 := \frac{\omega}{\omega(X_1)}$ so that $\omega_0(X_1) = 1$. Therefore, $\omega_0(L^i_{X_1} X_2) = 0$ (i.e. $L^i_{X_1} X_2 \in \mathcal{L}'$) for $i = 1$ or 2. Finally, after writing every vector field $Y \in \mathcal{L}$ into the form

$$Y = c_1 \cdot X_1 + c_2 \cdot X_2 + Y' \,,$$

with $c_1, c_2 \in \mathbb{C}$ and $Y' \in \ker(\phi)$ and applying Lemma 3.2 as in Section 3.1.1, we conclude that $\mathcal{F}$ is transversely affine.

3.1.3. *Third Case:* $\operatorname{codim} \ker(\phi) = 3$. We construct a basis X_1, X_2, X_3 of $\mathcal{L}/\ker(\phi)$ such that $\omega(X_1) = 1$, X_2, X_3 is a basis for $\mathcal{L}'/\ker(\phi)$ and, after composing ϕ by an automorphism of $sl(2, \mathbb{C})$,

$$\phi(X_1) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \phi(X_2) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \phi(X_3) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Therefore, we have

$$[X_1, [X_1, [X_1, X_3]]] \;=\; [X_1, [X_1, X_2]] \;=\; 0 \qquad \operatorname{mod} \;\; \ker(\phi)$$

and $\omega(L^3_{X_1}(Y)) = 0$ for all $Y \in \mathcal{L}$. Finally, $\mathcal{F}$ is transversely projective.

3.2. **Codimension one Lie subalgebras over $\mathbb{K}$.** Before proving Theorem 1.1 for arbitrary algebraic dimension $a(M)$, we need the following more complete statement for Lemma 1.8.

Lemma 3.4 (Tits [9]). *Let $\mathcal{L}$ be a finite dimensional Lie algebra over a field $\mathbb{K}$, $\text{char}(\mathbb{K}) = 0$, having a codimension one Lie subalgebra $\mathcal{L}'$: $[\mathcal{L}', \mathcal{L}'] \subset \mathcal{L}'$. Then $\mathcal{L}$ also has a codimension ≤ 3 Lie-ideal $\mathcal{I}$, $[\mathcal{L}, \mathcal{I}] \subset \mathcal{I}$, which is contained in $\mathcal{L}'$ and we are in one of the following 3 cases:*

(1) $\mathcal{L}/\mathcal{I} = \mathbb{K} \cdot X$ *and* $\mathcal{L}' = \mathcal{I}$;
(2) $\mathcal{L}/\mathcal{I} = \mathbb{K} \cdot X + \mathbb{K} \cdot Y$ *with* $[X, Y] = X$ *and* $\mathcal{L}'/\mathcal{I} = \mathbb{K} \cdot Y$;
(3) $\mathcal{L}/\mathcal{I} = \mathbb{K} \cdot X + \mathbb{K} \cdot Y + \mathbb{K} \cdot Z$ *with* $[X, Y] = X$, $[X, Z] = 2Y$ *and* $[Y, Z] = Z$ *and* $\mathcal{L}'/\mathcal{I} = \mathbb{K} \cdot Y + \mathbb{K} \cdot Z$.

In other words, there exists a non trivial morphism $\phi : \mathcal{L} \to sl(2, \mathbb{K})$ whose kernel $\ker(\phi) = \mathcal{I}$ is contained in $\mathcal{L}'$.

Applying this Lemma to a finite dimensional transitive subalgebra $\mathcal{L} \subset \mathcal{X}(\mathbb{C}, 0)$ and to the subalgebra $\mathcal{L}'$ of those vector fields fixing 0, we retrieve a part of Lie's Lemma 3.1. In this sense, Lemma 3.4 may be considered as an algebraic version of Lie's Lemma.

3.3. **Foliated Algebraic Reduction: the general case.** Let M be a pseudo-parallelizable compact manifold having algebraic dimension $a(M) < \dim M$ and let $\mathcal{F}$ be a foliation on M. We assume that $\mathcal{F}$ is generically transverse to the fibers given by the Algebraic Reduction Theorem, otherwise we conclude with Lemma 2.2 that we are actually in the second alternative of Theorem 1.1. The idea of the proof is to proceed as in section 3.1 along the fibers, but dealing only with objects (vector fields and functions) living on the ambient manifold M. Denote by $\mathcal{X}_0(M)$ the space of meromorphic vector fields that are tangent to the fibers. Recall that $\mathcal{L} := \mathcal{X}_0(M)$ is a Lie algebra of dimension $\dim(M) - a(M)$ over the field $\mathbb{K} := \mathcal{M}(M)$ of meromorphic functions on M. Consider $\mathcal{L}' \subset \mathcal{L}$ the Lie subalgebra of those vector fields that are tangent to the foliation $\mathcal{F}$. Clearly, $\mathcal{L}'$ has codimension 1 in $\mathcal{L}$. Applying Lemma 3.4 to this situation, we see that there is an ideal $\mathcal{I} \subset \mathcal{L}$ contained in $\mathcal{L}'$, and there is some $X \in \mathcal{L} \setminus \mathcal{L}'$ satisfying $L_X^3 V \in \mathcal{I}$ for any $V \in \mathcal{X}_0(M)$ (for instance, $L_X^3 X = L_X^3 Y = L_X^3 Z = 0$ modulo $\mathcal{I}$ in case (3) of Lemma 3.4). Let ω be the unique meromorphic 1-form defining the foliation $\mathcal{F}$ and satisfying $\omega(X) = 1$. Since $\mathcal{I} \subset \mathcal{L}'$, we deduce that $\omega(L_X^3 V) = 0$ for any $V \in \mathcal{X}_0(M)$.

Now, in order to conclude by means of Lemma 3.2, we have to prove that $\omega(L_X^3 V) = 0$ for any $V \in \mathcal{X}(M)$. It is enough to consider $V \in \mathcal{X}'(M)$, the subspace of meromorphic vector fields tangent to the foliation, since X together with $\mathcal{X}'(M)$ span $\mathcal{X}(M)$ over $\mathcal{M}(M)$. We now consider the three cases given by Lemma 3.4. We recall that, in any case, $[X, \mathcal{L}] \in \mathcal{L}$ and $[V, \mathcal{L}'] \in \mathcal{L}'$ (Frobenius integrability condition for $\mathcal{F}$).

3.3.1. *First Case: $\mathcal{L}'$ is an ideal of $\mathcal{L}$.* We have $[X, V] = f \cdot X$ modulo $\mathcal{I} = \mathcal{L}'$ for some $f \in \mathcal{M}(M)$. Therefore, $[X, [X, V]] = 0$ and $\omega(L_X^2 V) = 0$ by Lemma 3.2. We conclude by Lemma 3.3 that the foliation $\mathcal{F}$ is transversely affine.

3.3.2. *Second Case: $\mathcal{L}'/\mathcal{I}$ is generated by Y with $[X, Y] = X$ modulo I.* We have $[X, V] = f \cdot X + g \cdot Y$ mod I and $[Y, V] = h \cdot Y$ mod I for coefficients $f, g, h \in \mathcal{M}(M)$ (here, we use the fact that both Y and V are tangent to $\mathcal{F}$, whence their Lie bracket). Applying Jacobi identity to X, Y and Z yields:

$$[X, [Y, V]] + [V, [X, Y]] + [Y, [V, X]] = h \cdot X - g \cdot Y = 0$$

and we have $h = g = 0$. In particular, $[X, V] = f \cdot X$ and $[X, [X, V]] = 0$. We conclude as before that $\mathcal{F}$ is transversely affine.

3.3.3. *Third Case: $\mathcal{L}'/\mathcal{I}$ is generated by Y, Z with $[X, Y] = X$, $[X, Z] = 2Y$ and $[Y, Z] = Z$ modulo I.* We have:

$$\begin{cases} [X, V] &= & f \cdot X + g \cdot Y + h \cdot Z \\ [Y, V] &= & i \cdot Y + j \cdot Z \\ [Z, V] &= & k \cdot Y + l \cdot Z \end{cases} \quad \text{mod } \mathcal{I}$$

for some coefficients $f, g, h, i, j, k, l \in \mathcal{M}(M)$. Jacobi identities yield:

$$[X, [Y, V]] + [V, [X, Y]] + [Y, [V, X]] =$$

$$= i \cdot X + (2j - g) \cdot Y - 2h \cdot Z = 0$$

$$[X, [Z, V]] + [V, [X, Z]] + [Z, [V, X]] =$$

$$= k \cdot X + 2(f + l - i) \cdot Y + (g - 2j) \cdot Z = 0$$

$$[Y, [Z, V]] + [V, [Y, Z]] + [Z, [V, Y]] = -k \cdot Y + i \cdot Z = 0$$

modulo $\mathcal{I}$ and thus $h = i = k = 0$, $l = -f$ and $g = 2j$. In particular, $[X, [X, [X, V]]] = 0$ and $\mathcal{F}$ is transversely projective, thus proving the Theorem 1.1. Theorem 1.3 corresponds to the first 2 cases above and immediately follows.

3.4. **Proof of Proposition 1.9.** If the algebraic dimension $a(M)$ of M is n, the second alternative is trivially satisfied. Also, when $a(M) = 0$, any 1-form on M is closed since it is a linear combination of the given n closed ones with coefficients in $\mathcal{M}(M) = \mathbb{C}$; in particular, any 1-form defining $\mathcal{F}$ is closed.

Let $f_1, \ldots, f_q \in \mathcal{M}(M)$, $q = a(M)$, be such that $df_1 \wedge \ldots \wedge df_q \neq 0$. By our hypothesis we can find $p = n - q$ closed meromorphic 1-forms such that $\omega_1 \wedge \ldots \wedge \omega_p \wedge df_1 \wedge \ldots \wedge df_q \neq 0$. If ω is a 1-form defining $\mathcal{F}$ then we can write it as

$$\omega = \sum \lambda_i \omega_i + \sum \mu_j df_j \, ,$$

where the λ_i and the μ_j belong to $\mathcal{M}(M)$. If all the λ_i are zero then we are in the first case; if not we can suppose that $\lambda_1 = 1$.

Therefore

$$d\omega = \sum_{i=2}^{p} d\lambda_i \wedge \omega_i + \sum_j d\mu_j \wedge df_j \, ,$$

and the integrability condition writes:

$$\begin{aligned} 0 &= \omega_1 \wedge \left(\sum_{i \leq 2} d\lambda_i \wedge \omega_i + \sum_j d\mu_j \wedge df_j \right) \\ &+ \left(\sum_{i \leq 2} \lambda_i \omega_i + \sum_j \mu_j df_j \right) \wedge \left(\sum_{i \leq 2} d\lambda_i \wedge \omega_i + \sum_j d\mu_j \wedge df_j \right) . \end{aligned}$$

First suppose that $\dim(M) \geq 3$. Notice that the $d\lambda_i$ and $d\mu_j$ are in the $\mathcal{M}(M)$-vector space generated by the df_i. Since the meromorphic 3-forms $\omega_i \wedge \omega_j \wedge df_k$ together with $\omega_i \wedge df_j \wedge df_k$ are linearly independant over $\mathcal{M}(M)$, we deduce that the first term is zero: ω_1 does not occur when one developp the second term on the 3-forms above. Therefore

$$\sum_{i=2}^{q} d\lambda_i \wedge \omega_i + \sum_j d\mu_j \wedge df_j = 0 \, ,$$

and consequently ω is closed.

When $\dim M = 2$, we just have to consider the case where $a(M) = 1$. Then, $\omega = \omega_1 + \lambda_1 df_1$ and $d\omega = d\lambda_1 \wedge df_1 = 0$ since $a(M) = 1$.

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