Calculation of mixed Hodge structures, Gauss-Manin connections and Picard-Fuchs equations

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Abstract. In this article we introduce algorithms which compute iterations of Gauss-Manin connections, Picard-Fuchs equations of Abelian integrals and mixed Hodge structure of affine varieties of dimension n in terms of differential forms. In the case $n = 1$ such computations have many applications in differential equations and counting their limit cycles. For $n > 3$, these computations give us an explicit definition of Hodge cycles.

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1. Introduction

The theory of abelian integrals which arises in polynomial differential equations of the type $\dot{x} = P(x, y),\ \dot{y} = Q(x, y)$ is one of the most fruitful areas which needs a special attention from algebraic geometry and in particular singularity theory. The reader is referred to the articles [6], [10] and [3] for a history and applications of such abelian integrals in differential equations. The book [1] and its references contains the theory of such integrals in the local case. In this article we deal with computational aspects of such integrals. All polynomial objects which we use are defined over $\mathbb{C}$.

Let us be given a polynomial f in $n+1$ variables $x_1, x_2, \ldots, x_{n+1}$, a polynomial differential n-form ω and a continuous family of n-dimensional oriented cycles $\delta_t \subset L_t := f^{-1}(t)$. The protagonist of this article is the integral $\int_{\delta_t} \omega$, called the abelian integral. Computations related to these integrals become easier when we put a certain kind of tameness condition on f (see §2). For such a tame polynomial we can write $\int_{\delta_t} \omega$ as:

$$\sum_{\beta \in I} p_\beta(t) \int_{\delta_t} \eta_\beta, \tag{1.1}$$

where $\eta_\beta, \beta \in I$ is a class of differential n-forms constructed from a basis of the Milnor vector space of f and p_β's are polynomials in t (see §5 for the algorithm which produces p_β's). The Gauss-Manin connection $\nabla\omega$ has the following basic property

$$\frac{\partial}{\partial t}\int_{\delta_t}\omega = \int_{\delta_t}.\nabla\omega \tag{1.2}$$

The above term can be written in the form (1.1) with p_β's rational functions in t with poles in the critical values of f (see §6 for the algorithm which produces p_β's). The n-th cohomology of a smooth fiber L_t is canonically isomorphic to $\Omega^n_{L_t}/d\Omega^{n-1}_{L_t}$, where $\Omega^i_{L_t}$ is the restriction of polynomial differential i-forms to L_t, and carries two natural filtrations called the weight and the Hodge filtrations (a mixed Hodge structure consists of these filtrations and a real structure satisfying certain axioms). These filtrations are generalizations of classical notions of differential forms of the first, second and third type for Riemann surfaces in higher dimensional varieties. The reader who is not interested in the case $n > 1$ is invited to follow the article with $n = 1$ and with the usual notions of differential forms of the first, second and third type. How to calculate these filtrations by means of differential forms is the main theorem of [9] and related algorithms are explained in §7. Last but not least, our protagonist satisfies a Picard-Fuchs equation $\sum_{i=0}^k p_i(t)\frac{\partial^i}{\partial t^i} = 0$, where p_i's are polynomials in t. The algorithm which produces p_i's is explained in §8. The theory of abelian integrals can be studied even in the case $n = 0$, i.e. f is a polynomial in one variable. Since some open problems, for instance infinitesimal Hilbert Problem (see [6]), can be also stated in this case, we have included §9. All the algorithms explained in this article are implemented in a library of Singular. This together with some examples are explained in §10. Applications of our computations in differential equations and particularly in direction of the article [3] is a matter of future work.

2. Tame polynomials and Brieskorn modules

We start with a definition.

Definition 2.1. A polynomial $f \in \mathbb{C}[x]$ is called (weighted) tame if there exist natural numbers $\alpha_1, \alpha_2, \ldots, \alpha_{n+1} \in \mathbb{N}$ such that $Sing(g) = \{0\}$, where $g = f_d$ is the last homogeneous piece of f in the graded algebra $\mathbb{C}[x]$, $deg(x_i) = \alpha_i$.

The multiplicative group $\mathbb{C}^*$ acts on $\mathbb{C}^{n+1}$ in the following way:

$$\lambda^* : (x_1, x_2, \ldots, x_{n+1}) \to (\lambda^{\alpha_1}x_1, \lambda^{\alpha_2}x_2, \ldots, \lambda^{\alpha_{n+1}}x_{n+1}),\ \lambda \in \mathbb{C}^*.$$

The polynomial (resp. the polynomial form) ω in $\mathbb{C}^{n+1}$ is (weighted) homogeneous of degree $d \in \mathbb{N}$ if $\lambda^*(\omega) = \lambda^d\omega,\ \lambda \in \mathbb{C}^*$. Fix a homogeneous polynomial g of degree d and with an isolated singularity at $0 \in \mathbb{C}^{n+1}$. Let A_g be the affine space of all tame polynomials $f = f_0 + f_1 + \cdots + f_{d-1} + g$. The space A_g is parameterized by the coefficients of $f_i, i = 0, 1, \dots, d-1$. The multiplicative group $\mathbb{C}^*$ acts on A_g by

$$\lambda \bullet f = \frac{f \circ \lambda^*}{\lambda^d} = \lambda^{-d} f_0 + \lambda^{-d+1} f_1 + \cdots + \lambda^{-1} f_d + g.$$

The action of $\lambda \in \mathbb{C}^*$ takes $\lambda \bullet f = 0$ biholomorphically to $f = 0$.

Let $f \in \mathsf{A}_g$. We choose a basis $x^I := \{x^\beta \mid \beta \in I\}$ of monomials for the Milnor $\mathbb{C}$-vector space

$$V := \mathbb{C}[x]/\mathrm{jacob}(g).$$

Define

$$w_i := \frac{\alpha_i}{d},\ 1 \le i \le n+1,\ \eta := (\sum_{i=1}^{n+1} (-1)^{i-1} w_i x_i \widehat{dx_i}),\ L_t := f^{-1}(t), t \in \mathbb{C}, \quad (2.1)$$

$$A_\beta := \sum_{i=1}^{n+1} (\beta_i + 1) w_i,\ \eta_\beta := x^\beta \eta,\ \omega_\beta = x^\beta dx,\ (\beta \in I),$$

where $\widehat{dx_i} = dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_{n+1}$. Note that $A_\beta = \frac{\deg(x^{\beta+1})}{d}$. It turns out that x^I is also a basis of $V_f := \mathbb{C}[x]/\mathrm{jacob}(f)$ and so f and g have the same Milnor numbers (see the conclusion after Lemma 4 of [9]). We denote it by μ . We denote by P the set of critical points of f and by $C := f(P)$ the set of critical values of f. We will also use P for a polynomial in $\mathbb{C}[x]$. This will not make any confusion.

Let $\Omega^i, i = 1, 2, \dots, n+1$ (resp. $\Omega^i_j,\ j \in \mathbb{N} \cup \{0\}$) be the set of polynomial differential i-forms (resp. homogeneous degree j polynomial differential i-forms) in $\mathbb{C}^{n+1}$. The Milnor vector space of f can be rewritten in the form $V := \frac{\Omega^{n+1}}{df \wedge \Omega^n}$. The Brieskorn modules

$$H' = H'_f := \frac{\Omega^n}{df \wedge \Omega^{n-1} + d\Omega^{n-1}},\ H'' = H''_f = \frac{\Omega^{n+1}}{df \wedge d\Omega^{n-1}}$$

of f are $\mathbb{C}[t]$-modules in a natural way: $t.[\omega] = [f\omega],\ [\omega] \in H'$ resp. $\in H''$. They are defined in the case $n > 0$. The case $n = 0$ is treated separately in §9.

3. Mixed Hodge structures

In this section we assume that the reader is familiar with the notion of mixed Hodge structure in the cohomologies of an affine variety (see [7, 2]).

Definition 3.1. Let H be one of H' or H''. If $H = H''$ then by restriction of ω on $L_c,\ c \in \mathbb{C} \backslash C$ we mean the residue of $\frac{\omega}{f-c}$ in L_c and by $\int_\delta \omega,\ \delta \in H_n(L_c, \mathbb{Z})$ we mean $\int_\delta \mathrm{residue}(\frac{\omega}{f-c})$. It is natural to define the Hodge and weight filtrations of H as follows: $W_m H,\ m \in \mathbb{Z}$ (resp. $F^k H,\ k \in \mathbb{Z}$) consists of elements $\omega \in H$ such that the restriction of ω on all $L_c,\ c \in \mathbb{C} \backslash C$ belongs to $W_m H^n(L_c, \mathbb{C})$ (resp. $F^k H^n(L_c, \mathbb{C})$).

Each piece of the mixed Hodge structure of H is a $\mathbb{C}[t]$-module. In the same way we define the mixed Hodge structure of the localization of H over multiplicative subgroups of $\mathbb{C}[t]$. In the case $n = 1$ our definition can be simplified as follows: We have the filtrations $\{0\} = W_0 \subset W_1 \subset W_2 = H$ and $0 = F^2 \subset F^1 \subset F^0 = H$, where

$$W_1 = \{\omega \in H \mid \omega \text{ restricted to a regular fiber has not residue at infinity }\},$$

$$F^1 = \{\omega \in H \mid \omega \text{ restricted to a regular fiber has poles of order} \geq 1 \text{ at infinity}\}.$$

In particular, $W_1 \cap F^1$ is the set of all $\omega \in H$ such that ω restricted to a regular compactified fiber is of the first kind. For the notion of compactification of $\mathbb{C}^2$ and infinity see [3] and [8]. The projection of $F^\bullet$ in $\mathrm{Gr}_m^W H := W_m/W_{m-1}$ gives us the filtration $\bar{F}^\bullet$ in $\mathrm{Gr}_m^W H$ and we define $\mathrm{Gr}_F^k \mathrm{Gr}_m^W H = \bar{F}^k/\bar{F}^{k+1}$.

Definition 3.2. Suppose that H is a free $\mathbb{C}[t]$-module. The set $B = \cup_{m,k\in\mathbb{Z}} B_m^k \subset H$ is a basis of H compatible with the mixed Hodge structure if B_m^k form a basis of $\mathrm{Gr}_F^k \mathrm{Gr}_m^W H$.

For a $\mathbb{C}[t]$-module M and a set $C \subset \mathbb{C}$, we denote by M_C the localization of M on the multiplicative subset of $\mathbb{C}[t]$ generated by $\{t - c \mid c \in C\}$. The following theorem gives a basis of a localization of H which is compatible with the mixed Hodge structure. It is proved in [9]. Our aim in this article is to explain the algorithms which lead to the calculation of such a basis.

Theorem 3.3. *Let $b \in \mathbb{C}\backslash C$ be a regular value of $f \in \mathbb{C}[x]$. If f is a (weighted) tame polynomial then $\mathrm{Gr}_m H' = 0$ for $m \neq n, n+1$ and there exist a map $\beta \in I \to d_\beta \in \mathbb{N} \cup \{0\}$ and $C \subset \tilde{C} \subset \mathbb{C}$ such that $b \notin \tilde{C}$ and*

$$\nabla^k \eta_\beta,\ \beta \in I,\ A_\beta = k \tag{3.1}$$

form a basis of $\mathrm{Gr}_F^{n+1-k} \mathrm{Gr}_{n+1}^W H'_{\tilde{C}}$ and the forms

$$\nabla^k \eta_\beta, A_\beta + \frac{1}{d} \leq k \leq A_\beta + \frac{d_\beta}{d} \tag{3.2}$$

form a basis of $\mathrm{Gr}_F^{n+1-k} \mathrm{Gr}_n^W H'_{\tilde{C}}$. The same is true for $H''_{\tilde{C}}$ replacing $\nabla^k \eta_\beta$ with $\nabla^{k-1}\omega_\beta$.

In the above theorem $\nabla : H \to H_C$ is the Gauss-Manin connection associated to f (see §6).

4. Quasi-homogeneous singularities

Let $f = g$ be a weighted homogeneous polynomial with an isolated singularity at origin. It is well-known that H' (resp. H'') is freely generated by $\eta_\beta,\ \beta \in I$ (resp. $\omega_\beta, \beta \in I$). In this section we explain the algorithm which writes every element in H' (resp. H'') of g as a $\mathbb{C}[t]$-linear combination of η_β's (resp. ω_β's). Recall that

$$dg \wedge d(P\widehat{dx_i, dx_j}) = (-1)^{i+j+\epsilon_{i,j}} \Big(\frac{\partial g}{\partial x_j}\frac{\partial P}{\partial x_i} - \frac{\partial g}{\partial x_i}\frac{\partial P}{\partial x_j}\Big)dx,$$

where $\epsilon_{i,j} = 0$ if $i < j$ and $= 1$ if $i > j$ and $\widehat{dx_i, dx_j}$ is dx without dx_i and dx_j (we have not changed the order of $dx_1, dx_2, \ldots$ in dx).

Proposition 4.1. *For a monomial $P = x^\beta$ we have*

$$\frac{\partial g}{\partial x_i} P dx = \frac{d}{d \cdot A_\beta - \alpha_i} \frac{\partial P}{\partial x_i} g dx + dg \wedge d\Big(\sum_{j \neq i} \frac{(-1)^{i+j+1+\epsilon_{i,j}} \alpha_j}{d \cdot A_\beta - \alpha_i} x_j P \widehat{dx_i, dx_j}\Big). \quad (4.1)$$

Proof. The proof is a straightforward calculation.

$$\sum_{j \neq i} \frac{(-1)^{i+j+1+\epsilon_{i,j}} \alpha_j}{d \cdot A_\beta - \alpha_i} dg \wedge d(x_j P \widehat{dx_i, dx_j}) =$$

$$\frac{-1}{d \cdot A_\beta - \alpha_i} \sum_{j \neq i} \Big(\alpha_j \frac{\partial g}{\partial x_j} \frac{\partial (x_j P)}{\partial x_i} - \alpha_j \frac{\partial g}{\partial x_i} \frac{\partial (x_j P)}{\partial x_j}\Big) dx =$$

$$\frac{-1}{d \cdot A_\beta - \alpha_i} \Big(\big(d \cdot g - \alpha_i x_i \frac{\partial g}{\partial x_i}\big) \frac{\partial P}{\partial x_i} - P \frac{\partial g}{\partial x_i} \sum_{j \neq i} \alpha_j (\beta_j + 1)\Big) dx =$$

$$\frac{-1}{d \cdot A_\beta - \alpha_i} \Big(d \cdot g \frac{\partial P}{\partial x_i} - \alpha_i \beta_i P \frac{\partial g}{\partial x_i} - P \frac{\partial g}{\partial x_i} \sum_{j \neq i} \alpha_j (\beta_j + 1)\Big) dx.$$

In the above equalities ds means the differential of s and $d \cdot s$ means the multiplication of d, the degree of g, with s. □

We use the above Proposition to write every $P dx \in \Omega^{n+1}$ in the form

$$P dx = \sum_{\beta \in I} p_\beta(g) \omega_\beta + dg \wedge d\xi, \quad (4.2)$$

$$p_\beta \in \mathbb{C}[t],\ \xi \in \Omega^{n-1}, \deg(p_\beta(g)\omega_\beta, dg \wedge d\xi) \leq \deg(P dx).$$

- Input: The homogeneous polynomial g and $P \in \mathbb{C}[x]$ representing $[P dx] \in H''$. Output: $p_\beta, \beta \in I$ and ξ satisfying (4.2).

  We write

  $$P dx = \sum_{\beta \in I} c_\beta x^\beta . dx + dg \wedge \eta,\ \deg(dg \wedge \eta) \leq \deg(P dx). \quad (4.3)$$

  Then we apply (4.1) to each monomial component $\tilde{P} \frac{\partial g}{\partial x_i}$ of $dg \wedge \eta$ and then we write each $\frac{\partial \tilde{P}}{\partial x_i} dx$ in the form (4.3). The degree of the components which make $P dx$ not to be of the form (4.2) always decreases and finally we get the desired form.

To find a similar algorithm for H' we note that if $\eta \in \Omega^n$ is written in the form

$$\eta = \sum_{\beta \in I} p_\beta(g) \eta_\beta + dg \wedge \xi + d\xi_1,\ p_\beta \in \mathbb{C}[t], \xi, \xi_1 \in \Omega^{n-1}, \quad (4.4)$$

where each piece in the right hand side of the above equality has degree less than $\deg(\eta)$ then

$$d\eta = \sum_{\beta \in I} (p_\beta(g) A_\beta + p'_\beta(g) g) \omega_\beta - dg \wedge d\xi \quad (4.5)$$

and the inverse of the map $\mathbb{C}[t]\to\mathbb{C}[t],\ p\mapsto A_\beta.p+p'.t$ is given by $\sum_{i=0}^k a_it^i\mapsto \sum_{i=1}^k \frac{a_i}{A_\beta+i}t^i$.

Since in the case of a quasi-homogeneous singularity $f=g$ we have $\nabla(\omega_\beta)=\frac{A_\beta-1}{t}\omega_\beta$ and $\nabla(\eta_\beta)=\frac{A_\beta}{t}\eta_\beta$ (see §6), Theorem 3.3 in this case reduces to:

Theorem 4.2. *(Steenbrink, [11]) For a weighted homogeneous polynomial g, the set*

$$B=\cup_{k=1}^n B_{n+1}^k\cup\cup_{k=0}^n B_n^k$$

with

$$B_{n+1}^k=\{\eta_\beta\mid A_\beta=n-k+1\}, B_n^k=\{\eta_\beta\mid n-k<A_\beta<n-k+1\},$$

is a basis of H' compatible with the mixed Hodge structure. The same is true for H'' replacing η_β with ω_β.

5. A basis of H' and H''

Proposition 5.1. *For every tame polynomial $f\in\mathsf{A}_g$ the forms $\omega_\beta,\ \beta\in I$ (resp. $\eta_\beta,\ \beta\in I$) form a basis of the Brieskorn module H'' (resp. H') of f. More precisely, every $\omega\in\Omega^{n+1}$ (resp. $\omega\in\Omega^n$) can be written*

$$\omega=\sum_{\beta\in I}p_\beta(f)\omega_\beta+df\wedge d\xi,\ p_\beta\in\mathbb{C}[t],\ \xi\in\Omega^{n-1},\ \deg(p_\beta)\leq\frac{\deg(\omega)}{d}-A_\beta \quad (5.1)$$

(resp.

$$\omega=\sum_{\beta\in I}p_\beta(f)\eta_\beta+df\wedge\xi+d\xi_1,\ p_\beta\in\mathbb{C}[t],\ \xi\in\Omega^{n-1},\ \deg(p_\beta)\leq\frac{\deg(\omega)}{d}-A_\beta \quad (5.2)$$

).

This proposition is proved in [9] Proposition 1. The proof also gives us the following algorithm to find all the unknown data in the above equalities.

- Input: The tame polynomial f and $P\in\mathbb{C}[x]$ representing $[Pdx]\in H''$. Output: $p_\beta,\beta\in I$ and ξ satisfying (5.1).

  We use the algorithm of §4 and write an element $\omega\in\Omega^{n+1},\deg(\omega)=m$ in the form:

$$\omega=\sum_{\beta\in I}p_\beta(g)\omega_\beta+dg\wedge d\psi,\ p_\beta\in\mathbb{C}[t],\ \psi\in\Omega^{n-1},\ \deg(p_\beta(g)\omega_\beta),\deg(dg\wedge d\psi)\leq m$$

  This is possible because g is homogeneous. We have

$$\omega=\sum_{\beta\in I}p_\beta(f)\omega_\beta+df\wedge d\psi+\omega',\ \omega'=\sum_{\beta\in I}(p_\beta(g)-p_\beta(f))\omega_\beta+d(g-f)\wedge d\psi.$$

  The degree of ω' is strictly less than m and so we repeat what we have done at the beginning and finally we write ω as a $\mathbb{C}[t]$-linear combination of ω_β's.

The algorithm for H' is similar. The statement about degrees is the direct consequence of the proof and (4.2).

6. Gauss-Manin connection

Let $S(t)\in\mathbb{C}[t]$ such that

$$S(f)dx=df\wedge\eta_f,\ \eta_f=\sum_{i=1}^{n+1}(-1)^{i-1}p_i\widehat{dx_i}\in\Omega^{n-1}.$$

For instance one can take $S(t):=det(A_f-t.I)$, where A_f is the multiplication by f linear map from $V_f:=\mathbb{C}[x]/\mathrm{jacob}(f)$ to itself. The Gauss-Manin connection $\nabla=\nabla_{\frac{\partial}{\partial t}}$ associated to the fibration $f=t,\ t\in\mathbb{C}$ on H'' turns out to be the map

$$\nabla: H''\to H''_C, \nabla([Pdx])=\frac{[(Q_P-P.S'(f))dx]}{S},\ P\in\mathbb{C}[x],$$

where

$$Q_P=\sum_{i=1}^{n+1}(\frac{\partial P}{\partial x_i}p_i+P\frac{\partial p_i}{\partial x_i}) \tag{6.1}$$

satisfying the Leibniz rule, where for a set $\tilde{C}\subset\mathbb{C}$ by $H''_{\tilde{C}}$ we mean the localization of H'' on the multiplicative subgroup of H'' generated by $t-c,\ c\in\tilde{C}$. Using the Leibniz rule one can extend ∇ to a function from H''_C to itself and so the iteration $\nabla^k=\nabla\circ\nabla\cdots\nabla$ k times, makes sense. It is given by

$$\nabla^k=\frac{\nabla_{k-1}\circ\nabla_{k-2}\circ\cdots\circ\nabla_0}{S(t)^k}, \tag{6.2}$$

where

$$\nabla_k: H''\to H'',\ \nabla_k([Pdx])=[(Q_P-(k+1)S'(t)P)dx].$$

To calculate $\nabla: H'\to H'_C$ we use the fact that

$$\nabla^k\omega=\frac{\nabla^{k-1}d\omega}{df},\ \omega\in H',$$

where $d: H'\to H''$ is taking differential and is well-defined. The main property of ∇ is (1.2). Usually the iteration of the Gauss-Manin connection produces polynomial forms with huge number of monomials. But fortunately our Brieskorn module H'' (resp. H') has already the canonical basis $\omega_\beta,\ \beta\in I$ (resp. $\eta_\beta,\ \beta\in I$) and after writing ∇ the obtained coefficients are much more easier to read. In H'' one can write

$$S(t)\nabla(\omega_\beta)=\sum_{\beta'\in I}p_{\beta,\beta'}\omega_{\beta'},\ p_{\beta,\beta'}\in\mathbb{C}[t],\ \deg(p_{\beta,\beta'})\leq\deg(S)-1+A_\beta-A_{\beta'}. \tag{6.3}$$

The bound on degrees can be obtained as follows:

$$S(f)\omega_\beta=df\wedge\eta,\ \Rightarrow d\cdot\deg(S)+d\cdot A_\beta=d+\deg(\eta).$$

$$\deg(p_{\beta,\beta'})\leq\frac{\deg(d\eta)}{d}-A_{\beta'}=\deg(S)-1+A_\beta-A_{\beta'}.$$

The Gauss-Manin connection ∇ has two nice properties:

1. Griffiths transversality theorem: For all $i = 1, 2, \ldots, n+1$ we have
$$S(t)\nabla(F^i) \subset F^{i-1}.$$
2. Residue killer: For all $\omega \in H$ there exists a $k \in \mathbb{N}$ such that $\nabla^k\omega \in W_n$

For the first one see [5]. The second one for $n = 1$ is proved in Lemma 2.3 of [8]. The proof for $n > 1$ is similar and uses the fact that the residue as a function in t for a cycle around infinity is a polynomial in t.

7. The numbers $d_\beta,\ \beta \in I$

Let f be a tame polynomial with the last homogeneous part g, F be its homogenization and
$$V = \mathbb{C}[x, x_0]/ < \frac{\partial F}{\partial x_i} \mid i = 1, 2, \ldots, n+1 > .$$
We consider V as a $\mathbb{C}[x_0]$-module and it is easy to show that V is freely generated by $x^I := \{x^\beta, \beta \in I\}$. Let
$$A_F : V \to V,\ A_F(G) = \frac{\partial F}{\partial x_0} G,\ G \in V.$$

Proposition 7.1. *The matrix of A_F in the basis x^I is of the form $d \cdot [x_0^{K_{\beta,\beta'}} c_{\beta,\beta'}]$, where $K_{\beta,\beta'} := d - 1 + \deg(x^\beta) - \deg(x^{\beta'})$ and $A_f := [c_{\beta,\beta'}]$ is the multiplication by f in the Milnor vector space of f. In particular, if $A_{\beta'} - A_\beta \geq 1$ then $c_{\beta,\beta'} = 0$ and*
$$\det(A_F - t.x_0^{d-1} I) = \det(A_f - t.I) x_0^{(d-1)\mu}.$$

Proof. Since the polynomial F is weighted homogeneous, we have $\sum_{i=0}^{n+1} \alpha_i x_i \frac{\partial F}{\partial x_i} = d \cdot F$ and so $x_0 \frac{\partial F}{\partial x_0} = d.F$ in V (Note that $\alpha_0 = 1$ by definition). Let
$$F.x^\beta = \sum_{\beta' \in I} x^{\beta'} c_{\beta,\beta'}(x_0) + \sum_{i=1}^{n+1} \frac{\partial F}{\partial x_i} q_i,\ c_{\beta,\beta'}(x_0) \in \mathbb{C}[x_0],\ q_i \in \mathbb{C}[x_0, x]. \tag{7.1}$$
Since the left hand side is homogeneous of degree $d + \deg(x^\beta)$ we can assume that the pieces of the write hand side are also homogeneous of the same degree. This can be done by taking an arbitrary equation (7.1) and subtracting the unnecessary parts. □

Let $\tilde{C}$ be a finite subset of $\mathbb{C}$ and $\mathbb{C}[t]_{\tilde{C}}$ be the localization of $\mathbb{C}[t]$ on its multiplicative subgroup generated by $t - c,\ c \in \tilde{C}$ and $F_t = F - t.x_0^d$. From now on we work with $\mathbb{C}[t]_{\tilde{C}}[x_0, x]$ instead of $\mathbb{C}[x_0, x]$ and redefine V using $\mathbb{C}[t]_{\tilde{C}}[x_0, x]$. Let
$$V_{\tilde{C}} = \mathbb{C}[t]_{\tilde{C}}[x_0, x]/ < \frac{\partial F_t}{\partial x_0}, \frac{\partial F}{\partial x_i}, \mid i = 1, 2, \ldots, n+1 > .$$

It is useful to reformulate $V_{\tilde{C}}$ in the following way: Let $R := \mathbb{C}[t]_{\tilde{C}}[x_0]$ be the set of polynomials in x_0 with coefficients in $\mathbb{C}[t]_{\tilde{C}}$ and $A_t = A_F - t.d.x_0^{d-1}I$. We have

$$V_{\tilde{C}} = V/ < \frac{\partial F_t}{\partial x_0} q \mid q \in V >= R^\mu / A_t.R^\mu.$$

Here R^μ is the set of $\mu \times 1$ matrices with entries in R. We consider the statement:

$*(\tilde{C})$: There is a function $\beta \in I \to d_\beta \in \mathbb{N} \cup \{0\}$ such that the $\mathbb{C}[t]_{\tilde{C}}$-module $V_{\tilde{C}}$ is freely generated by

$$\{x_0^{\beta_0} x^\beta, 0 \leq \beta_0 \leq d_\beta - 1, \beta \in I\}. \tag{7.2}$$

To prove the statement $*(\tilde{C})$ we may introduce a kind of Gaussian elimination in A_t and simplify it. For this reason we introduce the operation $GE(\beta_1, \beta_2, \beta_3)$. For $\beta \in I$ let $(A_t)_\beta$ be the β-th row of A_t.

- Input: A_t, $\beta_1, \beta_2, \beta_3 \in I$ with $A_{\beta_1} \leq A_{\beta_2}$. Output: a matrix A'_t and a finite subset B of $\mathbb{C}$.

  We replace $(A_t)_{\beta_2}$ with

  $$-\frac{(A_t)_{\beta_2,\beta_3}}{(A_t)_{\beta_1,\beta_2}} * (A_t)_{\beta_1} + (A_t)_{\beta_2}$$

  and we set $B = zero(c(t))$, where $(A_t)_{\beta_1,\beta_2} = c(t).x_0^{K_{\beta_1,\beta_2}}$. Since for all $\beta_4 \in I$ we have

  $$K_{\beta_2,\beta_3} + K_{\beta_1,\beta_4} = K_{\beta_1,\beta_3} + K_{\beta_2,\beta_4}.$$

  The obtained matrix A'_t is of the form $[x_0^{K_{\beta,\beta'}} c'_{\beta,\beta'}]$ and $c'_{\beta_2,\beta_3} = 0$. If the matrix B_t is obtained from A_t by applying the above operation and $B \subset \tilde{C}$ then $A_t.R^\mu = B_t R^\mu$.

  We give an example of algorithm which calculates d_β's for for some finite set $\tilde{C} \subset \mathbb{C}$:

- Input: A_t. Output: $d_\beta, \beta \in I$ and a finite set $\tilde{C} \subset \mathbb{C}$.

  We identify I with $\{1, 2, \ldots, \mu\}$ and assume that

  $$\beta_1 \leq \beta_2 \Rightarrow A_{\beta_1} \geq A_{\beta_2}.$$

  The algorithm has μ steps indexed by $\beta = \mu, \mu - 1, \ldots, 1$. We define the set $\tilde{C}$ to be empty. In $\beta = \mu$ we have $A(\beta) = A_t$. In the step β we find the first β_1 such that $A(\beta)_{\beta,\beta_1} \neq 0$ and put $d_{\beta_1} = d - 1 + \deg(x^\beta) - \deg(x^{\beta_1})$. For $\beta_2 = \beta - 1, \ldots, 1$ we make $GE(\beta, \beta_2, \beta_1)$ and define $\tilde{C} = \tilde{C} \cup \cup_{\beta_2=1}^{\beta-1} B_{\beta_2}$, where B_{β_2} is obtained during $GE(\beta, \beta_2, \beta_1)$. The numbers d_β's obtained in this way proves the statement $*(\tilde{C})$.

The advantage of this algorithm is that in many cases it gives $\tilde{C} = C$. We do not have a proof for $*(C)$. One can also fix a value $c \in \mathbb{C} \backslash C$ and apply the above algorithm for A_c. In this case we do not care about $\tilde{C}$ during the algorithm. The obtained d_β's make the statement $*(\tilde{C})$ true for some $\tilde{C} \subset \mathbb{C}$ with $c \notin \tilde{C}$. We prove the following weak statements:

Proposition 7.2. *There is a function $\beta\in I\to d_\beta\in\mathbb{N}\cup\{0\}$ such that the $\mathbb{C}[t]_C$-module V' is generated by $\{x_0^{\beta_0}x^\beta, 0\leq\beta_0\leq d_\beta-1,\beta\in I\}$.*

Proof. We have

$$V'=R^\mu/A_tR^\mu\overset{b}{\cong}A_t^{-1}R^\mu/R^\mu=\frac{A_t^{\mathrm{adj}}R^\mu}{x_0^{\mu(d-1)}}/R^\mu.$$

The isomorphism b in the middle is obtained by acting A_t^{-1} from left on R^μ and a*dj* makes the adjoint of a matrix. Now for $\beta\in I$ let d_β be the pole order of β-th arrow of $\frac{A_t^{\mathrm{adj}}}{x_0^{\mu(d-1)}}$. The numbers d_β are the desired numbers. It is easy to see that $\{x_0^{\beta_0}x^\beta, 0\leq\beta_0\leq d_\beta,\beta\in I\}$ generates V'. □

Proposition 7.3. *There is a subset $\tilde{C}\subset\mathbb{C}$ such that the statement $*(\tilde{C})$ is true with $d_\beta=d-1,\beta\in I$.*

Proof. We identify I with $\{1,2,\dots,\mu\}$ and assume that

$$\beta_1\leq\beta_2\Rightarrow A_{\beta_1}\geq A_{\beta_2}.$$

By various use of operation GE on A_t we make all the entries of $(A_t)_{\beta,\mu}=0,\ \beta\in I\backslash\{\mu\}$. We repeat this for $(A_t)_{\beta,\mu-1}=0,\beta\in I\backslash\{\mu,\mu-1\}$ and after μ-times we get a lower triangular matrix. We always divide on a polynomial on t with leading coefficient one and so division by zero does not occur. □

Proposition 7.4. *Let $*(\tilde{C})$ be valid with $d_\beta,\beta\in I$. Then*

$$A_\beta<n+1,\ d_\beta<d(n+2-A_\beta),\ \sum_{\beta\in I}d_\beta=\mu(d-1).$$

Proof. The first one is already in Steenbrink's Theorem 4.2. The second inequality is obtained by applying the first inequality associated to $F-cx_0^d$ for some $c\in\mathbb{C}\backslash\tilde{C}$:

$$A_{(d_\beta-1,\beta)}=A_\beta+\frac{d_\beta-1+1}{d}<n+2.$$

The Milnor number of $F-cx_0^d$ is $\sum_{\beta\in I}d_\beta$ and equals to the Milnor number of $g-cx_0^d$ which is $\mu(d-1)$. □

Suppose that $*(\tilde{C})$ is valid with $d_\beta,\ \beta\in I$. Define

$$I_{n+1}^k=\{\beta\in I\mid A_\beta=n+1-k\},\ I_n^k=\{\beta\in I\mid A_\beta+\frac{1}{d}\leq n+1-k\leq A_\beta+\frac{d_\beta}{d}\}.$$

We can restate Theorem 3.3 in the following way: For a tame polynomial f, the set

$$B=\cup_{k=1}^nB_{n+1}^k\cup\cup_{k=0}^nB_n^k$$

with

$$B_{n+1}^k=\{\nabla^{n-k}\omega_\beta\mid\beta\in I_{n+1}^k\},B_n^k=\{\nabla^{n-k}\omega_\beta\mid\beta\in I_n^k\},$$

is a basis of $H''_{\tilde{C}}$ compatible with the mixed Hodge structure. The same is true for $H'_{\tilde{C}}$ replacing $\nabla^{n-k}\omega_\beta$ with $\nabla^{n+1-k}\eta_\beta$. Unfortunately, this theorem gives us a basis of a localization H compatible with the mixed Hodge structure. In §10 we have computed such bases for the Brieskorn module itself.

To handle easier the pieces of the mixed Hodge structure of $H_{\tilde{C}}$ we make the following table.

0		1		2		$\cdots$		n		$n+1$
	I_n^n	I_{n+1}^n	I_n^{n-1}	I_{n+1}^{n-1}	I_n^{n-2}	$\cdots$	I_n^1	I_{n+1}^1	I_n^0	

In the case $n = 1$ we have the table

0		1		2
	I_1^1	I_2^1	I_1^0	

$$I_1^1 = \{\beta \in I \mid A_\beta + \frac{1}{d} \leq 1 \leq A_\beta + \frac{d_\beta}{d}\},\ I_1^0 = \{\beta \in I \mid A_\beta + \frac{1}{d} \leq 2 \leq A_\beta + \frac{d_\beta}{d}\},$$

$$I_2^1 = \{\beta \in I \mid A_\beta = 1\}.$$

The forms $\omega_\beta,\ \beta \in I_1^1$ form a basis of $F^1 \cap W_1$ and the forms $\omega_\beta, \beta \in I_1^2$ form a basis of H''/W_1. Now to obtain a basis of $W_1/(F^1 \cap W_1)$ we must modify $\nabla\omega_\beta,\ \beta \in I_1^0$.

8. Picard-Fuchs equations

It is a well-known fact that for a polynomial $f \in \mathbb{C}[x]$ and $\omega \in H$ the integral $I(t) := \int_{\delta_t} \omega$ satisfies

$$(\sum_{i=0}^{k} p_i(t) \frac{\partial^i}{\partial t^i}) I_t = 0,\ p_i(t) \in \mathbb{C}[t] \tag{8.1}$$

called Picard-Fuchs equation, where $\delta_t \in H_n(L_t, \mathbb{Z})$ is a continuous family of topological cycles. When f is tame, it is possible to calculate p_i' as follows:

We write

$$\nabla^i(\omega) = \sum_{\beta \in I} p_{i,\beta}\omega_\beta$$

and define the $k \times \mu$ matrix $A = [p_{i,\beta}]$, where i runs through $1, 2, \ldots, k$ and $\beta \in I$. Let k be the smallest number such that the the rows of A_{k-1} are $\mathbb{C}(t)$-linear independent. Now, the rows of A_k are $\mathbb{C}(t)$-linear dependent and this gives us (after multiplication by a suitable element of $\mathbb{C}[t]$)

$$\sum_{i=0}^{k} p_i(t) \nabla^i(\omega) = 0,\ p_i(t) \in \mathbb{C}[t].$$

Using the formula (1.2) and integrating the above equality, we get the equation (8.1).

9. Polynomials in one variable, $n = 0$

The theory developed in §2 does not work for the case $n = 0$. For a polynomial of degree d in one variable $\dim(H^0(L_t, \mathbb{C})) = d$ but $\mu = d - 1$. However, if we use the following definition of homology and cohomology for a discrete topological space M,

$$H_0(M, \mathbb{Z}) = \{m = \sum_i a_i m_i \mid a_i \in \mathbb{Z},\ m_i \in M \mid deg(m) = \sum_i a_i = 0\},$$

$$H^0(M, \mathbb{C}) = \{f : H_0(M, \mathbb{Z}) \to \mathbb{C} \text{ linear}\}/\{f \mid \text{ f is constant on } M\},$$

then

$$H' = \mathbb{C}[x]/\mathbb{C}[f],\ H'' = \mathbb{C}[x]dx/f'\mathbb{C}[f]dx,\ I = \{1, x, x^2, \ldots, x^{d-2}\}, \mu = d - 1.$$

In this case

$$\int_\delta \omega = \sum_i a_i \omega(p_i), \quad \text{where } \delta = \sum_i a_i p_i,\ a_i \in \mathbb{Z},\ p_i \in f^{-1}(t),\ \omega \in H'.$$

If, for instance, $f' = 0$ has d distinct roots then every vanishing cycle in L_t is a difference of two points of L_t. The set $B = \{x, x^2, \ldots, x^{d-1}\}$ form a basis of H' and its ∇ which is $\{dx, xdx, \ldots x^{d-2}dx\}$ (up to multiplication by some constants) form a basis of H''. The first fact is easy to see. We write $f = a_d x^d + f_0$ and for a polynomial $p(x) \in \mathbb{C}[x]$ whenever we find some x^d we replace it with $\frac{f - f_0}{a_d}$ and at the end we get $p(x) = p_0(f) + \sum_{i=1}^{d-1} p_i(f)x^i$ or equivalently $p = \sum_{i=1}^{d-1} p_i(t)x^i$ in H'. There is no $\mathbb{C}[t]$-linear relation between the elements of B because B restricted to each regular fiber is of dimension d. We write

$$p(x)dx = \sum_{i=0}^{d-2} q_i(f)x^i dx + q_{d-1}(f)x^{d-1}dx$$

$$= (\sum_{i=0}^{d-2} q_i(f)x^i dx - \frac{q_{d-1}(f)f_0'}{d.a_d}dx) + \frac{q_{d-1}(f)f'}{d.a_d}dx$$

and this proves the statement for H''.

Proposition 4.1 can be stated in the case $n = 0$ as follows: The only case in which $dA_\beta - \alpha_i = 0$ is when $n = 0$ and $P = 1$. In the case $n = 0$ for $P \neq 1$ we have

$$\frac{\partial g}{\partial x_i}.Pdx = \frac{d}{d.A_\beta - \alpha_i}\frac{\partial P}{\partial x_i} g dx$$

and if $P = 1$ then $\frac{\partial g}{\partial x_i}.Pdx$ is zero in H''. The argument in (4.4) and (4.5) can be done also in the case $n = 0$. In this case if

$$\eta = \sum_{\beta \in I} p_\beta(g)\eta_\beta + p(g),\ p, p_\beta \in \mathbb{C}[t], \tag{9.1}$$

where each piece in the right hand side of the above equality has degree less than $\deg(\eta)$ then

$$d\eta = \sum_{\beta\in I}(p_\beta(g)A_\beta + p'_\beta(g)g)\omega_\beta + p'(g)dg. \quad (9.2)$$

In the case $n = 0$, we have only the set $I_0^0 = \{A_\beta + \frac{1}{d} \leq 1 \leq A_\beta + \frac{d_\beta}{d}\}$ and this is equal to I. We have $d_\beta < d.(n+2-A_\beta) = 2d - \beta - 1 =$ and $A_\beta = \frac{\beta+1}{d}$. We conclude that

$$d \leq d_\beta + \beta + 1 < 2d.$$

Now the infinitesimal Hilbert problem (see [6] Problem 7) can be stated in the case $n = 0$. Can one give an effective solution to this problem in this case? The positive answer to this question may give light into the the problem in the case $n = 1$. It is worked out in [4].

10. Examples

All the algorithms explained in this article are implemented in a library of Singular. It can be downloaded from the authors homepage. The procedure `okbase` makes a permutation on the output of `kbase` and gives us the set x^I with $\deg(x^\beta)$ decreasing. The algorithms in §4 after (4.2) are implemented in the procedures `linear1, linear2`. The procedures `linear` and `linearp` are for the algorithms in §5. Based on the observations in §9, these proceedures work also for the case $n = 0$. The procedure `nabla` uses the formulas (6.1) and (6.2) and computes ∇ and its iterations. The procedure `nablamat` calculates the matrix $\frac{1}{S(t)}[p_{\beta,\beta'}]$ in (6.3). The calculation of the polynomial S in §6 is implemented in the procedure `S`. Using Proposition 7.1, the procedure `muldF` calculates A_F. The algorithm for d_β's is implemented in the procedure `dbeta`. The procedure `changebase` calculates the matrix of the basis of the Brieskorn module $H''_{\tilde{C}}$ obtained in Theorem 3.3 in the canonical basis $\omega_\beta, \beta \in I$. The procedure `Imk` gives us $x^\beta, \beta \in I_m^k,\ m = n, n+1, k = 0, 1, \ldots n$ with the order $I_n^n, I_n^{n-1}, \ldots, I_n^0, I_{n+1}^n, I_{n+1}^{n-1}, \ldots, I_{n+1}^1$. The procedure `PFeq` calculates p_i's in (8.1).

Theorem 3.3 does not give a basis of the Brieskorn module compatible with the mixed Hodge structure. In the following examples we obtain such bases for some examples of f by modifying the one given in §3.3 (we do not have a general method for every f).

For all the examples below we download the author's library and `matrix.lib`.

10.1. Examples, $n = 0$

Example. $f = x^5 - 5x$, $P = \{\epsilon^i \mid i = 0, 1, 2, 3\}$, $C = \{-4\epsilon^i \mid i = 0, 1, 2, 3\}$, where $\epsilon = e^{\frac{2\pi i}{d-1}}$ is the d-th root of unity.

```
> ring r0=(0,t),x, dp;
> int d=5; poly f=x^d-d*x;
> poly Sf=S(f); Sf;
(t4-256)
> list l=nablamat(f,Sf);
> l[1]; print(l[2]);
1/(5t4-1280)
(-t3), 128,   (-48t),(16t2),
(4t2), (-2t3),192,   (-64t),
(-16t),(8t2), (-3t3),256,
64,    (-32t),(12t2),(-4t3)
//This is the matrix of nabla in the canonical
//basis x^3,x^2,x^1,1.
 >PFeq(f,1);
_[1,1]=6144
_[1,2]=(35625t)
_[1,3]=(33375t2)
```

```
_[1,4]=(8750t3)
_[1,5]=(625t4-160000)
```

The residues of $\frac{dx}{f-t}$ at its poles satisfy the Picard-Fuchs equation

$$6144+35625t\frac{\partial}{\partial t}+33375t^2\frac{\partial^2}{\partial t^2}+8750t^3\frac{\partial^3}{\partial t^3}+$$
$$(625t^4-160000)\frac{\partial^4}{\partial t^4}=0.$$

10.2. Examples, $n=1$

For the examples below we define

```
ring r1=(0,t), (x,y), dp;
```

Example. $f=xy(x+y-1)$.

```
 > poly f= x2y+xy2-xy ;
 > poly g=lasthomo(f); g;
x2y+xy2
 > okbase(std(jacob(g)));
_[1]=y2
_[2]=y
_[3]=x
_[4]=1
 > print(muldF(f-par(1)));
(-3t+1/18)*x2,-1/18*x3,    0,       0,
1/6*x,        (-3t-1/6)*x2,0,       0,
1/6*x,        -1/6*x2,     (-3t)*x2,0,
1/2,          -1/2*x,      0,       (-3t)*x2
 > poly Sf=S(f); Sf;
(t4+1/27t3)
//We can take Sf=t*(t+1/27);
 > list l1=nablamat(f,Sf);
 > l1[1]; " "; print(l1[2]);
1/(54t2+2t)

(18t+1),(-18t-1),0,(-2t),
1,      -1,      0,(-6t),
1,      -1,      0,(-6t),
3,      -3,      0,(-18t)
//--------------
 > dbeta(f,par(1));
0,2,2,4
 > Imk(f,par(1));
[1]:
   [1]:
      [1]:
         1
   [2]:
      [1]:
         1
[2]:
   [1]:
      [1]:
         x
      [2]:
         y
 > list l3=changebase(f,Sf,par(1));
 > print(l3[1]); " "; print(l3[2]); det(l3[2]);
1,3/(54t2+2t),1,1

0,0, 0,1,
1,-1,0,(-6t),
0,0, 1,0,
0,1, 0,0
1
//--------------
 > dbeta(f);
2,2,2,2
 > Imk(f);
[1]:
   [1]:
      [1]:
         1
   [2]:
      [1]:
         y2
[2]:
   [1]:
      [1]:
         x
      [2]:
         y
 > list l2=changebase(f,Sf);
 > print(l2[1]); " "; print(l2[2]); det(l2[2]);
1,1/(54t2+2t),1,1

0,      0,       0,1,
(18t+1),(-18t-1),0,(-2t),
0,      0,       1,0,
0,      1,       0,0
(18t+1)
//The obtained basis does not work for the
//fiber c=-1/18.
//--------------
 > PFeq(f,1, Sf);
_[1,1]=6
_[1,2]=(54t+1)
_[1,3]=(27t2+t)
_[1,4]=0
_[1,5]=0
```

We get the following basis of H'' compatible with mixed Hodge structure.

$f=xy(x+y-1)$

$\mathrm{Gr}_F^1\mathrm{Gr}_1^W H''$	$[1]$
$\mathrm{Gr}_F^0\mathrm{Gr}_1^W H''$	$[y^2]-[y]-6t[1]$
$\mathrm{Gr}_F^1\mathrm{Gr}_2^W H''$	$[x],[y]$

The integrals $I=\int_{\delta_t}\frac{dx\wedge dy}{f-t}$ satisfy the Picard-Fuchs equation

$$6+(54t+1)\frac{\partial I}{\partial t}+(27t^2+t)\frac{\partial^2 I}{\partial t^2}=0$$

Example. $f=2(x^3+y^3)-3(x^2+y^2)$, $P=\{(0,0),(0,1),(1,0),(1,1)\}$, $C=\{0,-1,-1,-2\}$,

```
 > poly f= 2*x3+2*y3-3*x2-3*y2 ;
 > poly g=lasthomo(f); g;
2*x3+2*y3
 > okbase(std(jacob(g)));
_[1]=xy
_[2]=y
_[3]=x
_[4]=1
 >S(f);
(t4+4t3+5t2+2t)
//We can put
 >poly Sf=t*(t+1)*(t+2);
 > list l2=changebase(f,Sf);
 > print(l2[1]); " "; print(l2[2]); det(l2[2]);
1,-1/(6t+12),1,1

0, 0,0,1,
-2,1,1,0,
0, 0,1,0,
0, 1,0,0
-2
```

$f=2(x^3+y^3)-3(x^2+y^2)$

$\mathrm{Gr}_F^1\mathrm{Gr}_1^W H''$	$[1]$
$\mathrm{Gr}_F^0\mathrm{Gr}_1^W H''$	$[2xy-x-y]$
$\mathrm{Gr}_F^1\mathrm{Gr}_2^W H''$	$[x],[y]$

Example. $f=x^4+y^4-x$.

```
 > poly f= x4+y4-x ;
 > poly g=lasthomo(f);
 > okbase(std(jacob(g)));
_[1]=x2y2
_[2]=xy2
_[3]=x2y
_[4]=y2
_[5]=xy
_[6]=x2
_[7]=y
_[8]=x
_[9]=1
 > poly Sf=S(f); Sf;
(t9+81/256t6+2187/65536t3+19683/16777216)
//We can take
 >Sf=t^3+27/256;
 > dbeta(f,par(1));
2,2,2,5,2,2,5,2,5
 > Imk(f,par(1));
[1]:
   [1]:
      [1]:
         1
      [2]:
         x
      [3]:
         y
   [2]:
      [1]:
         y
      [2]:
         y2
      [3]:
         x2y2
[2]:
   [1]:
      [1]:
         x2
      [2]:
         xy
      [3]:
         y2
 > list l3=changebase(f,Sf,par(1));
 > print(l3[1]); " "; print(l3[2]); det(l3[2]);
1,1,1,4/(256t3+27),24/(256t3+27),1/(256t3+27),1,1,1

0,      0,    0,0,0,0,0,      0,1,
0,      0,    0,0,0,0,0,      1,0,
0,      0,    0,0,0,0,1,      0,0,
0,      0,    9,0,0,0,(-16t2),0,0,
3,      (-2t),0,0,0,0,0,      0,0,
(128t2),9,    0,0,0,0,0,      0,0,
0,      0,    0,0,0,1,0,      0,0,
0,      0,    0,0,1,0,0,      0,0,
0,      0,    0,1,0,0,0,      0,0
(2304t3+243)  // 9*256*Sf;
 > matrix A=l3[2];
 > A[6,1..ncols(A)]=
((-128*t2)/3)*submat(A,5,1..ncols(A))+
submat(A,6,1..ncols(A));
 > A[5,1..ncols(A)]=
2*t*submat(A,6,1..ncols(A))+submat(A,5,1..ncols(A));
print(A);
 0,0,0,0,0,0,0,      0,1,
 0,0,0,0,0,0,0,      1,0,
 0,0,0,0,0,0,1,      0,0,
 0,0,9,0,0,0,(-16t2),0,0,
 1,0,0,0,0,0,0,      0,0,
 0,1,0,0,0,0,0,      0,0,
 0,0,0,0,0,1,0,      0,0,
 0,0,0,0,1,0,0,      0,0,
 0,0,0,1,0,0,0,      0,0
```

We obtain the following table

$f = x^4 + y^4 - x$

$\mathrm{Gr}^1_F \mathrm{Gr}^W_1 H''$	$[1], [x], [y]$
$\mathrm{Gr}^0_F \mathrm{Gr}^W_1 H''$	$9[x^2y] - 16t^2[y], [x^2y^2], [xy^2]$
$\mathrm{Gr}^1_F \mathrm{Gr}^W_2 H''$	$[x^2], [xy], [y^2]$

We make the following remark

```
 > reduce(9*x2*y-16*(f^2)*y, std(jacob(f)));
0
```

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