# On Ramanujan relations between Eisenstein series


Hossein Movasati
Instituto de Matemática Pura e Aplicada, IMPA,
Estrada Dona Castorina, 110,
22460-320, Rio de Janeiro, RJ, Brazil,
E-mail: hossein@impa.br
www.impa.br/∼hossein



**Abstract**

The Ramanujan relations between Eisenstein series can be interpreted as an ordinary differential equation in a parameter space of a family of elliptic curves. Such an ordinary differential equation is inverse to the Gauss-Manin connection of the corresponding period map constructed by elliptic integrals of first and second kind. In this article we consider a slight modification of elliptic integrals by allowing non-algebraic integrands and we get in a natural way generalizations of Ramanujan relations between Eisenstein series.


## 1 Introduction

In the inverse of the period map of the classical two parameter Weierstrass family of elliptic curves, we get the Eisenstein series of weight 4 and 6. In a more general context, the Schwarz triangle function with triangular parameters $\frac{1}{p}, \frac{1}{q}, \frac{1}{r}$, $p, q, r \in \mathbb{N}$, is the inverse of an automorphic function for the triangle group with signature $\langle p, q, r\rangle$. In all these examples the period maps of differential forms of the first kind are considered. If we consider periods of differential forms of the second kind we get differential automorphic functions which are solutions of certain ordinary differential equations (see [13]). In this way, it is not necessary to define (differential) automorphic functions by functional equations which they satisfy with respect to a Kleinian group, but as functions which are solutions of certain ordinary differential equations. To explain better this idea, let us state the main result of this paper:

**Theorem.** *Consider the multi-valued function*

$$\mathsf{pm} : \mathbb{C}^3 \backslash \{(t_1, t_2, t_3) \in \mathbb{C}^3 \mid 27t_3^2 - 4t_2^3 = 0\} \to \mathrm{SL}(2, \mathbb{C}) \tag{1}$$

$$t \mapsto \begin{pmatrix} \int_{\delta_1} \frac{dx}{y} & \int_{\delta_1} \frac{xdx}{y} \\ -\int_{\delta_2} \frac{dx}{y} & -\int_{\delta_2} \frac{xdx}{y} \end{pmatrix},$$

*where*

$$y = \gamma^{\frac{1}{2}} (27t_3^2 - 4t_2^3)^{\frac{1}{2}(\frac{1}{2}-a)} ((x - t_1)^3 - t_2(x - t_1) - t_3)^a, \ \ a \notin \mathbb{N} \cup \{0\} \tag{2}$$

*and $\delta_1$ and $\delta_2$ are two straight paths in the $x$-plane connecting a zero of $y$ (as a function in $x$) to the remaining zeros or they are Pochhammer cycles as defined in §3. Here $\gamma$ is a complex number depending only on $a$. It is taken in such a way that the image of* $\mathsf{pm}$ *is in* $\mathrm{SL}(2, \mathbb{C})$.

1. *For $a \neq \frac{2}{3}$ the map* pm *is a local biholomorphism and its local inverse restricted to $\begin{pmatrix} z & -1 \\ 1 & 0 \end{pmatrix}$, namely $(g_{1,a}(z), g_{2,a}(z), g_{3,a}(z))$, where $z$ is in some small open set $U$ in the image of the Schwarz map $t \mapsto -\frac{\int_{\delta_1} \frac{dx}{y}}{\int_{\delta_2} \frac{dx}{y}}$, satisfies the system of ordinary differential equations:*

$$\text{(3)} \qquad \begin{cases} \dot{t}_1 = t_1^2 + \frac{3a-1}{9a-6} t_2 \\ \dot{t}_2 = 4t_1t_2 + \frac{3}{3a-2} t_3 \\ \dot{t}_3 = 6t_1t_3 + \frac{2}{9a-6} t_2^2 \end{cases},$$

*where $\cdot$ is the derivation with respect to $z$.*

2. *The integrals $\int_\delta \frac{xdx}{y}$, where $\delta$ is a path connecting two roots of $y$, are constant along the solutions of (3).*

3. *The functions $g_{k,a}$, $k = 1, 2, 3$ with respect to the group*

$$\text{(4)} \qquad \Gamma := \langle M_1, M_2 \rangle \subset \mathrm{SL}(2, \mathbb{C}),$$

$$M_1 := \frac{i}{e^{\pi i a}} \begin{pmatrix} -e^{2\pi i a} & 0 \\ 1 & 1 \end{pmatrix}, M_2 := \frac{i}{e^{\pi i a}} \begin{pmatrix} 1 & e^{2\pi i a} \\ 0 & -e^{2\pi i a} \end{pmatrix},$$

*have the following automorphic properties: for every $A = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma$ and $z \in U$ such that $cz + d \neq 0$ there exists an analytic continuation $\tilde{g}_{k,a}$ of $g_{k,a}$, $k = 1, 2, 3$ along a path which connects $z$ to $Az$ such that*

$$\text{(5)} \qquad (cz + d)^{-2k} \tilde{g}_{k,a}(Az) = g_{k,a}(z), \ k = 2, 3,$$

$$\text{(6)} \qquad (cz + d)^{-2} \tilde{g}_{1,a}(Az) = g_{1,a}(z) + c(cz + d)^{-1}.$$

Note that for the third item in the above theorem we do not need that the action of $\Gamma$ to be properly discontinuous. One can show that

$$\text{(7)} \qquad g_{k,\frac{1}{2}} = a_k \Big(1 + (-1)^k \frac{4k}{B_k} \sum_{n \geq 1} \sigma_{2k-1}(n) e^{2\pi i z n}\Big), \quad k = 1, 2, 3, \ z \in \mathbb{H},$$

is the Eisenstein series of weight $2k$, where $\mathbb{H}$ is the upper half plane, $B_k$ is the $k$-th Bernoulli number ($B_1 = \frac{1}{6}$, $B_2 = \frac{1}{30}$, $B_3 = \frac{1}{42}$, $\ldots$), $\sigma_i(n) := \sum_{d|n} d^i$ and

$$\text{(8)} \qquad (a_1, a_2, a_3) = (\frac{2\pi i}{12}, 3(\frac{2\pi i}{12})^2, 2(\frac{2\pi i}{12})^3)$$

(see [13]). In the case $a = \frac{1}{2}$ the ordinary differential equations (3) are known as the Ramanujan relations between $g_{k,a}$, $k = 1, 2, 3$ because he noticed that in this case the series (7) satisfy the differential equation (3) (see for instance [11]). I do not know any explicit expressions like (7) for an arbitrary $a \in \mathbb{C}$.

The main result of the present text is in fact formulated for the family (9) which is a generalization of (2) by setting $a = b = c$. It has the advantage that it contains three exponent parameters corresponding to the three parameters of the Gauss hypergeometric function and Halphen's differential equation. The present text arose from the Ramanujan's case $a = \frac{1}{2}$. This very particular case leads to the theory of modular and quasi-modular forms with fruitful applications in number theory and mathematical physics, see for instance [10, 14] and the references within there. We are looking for the possible generalizations of such applications. These are the main reasons for announcing the main results of the present text for the family (2) and also choosing the title of the article. That is also why we have performed the calculations related to (2) separately, even so that they follow, without computer-assisted calculations, from the corresponding calculations for (9) .

The text is organized in the following way: In §2 we consider a more general family of transcendental curves. In §3 and §4 we fix up the paths of integration and calculate the monodromies. In §5 we calculate the derivation of the period map. The calculation is similar to the calculation of Gauss-Manin connections in the algebraic context. In §6 we calculate the determinant of the period map and according to this calculation in §7 we redefine the period map. In §8 we take the inverse of the period map and obtain Halphen's differential equation and Ramanujan type relations. §9 is devoted to the action of an algebraic group. In §10 we discuss the automorphic properties of the functions which appear in the inverse of the period map. Finally, §11 is dedicated to Lie theoretic aspects of Ramanujan and Halphen differential equations.

## 2 Families of transcendental curves

For $a, b, c \in \mathbb{C}$ fixed, we consider the following family of transcendental curves:

$$E_{t,a,b,c} = E_t : y = f(x), \tag{9}$$

$$f(x) := t_0^{\frac{1}{2}}(x - t_1)^a(x - t_2)^b(x - t_3)^c.$$

Here $t = (t_0, t_1, t_2, t_3) \in \mathbb{C}^4$ is a parameter. The discriminant of $E_t$ is defined to be

$$\Delta = \Delta(t) := t_0(t_1 - t_2)^2(t_2 - t_3)^2(t_3 - t_1)^2$$

We work with regular parameters, i.e.

$$t \in T := \{t \in \mathbb{C}^4 \mid \Delta(t) \neq 0\}.$$

The parameter $t_0$ is introduced to simplify the calculations related to the Gauss-Manin connection of the family (see §5). If $a, b$ and $c$ are rational numbers then the curves $E_t$ are algebraic. In this case one can use algebro-geometric methods in order to study the periods of $E_t$, see for instance [16]. In general, $E_t$ is a solution of the following logarithmic differential equation

$$\frac{dy}{y} = \frac{adx}{x - t_1} + \frac{bdx}{x - t_2} + \frac{cdx}{x - t_3}.$$

In order to prove our main theorem we also consider the family

$$\tilde{E}_t : y = f(x),\ f(x) = \tilde{t}_0^{\frac{1}{2}}((x - \tilde{t}_1)^3 - \tilde{t}_2(x - \tilde{t}_1) - \tilde{t}_3)^a. \tag{10}$$

In the case $a = b = c$ the curves (9) and (10) are the same with different parameter spaces. The map between the parameter spaces is given by:

$$\tilde{t}_0 = t_0,\ \tilde{t}_1 = \frac{t_1 + t_2 + t_3}{3},\ \tilde{t}_2 = (\tilde{t}_1 - t_1)(\tilde{t}_1 - t_2) + (\tilde{t}_1 - t_2)(\tilde{t}_1 - t_3) + (\tilde{t}_1 - t_1)(\tilde{t}_1 - t_3),$$

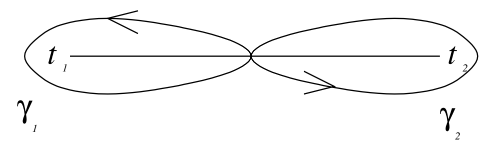


Figure 1: Pochhammer cycle

$$\tilde{t}_3 = (\tilde{t}_1 - t_1)(\tilde{t}_1 - t_2)(\tilde{t}_1 - t_3).$$

It is a degree 6 map. The inverse image of a point $\tilde{t}$ contains all possible permutations of the triple $(t_1, t_2, t_3)$. For simplicity we will also use $t$ instead of $\tilde{t}$; being clear which family of curves we are talking about. Throughout the text we will mainly work with the family (9). The similar discussion for (10) follows immediately. The main reason for considering the family (10) is that we first obtained the result of the present paper for such a family generalizing the classical case $a = \frac{1}{2}$. Another reason is that we are looking for possible number theoretic applications similar to those for Eisenstein series in the case $a = \frac{1}{2}$, see [19].

# 3 Paths of integration and Pochhammer cycles

Let us consider the family (9). We distinguish three, not necessarily closed, paths in $E_t$. In the $x$-plane let $\tilde{\delta}_i,\ i = 1, 2, 3$ be the straight path connecting $t_{i+1}$ to $t_{i-1}$, $i = 1, 2, 3$ (by definition $t_4 := t_1$ and $t_0 := t_3$). If $t_1, t_2, t_3$ are collinear then we may take curved paths such that we have a triangle with edges $\tilde{\delta}_i,\ i = 1, 2, 3$ and with vertices $t_i,\ i = 1, 2, 3$ (see Figure 3). There are many paths in $E_t$ which are mapped to $\tilde{\delta}_i$ under the projection on the $x$-plane. We choose one of them and call it $\delta_i$. For the case in which $\mathrm{Re}(a), \mathrm{Re}(b), \mathrm{Re}(c) < 0$ the paths $\delta_i$'s and $\tilde{\delta}_i$'s are depicted in Figure 2. We can choose $\delta_i$'s in such a way that $\delta_1 + \delta_2 + \delta_3$ is the limit of a closed and homotopic-to-zero path in $E_t$. Now, we have the convergent linear integral

$$\int_{\delta} \frac{p(x)dx}{y} = \int_{\tilde{\delta}} \frac{p(x)dx}{f(x)},\ p \in \mathbb{C}[x], \tag{11}$$

for

$$\mathrm{Re}(a), \mathrm{Re}(b), \mathrm{Re}(c) < 1, \tag{12}$$

where $\delta$ is one of the paths explained above. Our hypothesis on $\delta_i$'s implies that

$$\sum_{i=1}^{3} \int_{\delta_i} \frac{p(x)dx}{y} = 0.$$

By a linear change of the variable $x$ such integrals can be written in terms of the Gauss hypergeometric function (see [9]).

The linear integrals (11) have the disadvantage that they converge under the assumption (12). Using Pochhammer cycles we do not have the convergence problem and we can discard (12). For simplicity we explain it for the pairs $(t_1, t_2)$. The Pochhammer cycle associated to the points $t_1, t_2 \in \mathbb{C}$ and the path $\tilde{\delta}_3$ is the commutator

$$\tilde{\alpha}_3 = [\gamma_1, \gamma_2] = \gamma_1^{-1} \cdot \gamma_2^{-1} \cdot \gamma_1 \cdot \gamma_2,$$

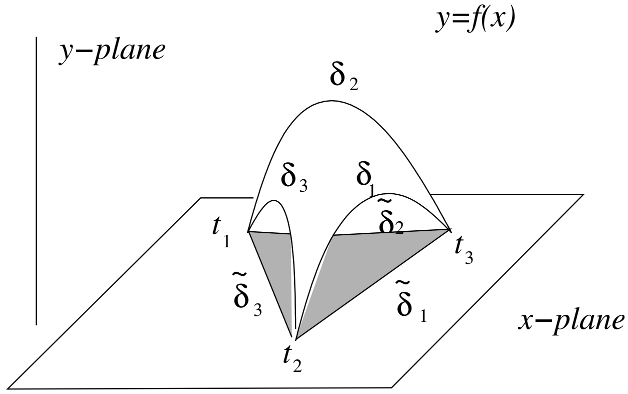


Figure 2: Paths of integration

where $\gamma_1$ is a loop along $\tilde{\delta}_3$ starting and ending at some point in the middle of $\tilde{\delta}_3$. It encircles $t_1$ once counterclockwise. The path $\gamma_2$ is a similar loop with respect to $t_2$ (see Figure 1). It is easy to see that the cycle $\tilde{\alpha}_3$ lifts up to a closed path $\alpha_3$ in $E_t$ and if $a, b \notin \mathbb{Z},\ \mathrm{Re}(a), \mathrm{Re}(b) < 1$ then

$$\int_{\alpha_3} \frac{p(x)dx}{y} = (1 - e^{-2\pi i a})(1 - e^{-2\pi i b}) \int_{\delta_3} \frac{p(x)dx}{f(x)} dx. \tag{13}$$

(see [9], Proposition 3.3.7).

For $a, b, c \in \mathbb{Z}$, integrals over Pochhammer cycles are identically zero. In summary, we only need to assume the hypothesis $a, b, c \notin \mathbb{N}$ in order to work with integrals; we may always use integrals over Pochhammer cycles except for $a, b, c \in \{0, -1, -2, \ldots\}$ which in this case we use linear integrals. Notice also that in order to have

$$\int_{\tilde{\delta}_i} d\left(\frac{p(x)}{f(x)}\right) = 0,\ \forall p \in \mathbb{C}[x],\ i = 1, 2, 3$$

we have to assume that $\mathrm{Re}(a), \mathrm{Re}(b), \mathrm{Re}(c) < 0$. But this is not necessary if we work with Pochhammer cycles.

# 4 The period map and the monodromy group

Let us consider the family (9). For a fixed $\mathsf{a} \in T$, let $(T, \mathsf{a})$ be a small neighborhood of $\mathsf{a}$ in $T$. The local period map is defined by:

$$\mathsf{pm} : (T, \mathsf{a}) \to \mathrm{Mat}(2, \mathbb{C}),\ t \mapsto \begin{pmatrix} \int_{\delta_1} \frac{dx}{y} & \int_{\delta_1} \frac{xdx}{y} \\ \int_{\delta_2} \frac{dx}{y} & \int_{\delta_2} \frac{xdx}{y} \end{pmatrix}, \tag{14}$$

where $\mathrm{Mat}(2, \mathbb{C})$ is the set of 2 by 2 matrices and $\delta_1, \delta_2$ are as in the previous section. Later in §6 we will see that its image is an open subset of $\mathrm{GL}(2, \mathbb{C})$. It is only defined in a neighborhood of $\mathsf{a}$. However, it can be extended along any path in $T$ with the starting point $\mathsf{a}$. In this way we can regard $\mathsf{pm}$ as a multi-valued function defined in $T$. Let $\mathcal{P}$ be the union of images of extensions of $\mathsf{pm}$:

$$\mathcal{P} := \{x \in \mathrm{Mat}(2, \mathbb{C}) \mid \exists \text{ a path } \gamma : [0, 1] \to T \text{ with } \gamma(0) = \mathsf{a},\ \mathsf{pm}(\gamma(1)) = x\},$$

where $\mathsf{pm}(\gamma(1))$ is obtained by the analytic continuation of $\mathsf{pm}$ along $\gamma$. It is called the period domain.

**Remark 1.** Using an algebraic group action on the space $T$ (see §9), the study of $\mathcal{P}$ is reduced to the study of the image of the classical Schwarz map of the Gauss hypergeometric function (set $t_0 = 1,\ t_1 = 0,\ t_2 = 1,\ t_3 = z$). For the latter the reader is referred to [9]. For arbitrary $a, b, c$, it is hard to describe $\mathcal{P}$ or the image of Schwarz map precisely. Using computer graphics it is possible to observe how strange its boundary can be (see for instance [17]). For the elliptic curve case $a = b = c = \frac{1}{2}$, $\mathcal{P}$ is the set of of matrices $x = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \in \mathrm{GL}(2, \mathbb{C})$ such that $x_1 \neq 0,\ x_3 \neq 0$ and $\mathrm{Im}(\frac{x_1}{x_3}) > 0$ (see for instance [13]).

In order to study the analytic extensions of $\mathsf{pm}$, we have to calculate the monodromy group $\Gamma$. By definition $\Gamma$ is the set of all matrices $A \in \mathrm{GL}(2, \mathbb{C})$ such that $A\mathsf{pm}$ is an analytic continuation of $\mathsf{pm}$ along some closed path starting and ending at the point $\mathsf{a} \in T$.

**Theorem 1.** *The monodromy group $\Gamma$ for the family (9) is generated by the matrices*

$$M_1 = \begin{pmatrix} BC & 0 \\ 1-B & 1 \end{pmatrix}, M_2 = \begin{pmatrix} 1 & C-CA \\ 0 & CA \end{pmatrix},\ M_3 = \begin{pmatrix} A & A-1 \\ A(B-1) & A(B-1)+1 \end{pmatrix}, \tag{15}$$

*where*

$$A = e^{2\pi i a},\ B = e^{2\pi i b},\ C = e^{2\pi i c}.$$

*Proof.* In what follows we use the following convention: Two paths in $E_t$ are equal if the integration of any differential form $\frac{p(x)dx}{y}$, $p \in \mathbb{C}[x]$ over them is equal. For instance, using this convention we have

$$\delta_1 + \delta_2 + \delta_3 = 0. \tag{16}$$

We also work with the $\mathbb{C}$-vector spaces generated by the paths in $E_t$. We fix $t_2$ and $t_3$ and let $t_1$ turn around $t_2$ counterclockwise. We obtain three new paths $h_3(\delta_1)$, $h_3(\delta_2)$ and $h_3(\delta_3)$ in $E_t$ such that $h_3(\delta_1) + h_3(\delta_2) + h_3(\delta_3) = 0$ (this follows from (16)). Notice that in the $x$-plane (resp. in $E_t$) the triangle formed by $h_3(\tilde{\delta}_i)$'s (resp. $h_3(\delta_i)$'s) does not intersect itself. We have

$$h_3(\delta_2) = \delta_2 + (A - AB)\delta_3,\ h_3(\delta_1) = -\delta_2 - A\delta_3 = \delta_1 + (1-A)\delta_3,\ h_3(\delta_3) = AB\delta_3$$

(see Figure 3, A). We call $h_3$ the monodromy around the hyperplane $t_1 = t_2$. These formulas are compatible with the Picard-Lefschetz formula in the case $a = b = c = \frac{1}{2}$. In a similar way

$$h_1(\delta_3) = \delta_3 + B\delta_1 - BC\delta_1, h_1(\delta_2) = -\delta_3 - B\delta_1,\ h_1(\delta_1) = BC\delta_1$$

and

$$h_2(\delta_1) = \delta_1 + C\delta_2 - CA\delta_2, h_2(\delta_3) = -\delta_1 - C\delta_2,\ h_2(\delta_2) = CA\delta_2.$$

Therefore, the monodromies with respect to the basis $(\delta_1, \delta_2)$ have the form (15).

When using Pochhammer cycles, these are deformed under the monodromy $h_i$ in a similar way. However, note that due to the equalities of type (13), the monodromy group written in the Pochhammer cycles $(\alpha_1, \alpha_2)$ is conjugated to $\Gamma$ by a diagonal matrix. $\square$

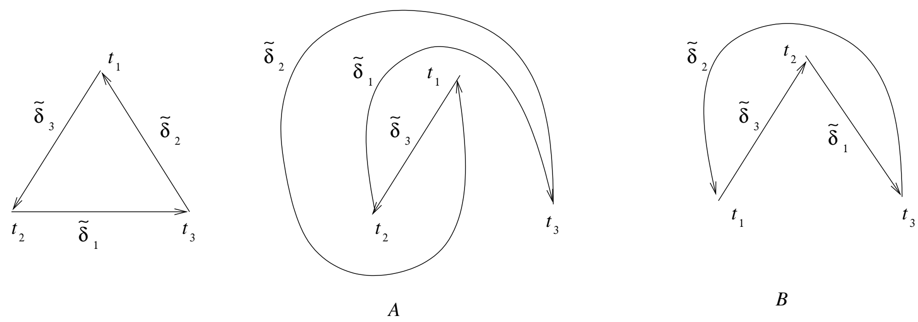


Figure 3: Monodromy

**Remark 2.** Notice that

$$M_1M_2M_3 = \begin{pmatrix} ABC & 0 \\ 0 & ABC \end{pmatrix},$$

and that for $n \in \mathbb{N}$

$$h_3^n(\delta_2) = \delta_2 + (A - AB)\frac{(AB)^n - 1}{AB - 1}\delta_3, h_3^n(\delta_1) = \delta_1 + (1 - A)\frac{(AB)^n - 1}{AB - 1}\delta_3,\ h_3^n(\delta_3) = (AB)^n\delta_3.$$

Since the choice of $\delta_1, \delta_2$ is not canonical, the symmetry between $A$, $B$ and $C$ cannot be seen in the $2 \times 2$ matrices (15). It can be seen if we write them as $3 \times 3$ matrices using the three elements $\delta_1, \delta_2, \delta_3$.

For $a = b = c = \frac{1}{2}$ we have

$$M_3 = \begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix},\ M_1 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, M_2 = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$$

and it is easy to see that $\Gamma = \Gamma(2) := \{A \in \mathrm{SL}(2,\mathbb{Z}) \mid A \equiv_2 \mathrm{Id}\}$.

**Theorem 2.** *The monodromy group $\Gamma$ for the family (10) is generated by the matrices*

$$M_1 = \begin{pmatrix} -A & 0 \\ 1 & 1 \end{pmatrix},\ M_2 = \begin{pmatrix} 1 & A \\ 0 & -A \end{pmatrix},\ M_3 = \begin{pmatrix} 0 & -1 \\ -A & 1 - A \end{pmatrix}, \tag{17}$$

*where $A = e^{2\pi i a}$.*

*Proof.* For the family (10) the monodromy group $\Gamma$ is obtained by the permutation of the roots of $f$. The element $h_3$ of $\Gamma$ obtained by the permutation $(t_1, t_2, t_3) \mapsto (t_2, t_1, t_3)$ is given by:

$$h_3(\delta_1) = -\delta_2 = \delta_1 + \delta_2,\ h_3(\delta_2) = \delta_2 + A\delta_3,\ h_3(\delta_3) = -A\delta_3$$

(see Figure 3 B). For the other monodromies in a similar way we have:

$$h_1(\delta_2) = -\delta_3,\ h_1(\delta_3) = \delta_3 + A\delta_1,\ h(\delta_1) = -A\delta_1,$$

$$h_2(\delta_3) = -\delta_1,\ h_2(\delta_1) = \delta_1 + A\delta_2,\ h_2(\delta_2) = -A\delta_2.$$

Therefore, in the basis $(\delta_1, \delta_2)$, the monodromies have the form (17). □

Notice that

$$M_3^{-1} M_1 M_3 = M_2.$$

For $a = \frac{1}{2}$ we have

$$M_3 = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix},\ M_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},\ M_2 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

and so $\Gamma = \mathrm{SL}(2, \mathbb{Z})$. Note that $T := M_2^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S := M_2 M_1 M_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ are the classical generators of $\mathrm{SL}(2, \mathbb{Z})$ with $S^2 = (ST)^3 = -I$ and no other relations between $S$ and $TS$ in $\mathrm{SL}(2, \mathbb{Z})/(\pm I)$.

**Remark 3.** The group $\Gamma$ acts on $\mathcal{P}$ from the left by usual multiplication of matrices and it is a hard problem to classify parameters $a, b, c$ for which this action is properly discontinuous in some open subset of $\mathcal{P}$. Using the algebraic group action in §9, this problem is equivalent to the following: Let $\Gamma$ act on $\mathbb{C} \cup \{\infty\}$ by the classical Möbius transformation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z \quad \mapsto \quad \frac{az + b}{cz + d}.$$

For which parameters $a, b, c$, the group $\Gamma$ is Kleinian, i.e it acts properly discontinuously in some open subset of $\mathbb{C} \cup \{\infty\}$? There is a necessary condition for such groups called Jorgensen's inequality (see [2]) but it is not sufficient[1]. For $\nu_0 := 1 - a - c = \frac{1}{p},\ \nu_1 := 1 - b - c = \frac{1}{q},\ \nu_\infty := 1 - a - b = \frac{1}{r}$, where $p, q, r$ are positive integers, the group $\Gamma$ is the triangular group of type $\langle p, q, r \rangle$ and it is Kleinian (see [2, 12, 16]). If $(\nu_0, \nu_1, \nu_\infty)$ is sufficiently near to a point with pure imaginary coordinates, then $\Gamma$ is a genus two Schottky group and hence it is Kleinian (see [15] and [18]). In the elliptic curve case $a = b = c = \frac{1}{2}$ the group $\Gamma$ is $\mathrm{SL}(2, \mathbb{Z})$ for the family (2) and it is $\Gamma(2)$ for the family (9). Both these groups are Kleinian.

## 5 A kind of Gauss-Manin connection

The Gauss-Manin connection is the art of derivation of differential forms on families of algebraic varieties and then simplifying the result. Despite the fact that the varieties considered in this article are not algebraic, the process of derivation and simplification is similar to the algebraic case (see for instance [13]). In what follows, derivation with respect to $x$ is denoted by $'$. We have used the word connection because the linear differential system (18) that we calculate can be considered as a connection on the trivial bundle on the $t$ space. In the Ramanujan's case $a = b = c = \frac{1}{2}$ this bundle is the first cohomology bundle of the corresponding family of elliptic curves and its sections with images in the cohomology with integer coefficients generate the set of flat sections.

Let us consider the family (9). First of all we have to simplify the integral (11). More precisely we want to reduce it to the integrals with $p = 1, x$. Let $\mathsf{R} = \mathbb{C}(t)$ and

$$g = (x - t_1)(x - t_2)(x - t_3).$$

[1] I would like to thank Katsuhiko Matsuzaki who informed me about the mentioned fact.

**Proposition 1.** *For all $p \in \mathsf{R}[x]$, there is $\tilde{p} \in \mathsf{R}[x]$, $\deg(\tilde{p}) \leq 1$ such that*

$$\int_\delta \frac{pdx}{y} = \int_\delta \frac{\tilde{p}dx}{y},$$

*where $\delta$ is any closed path in $E_t$ or a path which connects two points of $\{t_1, t_2, t_3\}$ and does not cross it elsewhere.*

*Proof.* For $n > 1$ modulo exact forms we have

$$0 = d\left(\frac{x^{n-2}g}{f}\right) = \left(-x^{n-2}g\frac{f'}{f} + (x^{n-2}g)'\right)\frac{dx}{f}.$$

Notice that $g\frac{f'}{f}$ is a polynomial in $x$. We set $p_n = b_n x^n + r_n(x),\ b_n \in \mathbb{C},\ \deg(r_n) \leq n-1$ the polynomial in the parenthesis. We have $b_n \neq 0$ and so

$$x^n\frac{dx}{f} = \frac{-1}{b_n} r_{n-1}\frac{dx}{f}$$

modulo exact forms. By various applications of the above equality in $\int_\delta \frac{pdx}{y}$ we finally get the desired equality. $\square$

Let us now take the derivatives of integrals:

**Proposition 2.** *Let $t$ be one of the parameters $t_i,\ i = 0, 1, 2, 3$. We have*

$$\frac{\partial}{\partial t}\int_{\tilde{\delta}} \frac{pdx}{f} = \int_{\tilde{\delta}} \nabla_{\frac{\partial}{\partial t}} \frac{pdx}{f},\ p \in \mathbb{C}[x],$$

*where*

$$\nabla_{\frac{\partial}{\partial t}} \frac{pdx}{f} := \frac{1}{\Delta}\left(\left(a_1 \frac{-\frac{\partial f}{\partial t} g}{f} p\right)' + a_2 \frac{-\frac{\partial f}{\partial t} g}{f} p + \Delta \frac{\partial p}{\partial t}\right)\frac{dx}{f}.$$

*and $\delta$ is any closed path in $E_t$. Here $a_1$ and $a_2$ are two polynomials in $\mathsf{R}[x]$ such that*

$$g\frac{f'}{f}a_1 + ga_2 = \Delta.$$

*Proof.* We have

$$
\begin{aligned}
\frac{\partial}{\partial t}\int_{\tilde\delta}\frac{pdx}{f} &= \int_{\tilde\delta}\left(\frac{-\frac{\partial f}{\partial t}p}{f}+\frac{\partial p}{\partial t}\right)\frac{dx}{f}\\
&= \int_{\tilde\delta}\left(\frac{\frac{-\frac{\partial f}{\partial t}gp}{f}}{g}+\frac{\partial p}{\partial t}\right)\frac{dx}{f}\\
&= \int_{\tilde\delta}\left(\frac{\tilde p}{g}+\frac{\partial p}{\partial t}\right)\frac{dx}{f}, \qquad \tilde p=\frac{-\frac{\partial f}{\partial t}g}{f}p\\
&= \frac{1}{\Delta}\int_{\tilde\delta}\left(\frac{\left(g\frac{f'}{f}a_1+ga_2\right)\tilde p}{g}+\Delta\frac{\partial p}{\partial t}\right)\frac{dx}{f}\\
&= \frac{1}{\Delta}\int_{\tilde\delta}\left(\frac{df}{f^2}a_1\tilde p+a_2\tilde p\frac{dx}{f}+\Delta\frac{\partial p}{\partial t}\frac{dx}{f}\right)\\
&= \frac{1}{\Delta}\int_{\tilde\delta}\left(\frac{1}{f}d(a_1\tilde p)+a_2\tilde p\frac{dx}{f}+\Delta\frac{\partial p}{\partial t}\frac{dx}{f}\right)\\
&= \frac{1}{\Delta}\int_{\tilde\delta}\left((a_1\tilde p)'+a_2\tilde p+\Delta\frac{\partial p}{\partial t}\right)\frac{dx}{f}.
\end{aligned}
$$

□

For the implementation of the algorithms of this section in SINGULAR [4] see the author's web page. For the family (9) we have used these algorithms and we have obtained:

$$\nabla_{\frac{\partial}{\partial t_1}}\omega = \mathrm{A}_{\frac{\partial}{\partial t_1}}\omega \tag{18}$$

$$A_{\frac{\partial}{\partial t_1}} = \frac{1}{(t_1-t_2)(t_1-t_3)}\begin{pmatrix} -at_1+(a+c-1)t_2+(a+b-1)t_3 & -a-b-c+2\\ at_2t_3+(b-1)t_1t_3+(c-1)t_1t_2 & (-a-b-c+2)t_1\end{pmatrix},$$

where

$$\omega=\begin{pmatrix}\frac{dx}{y}\\ \frac{xdx}{y}\end{pmatrix}.$$

The derivation with respect to $t_2$ (resp $t_3$) is obtained by permutation of $t_1$ with $t_2$ and $a$ with $b$ (resp. $t_1$ with $t_3$ and $a$ with $c$). It is also easy to check by hand that

$$\nabla_{\frac{\partial}{\partial t_0}}\omega = A_{\frac{\partial}{\partial t_0}}\omega,\ A_{\frac{\partial}{\partial t_0}}=\frac{1}{t_0}\begin{pmatrix}-\frac{1}{2} & 0\\ 0 & -\frac{1}{2}\end{pmatrix}$$

Let $\mathsf{R}=\mathbb{C}(t_0,t_1,t_2,t_3)$ and $V$ be the $\mathsf{R}$-vector space generated by the differential forms $\frac{x^idx}{y},\ i=0,1$. Let also $\Omega^1_{\mathsf{R}}$ be the set of rational differential 1-forms in the variables $t_0,t_1,t_2$ and $t_3$. We have the connection

$$\nabla: V\to\Omega_{\mathsf{R}}\otimes_{\mathsf{R}}V$$

on $V$ which is defined uniquely by its image on the basis $\frac{x^idx}{y},\ i=0,1$:

$$\nabla\omega=\sum_{i=0}^{3}dt_i\otimes\nabla_{\frac{\partial}{\partial t_i}}\omega.$$

For a vector field $X = \sum_{i=0}^{3} X_i \frac{\partial}{\partial t_i}$ we have the composition $\nabla_X : V \to \Omega^1_{\mathsf{R}} \otimes_{\mathsf{R}} V \to V$, where the second map is given by $\omega \otimes v \mapsto \omega(X)v$. For $X = \frac{\partial}{\partial t_i}$ we obtain the same maps as before.

For the family (10) we use $g = (x-t_1)^3 - t_2(x-t_1) - t_3$ and $\Delta = t_0(4t_2^3 - 27t_3^2)$ and we have the matrix of all $\nabla_{\frac{\partial}{\partial t_i}}$ written in in the basis $\frac{x^i dx}{y}$, $i = 0, 1$:

$$A_{\frac{\partial}{\partial t_0}} = \frac{1}{t_0}\begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix},\ A_{\frac{\partial}{\partial t_1}} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$A_{\frac{\partial}{\partial t_2}} = \frac{1}{(4t_2^3 - 27t_3^2)}\begin{pmatrix} -27at_1t_3 - 6at_2^2 + 18t_1t_3 + 2t_2^2 & 27at_3 - 18t_3 \\ -27at_1^2t_3 + 9at_2t_3 + 18t_1^2t_3 - 2t_1t_2^2 - 3t_2t_3) & 27at_1t_3 - 6at_2^2 - 18t_1t_3 + 4t_2^2 \end{pmatrix},$$

$$A_{\frac{\partial}{\partial t_3}} = \frac{1}{(4t_2^3 - 27t_3^2)}\begin{pmatrix} 18at_1t_2 + 27at_3 - 12t_1t_2 - 9t_3 & -18at_2 + 12t_2 \\ 18at_1^2t_2 - 6at_2^2 - 12t_1^2t_2 + 9t_1t_3 + 2t_2^2 & -18at_1t_2 + 27at_3 + 12t_1t_2 - 18t_3 \end{pmatrix}.$$

# 6 Determinant of the period matrix

Let us consider the family (9). It follows from Proposition 2 that the period map satisfies the differential equation

$$d(\mathsf{pm}) = \mathsf{pm}A^{\mathsf{tr}},\ \text{where } A = \sum_{i=0}^{3} A_{\frac{\partial}{\partial t_i}} dt_i.$$

This and (18) imply that $\det := \det(\mathsf{pm})$ satisfies

$$\frac{\partial \det}{\partial t_1} = \frac{1}{(t_1 - t_2)(t_1 - t_3)}((a+c-1)t_2 + (a+b-1)t_3 + (-2a-b-c+2)t_1)\det.$$

We solve this differential equation and conclude that det is of the form $C(t_1-t_3)^{1-a-c}(t_1-t_2)^{1-a-b}$, where $C$ does not depend on $t_1$. Repeating the same argument for $t_0, t_2, t_3$ we conclude that

$$\det(\mathsf{pm}) = \gamma \cdot t_0^{-1}(t_1 - t_3)^{1-a-c}(t_1 - t_2)^{1-a-b}(t_2 - t_3)^{1-b-c}, \tag{19}$$

where $\gamma$ is a constant depending only on $a, b$ and $c$. For the family (10) in a similar way we get

$$\det(\mathsf{pm}) = \gamma \cdot t_0^{-1}(27t_3^2 - 4t_2^3)^{\frac{1}{2}-a}.$$

# 7 Redefining the period map and the monodromy group

Let us consider the family (9). We have calculated the determinant of the period map in (19). It depends on $t_1, t_2, t_3$ except for the case $a = b = c = \frac{1}{2}$. We multiply (14) by

$$\kappa := \gamma^{-\frac{1}{2}}(t_1 - t_3)^{-\frac{1}{2}(1-a-c)}(t_1 - t_2)^{-\frac{1}{2}(1-a-b)}(t_2 - t_3)^{-\frac{1}{2}(1-b-c)}.$$

The determinant of the new period map is equal to $t_0^{-1}$ and the monodromy group is a subgroup of $\mathrm{SL}(2,\mathbb{C})$. In other words, we redefine

$$f(x) := \gamma^{\frac{1}{2}} t_0^{\frac{1}{2}}(t_1 - t_3)^{\frac{1}{2}(1-a-c)}(t_1 - t_2)^{\frac{1}{2}(1-a-b)}(t_2 - t_3)^{\frac{1}{2}(1-b-c)}(x - t_1)^a(x - t_2)^b(x - t_3)^c$$

for the family (9). We have to calculate the corresponding connection.

$$\nabla(\kappa\omega) = (d\kappa) \otimes \omega + \kappa \cdot A \otimes \omega = (\frac{d\kappa}{\kappa} I_{2\times 2} + A) \otimes (\kappa\omega)$$

and

$$\frac{d\kappa}{\kappa} = \frac{1}{2}(a+b-1)\frac{dt_1 - dt_2}{t_1 - t_2} + \cdots = (\frac{1}{2}(a+b-1)\frac{1}{t_1-t_2} + \frac{1}{2}(a+c-1)\frac{1}{t_1-t_3})dt_1 + \cdots$$

After redefining the period map the monodromy matrices are changed as follows:

$$M_3 = \frac{-1}{\sqrt{AB}}\begin{pmatrix} A & A-1 \\ A(B-1) & A(B-1)+1 \end{pmatrix},\ M_1 = \frac{-1}{\sqrt{BC}}\begin{pmatrix} BC & 0 \\ 1-B & 1 \end{pmatrix}, M_2 = \frac{-1}{\sqrt{CA}}\begin{pmatrix} 1 & C-CA \\ 0 & CA \end{pmatrix}$$

Notice that $\sqrt{A} = e^{\pi i a}$, $B = \cdots$ are well-defined and $\Gamma := \langle M_1, M_2, M_3\rangle = \langle M_1, M_2\rangle \subset \mathrm{SL}(2,\mathbb{C})$.

For the family (10) we redefine

$$f(x) = \gamma^{\frac{1}{2}} t_0^{\frac{1}{2}} (27t_3^2 - 4t_2^3)^{\frac{1}{2}(\frac{1}{2}-a)}((x-t_1)^3 - t_2(x-t_1) - t_3)^a,\ t = (t_0, t_1, t_2, t_3) \in \mathbb{C}^4$$

which is the one in (1) with $t_0 = 1$. For $\kappa = (27t_3^2 - 4t_2^3)^{-\frac{1}{2}(\frac{1}{2}-a)}$ we have

$$\frac{d\kappa}{\kappa} = \frac{1}{2}(a - \frac{1}{2})\frac{54t_3dt_3 - 12t_2^2dt_2}{27t_3^2 - 4t_2^3}$$

The new monodromy group is (4). For both families we conclude that $\det(\mathsf{pm}) = t_0^{-1}$.

# 8 The inverse of the period map

Let us consider the family (9). First we notice that the local period map $\mathsf{pm} : (T, \mathsf{a}) \to \mathrm{GL}(2,\mathbb{C})$ is a biholomorphism. We consider $\mathsf{pm}$ as a map sending the point $(t_0, t_1, t_2, t_3)$ to $(x_1, x_2, x_3, x_4)$ with $x = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \in \mathcal{P}$. Its derivative at $t$ is a $4\times 4$ matrix for which the $i$-th column constitutes of the first and second row of $x(\nabla_{\frac{\partial}{\partial t_i}})^{\mathsf{tr}}$. For $s := a+b+c-2 \neq 0$ this is an invertible matrix. More precisely, we have

$$(dF)_x = (d\mathsf{pm})_t^{-1} = \frac{1}{\det(x)}\cdot$$

$$\begin{pmatrix} -t_0x_4 & t_0x_3 & t_0x_2 & -t_0x_1 \\ (at_1t_2x_3 + at_1t_3x_3 - at_1x_4 - at_2t_3x_3 + bt_1^2x_3 - bt_1x_4 + ct_1^2x_3 - ct_1x_4 - t_1^2x_3 - t_1t_2x_3 - t_1t_3x_3 + 2t_1x_4 + t_2t_3x_3)/s & (-t_1x_3 + x_4) & (-at_1t_2x_1 - at_1t_3x_1 + at_1x_2 + at_2t_3x_1 - bt_1^2x_1 + bt_1x_2 - ct_1^2x_1 + ct_1x_2 + t_1^2x_1 + t_1t_2x_1 + t_1t_3x_1 - 2t_1x_2 - t_2t_3x_1)/s & (t_1x_1 - x_2) \\ (at_2^2x_3 - at_2x_4 + bt_1t_2x_3 - bt_1t_3x_3 + bt_2t_3x_3 - bt_2x_4 + ct_2^2x_3 - ct_2x_4 - t_1t_2x_3 + t_1t_3x_3 - t_2^2x_3 - t_2t_3x_3 + 2t_2x_4)/s & (-t_2x_3 + x_4) & (-at_2^2x_1 + at_2x_2 - bt_1t_2x_1 + bt_1t_3x_1 - bt_2t_3x_1 + bt_2x_2 - ct_2^2x_1 + ct_2x_2 + t_1t_2x_1 - t_1t_3x_1 + t_2^2x_1 + t_2t_3x_1 - 2t_2x_2)/s & (t_2x_1 - x_2) \\ (at_3^2x_3 - at_3x_4 + bt_3^2x_3 - bt_3x_4 - ct_1t_2x_3 + ct_1t_3x_3 + ct_2t_3x_3 - ct_3x_4 + t_1t_2x_3 - t_1t_3x_3 - t_2t_3x_3 - t_3^2x_3 + 2t_3x_4)/s & (-t_3x_3 + x_4) & (-at_3^2x_1 + at_3x_2 - bt_3^2x_1 + bt_3x_2 + ct_1t_2x_1 - ct_1t_3x_1 - ct_2t_3x_1 + ct_3x_2 - t_1t_2x_1 + t_1t_3x_1 + t_2t_3x_1 + t_3^2x_1 - 2t_3x_2)/s & (t_3x_1 - x_2) \end{pmatrix}$$

and

$$F = (F_0, F_1, F_2, F_3) : (\mathcal{P}, x_0) \to (T, \mathsf{a})$$

is the local inverse of $\mathsf{pm}$, where $x_0 = \mathsf{pm}(\mathsf{a})$. From $\det(\mathsf{pm}) = t_0^{-1}$ it follows that $F_0(x) = \det(x)^{-1}$. Let us take $\mathsf{a}$ in such a way that $x_0$ is of the form $\begin{pmatrix} z_0 & -1 \\ 1 & 0 \end{pmatrix}$. In the next section we will see that such $x_0$'s exist. Let $g_i(z)$ be the restriction of $F_i$ to $\begin{pmatrix} z & -1 \\ 1 & 0 \end{pmatrix}$, where $z$ is in a neighborhood of $z_0$ in $\mathbb{C}$. Considering the equations related to the entries $(i,1), i = 2,3,4$, we conclude that $(g_1(z), g_2(z), g_3(z))$ satisfies the ordinary differential equation:

$$
\left\{\begin{array}{l}
\dot{t}_1 = \frac{a-1}{a+b+c-2}(t_1t_2 + t_1t_3 - t_2t_3) + \frac{b+c-1}{a+b+c-2}t_1^2 \\
\dot{t}_2 = \frac{b-1}{a+b+c-2}(t_2t_1 + t_2t_3 - t_1t_3) + \frac{a+c-1}{a+b+c-2}t_2^2 \\
\dot{t}_3 = \frac{c-1}{a+b+c-2}(t_3t_1 + t_3t_2 - t_1t_2) + \frac{a+b-1}{a+b+c-2}t_3^2
\end{array}\right. . \tag{20}
$$

This ordinary differential equation is discovered by G. Halphen [8, 7, 6] in his study of hyper-geometric functions. In a similar way for the family (10), we get (3) and so the first part of our theorem is proved. Let Ra be the vector field in $\mathbb{C}^4$ corresponding to (20) together with $\dot{t}_0 = 0$. It is a mere calculation to see that

$$\nabla_{\mathrm{Ra}} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}.$$

This means that $d(\mathsf{pm})(\mathrm{Ra}) = \begin{pmatrix} * & 0 \\ * & 0 \end{pmatrix}$ and so $\int_\delta \frac{xdx}{y}$ is constant along the solutions of Ra. A similar argument works for the family (10) and so the second part of our theorem is proved.

**Remark 4.** In this section we have considered the local period map and hence its inverse $F : (\mathcal{P}, x_0) \to (T, a)$ is a one valued holomorphic function defined in a neighborhood of $x_0$ in $\mathcal{P}$. Since the period map can be extended analytically to any region in $T$, its inverse can be also extended analytically to any region in $\mathcal{P}$. Since we do not know whether $\mathcal{P}$ is simply connected or not and whether the extended $\mathsf{pm}$ is injective or not, analytic extensions of $F$ lead to a priori a multivalued function. The functions $g_i(z),\ i = 1,2,3$ that we get in the inverse of the period map are well-defined one valued holomorphic functions in a small neighborhood of $z_0$ and they can be also extended to regions far from $z_0$. The domain of definition of $g_i$'s is exactly the image of the Schwarz map and as it is explained at the beginning of §4, it is in general hard to describe the image of the Schwarz map. For arbitrary $a, b, c$ the Schwarz map may not be injective and so analytic extensions of $g_i$'s may lead to multi-valued functions. For the elliptic curve case $a = b = c = \frac{1}{2}$, the analytic extensions of $g_i$'s result in one valued Eisenstein series defined in the upper half plane (see for instance [13]).

## 9 Action of an algebraic group

Let us consider the family (9). The algebraic group

$$G_0 := \left\{ \begin{pmatrix} k_1 & k_3 \\ 0 & k_2 \end{pmatrix} \mid\ k_3 \in \mathbb{C}, k_1, k_2 \in \mathbb{C}^* \right\} \tag{21}$$

acts on $\mathrm{GL}(2,\mathbb{C})$ from the right by the usual multiplication of matrices. In Proposition 3 we will see that $\mathcal{P}$ is invariant under this action and so $G_0$ acts on $\mathcal{P}$ (this is not clear from the definition of $\mathcal{P}$ in §4). The algebraic group $G_0$ also acts on $\mathbb{C}^4$ as follows:

$$t \bullet g := (t_0(k_1k_2)^{-1}, t_1k_1^{-1}k_2 + k_3k_1^{-1}, t_2k_1^{-1}k_2 + k_3k_1^{-1}, t_3k_1^{-1}k_2 + k_3k_1^{-1})$$

$$t = (t_0, t_1, t_2, t_3) \in \mathbb{C}^4, g = \begin{pmatrix} k_1 & k_3 \\ 0 & k_2 \end{pmatrix} \in G_0. \tag{22}$$

For a topological space $X$ and $x \in X$, let $(X, x)$ be a small neighborhood of $x$ in $X$. The relation between the two actions of $G_0$ is given by:

**Proposition 3.** *We have*

$$\mathsf{pm}(t \bullet g) = \mathsf{pm}(t) \cdot g, \ \ t \in T, \ \ g \in (G_0, I), \tag{23}$$

*where $I$ is the identity $2 \times 2$ matrix.*

Note that the equality (23) implies that $\mathsf{pm}(t) \cdot g$ is in the image of the period map and so it is in $\mathcal{P}$. If $t_s, s \in [0,1]$, is a path in $T$ and $g_s$, $s \in [0,1]$, is a path in $G_0$ which connects $I$ to $g \in G_0$, then by analytic continuation of both sides of the equality (23) it makes sense to say that (23) is true for an arbitrary $g \in G_0$.

*Proof.* Let

$$\alpha : \mathbb{C}^2 \to \mathbb{C}^2, \ (x, y) \mapsto (k_2^{-1}k_1 x - k_3 k_2^{-1}, k_2^{-1}k_1^2 y).$$

Then

$$k_2k_1^{-2}\alpha^{-1}(y - f(x)) = y - (\gamma t_0)^{\frac{1}{2}} k_2 k_1^{-2} (t_2 - t_3)^{\frac{1}{2}(1-b-c)} (k_2^{-1}k_1 x - k_3 k_2^{-1} - t_1)^a \cdots = y -$$
$$(\gamma t_0)^{\frac{1}{2}} k_2^{1-a-b-c} k_1^{a+b+c-2} (k_2^{-1}k_1)^{\frac{1}{2}(3-2(a+b+c))} (k_2k_1^{-1}t_2 - k_2k_1^{-1}t_3)^{\frac{1}{2}(1-b-c)} (x - (k_2k_1^{-1}t_1 + k_3k_1^{-1}))^a$$
$$\cdots = y - (\gamma t_0)^{\frac{1}{2}} (k_2k_1)^{-\frac{1}{2}} (k_2k_1^{-1}t_2 + k_3k_1^{-1} - (k_2k_1^{-1}t_3 + k_3k_1^{-1}))^{\frac{1}{2}(1-b-c)} (x - (k_2k_1^{-1}t_1 + k_3k_1^{-1}))^a \cdots$$

This implies that $\alpha$ induces an isomorphism

$$\alpha : E_{t \bullet g} \to E_t.$$

Now

$$\alpha^{-1}\omega = \begin{pmatrix} k_1^{-1} & 0 \\ -k_3k_2^{-1}k_1^{-1} & k_2^{-1} \end{pmatrix} \omega = \begin{pmatrix} k_1 & 0 \\ k_3 & k_2 \end{pmatrix}^{-1} \omega,$$

where $\omega = (\frac{dx}{y}, \frac{xdx}{y})^{\mathsf{tr}}$, and so

$$\mathsf{pm}(t) = \mathsf{pm}(t \bullet g).g^{-1}$$

which proves (23). $\square$

In a similar way for the family (10) we have the action

$$t \bullet g := (t_0k_1^{-1}k_2^{-1}, t_1k_1^{-1}k_2 + k_3k_1^{-1}, t_2k_1^{-3}k_2, t_3k_1^{-4}k_2^2)$$

$$t = (t_0, t_1, t_2, t_3) \in \mathbb{C}^4, g = \begin{pmatrix} k_1 & k_3 \\ 0 & k_2 \end{pmatrix} \in G_0 \tag{24}$$

with the property (23).

Using the equality

$$x := \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = \begin{pmatrix} \frac{x_1}{x_3} & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_3 & x_4 \\ 0 & \frac{-x_2x_3 + x_1x_4}{x_3} \end{pmatrix},$$

Proposition (3) and the analytic continuation argument along a path which connects the identity matrix to $\begin{pmatrix} x_3 & x_4 \\ 0 & \frac{-x_2x_3+x_1x_4}{x_3} \end{pmatrix}^{-1}$ in $G_0$, we conclude that for any element $x$ in the image of the period map, the matrix $\begin{pmatrix} \frac{x_1}{x_3} & -1 \\ 1 & 0 \end{pmatrix}$ is also in the image of the period map.

## 10 Automorphic properties of $g_i$'s

Let us consider the family (9). We keep the notation introduced in §8. Let

$$F = (F_0, F_1, F_2, F_3) : (\mathcal{P}, x_0) \to (T, \mathsf{a})$$

be the local inverse of the period map. Taking $F$ of (23) we conclude that

$$F(xg) = F(x) \bullet g,\ \ g \in (G_0, I). \tag{25}$$

We get

$$F_0(xg) = F_0(x)k_1^{-1}k_2^{-1},$$

$$F_i(xg) = F_1(x)k_1^{-1}k_2 + k_3k_1^{-1},\ \ i = 1, 2, 3. \tag{26}$$

The first equality also follows from $F_0(x) = \det(x)^{-1}$.

For any $A = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma$ there is a path $\gamma \in \pi_1(T, \mathsf{a})$ such that if $\tilde{\mathsf{pm}} : (T, \mathsf{a}) \to \mathcal{P}$ is the analytic continuation of $\mathsf{pm}$ along $\gamma$ then

$$\tilde{\mathsf{pm}}(t) = A\mathsf{pm}(t),\ \ \forall t \in (T, \mathsf{a}).$$

This implies that the analytic continuation $\tilde{F}$ of $F$ along the path $\delta = \mathsf{pm}(\gamma)$, which connects $\mathsf{pm}(\mathsf{a})$ to $A\mathsf{pm}(\mathsf{a})$, satisfies

$$F(x) = \tilde{F}(Ax),\ \ x \in (\mathcal{P}, x_0). \tag{27}$$

Using the Schwarz function

$$D(t) = \frac{\int_{\delta_1} \frac{dx}{f}}{\int_{\delta_2} \frac{dx}{f}}$$

we define the path $\sigma = D(\gamma)$. If $cz_0 + d \neq 0$ then $Az_0$ is well-defined and the path $\sigma$ connects $z_0$ to $Az_0$ in $\mathbb{C}$. We claim that there are analytic continuations $\tilde{g}_i$ of $g_i$'s along $\sigma$ such that

$$(cz + d)^{-2}\tilde{g}_i(Az) = g_i(z) + c(cz + d)^{-1},\ \ i = 1, 2, 3,\ \ A = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma, z \in (\mathbb{C}, z_0).$$

We have

$$\begin{aligned}
(1, g_1(z), g_2(z), g_3(z)) &= F\begin{pmatrix} z & -1 \\ 1 & 0 \end{pmatrix} \\
&\overset{(27)}{=} \tilde{F}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} z & -1 \\ 1 & 0 \end{pmatrix}\right) \\
&= \tilde{F}\left(\begin{pmatrix} Az & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} cz + d & -c \\ 0 & (cz + d)^{-1}\det(A) \end{pmatrix}\right) \\
&\overset{(25)}{=} \tilde{F}\left(\begin{pmatrix} Az & -1 \\ 1 & 0 \end{pmatrix}\right) \bullet \begin{pmatrix} cz + d & -c \\ 0 & (cz + d)^{-1} \end{pmatrix} \\
&= \left(1, (cz + d)^{-2}\tilde{g}_1(Az) - c(cz + d)^{-1}, \cdots\right)
\end{aligned}$$

The fourth equality makes sense in the following way: let

$$x_s := \begin{pmatrix} D(\gamma_s) & -1 \\ 1 & 0 \end{pmatrix} \in \mathcal{P},\ \tau_s := x_s^{-1}\mathsf{pm}(\gamma_s) \in G_0,\ s \in [0,1].$$

The path $\tau_s$ in $G_0$ connects $I$ to $\begin{pmatrix} cz+d & -c \\ 0 & (cz+d)^{-1} \end{pmatrix}$. For $s$ near enough to 0 we have $F(x_s\tau_s) = F(x_s) \bullet \tau_s$ and so by analytic continuation we have it for $s = 1$.

In a similar way we prove the third part of our main theorem. Notice that for the family (10), $F_2$ and $F_3$ satisfy:

$$F_2(xg) = F_2(x)k_1^{-3}k_2,\ F_3(xg) = F_3(x)k_1^{-4}k_2^2,\ \forall x \in \mathcal{P},\ g \in G_0.$$

It is well-known, see for instance [5], that if $(t_1(z), t_2(z), t_2(z))$ is a solution of Halphen's differential equation then

$$\left((cz+d)^{-2}t_i(Az) - c(cz+d)^{-1},\ i = 1,2,3\right),\quad A = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C})$$

is also a solution. Therefore, $\mathrm{SL}(2,\mathbb{C})$ acts on the solution space and the monodromy group $\Gamma$ is contained in the stablizer of the solution $(g_1, g_2, g_3)$. I do not know whether the equality holds.

## 11 Final comments

From Lie theoretic point of view Halphen's differential equation should not be considered on its own but together with the attached $\mathfrak{sl}(2,\mathbb{C})$ structure. For more details on this topic the reader is referred to Guillot's article [5]. Here we explain this briefly.

We work with the family (9) with $t_0 = 1$. Let $H$ be the vector field in $\mathbb{C}^3$ corresponding to the Halphen's differential equation (20) and

$$E = \sum_{i=1}^{3} t_i \frac{\partial}{\partial t_i},\ Z = \sum_{i=1}^{3} \frac{\partial}{\partial t_i}.$$

The vector fields $H, E$ and $Z$ are linearly independent and satisfy the Lie bracket relations

$$[E,H] = H,\ [E,Z] = -Z,\ [Z,H] = 2E$$

and so the Lie algebra they generate is isomorphic to $\mathfrak{sl}(2,\mathbb{C})$. The isomorphism is in fact given by $X \mapsto A_X$, where the matrix $A_X$ is defined by $\nabla_X\omega = A_X\omega$ and $\omega$ and $\nabla$ are defined in §5. This follows from the equalities:

$$A_H = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix},\ A_E = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix},\ A_Z = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Let $\delta$ be a path as in §3 and $X := \int_\delta \frac{dx}{y}$ and $Y := \int_\delta \frac{xdx}{y}$. Using $[dX, dY]^{\mathsf{tr}} = A[X,Y]^{\mathsf{tr}}$ and the above equalities we conclude that under the local map $(t_1,t_2,t_3) \mapsto (X,Y)$ the vector fields $H, E$ and $Z$ are mapped respectively to

$$-Y\frac{\partial}{\partial X},\ -\frac{1}{2}X\frac{\partial}{\partial X} + \frac{1}{2}Y\frac{\partial}{\partial Y},\ X\frac{\partial}{\partial Y}.$$

Notice that these vector fields satisfy the same Lie bracket relations and that they generate the standard (and unique) action of $\mathrm{SL}(2,\mathbb{C})$ on $\mathbb{C}^2$. From the Lie theoretic point of view, the Halphen's differential equation can be characterized as the vector field $H$ that satisfies the above Lie bracket relations with respect to the vector fields $E$ and $Z$ (see [5]). In a similar way the vector field (3) is uniquely determined through the same Lie bracket relations with respect to $E := \sum_{i=1}^{3} it_i dt_i$ and $Z := \frac{\partial}{\partial t_1}$. In other words (3) is a natural deformation of Ramanujan's differential equation between Eisenstein series and not an arbitrary one.

Finally, the differential equation (3) is as natural and as historical as Halphen's differential equation. If we eliminate the variables $t_2$ and $t_3$ in (3) and set $a = \frac{1}{2} - \frac{1}{n}$ then we get the Chazy differential equation

$$t_1''' = 12t_1''t_1 - 18(t_1')^2 + \frac{1728}{2(36-n^2)}(t_1' - t_1^2)^2,$$

see [3].

# References


[1] Tom M. Apostol. *Modular functions and Dirichlet series in number theory*, volume 41 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1990.

[2] A. Beardon, The geometry of discrete groups. Graduate Texts in Mathematics, 91. Springer-Verlag, New York, 1995.

[3] J. Chazy, Sur les équations différentielles du troisième ordre et d'ordre supérieur dont l'intégrale générale a ses points critiques fixes. Acta Math. 34 (1911), no. 1, 317-385.

[4] G.-M. Greuel, G. Pfister, and H. Schönemann. Singular 2.0. A Computer Algebra System for Polynomial Computations, Centre for Computer Algebra, University of Kaiserslautern, 2001. http://www.singular.uni-kl.de.

[5] Adolfo Guillot. Sur les équations d'Halphen et les actions de $\mathrm{SL}_2(\mathbf{C})$. *Publ. Math. Inst. Hautes Études Sci.*, (105):221–294, 2007.

[6] G. Halphen. Sur certains systéme d'équations difféetielles. *C. R. Acad. Sci Paris*, 92:1404–1407, 1881.

[7] G. Halphen. Sur des fonctions qui proviennent de l'équation de gauss. *C. R. Acad. Sci Paris*, 92:856–859, 1881.

[8] G. Halphen. Sur une systéme d'équations difféetielles. *C. R. Acad. Sci Paris*, 92:1101–1103, 1881.

[9] Katsunori Iwasaki, Hironobu Kimura, Shun Shimomura, and Masaaki Yoshida. *From Gauss to Painlevé*. Aspects of Mathematics, E16. Friedr. Vieweg & Sohn, Braunschweig, 1991. A modern theory of special functions.

[10] Maxim Kontsevich. Homological algebra of mirror symmetry. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)*, pages 120–139, Basel, 1995. Birkhäuser.

[11] Serge Lang. *Introduction to modular forms*, volume 222 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, 1995. With appendixes by D. Zagier and Walter Feit, Corrected reprint of the 1976 original.

[12] J. Lehner Discontinuous groups and automorphic functions. Mathematical Surveys, No. VIII American Mathematical Society, Providence, R.I. 1964.

[13] H. Movasati. On Differential modular forms and some analytic relations between Eisenstein series, Ramanujan Journal, 17:53-76, 2008.

[14] H. Movasati, Quasi-modular forms attached to elliptic curves, I, Preprint 2010.

[15] F. Schilling, Beiträge zur geometrischen Theorie der Schwarz'schen $s$-Function. Math. Ann. 44 (1894), no. 2-3, 161–260.

[16] H. Shiga and J. Wolfart. Algebraic values of triangle schwarz functions. Arithmetic and geometry around hypergeometric functions, 287–312, Progr. Math., 260, Birkhuser, Basel, 2007.

[17] H. Shiga and T. Tsutsui and J. Wolfart. Triangle Fuchsian differential equations with apparent singularities. With an appendix by Paula B. Cohen. Osaka J. Math. 41 (2004), no. 3, 625658.

[18] M. Yoshida, From the power function to the hypergeometric function. Arithmetic and geometry around hypergeometric functions, 407–429, Progr. Math., 260, Birkhuser, Basel, 2007.

[19] Don Zagier. Elliptic modular forms and their applications. In *The 1-2-3 of modular forms*, Universitext, pages 1–103. Springer, Berlin, 2008.