Modular foliations and periods of hypersurfaces

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Chapter 0

Introduction

After studying algebraic numbers, one naturally starts to study transcendent numbers and among them the numbers obtained by integration. Of particular interest is the case in which the integrand is a differential form obtained by algebraic operations and the integration takes place over a topological cycle of an affine variety. The first non trivial class of such integrals are elliptic integrals $\int R(x, \sqrt{f(x)})$, where $f(x)$ is a polynomial of degree 3 or 4 and $R(x, y)$ is a rational function in x, y. Since the 19th century, many people have worked on the theory of elliptic integrals, including Gauss, Abel, Bernoulli, Ramanujan and many others, and still it is an active area due to its application on the arithmetic of elliptic curves (see for instance [73] for a historical account on this). Going to higher genus one has the theory of Jacobian and Abelian varieties and in higher dimension one has the Hodge theory. However, with the development of all these elegant areas it has become difficult to relate them to some simple classical integrals. "... students who have sat through courses on differential and algebraic geometry (read by respected mathematicians) turned out to be acquainted neither with the Riemann surface of an elliptic curve $y^2 = x^3 + ax + b$ nor, in fact, with the topological classification of surfaces (not even mentioning elliptic integrals of first kind and the group property of an elliptic curve, that is, the Euler-Abel addition theorem). They were only taught Hodge structures and Jacobi varieties!" [1].

The objective of the present text is to give a unified approach to Abelian integrals or periods in different areas of mathematics. We allow an integral to depend on many parameters and look for local analytic subvarieties in the parameter space where the integral is constant for any choice of the underlying topological cycle. It turns out that such varieties make part of the leaves of an algebraic foliation in the parameter space. We call them modular foliations. We develop machineries and tools, such as de Rham cohomology of affine varieties, Picard-Lefschetz theory, Hodge structures and Gauss-Manin connections for studying the dynamics and arithmetic of modular foliations. One must always bear in mind that modular foliations do not live in the classical parameter spaces for algebraic objects, for instance the parameter space of a versal deformation of a singularity (see [1]), or Teichmüller spaces. From algebraic geometric point of view, the moduli spaces or parameter spaces considered in this text are the moduli of varieties with some marked elements in their de-Rham cohomologies. To motivate the reader, we first collect some aspects of Abelian integrals in the literature.

[1] V.I. Arnold, On teaching mathematics, Palais de Découverte in Paris, 7 March 1997.

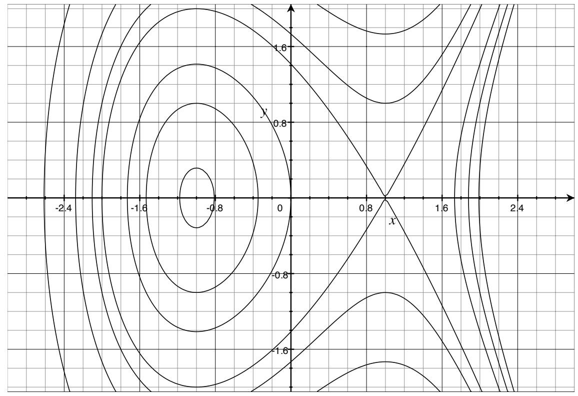

Figure 1: Elliptic curves: $y^2 - x^3 + 3x - t,\ t = -1.9, -1, 0, 2, 3, 5, 10$

0.1 Some aspects of Abelian integrals

Let us first clarify what we mean by an Abelian integral or a period. We will fairly use some elementary notations related to algebraic varieties over the field of complex numbers.

Let f be a polynomial in $(n+1)$-variables $x = (x_1, x_2, \ldots, x_{n+1})$, $L_0 := \{f = 0\} \subset \mathbb{C}^{n+1}$ be the corresponding affine variety, ω be a polynomial n-form in $\mathbb{C}^{n+1}$ and $\delta_0 \cong \mathbb{S}^n$ be an n-dimensional sphere C^∞-embedded in L_0 (we call it a cycle). For simplicity we assume that L_0 is smooth. The protagonist of the present text is the number obtained by the integration $\int_\delta \omega$, which we call it an Abelian integral. In fact one can take δ any element in the n-th homology of L_0. Such a number is also called a period of ω (in the literature the name Abelian integral is mainly used for the case $n = 1$). If $f = f_t$ depends on a parameter $t \in T$ with $0 \in T$ then L_0 is a member of the family $L_t := \{f_t = 0\}, t \in T$ and we can talk about the continuous family of cycles $\delta_t \subset L_t$ obtained by the monodromy of δ_0 in the nearby fibers. Therefore, the Abelian integral $\int_{\delta_t} \omega$ is a holomorphic function in a neighborhood of $0 \in T$. To carry an example in mind, take the polynomial $f = y^2 - x^3 + 3x$ in two variables x and y and $f_t := f - t, t \in \mathbb{C}$. Only for $t = -2, 2$ the affine variety L_t is singular and for other values of t, L_t is topologically a torus minus one point (point at infinity). For t a real number between 2 and -2 the level surface of f intersects the real plane $\mathbb{R}^2$ in two connected pieces which one of them is an oval and we can take it as δ_t (with an arbitrary orientation). In this example as t moves from -2 to 2, δ_t is born from the critical point $(-1, 0)$ of f and end up in the α-shaped piece of the fiber $f^{-1}(2) \cap \mathbb{R}^2$ (See Figure 1).

Planar differential equations and holomorphic foliations: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a polynomial mapping and $\delta_t \cong \mathbb{S}^1,\ t \in (\mathbb{R}, 0)$ be a continuous family of ovals in the fibers of f. The level surfaces of f are the images of the solutions of the ordinary differential equation

$$(1) \qquad \mathcal{F}_0 : \begin{cases} \dot{x} = f_y \\ \dot{y} = -f_x \end{cases} .$$

We make a perturbation of $\mathcal{F}_0$

$$(2) \qquad \mathcal{F}_\epsilon : \begin{cases} \dot{x} = f_y + \epsilon P(x, y) \\ \dot{y} = -f_x + \epsilon Q(x, y) \end{cases} , \epsilon \in (\mathbb{R}, 0),$$

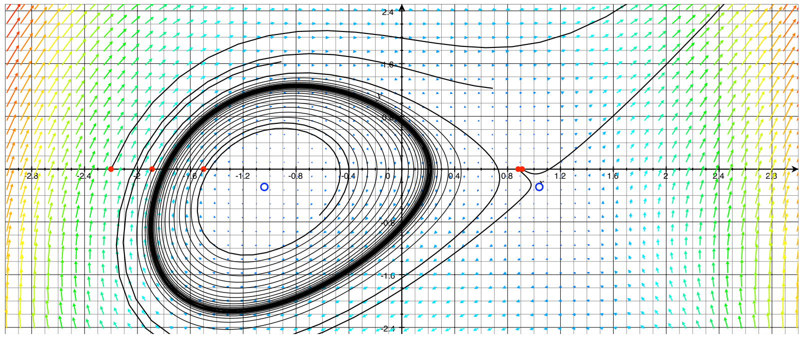

Figure 2: A limit cycle crossing $(x, y) \sim (-1.79, 0)$

where P and Q are two polynomials with real coefficients. Usually one expects that in the new ordinary differential equation the cycle δ_0 breaks and accumulates, in positive or negative time, in some part of the real plane or infinity. However, if the Abelian integral $\int_{\delta_t}(Pdy - Qdx)$ is zero for $t = 0$, but not identically zero, then for any small ϵ there will be a limit cycle of $\mathcal{F}_\epsilon$ near enough to δ_0 (see for instance [44, 64, 62]). In other words, δ_0 persists as a limit cycle in the perturbed differential equation. If the Abelian integral is identically zero (for instance if δ_t is homotopic to zero in the complex fiber of f) then the birth of limit cycles is controlled by iterated integrals (see for instance [27, 69]). In our main example take the ordinary differential equation

$$\mathcal{F}_\epsilon : \begin{cases} \dot{x} = 2y + \epsilon \frac{x^2}{2} \\ \dot{y} = 3x^2 - 3 + \epsilon s y \end{cases}, \epsilon \in (\mathbb{R}, 0). \tag{3}$$

If $\int_{\delta_0}(\frac{x^2}{2}dy - sydx) = 0$ or equivalently

$$s := \frac{-\int_{\Delta_0} xdx \wedge dy}{\int_{\Delta_0} dx \wedge dy} = \frac{5}{7}\frac{\Gamma(\frac{5}{12})\Gamma(\frac{13}{12})}{\Gamma(\frac{7}{12})\Gamma(\frac{11}{12})} \sim 0.9025,$$

where Δ_0 is the bounded open set in $\mathbb{R}^2$ with the boundary δ_0, then for ϵ near to 0, $\mathcal{F}_\epsilon$ has a limit cycle near δ_0. In fact for $\epsilon = 1$ and $s = 0.9$ such a limit cycle still exists and it is depicted in Figure (2). The origin of the above discussion comes form the second part of the Hilbert sixteen problem (shortly H16). A weaker version of H16, known as the infinitesimal Arnold-Hilbert problem asks for a reasonable bound for the number of zeros of real Abelian integrals when the degrees of f, P and Q are bounded. There are some partial solutions to this problem but it is still open in its generality (see [45, 26]). Even the zero dimensional version of this problem, in which Abelian integrals are algebraic functions, is not completely solved(see [28]).

De Rham cohomologies: A combination of Atiyah-Hodge theorem and Kodaira vanishing theorem implies that the n-th de Rham cohomology (see [71, 68]) of the affine variety L_0 is finite dimensional and it is given by polynomial differential n-forms in $\mathbb{C}^{n+1}$ modulo relative to L_0 exact n-forms. This implies that every Abelian integral $\int_{\delta_t}\omega$ can be written as a $\mathbb{C}(t)$-linear combination of $\int_{\delta_t}\omega_i,\ i = 1, 2, \ldots$, where ω_i's form a basis of the n-th de Rham cohomology of L_0. In our example, the arithmetic algebraic geometers usually uses the differential forms $\frac{dx}{y}$, $\frac{xdx}{y}$, which restricted to the regular fibers of f are holomorphic and form a basis of the corresponding de Rham cohomology. The relation of these differential forms and those in the previous paragraph is given by:

$$\int_{\delta_t}(\frac{x^2}{2}dy - sydx) = (-\frac{3}{5}st + \frac{6}{7})\int_{\delta_t}\frac{dx}{y} + (\frac{6}{5}s - \frac{3}{7}t)\int_{\delta_t}\frac{xdx}{y} \tag{4}$$

(see Chapter 3).

Picard-Fuchs equations and Gauss-Manin connections: The Abelian integral $\int_{\delta_t}\frac{dx}{y}$ (resp. $\int_{\delta_t}\frac{xdx}{y}$) satisfies the differential equation

$$\frac{5}{36}I + 2tI' + (t^2-4)I'' = 0 \qquad (\text{ resp. } \frac{-7}{36}I + 2tI' + (t^2-4)I'' = 0) \tag{5}$$

which is called a Picard-Fuchs equation. If we choose another cycle $\delta_t' \in H_1(L_t, \mathbb{Z})$ which together with δ_t form a basis of $H_1(L_t, \mathbb{Z})$ then the matrix $Y = \begin{pmatrix} \int_{\delta_t}\frac{dx}{y} & \int_{\delta_t'}\frac{dx}{y} \\ \int_{\delta_t}\frac{xdx}{y} & \int_{\delta_t'}\frac{xdx}{y} \end{pmatrix}$ form a fundamental system of the linear differential equation:

$$Y' = \frac{1}{t^2-4}\begin{pmatrix} \frac{-1}{6}t & \frac{1}{3} \\ \frac{-1}{3} & \frac{1}{6}t \end{pmatrix} Y \tag{6}$$

which we call it the Gauss-Manin connection of the family $L_t,\ t \in \mathbb{C}$. The main point behind the calculation of Picard-Fuchs equations and Gauss-Manin connections is the techniques of derivation of an integral with respect to a parameter and simplifying the result in a similar way as in (4). For more details see Chapter 3.

Special functions: The reader may transfer the singularities $-2, 2$ of (5) to 0 and 1 and obtain a recursive formula for the coefficients of the Taylor series around 0 of its solutions. Since the integrals $\int_{\delta_t}\frac{dx}{y}$ and $\int_{\delta_t}\frac{xdx}{y}$ are holomorphic around $t = -2$ (this follows from (4)), doing in this way we get:

$$\int_{\delta_t}\frac{dx}{y} = \frac{-2\pi}{\sqrt{3}}F(\frac{1}{6}, \frac{5}{6}, 1|\frac{t+2}{4}), \int_{\delta_t}\frac{xdx}{y} = \frac{2\pi}{\sqrt{3}}F(\frac{-1}{6}, \frac{7}{6}, 1|\frac{t+2}{4})$$

[2] where

$$F(a,b,c|z) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}z^n,\ c \notin \{0,-1,-2,-3,\ldots\},$$

is the Gauss hypergeometric function and $(a)_n := a(a+1)(a+2)\cdots(a+n-1)$. An elegant way to prove the statement

$$F(\frac{5}{6}, \frac{1}{6}, 1| \pm \frac{\sqrt{\frac{54001}{15}}}{120} + \frac{1}{2})\frac{\pi^2}{\Gamma(\frac{1}{3})^3} \in \bar{\mathbb{Q}}, \tag{7}$$

is as follows: The elliptic curve L_t has the j invariant $\frac{4}{t^2-4}$. For the values of t such that $j = 2^4 \cdot 3^3 \cdot 5^3$, L_t admits a complex multiplication by the field $\mathbb{Q}(\sqrt{-3})$ (see [81], p. 483). Now one uses the Chowla-Selberg Theorem (see [38]) on the periods of differential forms of the first kind on elliptic curves with complex multiplication. The reader is referred to [4, 80, 79] and the references there for similar topics. In the next paragraph we give another interpretation of (7) in terms of a Hodge cycle of a four dimensional cubic hypersurface.

[2] In order to calculate $a_i = \int_{\delta_{-2}}\frac{x^i dx}{y}$, $i = 0, 1$, we use (6) and obtain $a_0 + a_1 = 0$. We also use $\int_2^{\infty}\frac{dx}{(x+1)\sqrt{x-2}} = -\frac{\pi}{\sqrt{3}}$.

Hodge cycles: Let us consider the affine hypersurface

$$U_c: \ x_1^3 + x_2^3 + \cdots + x_5^3 - x_1 - x_2 - c = 0, \ c \in \mathbb{C} - \{\pm\frac{4}{3\sqrt{3}}, 0\}$$

in $\mathbb{C}^5$ and its compactification M_c in the projective space of dimension 5. The Hodge decomposition of the 4-th primitive cohomology of M_c has the Hodge numbers $0, 1, 20, 1, 0$ and a generator of $H^{3,1}$ piece restricted to U_c is represented by the differential 4-form

$$\alpha := \big((972c^2 - 192)x_1x_2 + (-405c^3 - 48c)x_2 + (-405c^3 - 48c)x_1 + (243c^4 - 36c^2 + 64)\big)$$

$$\cdot \sum_{i=1}^{5} (-1)^{i-1} x_i dx_1 \wedge \cdots dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_5$$

(see [65] and Chapter 5). Therefore, a cycle $\delta \in H_4(M_c, \mathbb{Z})$ with support in U_c is a Hodge cycle if and only if $\int_\delta \alpha = 0$. It turns out that the $\mathbb{Q}$-vector space of the periods of α is spanned over $\mathbb{Q}$ by $\Gamma(\frac{1}{3})^3\mathbb{Q}(\zeta_3)$ times the periods of $\frac{dx}{y}$ over the elliptic curve $L_t : y^2 - x^3 + 3x - t, \ t = 2 - \frac{27}{4}c^2$. For $j = \frac{1}{t^2-4} = 2^4 \cdot 3^4 \cdot 5^3$, L_t has a complex multiplication by $\mathbb{Q}(\zeta_3)$ and this gives us a Hodge cycle δ in $H_4(M_c, \mathbb{Q})$. One of the consequences of the Hodge conjecture is that for $c \in \bar{\mathbb{Q}}$ the integration over δ of any 4-differential form in $\mathbb{C}^5$, which is defined over $\bar{\mathbb{Q}}$ and is without residue at infinity, belongs to $\pi^2\bar{\mathbb{Q}}$. Since the Hodge conjecture is proved for cubic hypersurfaces of dimension 4 (see for instance [86]), we get another interpretation of (7). For more details, see [65] and Chapter 5.

Contraction of curves Let S be a curve of genus two and ω be a differential form of the first kind on S, both defined over a number field. A direct consequence of the Abelian Subvariety Theorem (see [7] and the references there) says that if $\int_\delta \omega = 0$ for some homologically non trivial topological cycle δ, then there exists a morphism $f : S \to E$ from S into an elliptic curve E, where f and E are defined over a finite extension of the original number field, such that δ is mapped to zero under f and ω is the pull-back of some differential form of the first kind on E. This statement is trivially false when we do not assume that our objects are defined over a number field. For an arbitrary genus, one can say that the Jacobian of S is not simple but to obtain the contraction of S we need more hypothesis.

Modular foliations Now, I hope that the reader is enough motivated to study Abelian integrals. A classical approach to study a mathematical object is to put it inside a good family and then study it as a member of the family. This is also the case of Abelian integrals. If the parameter space T is 'good' enough then the locus of parameters t, for which $\int_{\delta_t} \omega, \forall \delta_t \in H_n(L_t, \mathbb{Z})$ is constant, is a local holomorphic foliation and one can show that it is a part of a global algebraic foliation in T. For instance for the family of elliptic curves

$$y^2 - 4(x - t_1)^3 + t_2(x - t_1) + t_3, \ t \in T := \mathbb{C}^3 \backslash \{27t_3^2 - t_2^3 = 0\} \tag{8}$$

and the differential form $\frac{xdx}{y}$ the corresponding foliation is given by the ordinary differential equation

$$\begin{cases} \dot{t_1} = t_1^2 - \frac{1}{12}t_2 \\ \dot{t_2} = 4t_1t_2 - 6t_3 \\ \dot{t_3} = 6t_1t_3 - \frac{1}{3}t_2^2 \end{cases} . \tag{9}$$

In the literature it is known as the Ramanujan relations, because Ramanujan observed that the Eisenstein series form a solution of (9) (for more details on this example see §5.3 and [66, 67]).

0.2 Synopsis of the contents of this text

Let us now explain the content of each chapter.

Chapter 1 is dedicated to zero dimensional Abelian integrals. Almost all the machineries which we need in general case can be introduced for zero dimensional Abelian integrals and this is the main reason why this chapter is written. The corresponding de Rham cohomologies, Picard-Lefschetz theory, Gauss-Manin connection and monodromy groups are explained in this chapter. Specially, this chapter is recommended to the reader who is not familiar with the mentioned machineries. There are some theorems in this chapter whose generalizations for an arbitrary dimension are not known. Relating the zeros of Abelian integrals to contraction of varieties is one of them.

In Chapter 2 we introduce a general notion of a modular foliation associated to a connection. First we collect some machineries for dealing with connections. Then, we give a list of examples of modular foliations which various authors have worked out. The reader can skip this chapter if he is only interested to the modular foliations comming from geometry, i.e. associated to the Gauss-Manin connection of a fibration.

In Chapter 3 we introduce the main protagonist of the present text, namely a tame polynomial in $n+1$ variables which depends on many parameters and the corresponding affine variety. We find a canonical basis of the de Rham cohomology of the affine variety, explain the algorithms for calculating the discriminant and the Gauss-Manin connection.

The topological study of the affine variety associated to a tame polynomial is done in Chapter 4. A good source for the materials of this chapter is the book [1]. Since this book has mainly discussed the local theory of tame polynomials, we have collected and reproved many theorems on the topology of tame polynomials. In particular, our approach to calculations of the intersection matrices of tame polynomials, joint cycles and reduction of integrals has a slightly new feature.

In Chapter 5 we consider modular foliations associated to tame polynomials. We recall the fact that the period matrix is completely determined through its value in one point and the Gauss-Manin connection matrix. We recall Kodaira-Spencer theory of deformation of complex manifolds and its consequence that the deformation of a hypersurface remains again a hypersurface, except for some exceptional cases. We calculate some modular foliations for tame polynomials in two variables. The notion of a mixed Hodge structure associated to a tame polynomial, Hodge cycles and the locus of Hodge cycles is covered in this chapter. It is shown that the locus of Hodge cycles is invariant under certain modular foliations.

In Chapter 6 we introduce the abstract notion of Abelian integrals, namely a polarized Hodge structure. We construct the moduli of polarized Hodge structures which is a complex manifold living on the Griffiths domain D. The modular foliations appear in this new space and not in D.

In Appendix A for the convenience of the reader we review the classical Hodge theory of affine varieties. Logarithmic differential forms, pole filtrations and the Griffiths theorem on the Hodge filtration of the complement of a smooth hypersurface are explained in this Appendix.

Now, let us say some words on the dependence of the chapters of this text to each other. Chapters 1, 2, 3, 4, and 6 are independent from each other and can be read separately. Chapter 5 is a continuation of Chapter 3. One also needs the material of Chapter 4 for Chapter 5. The reader who has difficulties in Chapters 3, 5 is recommended to read first Chapter 1. In Chapters 5 and 6 we have used Appendix A.

The aim of this text is to collect enough machinery for studying modular foliations comming from Gauss-Manin connections. We have always in mind the Weierstrass family of elliptic curves (8) which depends on three parameters t_1, t_2, t_3 insteed of the classical two parameters t_2, t_3. The main modular foliation that we get is given by Ramanujan relations (9). There are many new results in this text. The calculation of de Rham cohomologies, Gauss-Manin connections, Picard-Fuchs equations and constructing a complex manifold over Griffiths domain are among these new results. The arithmetic properties of modular foliations is a vast and difficult arena yet to be discovered. For instance, in [67] it is proved that each transcendent leaf of (9) crosses a point with algebraic coordinates at most once. Another important aspect of modular foliations is algebraic varieties invariant by them. In all the cases which I know, such varieties have geometric interpretations for the fibration given by the tame polynomial.

The algorithms of the present text are implemented in the library `foliation.lib` of SINGULAR(see [31]) which can be downloaded from the author's web page. However, I have tried to write the text in such a way that the reader can do the calculations by any software in Commutative Algebra. A very important observation is that the calculations in the coefficient space of tame polynomials is not a matter of working with polynomials with small size which fit into a mathematical text. Even for a simple example like a hyperelliptic polynomial of degree 5 each entry of the Gauss-Manin connection matrix occupy half a page. For the mentioned example the modular foliations have simple expressions (see §5.3). Therefore, some of the proofs in this text highly uses the computer and it is almost impossible to follow the proof by hand calculations.

0.3 Terminology

Throughout the present text, we consider a commutative ring R with multiplicative identity element 1. We assume that R is without zero divisors, i.e. if for some $a, b \in \mathsf{R}$, $ab = 0$ then $a = 0$ or $b = 0$. We also assume that R is Noetherian, i.e. it does not contain an infinite ascending chain of ideals (equivalently every ideal of R is finitely generated/every set of ideals contains a maximal element).

A multiplicative system in a ring R is a subset S of R containing 1 and closed under multiplication. The localization M_S of an R-module M is defined to be the R-module formed by the quotients $\frac{a}{s}$, $a \in M$, $s \in S$. If $S = \{1, a, a^2, \ldots\}$ for some $a \in \mathsf{R}$, $a \neq 0$ then the corresponding localization is denoted by M_a. Note that by this notation $\mathbb{Z}_a$, $a \in \mathbb{Z}, a \neq 0$ is no more the set of integers modulo $a \in \mathbb{N}$. By $\check{M}$ we mean the dual of the R-module

$$\check{M} := \{a : M \to \mathsf{R},\ a \text{ is } \mathsf{R} \text{ linear }\}.$$

Usually we denote a basis/set of generators of M as a column matrix with entries in M.

We denote by k the field obtained by the localization of R over $\mathsf{R} \setminus \{0\}$ and by $\bar{\mathsf{k}}$ the algebraic closure of k. In many arguments we need that the characteristic of k to be zero. If this is the case then we mention it explicitly.

We denote by $\mathsf{R}[x] := \mathsf{R}[x_1, x_2, \ldots, x_{n+1}]$, the weighted polynomial ring in variables $x_1, x_2, \ldots, x_{n+1}$ and coefficients in R with $\deg(x_i) = \alpha_i \in \mathbb{N}$, $i = 1, 2, \ldots, n+1$. The degree of $a \in \mathsf{R}[x]$ with respect to a subset of variables y of x is denoted by $\deg_y(a)$. We also use $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{n+1})$. The protagonist of this text is a tame polynomial $f \in \mathsf{R}[x]$ defined in §3. We will frequently assume that the degree of f, namely $d := \deg(f)$, is invertible in R. The letter d is also used to denote the differential operator. I hope that this will not make any confusion. The last homogeneous piece of f is denoted by g. We use sometimes $f = f_t$ in order to stress the fact that f depends on the parameter t. We frequently use a localization R of $\mathbb{Q}[t] = \mathbb{Q}[t_0, t_1, \ldots, t_s]$ instead of the general ring. For c a fixed value for the parameter t, we denote by f_c the specialization of f at c, i.e. we substitute c for t in f. Using topological arguments, we may prove an equality for R and then we replace t by elements in an arbitrary ring and obtain the same equality for the arbitrary ring.

All the objects, like variety, connection and so on, in the present text are explicitly defined and so when we talk about an object defined over R we mean that the ingredient coefficients of the object are in R. In this way we have avoided to use the scheme machinery for our purposes. For instance many objects in this text carry the symbols $\mathbb{U}_0, \mathbb{U}_1$. From scheme theory point of view $\mathbb{U}_0 := \mathrm{Spec}(\mathsf{R})$, $\mathbb{U}_1 := \mathrm{Spec}(\mathsf{R}[x])$ and $\mathbb{U}_1/\mathbb{U}_0$ means the scheme $\mathbb{U}_1$ over $\mathbb{U}_0$. If someone is interested to arithmetic properties of modular foliations then the usage of schemes is indispensable. We also use $\{f = 0\}$ or $Z(f)$ or L_f to denote $\mathrm{Spec}(\frac{\mathsf{R}[x]}{f\mathsf{R}[x]})$ for $f \in \mathsf{R}[x]$. The discriminant of f is denoted by Δ_f, or simply Δ, and $T = \mathbb{U}_0 \backslash \{\Delta = 0\} := \mathrm{Spec}(\mathsf{R}_\Delta)$. The notion L_a is also used to denote the leaf of a foliation passing through a. I hope that this will not make any confusion.

We use $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\zeta_d = e^{\frac{2\pi i}{d}}$ the d-th primitive root of 1, $\mathrm{Mat}^{n\times m}(\mathsf{R})$ the set of $n \times m$ matrices with entries in the ring R, $I_{n\times n}$ the identity $n \times n$ matrix and $\bar{\mathbb{Q}}$ the field of algebraic numbers. For a holomorphic map h of complex manifolds we denote by h_* (resp. h^*) the map induced in homology (resp. cohomology). If h is a biholomorphism then we use also h_* to denote $(h^{-1})^*$.

Chapter 1

Zero dimensional Abelian integrals

1.1 Introduction

To understand better the problems related to the zeros of Abelian integrals which arise in Algebraic Geometry and Differential Equations, one may try to solve the similar problems in the case of zero dimensional Abelian integrals. These integrals are in fact algebraic functions and the word integral is used just because of their similarities with higher dimensional integrals. Surprisingly, all the topics which we are going to discuss in an arbitrary dimension fit well into the dimension zero. Since in this case we do not need the topology of varieties, this chapter can be understood without any advanced information in (co)homology theory of varieties. Our objective in this chapter is to analyze some problems for zero dimensional integrals whose counterparts in higher dimensional cases are difficult to treat. Our observation is that zero dimensional integrals can be studied in a more arithmetic context and this helps us to understand their behavior better. The idea of this chapter comes from my joint work [28] with Lubomir Gavrilov. While the mentioned paper mainly discusses the infinitesimal Hilbert problem in zero dimension, this chapter emphasizes the arithmetic properties of such integrals. A basic knowledge of the classical Galois theory will be useful for understanding the contents of this chapter. In this direction, we have used the book [58].

1.2 Zero dimensional Abelian integrals

For a finite discrete set M we denote by $\mathbb{Z}[M]$ the free $\mathbb{Z}$-module generated by the elements of M. The degree of $\delta = \sum_i r_i x_i \in \mathbb{Z}[M], r_i \in \mathbb{Z},\ x_i \in M$ is

$$\deg(\delta) := \sum_i r_i.$$

We use the reduced 0-th homology and cohomology for the set M:

$$H_0(M,\mathbb{Z}) = \{\delta \in \mathbb{Z}[M] \mid \deg(\delta) = 0\},\ H^0(M,\mathbb{Z}) := \check{H}_0(M,\mathbb{Z}),$$

where $\check{}$ means dual. In $H_0(M,\mathbb{Z})$ we have the intersection form induced by

$$\langle x, y\rangle = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases} \quad x, y \in M.$$

By definition $\langle \cdot, \cdot \rangle$ is a symmetric form in $H_0(M, \mathbb{Z})$, i.e. for all $\delta_1, \delta_2 \in H_0(M, \mathbb{Z})$ we have $\langle \delta_1, \delta_2 \rangle = \langle \delta_2, \delta_1 \rangle$.

Let R, k, $\bar{\mathsf{k}}$ be as in the Introduction and

$$f = t_d x^d + t_{d-1} x^{d-1} + \cdots + t_1 x + t_0 \tag{1.1}$$

be a polynomial of degree d in variable x and with coefficient $t := (t_0, \ldots, t_{d-1}, t_d)$ in R and

$$\{f = 0\} = L_f = L_t := \{x \in \bar{\mathsf{k}} \mid f(x) = 0\} = \{x_1, x_2, \ldots, x_d\}, \ \mu := d - 1.$$

We assume that t_d is invertible in R. An element of $H_0(L_t, \mathbb{Z})$ is called a cycle. A canonical basis of $H_0(L_t, \mathbb{Z})$ is given by

$$\delta_i = x_{i+1} - x_i, \ i = 1, 2, \ldots, d - 1.$$

The intersection matrix of f with respect to the above basis is:

$$\Psi_0 := \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}. \tag{1.2}$$

For a cycle δ and $\omega \in \mathsf{R}[x]$ we define

$$\int_\delta \omega := \sum_i r_i \omega(x_i), \quad \text{where } \delta = \sum_i r_i x_i, \ r_i \in \mathbb{Z}, \ x_i \in L_t,$$

and call them (zero dimensional Abelian) integrals/periods. Following the terminology in higher dimension, we call ω a 0-form and denote by $\Omega^0_{\mathbb{U}_1/\mathbb{U}_0} := \mathsf{R}[x]$ the R-module of 0-forms. Particularly we are interested on simple cycles $\delta = x_1 - x_2$, where x_1 and x_2 are two simple roots of $f(x) = 0$. We look at

$$\mathsf{P}(\{f = 0\}, \omega) := \{\int_\delta \omega \mid \delta \in H_0(\{f = 0\}, \mathbb{Z})\}$$

as a $\mathbb{Z}$-module and call it the period module.

Remark 1.1. When there is a danger of confusion between the complex number $x_1 - x_2$ and the cycle $\delta = x_1 - x_2$, we will write $\delta = [x_1] - [x_2]$. The first one can be obtained by integration of the 0-form x on δ.

Remark 1.2. We frequently use $\mathbb{Z}_a[t, \frac{1}{t_d}] = \mathbb{Z}_a[t_0, t_1, \ldots, t_{d-1}, t_d, \frac{1}{t_d}]$, $a \in \mathbb{N}$, instead of the general ring R and we consider t_i's in (1.1) as parameters. Using topological arguments, we may prove an statement for $\mathbb{Z}_a[t, \frac{1}{t_d}]$ and then we replace t by elements in an arbitrary ring R and obtain the same statement for arbitrary R (a must be invertible in R). It is useful in this case to consider $\mathbb{Z}_a[t, \frac{1}{t_d}]$ as a weighted ring with $\deg(t_i) = d - i$, $i = 0, 1 \ldots, d - 1$. We need the localization $\mathbb{Z}_a$ of $\mathbb{Z}$ over a because for some arguments we need to divide on a. In many cases we put also $t_d = 1$.

Remark 1.3. The notions irreducibility, irreducible decomposition, division and so on will be used in the ring $\mathsf{k}[x]$. For instance when we write $g \mid f$, $f, g \in \mathsf{R}[x]$, we mean that there exists $q \in \mathsf{k}[x]$ such that $f = gq$.

1.3 Discriminant of a polynomial

For a monic polynomial $f = x^d + t_{d-1}x^{d-1} + \cdots + t_1 x + t_0 \in \mathsf{R}[x]$ we define the discriminant of f

$$\Delta = \Delta_f := \prod_{1\leq i\neq j\leq d} (x_i - x_j) = \prod_{i=1}^{d} f'(x_i) \in \mathsf{R},$$

where $f' = \frac{\partial f}{\partial x}$ is the derivative of f. The discriminat of $af, a \in \mathsf{R},\ a \neq 0$ is defined to be the discriminant of f. Recall Remark (1.2). For $\mathsf{R} = \mathbb{Z}[t]$ the discriminant Δ_f is a homogeneous polynomial of degree $d(d-1)$ with $\mathbb{Z}$ coefficients in the graded ring $\mathbb{Z}[t],\ \deg(t_i) = d - i$.

Proposition 1.1. *If $\mathsf{R} = \mathbb{Z}[t]$ and $f = x^d + t_{d-1}x^{d-1} + \cdots + t_1 x + t_0$ then Δ_f is an irredecible polynomial in $\mathbb{C}[t]$.*

Proof. Consider the map

$$\alpha : \mathbb{C}^d \to \mathbb{C}^d$$

$$\alpha(x_1, x_2, \ldots, x_d) = (t_0, \ldots, t_{d-2}, t_{d-1}) = ((-1)^d x_1 x_2 \cdots x_d, \cdots, \sum_{i\neq j} x_i x_j, -\sum x_i).$$

α maps $\{x_1 = x_2\}$ onto $\{\Delta = 0\}$. Since the first variety is irreducible, the second one is also irreducible. □

Example 1.1. For $f = x^d - t$ we have

$$\Delta_f = t^{d-1} \prod_{0\leq i\neq j\leq d-1} (\zeta_d^i - \zeta_d^j) = d^d(-t)^{d-1}.$$

The Milnor module associated to f is the quotient

$$\mathsf{V}_f := \frac{\mathsf{R}[x]}{f' \cdot \mathsf{R}[x]}.$$

It is also useful to define the quotient

$$\mathsf{W}_f := \frac{\mathsf{R}[x]}{f' \cdot \mathsf{R}[x] + f \cdot \mathsf{R}[x]}.$$

If the charachteristic of k does not divide d then d is invertible in k.

Proposition 1.2. *Assume that $d = \deg(f)$ is invertible in R. Then*

1. *V_f is a free R-module with the basis*

$$I := \{1, x, x^2, \ldots, x^{d-2}\}.$$

2. *Let A be the multiplication by f R-linear map in V_f. We have the following identity*

$$\Delta_f = d^d \cdot \det(A).$$

3. *Δ_f is a zero divisor of W_f, i.e.*

$$\Delta_f \cdot \mathsf{W}_f = 0.$$

Proof. The first part of the proposition is easy and is left to the reader. Recall Remark (1.2). For the second part, it is enough to prove it for the case

$$\mathsf{R} = \mathbb{Z}_d[t],\ f = x^d + t_{d-1}x^{d-1} + t_{d-2}x^{d-2} + \cdots + t_0.$$

We first prove that Δ_f and $\det(A)$ have the same zero set in $\bar{\mathbb{Q}}^d$. The polynomial f has multiple roots in $\bar{\mathbb{Q}}$ if and only if there are polynomials $p(x), q(x) \in \bar{\mathbb{Q}}[x],\ \deg(p) \leq d-2$ such that $f \cdot p = f' \cdot q$. This is equivalent to the fact that there is a $\bar{\mathbb{Q}}$ linear relation between $f, xf, \ldots, x^{d-2}f$ in V_f and so $\det(A) = 0$.

By Proposition 1.1, we have $\det(A) = a \cdot \Delta_f^n$ for some $a \in \bar{\mathbb{Q}}$ and $n \in \mathbb{N}$. It remains to prove that $a = d^{-d}$ and $n = 1$. Since $\Delta_f := \prod_{i=1}^d f'(x_i)$, Δ_f as apolynomial in t_0 is of degree $d-1$ and it is with the leading coefficient d^d. From another side, we look the matrix of A in the basis I and see that the term t_0 appears only in the diagonal entries of of A and it has the leading coefficient 1. Therefore $\det(A)$, as a polynomial in t_0, is of degree $d-1$ and it is with the leading coefficient 1.

The third part follows from

$$\det(A - f \cdot I_{\mu\times\mu}) \cdot \mathsf{V}_f = 0.$$

□

Note that if $f = t_d x^d + t_{d-1}x^{d-1} + \cdots + t_0$ is not monic then the the corresponding multiplication by f linear map has determinant $(dt_d)^d \Delta_{\frac{f}{t_d}}$. Bellow there is a table of discriminants for $d \leq 4$ and $\mathsf{R} = \mathbb{Z}[t]$.

$$f = x^d + t_{d-1}x^{d-1} + t_{d-2}x^{d-2} + \cdots + t_0$$

d	Δ
2	$4t_0 - t_1^2$
3	$27t_0^2 - 18t_0t_1t_2 + 4t_0t_2^3 + 4t_1^3 - t_1^2t_2^2$
4	$256t_0^3 - 192t_0^2t_1t_3 - 128t_0^2t_2^2 + 144t_0^2t_2t_3^2 - 27t_0^2t_3^4 + 144t_0t_1^2t_2 - 6t_0t_1^2t_3^2 - 80t_0t_1t_2^2t_3 + 18t_0t_1t_2t_3^3 + 16t_0t_2^4 - 4t_0t_2^3t_3^2 - 27t_1^4 + 18t_1^3t_2t_3 - 4t_1^3t_3^3 - 4t_1^2t_2^3 + t_1^2t_2^2t_3^2$

The above table is obtained by the command `discriminant` from `foliation.lib`. Note that this proceedure calculates $\det(A)$ and so in order to obtain the above table, we have to multiply its output with d^d .

Remark 1.4. Throughout this chapter we assume that d is invertible in R. Therefore, we will freely use Proposition 1.2.

1.4 Gelfand-Leray form

We denote by $\Omega^1_{\mathbb{U}_1/\mathbb{U}_0}$ the space of 1-forms $\omega := pdx,\ p \in \mathsf{R}[x]$. According to Proposition 1.2 there are $q_1, q_2 \in \mathsf{R}[x]$ such that

$$\Delta \cdot dx = df \cdot q_1 + f \cdot q_2 dx.$$

The Gelfand-Leray form of $\omega = pdx$ is a 0-form given by

$$\frac{\omega}{df} := \frac{pq_1}{\Delta} \in \mathsf{R}[x]_\Delta.$$

By integration of a 1-form ω on a cycle $\delta \in H_0(\{f=0\},\mathbb{Z})$ we mean the integration of the Gelfand-Leray form $\frac{\omega}{df}$ on δ, i.e.

$$\int_\delta \omega := \int_\delta \frac{\omega}{df}.$$

1.5 De Rham cohomology/Brieskorn modules

The global Brieskorn modules associated to the polynomial $f \in \mathsf{R}[x]$ are the quotients

$$\mathsf{H}' = \mathsf{H}'_f := \frac{\mathsf{R}[x]}{f \cdot \mathsf{R}[x] + \mathsf{R}}$$

and

$$\mathsf{H}'' = \frac{\Omega^1_{\mathbb{U}_1/\mathbb{U}_0}}{f \cdot \Omega^1_{\mathbb{U}_1/\mathbb{U}_0} + \mathsf{R} \cdot df}.$$

They play the role of the de Rham cohomology of the zero dimensional variety $\{f=0\}$ [1]. More precisely, the map $\mathsf{H}' \to \mathsf{H}''$, $\omega \mapsto \omega df$ is an inclusion which gives us the isomorphism $\mathsf{H}' \otimes_{\mathsf{R}} \mathsf{k} \to \mathsf{H}'' \otimes_{\mathsf{R}} \mathsf{k}$ of k-vector spaces. Its inverse is given by the Gelfand-Leray map $\omega \mapsto \frac{\omega}{df}$. The mentioned k-vector space is the de Rham cohomology of $\{f=0\}$.

The sets H' and H'' are R-modules in a canonical way. By H we mean one of H' or H''. It turns out that the integrals $\int_\delta \omega$, $\delta \in H_0(L_f,\mathbb{Z})$, $\omega \in \mathsf{H}$ are well-defined.

Proposition 1.3. *The R-module H' (resp. H'') is a free R-module of rank $d-1$ generated by $x, x^2, \ldots, x^{d-1}$ (resp. $dx, xdx, \cdots, x^{d-2}dx$).*

Proof. The proof is easy and is left to the reader. □

The basis of H given in Proposition 1.3 is called the canonical basis of H.

1.6 The operation $\omega * f$

For polynomials $f, \omega \in \mathsf{R}[x]$ we define the following polynomial

$$\omega * f(x) := (x-\omega(x_1))(x-\omega(x_2))\cdots(x-\omega(x_d)) \in \mathsf{R}[x], \tag{1.3}$$

where $f(x) = (x-x_1)(x-x_2)\cdots(x-x_d)$. For $\omega, \omega_1, \omega_2, f, f_1, f_2 \in \mathsf{R}[x]$, $c \in \mathsf{R}$, $p, q \in \mathbb{N}$ we have the following trivial identities:

$$\omega_1 * (\omega_2 * f) = (\omega_1 \circ \omega_2) * f,\ \omega * (f_1 \cdot f_2) = (\omega * f_1)\cdot(\omega * f_2), \tag{1.4}$$

$$(\omega_1 \cdot f + \omega_2) * f = \omega_2 * f,\ c * f = (x-c)^{\deg(f)},\ x^p * (x^q - 1) = (x^{\frac{q}{(p,q)}} - 1)^{(p,q)}.$$

Proposition 1.4. *Suppose that $\Delta_f \neq 0$. Then there are $D_\omega \in \mathsf{R}$ and $E_\omega \in \mathsf{R}[x]$ such that*

$$(\omega * f) \circ \omega = E_\omega \cdot f \tag{1.5}$$

and

$$\Delta_f \cdot D_\omega^2 = \Delta_{\omega * f}.$$

[1] The classical defintion of Brieskorn modules is $H' := \frac{\mathsf{R}[x]}{\mathsf{R}[f]}$, $H'' = \frac{\Omega^1_{\mathbb{U}_1/\mathbb{U}_0}}{\mathsf{R}[f]\cdot df}$ for the case $\mathsf{R} = \mathbb{C}$. These are $\mathsf{R}[f]$-modules and are introduced for the study of the monodromy of the fibration $\{f - s = 0\}$.

Proof. For the first part note that $(\omega * f) \circ \omega(x_i) = 0,\ i = 1, 2, \cdots, d$ and the multiplicity of $(\omega * f) \circ \omega$ at x_i is at least the multiplicity of f at x_i. In the second part D_ω is explicitly given by

$$D_\omega := \prod_{1 \le i < j \le d} \frac{\int_{\delta_{ij}} \omega}{\int_{\delta_{ij}} x} \in \mathsf{R}, \tag{1.6}$$

where $\delta_{ij} = x_i - x_j \in H_0(L_t, \mathbb{Z})$. □

Proposition 1.5. *Let $f \in \mathsf{R}[x]$ be an irreducible polynomial, d be invertible in R and $\omega \in \mathsf{R}[x]$. Then $\omega * f = g^k$ for some $k \in \mathbb{N}$ and irreducible polynomial $g \in \mathsf{R}[x]$. Moreover, if for some simple cycle $\delta \in H_0(\{f = 0\}, \mathbb{Z})$ we have $\int_\delta \omega = 0$ then $k \ge 2$.*

Proof. We define the equivalence relation $\sim$ on L_f:

$$x_i \sim x_j \Leftrightarrow \omega(x_i) = \omega(x_j).$$

Let G_f be the Galois group of the splitting field of f. For $\sigma \in G_f$ we have

$$x_i \sim x_j \Rightarrow \sigma(x_i) \sim \sigma(x_j) \tag{1.7}$$

Since f is irreducible over k, the action of G_f on I is transitive (see for instance [58] Prop. 4.4). This and (1.7) imply that G_f acts on $I/\sim$ and each equivalence class of $I/\sim$ has the same number of elements as others. Let $I/\sim = \{v_1, v_2, \ldots, v_e\},\ e \mid d$ and $c_i := \omega(v_i)$. Define

$$g(x) := (x - c_1)(x - c_2) \cdots (x - c_e).$$

We have

$$g^k = f * \omega \in \mathsf{R}[x],$$

where $k = \frac{d}{e}$. Let $g = x^e + a_1 x^{e-1} + \cdots + a_e$. We have $ka_1 \in \mathsf{R}$ and we calculate the coefficients of g in terms of the coefficients of the right hand side of the above equality. A simple induction implies that all the coefficients of g lies in R. Note that here we use the fact that d, and hence k, is invertible in R. Since G_f acts transitively on the roots of g, we conclude that g is irreducible over k. □

Following the notations of Proposition 1.5, we have the morphism

$$\{f = 0\} \overset{\alpha_\omega}{\to} \{g = 0\},\ \alpha_\omega(x) = \omega(x)$$

defined over R. Since $f \mid g \circ \omega$ over R, it defines a well-defined map

$$\alpha_\omega^* : \mathsf{H}'_g \to \mathsf{H}'_f,\ \alpha_\omega(\omega') = \omega' \circ \omega.$$

Suppose that f is irreducible and there is no simple cycle $\delta \in H_0(\{f = 0\}, \mathbb{Z})$ such that $\int_\delta \omega = 0$. According to Proposition 1.5 the polynomial $g := \omega * f$ is also irreducible and the morphism α_ω is topologically an isomorphism.

Proposition 1.6. *Assume that D_ω in (1.6) is invertible in R. Under the above hypothesis, there is an $\eta \in \mathsf{R}[x]$, such that*

$$\eta \circ \omega = f \cdot q + x,\ \textit{for some } q \in \mathsf{R}[x]$$

and hence the inverse of α_ω is given by α_η.

Proof. Let $x_i,\ i=1,2,\ldots,d$ be the roots of f and $\omega(x_i)=c_i$. We are looking for a polynomial $\eta=r_0+r_1x+\cdots+r_{d-1}x^{d-1},\ r_i\in \mathsf{R}$ such that $\eta(c_i)=x_i$. This gives the equation $A(r_0,r_1,\cdots,r_{d-1})^{\mathsf{t}}=(x_1,x_2,\cdots,x_d)^{\mathsf{t}}$, where A is the Vandermonde matrix formed by c_i's. Since $\det(A)^2=\Delta_f D_\omega^2$ and $A^{-1}(x_1,x_2,\cdots,x_d)^{\mathsf{t}}$ is symmetric in $x_1,x_2,\cdots,x_d$, we conclude that $r_i\in\mathsf{R}$. Now, the facts that $\eta\circ\omega(x_i)=x_i,\ i=1,2,\ldots,d$ and f is irreducible finishes the proof. $\square$

Note that the topologically identity map $\{f^n=0\}\to\{f=0\},\ n\geq 2,\ x\mapsto x$ does not induce an isomorphim between the corresponding Brieskorn modules.

1.7 Zero dimensional Fermat variety

Let
$$f:=x^d-1\in\mathbb{Z}[x].$$
We call $\{f=0\}$ the Fermat variety of dimension zero. Let also
$$f:=x^d-1=\prod_{i|d}p_i(x), \tag{1.8}$$
be the decomposition of x^d-1 into irreducible component over $\mathbb{Q}$. We have
$$p_i(x)=\prod_{\gcd(a,i)=1,\ 1\leq a\leq i}(x-\zeta_d^a)\in\mathbb{Z}[x],$$
$p_1(x)=x-1$. The polynomial p_i is called the i-th cyclotomic polynomial. Using Proposition 1.5 one concludes that for all $i\mid d$ the morphisms
$$\{p_d(x)=0\}\to\{p_{\frac{d}{i}}(x)=0\},\ x\mapsto x^i$$
are well-defined over $\mathbb{Z}$.

Let $\phi(d):=p_d(1)$, the sum of the coefficients of p_d. Derivating (1.8) and putting $x=1$, we concludes that
$$\prod_{i|d,i\neq 1}\phi(i)=d.$$
This implies that
$$\phi(d)=\begin{cases} p, & \text{if for some prime } p, d=p^\alpha\\ 1, & \text{otherwise.}\end{cases}.$$
The following function
$$\sigma_d:\mathbb{Z}[\zeta_d]\to\mathbb{Z}/\phi(d)\mathbb{Z},\ \sum_{i=0}^{d-1}a_i\zeta_d^i\mapsto\sum_{i=0}^{d-1}a_i$$
is well defined, where $\mathbb{Z}[\zeta_d]$ is the ring of integers of $\mathbb{Q}(\zeta_d)$ (the sum of coefficients of any $p_d(x)q(x),\ q\in\mathbb{Z}[x]$ is congruent to 0 modulo $\phi(d)$). We conclude that:
$$\mathsf{P}(\{x^d-1=0\},x)=\ker(\sigma_d).$$
Note that if $a=\sum_i a_i\zeta_d^i\in\mathbb{Z}[\zeta_d]$ with $\sum_i a_i=\phi(d)k,\ k\in\mathbb{N}_0$ then $a=\sum_i a_i\zeta_d^i-p_{d,1}(\zeta_d)k=\sum_i b_i\zeta_d^i$ with $\sum_i b_i=0$.

For a $n \in \mathbb{N}$ we want to determine $\mathsf{P}(\{f=0\},\omega)$, where $\omega = x^n$ or $= x^{n-1}dx$. We note that if $d' := \frac{d}{\gcd(n,d)}$ and $n' = \frac{n}{\gcd(n,d)}$ then the morphisim

$$\alpha : \{x^d - 1 = 0\} \to \{x^{d'} - 1 = 0\},\ \alpha(x) = x^{(n,d)}$$

is defined over $\mathbb{Z}$ and has the property $\alpha^*(x^{n'}) = x^n$. Therefore,

$$\mathsf{P}(\{x^d - 1 = 0\}, x^n) = \mathsf{P}(\{x^{d'} - 1 = 0\}, x^{n'}).$$

For $\gcd(d',n') = 1$, the automorphism $\beta : \{x^{d'} - 1 = 0\} \to \{x^{d'} - 1 = 0\},\ \beta(x) = x^{n'}$ is an isomorphism. We conclude that:

Proposition 1.7. *We have*

$$\mathsf{P}(\{x^d - 1 = 0\}, x^n) = \ker(\sigma_{d'}),$$

where $d' = \frac{d}{\gcd(n,d)}$. *In particular*

$$\mathsf{P}(\{x^d - 1 = 0\}, x^n) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}(\zeta_{d'})$$

and if two distinct prime numbers divide d' *then* $\mathsf{P}(\{x^d - 1 = 0\}, x^n) = \mathbb{Z}[\zeta_{d'}]$.

For $f = x^d - t \in \mathbb{Z}[t][x]$ we have

$$\mathsf{P}(x^d - t = 0,\ x^n) = t^{\frac{n}{d}} \mathsf{P}(x^d - 1 = 0,\ x^n).$$

We have $f' \cdot x + f \cdot (-d) = d \cdot t$ and so we have

$$\int \frac{x^{n-1}dx}{df} = \frac{1}{d \cdot t} \int x^n. \tag{1.9}$$

Therefore,

$$\mathsf{P}(x^d - t = 0,\ x^{n-1}dx) = \frac{1}{d} t^{\frac{n}{d}-1} \mathsf{P}(x^d - 1 = 0,\ x^n).$$

1.8 Zeros of Abelian integrals and contraction of varieties

Let $f, g, \omega \in \mathsf{R}[x]$ be such that

$$g \circ \omega = q \cdot f, \quad \text{for some } q \in \mathsf{k}[x]. \tag{1.10}$$

We have the morphism

$$\{f = 0\} \overset{\alpha_\omega}{\to} \{g = 0\},\ \alpha_\omega(x) = \omega(x)$$

defined over R. Let $\delta \in H_0(\{f = 0\}, \mathbb{Z})$ such that $(\alpha_\omega)_*(\delta) = 0$, where $(\alpha_\omega)_*$ is the induced map in homology. For instance, if $\deg(g) < \deg(f)$ then because of (1.10), there exist two zeros x_1, x_2 of f such that $\int_\delta \omega = \omega(x_1) - \omega(x_2) = 0$ and so the topological cycle $\delta := x_1 - x_2$ has the desired property. Note that and the 0-form ω on $\{f = 0\}$ is the pull-back of the 0-form x by α_ω. The following theorem discusses the inverse of the above situation:

Theorem 1.1. *Let $f, \omega \in \mathsf{R}[x]$. Assume that f is monic, the degree of each irreducible component of f is invertible in R and*

$$\int_\delta \omega = 0 \tag{1.11}$$

for some simple cycle $\delta \in H_0(\{f = 0\}, \mathbb{Z})$. Then there exists a polynomial $g \in \mathsf{R}[x]$ such that

1. $\deg(g) < \deg(f)$*;*
2. *the degree of each irreducible components of g divides the degree of some irreducible component of f;*
3. *$g \circ \omega = fq$ for some $q \in \mathsf{k}[x]$, the morphism $\alpha_\omega : \{f = 0\} \to \{g = 0\}$ is surjective and $(\alpha_\omega)_*(\delta) = 0$.*

Proof. Let $f = f_1^{\alpha_1} f_2^{\alpha_2} \cdots f_r^{\alpha_r}$ (resp. $\omega * f = g_1^{\beta_1} g_2^{\beta_2} \cdots g_s^{\beta_s}$) be the decomposition of f (resp. $\omega * f$) into irreducible components (in $\mathsf{k}[x]$). By Proposition 1.5 and the second equality in (1.4), we have $s \leq r$ and we can assume that $\omega * f_i = g_i^{k_i}$ for $i = 1, 2, \dots, s$ and some $k_i \in \mathbb{N}$. For some $a \in \mathsf{R}$ the polynomial $g := a g_1^{\alpha_1} g_2^{\alpha_2} \cdots g_s^{\alpha_s}$ is in $\mathsf{R}[x]$ and we claim that it is the desired one. Except the first item and $(\alpha_\omega)_*(\delta) = 0$, all other parts of the theorem are satisfied by definition.

Let $\delta = x_1 - x_2$. We consider two cases: First let us assume that x_1 and x_2 are two distinct roots of an irreducible component of f, say f_1. By Proposition 1.5 we have $\omega * f_1 = g_1^{k_1}$, $k_1 > 1$ and so $\deg(g) < \deg(f)$. Now assume that x_1 is a zero of f_1 and x_2 is a zero of f_2. Let $\omega * f_1 = g_1^{k_1}$, $\omega * f_2 = g_2^{k_2}$, $k_1, k_2 \in \mathbb{N}$. The number $\omega(x_1) = \omega(x_2)$ is a root of both g_i, $i = 1, 2$ and G_f acts transitively on the roots of both g_i, $i = 1, 2$. This implies that $g_1 = b g_2$ for some $b \in \mathsf{k}$ and so $\deg(g) < \deg(f)$. □

Remark 1.5. Let $f \in \mathsf{R}[x]$ as before and $\delta \in H_0(\{f = 0\}, \mathbb{Z})$. We define

$$\Omega_\delta := \{\omega \in \mathsf{H} \mid \int_\delta \omega = 0\}.$$

It is a left $\mathsf{R}[x]$-module by the usual composition of polynomials:

$$\omega \in \Omega_\delta,\ p \in \mathsf{R}[x] \Rightarrow p \circ \omega \in \Omega_\delta.$$

For d a prime number, f irreducible and δ simple, Theorem 1.1 implies that the integral $\int_\delta \omega$, $0 \neq \omega \in \mathsf{H}$ never vanishes and so $\Omega_\delta = 0$.

We may want to formulate theorems like Theorem 1.1 for the collection of 0-forms Ω_δ. Since H is a freely generated R-module, its subset Ω_δ is finitly generated. Let ω_i, $i = 1, 2, \dots, s$ generate the R-module Ω_δ. Applying Theorem 1.1 to each ω_i we find varieties $\{g_i = 0\}$, $i = 1, 2, \dots, s$. Now the morphism

$$\alpha : \{f = 0\} \to Y := \{g_1 = 0\} \times \{g_2 = 0\} \times \cdots \times \{g_s = 0\},\ \alpha = (\alpha_{\omega_1}, \alpha_{\omega_2}, \dots, \alpha_{\omega_s})$$

has the property that $\alpha_*(\delta) = 0$ and Ω_δ is the pull-back of a set of 0-forms on Y.

Remark 1.6. Starting from a field k, polynomials f, ω over k and $\delta \in H_0(\{f=0\}, \mathbb{Z})$, we may integrate and obtain an element $\int_\delta \omega$ in $\bar{\mathsf{k}}$. A natural question is that whether $\int_\delta \omega$ can be a non-zero element of k. The answer is no for $\mathrm{char}(\mathsf{k}) = 0$, an irreducible polynomial f and a simple cycle δ (this is a part of a general philosophy that by integrating over topological cycles either we get zero or some element beyond the base filed). The reason is as follows: If $0 \neq \omega(x_1) - \omega(x_2) = r \in \mathsf{k}$ then we replace f with g, where $g^k = \omega * f$ is as in Proposition 1.5, and assume that $\omega = x$ and so $x_1 - x_2 = r$. Since the action of the Galois group G_f of f on the roots of f is transitive, there is a sequence $x_1, x_2, x_3, \dots$ of roots of f such that $x_i - x_{i+1} = r$ and some $\sigma_i \in G_f$ sends x_i to x_{i+1} and x_{i+1} to x_{i+2} for all $i = 1, 2, 3, \dots$. All x_i's are not distinct and at the end one get $nr = 0$ for some $n \in \mathbb{N}$. Since $\mathrm{char}(\mathsf{k}) = 0$ we obtain $r = 0$.

Remark 1.7. Let f be a polynomial over k without multiple roots. If $\int_\delta \omega = 0$ for some $0 \neq \omega \in \mathsf{H}$ and a cycle $0 \neq \delta \in H_0(L_f, \mathbb{Z})$ (not necessarily simple) then the Galois group G_f of f is not the full permutation group of the roots of f. The reason is as follows: For δ a simple cycle an argument similar to the one in the proof of Proposition 1.5 implies that there is a partition $\{x_1, x_2, \cdots, x_d\} = A_1 \cup A_2 \cup \cdots \cup A_s, s > 1$ of the roots of f such that the action of G_f on x_i's induces an action on each $A_j,\ j = 1, 2, \dots, s$. Therefore, G_f does not contain all possible permutations of x_i's. For an arbitrary cycle let us assume by contradiction that G_f contains all basic permutations $\sigma_{i,j}$: σ_{ij} permutes x_i and x_j and fixes other roots. We write $\delta = \sum_{i=1}^d a_i x_i,\ a_i \in \mathbb{Z}$ and let σ_{ij} to act on $\sum_{i=1}^d a_i \omega(x_i) = 0$. We conclude that either $a_i = a_j$ or $\int_{[x_i]-[x_j]} \omega = 0$. The second case is already treated and so we get $\delta = a_1 \sum_{i=1}^d x_i$ which is in contradiction with the definition of the 0-th reduced homology of f.

1.9 Zero locus of integrals

In this section we work with $\mathsf{R} = \mathbb{Z}_d[t]$ and $f = x^d + t_{d-1}x^{d-1} + \cdots + t_1 x + t_0$. For $\omega \in \mathsf{R}[x]$ we have defined the element

$$D_\omega := \sqrt{\frac{\Delta_{\omega * f}}{\Delta_f}} \in \mathsf{R}$$

in §1.6. We have

$$\{D_\omega = 0\} = \{t \in \mathbb{C}^d \mid \int_\delta \omega \text{ for some simple cycle } \delta \in H_0(\{f=0\}, \mathbb{Z})\}.$$

Let $s = (s_1, s_2, \dots, s_k)$ be a parameter, Ω be a R-submodule of H generated by $\omega_i \in \mathsf{R}[x],\ i = 1, 2, \dots, k$ and $\omega := s_1\omega_1 + s_2\omega_2 + \cdots + s_k\omega_k$. We write the polynomial expansion of D_ω in the variable s

$$D_\omega = \sum_\alpha D_\alpha \cdot s^\alpha,\ D_\alpha \in \mathsf{R},$$

where α runs through

$$S := \{\alpha := (\alpha_1, \alpha_2, \dots, \alpha_k) \in \mathbb{N}_0^k,\ \sum_{i=1}^k \alpha_i = \frac{d(d-1)}{2}\}.$$

Let us define the ideal

$$I_\Omega := \langle D_\alpha \mid \alpha \in S \rangle \subset \mathsf{R}.$$

We conclude that

Proposition 1.8. *We have*

$$Z(I_\Omega) := \{t \in \mathbb{C}^d \mid \exists \delta \in H_0(\{f=0\}, \mathbb{Z}) \text{ simple s.t. } \int_\delta \Omega = 0\} \tag{1.12}$$

The algebraic group $\mathbb{C}^*$ acts on $\mathbb{C}^d$ in the following way

$$k \bullet t = (k^d t_0, k^{d-1} t_1, \dots, k t_{d-1})$$

and $x, x^2, \dots, x^{d-1}$ are eigen 0-forms under this action. This implies that if Ω is generated by a subset of $\{x, x^2, \dots, x^{d-1}\}$ then $Z(I_\omega)$ is invariant under the action of $\mathbb{C}^*$. Note also that $0 \in Z(I_\Omega)$ and each irredcible component of $Z(I_\Omega)$ passes through 0.

For any subset $0 \neq \Omega \subset H$ is the algebraic set $Z(I_\Omega)$ irreducible over $\mathbb{C}$? For a moment assume that the answer is yes. Let $\Omega_i,\ i = 1, 2$ be two sub module of H and $\Omega = \Omega_1 \oplus \Omega_2$. For instance take $\Omega_i = \mathsf{R} \cdot \omega_i$. We have

$$Z(I_\Omega) \subset Z(I_{\Omega_1}) \cap Z(I_{\Omega_2}) = Z(D_{\omega_1}, D_{\omega_2}).$$

The irreducible component $Z(I_\omega)$ of $Z(I_{\Omega_1}) \cap Z(I_{\Omega_2})$ is described by integrals. Find similar descriptions for other irredicible components of $Z(I_{\Omega_1}) \cap Z(I_{\Omega_2})$ (see Example 1.3).

Example 1.2. For the case $f = x^3 + t_2 x^3 + t_1 x + t_0$ the integral $\int_\delta x^2$ is zero when

$$D_{x^2} = t_0 - t_1 t_2 = 0.$$

In such parameters, we have the contraction

$$\alpha : \{f = 0\} \to \{x + t_1 = 0\},\ \alpha(x) = x^2.$$

Example 1.3. For $d = 4$ we have the following table:

$$f = x^4 + t_3 x^3 + t_2 x^2 + t_1 x + t_0$$

ω	D_ω
x^2	$-t_1^2 + t_1 t_2 t_3 - t_0 t_3^2$
x^3	$t_0^3 - 2t_0^2 t_2^2 - t_1^2 t_2^3 + t_0 t_2^4 - 3t_0^2 t_1 t_3 + t_0 t_1 t_2^2 t_3 + 3t_0 t_1^2 t_3^2 + t_1^2 t_2^2 t_3^2 - t_0 t_2^3 t_3^2 - t_1^3 t_3^3$

The ideal $I_{\langle x^3 - x, x^2\rangle}$ is generated by 7 polynomials $p_i,\ i = 0, 2, \dots, 6$ which can be calculated by a computer. Note that for $f = x^4 - 1$ we have $\int_{[1]-[-1]} \{x^2, x^3 - x\} = 0$ and so $(0, 0, 0, -1) \in Z(I_{\langle x^3 - x, x\rangle}) = Z(p_0, p_1 \dots, p_6)$.

1.10 The connection of H

In this section we work with $\mathsf{R} = \mathbb{Z}_d[t]$ and $f = x^d + t_{d-1} x^{d-1} + \cdots + t_1 x + t_0$. We construct a connection on the R-module H'. A similar construction for H'' can be done easily and is left to the reader. A zero $x(t)$ of f can be seen as a holomorphic multi valued function on the affine space $\mathbb{C}^d \backslash \{\Delta = 0\}$, where $\Delta = \Delta_f$ is the discriminant of f. In particular, it is common to say that $\delta = \delta_t = x_1(t) - x_2(t) \in H_0(\{f = 0\}, \mathbb{Z})$ is a continuous family of simple cycles.

Consider the differential map

$$d : \mathsf{R} \to \Omega^1_{\mathbb{U}_0},$$

where

$$\Omega^1_{\mathbb{U}_0} = \Omega^1_{\mathsf{R}} := \{p_0 dt_0 + p_1 dt_1 + \cdots + p_{d-1} dt_{d-1} \mid p_i \in \mathsf{R},\ i = 0, 1, \ldots, d-1\}$$

is the set of differential 1-forms of R (for simplicity we have written Ω^1_{R} instead of $\Omega^1_{\mathsf{R}/\mathbb{Z}_d}$, see [41], p.17). The set of vector fields is given by

$$\mathcal{D} = \mathcal{D}_{\mathbb{U}_0} := \{p_0 \frac{\partial}{\partial t_0} + p_1 \frac{\partial}{\partial t_1} + \cdots + p_{d-1} \frac{\partial}{\partial t_{d-1}} \mid p_i \in \mathsf{R},\ i = 0, 1, \ldots, d-1\}$$

and we have the canonical R-bilinear map

$$\mathcal{D}_{\mathbb{U}_0} \times \Omega^1_{\mathbb{U}_0} \to \mathsf{R},\ (\partial, \eta) \mapsto \eta(\partial)$$

defined by the rule $dt_i(\frac{\partial}{\partial t_j}) = 1$ if $i = j$ and $= 0$ otherwise. One can look R as a (left) $\mathcal{D}$-module (differential module) in the following way:

$$\partial p := dp(\partial),\ \partial \in \mathcal{D},\ p \in \mathsf{R}.$$

The differential $d : \mathsf{R} \to \Omega^1_{\mathbb{U}_0}$ extends to $d : \mathsf{k} \to \Omega_{\mathsf{k}}$. We can consider k as a (left) $\mathcal{D}_{\mathbb{U}_0}$-module in a canonical way. Let $x(t)$ be a root of the polynomial f, $f(x(t)) = 0$. Then

$$d(x(t)) \cdot f'(x(t)) + (dt_{d-1}) \cdot x^{d-1} + (dt_{d-2}) \cdot x^{d-2} + \cdots + dt_0 = 0. \tag{1.13}$$

According to the third part of Proposition 1.2, there exists polynomial $p \in \mathsf{R}[x]$ such that

$$\Delta = p \cdot f' \text{ in } \mathsf{H}'. \tag{1.14}$$

This combined with (1.13) suggests to define the connection

$$\nabla : \mathsf{H}' \to \Omega_T \otimes_{\mathsf{R}} \mathsf{H}'$$

$$\omega \mapsto \frac{-1}{\Delta} \cdot (dt_{d-1} \otimes x^{d-1} + dt_{d-2} \otimes x^{d-2} + \cdots + dt_0 \otimes 1)\omega' \cdot p,$$

where

$$T := \mathbb{U}_0 \backslash \{\Delta = 0\}.$$

Our motivation of the definition of ∇ is the following identity:

$$d(\int_\delta \omega) = \int_\delta \nabla\omega,\ \delta \in H_0(\{f = 0\}, \mathbb{Z}), \tag{1.15}$$

which follows from (1.13) and (1.14). The operator ∇ satisfies the Leibniz rule, i.e.

$$\nabla(p \cdot \omega) = p \cdot \nabla(\omega) + \omega \otimes dp,\ p \in \mathsf{R},\ \omega \in \mathsf{H}'$$

and so it is a connection on the module H'. It defines the operators

$$\nabla_i = \nabla : \Omega^i_T \otimes_{\mathsf{R}} \mathsf{H}' \to \Omega^{i+1}_T \otimes_{\mathsf{R}} \mathsf{H}'.$$

If there is no danger of confusion we will use the symbol ∇ for these operators too. The connection ∇ is an integrable connection, i.e.

$$\nabla \circ \nabla = 0.$$

Using the connection ∇, the Brieskorn module H'_Δ turns into a $\mathcal{D}$-module. The notion of integration extends to the elements of $\Omega^i_T \otimes_T \mathsf{H}'$ in a trivial way:

$$\int_\delta \eta \otimes \omega = \eta \cdot (\int_\delta \omega) \in \Omega^i_{\mathsf{k}},\ \omega \in \mathsf{H}',\ \eta \in \Omega^i_T,\ \delta \in H_0(\{f=0\}, \mathbb{Z}).$$

The construction of the connection ∇ for the polynomial $f = t_d x^d + \cdots + t_0 \in \mathbb{Z}_d[t, \frac{1}{t_d}]$ is similar. Every element of H defines a section of the cohomology bundle of f. By (1.15) every continuous family of cycles is a locally constant section of the homology bundle, which means that ∇ coincides with the Gauss-Manin connection. Therefore, we will call ∇ the Gauss-Manin connection.

Remark 1.8. The construction of ∇ works essentially for the general ring R. If $\mathsf{R} = \mathbb{Z}[t]$ then apart from derivations with respect to the parameters in t we have the map $\delta : \mathbb{Z} \to \mathbb{Z},\ a \mapsto \frac{a-a^p}{p}$ p a prime number, which is called the Fermat quotient operator and can be considered as the derivation of integers because it satisfies $\delta(ab) = a\delta(b) + b\delta(a)$ mod $p,\ a, b \in \mathbb{Z}$. For more information the reader is referred to [10].

Note also that if $\mathsf{R} = \mathbb{Q}[e] \subset \mathbb{C}$ is a transcendent extension of $\mathbb{Q}$, where e is a collection of algebraically independent transcendent numbers, we have the derivation with respect to each transcendent number and so we can define again the connection ∇.

Example 1.4. Let[2]

$$f = 4x^3 - g_2 x - g_3.$$

A straightforward and elementary computation implies: In the Brieskorn module H' the following identity holds

$$\nabla \begin{pmatrix} x \\ x^2 \end{pmatrix} = \frac{1}{\Delta} \begin{pmatrix} \frac{d\Delta}{6} & -3\delta \\ -\frac{g_2\delta}{2} & \frac{d\Delta}{3} \end{pmatrix} \begin{pmatrix} x \\ x^2 \end{pmatrix},$$

where

$$\delta = 3g_3 dg_2 - 2g_2 dg_3,\ \Delta = g_2^3 - 27g_3^2.$$

Example 1.5. Let

$$f = t_3x^3 + t_2x^2 + t_1x + t_0.$$

The Gauss-Manin connection in the basis $\omega = (dx, xdx)^{\mathsf{t}}$ of H'' is given by $\nabla\omega = \frac{1}{\tilde{\Delta}}(\sum_{i=0}^3 A_i dt_i) \otimes \omega$, where $\tilde{\Delta} = 27t_0^2t_3^2 - 18t_0t_1t_2t_3 + 4t_0t_2^3 + 4t_1^3t_3 - t_1^2t_2^2$ and

$$A_0 = \begin{pmatrix} -18t_0t_3^2 + 8t_1t_2t_3 - 2t_2^3 & 6t_1t_3^2 - 2t_2^2t_3 \\ 3t_0t_2t_3 - 4t_1^2t_3 + t_1t_2^2 & -9t_0t_3^2 + t_1t_2t_3 \end{pmatrix},$$

$$A_1 = \begin{pmatrix} 3t_0t_2t_3 - 4t_1^2t_3 + t_1t_2^2 & -9t_0t_3^2 + t_1t_2t_3 \\ 6t_0t_1t_3 - 2t_0t_2^2 & 6t_0t_2t_3 - 2t_1^2t_3 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 6t_0t_1t_3 - 2t_0t_2^2 & 6t_0t_2t_3 - 2t_1^2t_3 \\ -9t_0^2t_3 + t_0t_1t_2 & 3t_0t_1t_3 - 4t_0t_2^2 + t_1^2t_2 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} -9t_0^2t_3 + t_0t_1t_2 & 3t_0t_1t_3 - 4t_0t_2^2 + t_1^2t_2 \\ 6t_0^2t_2 - 2t_0t_1^2 & -18t_0^2t_3 + 8t_0t_1t_2 - 2t_1^3 \end{pmatrix}.$$

[2]The notation g_i instead of t_i has historical reasons. They appear as Eisenstein series in the Weierstrass uniformization theorem for the family of elliptic curve $y^2 - f(x) = 0$.

1.11 Period map

In this section we work with $\mathsf{R} = \mathbb{Z}[t]$ and $f = x^d + t_{d-1}x^{d-1} + \cdots + t_1x + t_0$. Let $\omega = (\omega_1, \omega_2, \ldots, \omega_{d-1})^{\mathsf{t}}$ be the canonical basis of H. In this basis we can write the matrix of the connection ∇[3]:

$$\nabla\omega = A \otimes \omega,\ \ A \in \mathrm{Mat}^{\mu\times\mu}(\Omega_T^1). \tag{1.16}$$

We will call A the Gauss-Manin connection matrix of f in the basis ω. A fundamental matrix of solutions for the linear differential equation

$$dY = A \cdot Y$$

(with Y unknown) is given by $Y = \mathsf{pm}^{\mathsf{t}}$, where

$$\mathsf{pm} = [\int_\delta \omega^{\mathsf{t}}] = \begin{pmatrix} \int_{\delta_1}\omega_1 & \int_{\delta_1}\omega_2 & \cdots & \int_{\delta_1}\omega_{d-1} \\ \int_{\delta_2}\omega_1 & \int_{\delta_2}\omega_2 & \cdots & \int_{\delta_2}\omega_{d-1} \\ \vdots & \vdots & \vdots & \vdots \\ \int_{\delta_{d-1}}\omega_1 & \int_{\delta_{d-1}}\omega_2 & \cdots & \int_{\delta_{d-1}}\omega_{d-1} \end{pmatrix}$$

is the period matrix and $\delta = (\delta_1, \delta_2, \ldots, \delta_{d-1})^{\mathsf{t}}$ is a basis of $H_0(\{f = 0\}, \mathbb{Z})$. This follows form the equalities (1.15) and (1.16). We look at $\delta_i = \delta_{i,t}$ as continuous family of cycles. In this way $\mathsf{pm} = \mathsf{pm}(t)$ is a multi $\mathrm{Mat}^{d\times d}(\mathbb{C})$ valued holomorphic function defined in $\mathbb{C}^d \setminus \{\Delta = 0\}$ and we call it also a period map.

Example 1.6. For a natural number d let

$$\mathsf{pm}_d := \frac{1}{d}\left(\zeta_d^{(\beta_1+1)(\beta_1'+1)} - \zeta_d^{\beta_1(\beta_1'+1)}\right)_{0\leq\beta_1,\beta_1'\leq d-2} =$$

$$\frac{1}{d}\begin{pmatrix} \zeta_d - 1 & \zeta_d^2 - 1 & \zeta_d^3 - 1 & \cdots & \zeta_d^{d-1} - 1 \\ \zeta_d^2 - \zeta_d & \zeta_d^4 - \zeta_d^2 & \zeta_d^6 - \zeta_d^3 & \cdots & \zeta_d^{2(d-1)} - \zeta_d^{d-1} \\ \zeta_d^3 - \zeta_d^2 & \zeta_d^6 - \zeta_d^4 & \zeta_d^9 - \zeta_d^6 & \cdots & \zeta_d^{3(d-1)} - \zeta_d^{2(d-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \zeta_d^{d-1} - \zeta_d^{d-2} & \zeta_d^{(d-1)2} - \zeta_d^{(d-2)2} & \zeta_d^{(d-1)3} - \zeta_d^{(d-2)3} & \cdots & \zeta_d^{(d-1)(d-1)} - \zeta_d^{(d-2)(d-1)} \end{pmatrix}.$$

According to the equality (1.9), the period matrix of $f = x^d - t$ associated to $\omega = (dx, xdx, \ldots, x^{d-1}dx)$ and $\delta = (x_2 - x_1, x_3 - x_2, \ldots, x_d - x_{d-1})$ is given by $\frac{1}{t}\mathsf{pm}_d$.

Example 1.7. Let us take $f = x^2 + t_1x + t_0$. Then $\Delta = 4t_0 - t_1^2$. We take the cycle $\delta = [-\frac{1}{2}t_1 + \sqrt{\frac{1}{4}t_1^2 - t_0}] - [-\frac{1}{2}t_1 - \sqrt{\frac{1}{4}t_1^2 - t_0}]$ and we have

$$\int_\delta x = \sqrt{-\Delta},$$

$$\nabla([x]) = \frac{1}{\Delta}(t_1dt_1 - \frac{1}{2}dt_0) \otimes [x] = \frac{1}{2}\frac{d\Delta}{\Delta} \otimes [x].$$

[3]Note that in the zero dimensional case the entries of the Gauss-Manin connection matrix are in the localization of $\mathbb{Z}[t]$ over Δ. In higher dimensions to obtain such a matrix we will need that every integer to be invertible in R.

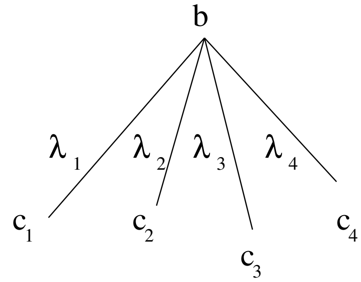

Figure 1.1: A distinguished set of paths

1.12 Monodromy group

We continue the notations of the previous section. For a fixed $p \in T := \mathbb{U}_0 \backslash \{\Delta = 0\}$, we have a canonical action

$$\pi_1(T, p) \times H_0(L_p, \mathbb{Z}) \to H_0(L_p, \mathbb{Z})$$

of the homotopy group $\pi_1(T, p)$ on the $\mathbb{Z}$-module $H_0(L_p, \mathbb{Z})$, defined by the continuation of the roots of f along a path in $\pi_1(T, p)$. The image $\Gamma_{\mathbb{Z}}$ of $\pi_1(T, p)$ in $\mathrm{Aut}_{\mathbb{Z}}(H_0(L_p, \mathbb{Z}))$ is usually called the monodromy group. To calculate it we proceed as follows:

The polynomial $f = (x-1)(x-2)\cdots(x-d)$ has $\mu := d-1$ distinct critical values, namely $c_1, c_2, \ldots, c_\mu$. We consider f as a function from $\mathbb{C}$ to itself and take a distinguished set of paths $\lambda_i,\ i = 1, 2, \ldots, \mu$ in $\mathbb{C}$ which connects $b := \sqrt{-1}$ to the critical values of f (see Figure 1.1). This mean that the paths λ_i do not intersect each other except at b and the order $\lambda_1, \lambda_2, \ldots, \lambda_\mu$ around b is anti-clockwise. The cycle $\delta_i = x_{i+1} - x_i, i = 1, 2, \ldots, \mu$ vanishes along the path λ_i and $\delta = (\delta_1, \delta_2, \ldots, \delta_\mu)$ is called a distinguished set of vanishing cycles in $H_0(L_f, \mathbb{Z})$. Now, the monodromy around the critical value c_i is given by

$$\delta_j \mapsto \begin{cases} \delta_j & j \neq i-1, i, i+1 \\ -\delta_j & j = i \\ \delta_j + \delta_i & j = i-1, i+1 \end{cases}.$$

For example, the monodromy group in the canonical basis $\delta_i, i = 1, 2, \ldots, d-1,\ d = 5$ is generated by the matrices:

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Let Ψ_0 be the intersection matrix in the basis δ. It is given in (1.2). The monodromy group keeps the intersection form in $H_0(L_p, \mathbb{Z})$. In other words:

$$\Gamma_{\mathbb{Z}} \subset \{A \in \mathrm{GL}(\mu, \mathbb{Z}) \mid A\Psi_0 A^{\mathrm{t}} = \Psi_0\}. \tag{1.17}$$

Example 1.8. Consider the case $d = 3$. We choose the basis $\delta_1 = x_2 - x_1,\ \delta_2 = x_3 - x_2$ for $H_0(L_f, \mathbb{Z})$. In this basis the intersection matrix is given by

$$\Psi_0 := \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

There are two critical points for f for which the monodromy is given by:

$$\delta_1 \mapsto -\delta_1,\ \delta_2 \mapsto \delta_2+\delta_1,$$

$$\delta_2 \mapsto -\delta_2,\ \delta_1 \mapsto \delta_2+\delta_1.$$

Let $g_1 = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$, $g_2 = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$. The monodromy group satisfies the equalities:

$$\Gamma_{\mathbb{Z}} = \langle g_1, g_2 \mid g_1^2 = g_2^3 = I, g_1g_2g_1 = g_2g_1g_2\rangle = \{I, g_1, g_2, g_1g_2g_1, g_2g_1, g_1g_2\} =$$

$$\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}\}.$$

For this example (1.17) turns out to be an equality (one obtains equations like $(a-b)^2 + a^2+b^2 = 2$ for the entries of the matrix A and the calculation is explicit).

Remark 1.9. For a polynomial f over a field k, the Galois group $\mathrm{Gal}(\bar{\mathsf{k}}/\mathsf{k})$ acts on $H_0(L_f, \mathbb{Z})$ in a canonical way:

$$\sigma \cdot \delta = \sigma(x_i) - \sigma(x_j),\ \sigma \in \mathrm{Gal}(\bar{\mathsf{k}}/\mathsf{k}),\ \delta = x_i - x_j \in H_0(L_f, \mathbb{Z}).$$

We denote the image of $\mathrm{Gal}(\bar{\mathsf{k}}/\mathsf{k})$ in $\mathrm{Aut}_{\mathbb{Z}}(H_0(L_f,\mathbb{Z}))$ by Γ_f. By definition we have:

$$\sigma(\int_\delta \omega) = \int_{\sigma\cdot\delta} \omega,\ \omega \in \mathsf{H},\ \delta \in H_0(L_f,\mathbb{Z}).$$

Therefore, if for some cycle δ we have $\int_\delta \omega = 0$ then $\int_{\sigma\cdot\delta} = 0$.

For $\mathsf{k} = \mathbb{Q}(t)$ we have the inclusion $\Gamma_{\mathbb{Z}} \subset \Gamma_f$ obtained in the following way: The action of a homotopy class $\gamma \in \pi_1(T, b)$ on the roots of f is obtained by analytic continuation of the roots of f along γ and so this action extends as an automorphism of the splitting field of f. Any such automorphism extends to an element of $\mathrm{Gal}(\bar{\mathsf{k}}/\mathsf{k})$.

1.13 Modular foliations

In this section we take $f = x^d + t_{d-1}x^{d-1} + \cdots + t_1x + t_0$ and $\mathsf{R} = \mathbb{Q}[t]$.

Definition 1.1. A modular foliation $\mathcal{F}_\eta$ associated to $\eta \in \mathsf{H}$ is a foliation in T given locally by the constant locus of the integrals $\int_{\delta_t} \eta, \delta_t \in H_1(L_t,\mathbb{Z})$, i.e. along the leaves of $\mathcal{F}_\eta$ the integral $\int_{\delta_t} \eta$ as a holomorphic function in t is constant.

By definition the period sets $\mathsf{P}(L_t,\eta)$ associated to to the points of a leaf of a modular foliation are the same. The algebraic description of a modular foliation $\mathcal{F}_\eta$ is as follows: We write $\nabla\eta = [\eta_1, \eta_2, \cdots, \eta_\mu]\omega,\ \eta_i \in \Omega_T^1,\ i = 1, 2, \ldots, \mu$, where $\omega = (\omega_1, \omega_2, \ldots, \omega_\mu)^{\mathsf{t}}$ is a basis of H. It is left to the reader to verify that:

$$\mathcal{F}_\eta : \eta_1 = 0, \eta_2 = 0, \cdots, \eta_\mu = 0.$$

Therefore, a modular foliation extends to an algebraic singular foliation in $\mathbb{U}_0$. The singular set of $\mathcal{F}_\eta$ is defined as follows:

$$\mathrm{Sing}(\mathcal{F}_\eta) := \{a \in \mathbb{U}_0 \mid \eta_1 \wedge \eta_2 \wedge \cdots \wedge \eta_\mu \mid_{\{a\}} = 0\}.$$

In practice one does as follows: Let us write $\eta = p\omega$, $p = (p_1, p_2, \ldots, p_\mu) \in \mathsf{R}^\mu$. If $\nabla\omega = A\omega$ is the Gauss-Manin connection of the polynomial f with respect to the basis ω then

$$\nabla(\eta) = \nabla(p\omega) = (dp + pA)\omega$$

and so

$$\mathcal{F}_\eta : dp_j + \sum_{i=1}^{\mu} p_i \omega_{ij} = 0,\ j = 1, 2, \ldots, \mu. \tag{1.18}$$

where $A = [\omega_{ij}]_{1\leq i,j\leq\mu}$. In particular, the foliation $\mathcal{F}_{\omega_i}$ is given by the differential forms of the i-th row of A.

Since $\mu = d-1$ differential forms define a modular foliation $\mathcal{F}_\eta$ in $\mathbb{U}_0$ with $\dim(\mathbb{U}_0) = d$, there is a vector-field $X_\eta = \sum_{i=0}^{d-1} p_i \frac{\partial}{\partial t_i}$, $p_i \in \mathsf{R}$, where p_i's have no common factors, which is tangent to the leaves of $\mathcal{F}_\eta$. For many examples, it is possible to prove that $\mathcal{F}_\eta$ is a foliation by curves and so $\mathcal{F}_\eta$ is given by the solutions of the vector field X_η.

Example 1.9. For the polynomial $f = x^d + t_{d-1}x^{d-1} + \cdots + t_1 x + t_0$, a leaf of the foliation $\mathcal{F}_x, x \in \mathsf{H}'$ is given by the coefficients of x^i's in $(x+s)^d + a_{d-2}(x+s)^{d-2} + \cdots + a_1(x+s) + a_0$, where a_i's are some constant complex numbers and s is a parameter. In fact, $\mathcal{F}_x$ is given by the solutions of the vector field:

$$t_1\frac{\partial}{\partial t_0} + 2t_2\frac{\partial}{\partial t_1} + 3t_3\frac{\partial}{\partial t_2} + \cdots + (d-1)t_{d-1}\frac{\partial}{\partial t_{d-2}} + d\frac{\partial}{\partial t_{d-1}}.$$

Example 1.10. For $x^3 + t_2x^2 + t_1x + t_0$ we have

$$\mathcal{F}_{x^2} : (-t_1)\frac{\partial}{\partial t_2} + (-t_2t_1 + 3t_0)\frac{\partial}{\partial t_1} + (2t_2t_0 - t_1^2)\frac{\partial}{\partial t_0}, \tag{1.19}$$

$\mathrm{Sing}(\mathcal{F}_{x^2}) = \{t \in \mathbb{C}^3 \mid t_1 = t_0 = 0\}$ and and the zero locus $\int x^2 = 0$ is given by $t_0 - t_2t_1 = 0$ which is $\mathcal{F}_{x^2}$-invariant.

Example 1.11. For $x^4 + t_3x^3 + t_2x^2 + t_1x + t_0$ we have

$$\mathcal{F}_{x^2} : (-2t_0t_2 + t_1^2)\frac{\partial}{\partial t_0} + (-3t_0t_3 + t_1t_2)\frac{\partial}{\partial t_1} + (-4t_0 + t_1t_3)\frac{\partial}{\partial t_2} + t_1\frac{\partial}{\partial t_3}$$

Example 1.12. We can take an arbitrary polynomial in some function field and define a modular foliation. For example, let $\mathsf{R} = \mathbb{Q}[s_1, s_2, \cdots, s_{d-1}, t]$, $f = g - t, g \in \mathbb{Q}[x]$, $\deg(g) = d$ and $\omega = s_1x + s_2x_2 + \cdots + s_{d-1}x^{d-1}$. The foliation $\mathcal{F}_\omega$ in $\mathbb{U}_0$ is given by the vector field:

$$p_1\frac{\partial}{\partial s_1} + p_2\frac{\partial}{\partial s_2} + \cdots + p_{d-1}\frac{\partial}{\partial s_{d-1}} - \frac{\partial}{\partial t},$$

where $\nabla_{\frac{\partial}{\partial t}}\omega = p_1x + p_2x^2 + \cdots + p_{d-1}x^{d-1}$.

Remark 1.10. For $\mathsf{R} = \mathbb{Q}[t]$, $f = x^d + t_{d-1}x^{d-1} + \cdots + t_1x + t_0$ and $\eta \in \mathsf{H}$, $\mathcal{F}_\eta$ is defined over $\mathbb{Q}$ and so the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts on the leaf space of $\mathcal{F}_\eta$. This implies that for a $\mathbb{Q}$-rational point $p \in \mathbb{U}_0\backslash\mathrm{Sing}(\mathcal{F}_\eta)$ the closure $\bar{L}_p$ of the leaf of $\mathcal{F}_\eta$ through p, which is an affine subvariety of $\mathbb{U}_0$, is defined over $\mathbb{Q}$. The reason to this fact is as follows: Since p is $\mathbb{Q}$-rational, $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ sends p to p and so it sends $\bar{L}_p$ to another leaf of $\mathcal{F}_\eta$ which crosses p. By our choice, p is not a singular point of $\mathcal{F}_\eta$ and so there is a unique leaf of $\mathcal{F}_\eta$ through p. This implies that $\bar{L}_p$ is mapped to itself under the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ and so $\bar{L}_p$ is defined over $\mathbb{Q}$.

1.14 Period domain and the inverse of the period map

Let $f = x^d + t_{d-1}x^{d-1} + \cdots + t_1x + t_0$, $\mathsf{R} = \mathbb{Q}[t]$ and $\mathcal{P} := \mathrm{GL}(\mu, \mathbb{C})$. The period map defines

$$\mathsf{pm} : T \to \mathcal{L} := \Gamma_{\mathbb{Z}} \backslash \mathcal{P}$$

which we call it again the period map and use the same notation pm as before. Here $\Gamma_{\mathbb{Z}}$ is the monodromy group. Without the danger of confusion, both complex manifolds $\mathcal{L}$ and $\mathcal{P}$ are also called the period domain. The complex manifold $\mathcal{L}$ is of dimension $(d-1)^2$ and the affine variety $\mathbb{U}_0$ is of dimension d. Therefore, the image of the period map is of codimension $\geq (d-1)^2 - d$. One of the problems which we will face in higher dimensions is to determine the ideal of the of image of pm. Every holomorphic global function in $\mathcal{L}$ which vanishes on the image of pm gives us a relation between the periods of f. To begin with, we have to construct holomorphic functions on $\mathcal{L}$.

Let $\mathcal{O}_{\mathcal{P}}$ (resp. $\mathcal{O}_{\mathcal{L}}$) be the space of global holomorphic functions in $\mathcal{P}$ (resp. $\mathcal{L}$). We want to construct some elements of $\mathcal{O}_{\mathcal{L}}$. The relation (1.17) implies that $\det(A) = \pm 1,\ A \in \Gamma_{\mathbb{Z}}$ and so the function $det(x)^2$ is a one valued function on $\mathcal{L}$. There are two methods of constructing elements in $\mathcal{O}_{\mathcal{L}}$. The first method is as follows:

Using the relation (1.17), one can see easily that for $[x] \in \mathcal{P}$ the entries of $g = x^{\mathsf{t}}\Psi_0^{-\mathsf{t}}x$ does not depend depend on the choice of x in the class $[x]$, where Ψ_0 is the intersection matrix in (1.2). Therefore, the entries of g are global holomorphic functions in $\mathcal{L}$. In the same way, the entries of the matrix $\tilde{g} = x^{\mathsf{t}}\Psi_0^{-\mathsf{t}}\bar{x}$ are real analytic functions on $\mathcal{L}$.

The second method for producing holomorphic functions on $\mathcal{L}$ is as follows: Define

$$\tilde{\cdot} : \mathcal{O}_{\mathcal{P}} \to \mathcal{O}_{\mathcal{P}},\ \tilde{p}(x) := \sum_{A \in \Gamma_{\mathbb{Z}}} p(Ax).$$

The new function $\tilde{p}$ is $\Gamma_{\mathbb{Z}}$ invariant and hence induce an element in $\mathcal{O}_{\mathcal{L}}$, which we denote it again with $\tilde{p}$. Note that the sum is finite and so we do not have the convergence problem. For higher dimensional abelian integrals the monodromy group will be an infinite group and so one has to verify the convergency.

Example 1.13. For Example 1.8, the functions on $\mathcal{L}$ obtained by the first method are given bellow:

$$x^{\mathsf{t}}\Psi_0^{-\mathsf{t}}x = \frac{1}{3}\begin{pmatrix} 2{x_1}^2 + 2x_1x_3 + 2{x_3}^2 & 2x_1x_2 + x_1x_4 + x_2x_3 + 2x_3x_4 \\ 2x_1x_2 + x_1x_4 + x_2x_3 + 2x_3x_4 & 2{x_2}^2 + 2x_2x_4 + 2{x_4}^2 \end{pmatrix},$$

$$x^{\mathsf{t}}\Psi_0^{-\mathsf{t}}\bar{x} = \frac{1}{3}\begin{pmatrix} 2x_1\bar{x}_1 + x_1\bar{x}_3 + x_3\bar{x}_1 + 2x_3\bar{x}_3 & 2x_1\bar{x}_2 + x_1\bar{x}_4 + x_3\bar{x}_2 + 2x_3\bar{x}_4 \\ 2x_2\bar{x}_1 + x_2\bar{x}_3 + x_4\bar{x}_1 + 2x_4\bar{x}_3 & 2x_2\bar{x}_2 + x_2\bar{x}_4 + x_4\bar{x}_2 + 2x_4\bar{x}_4 \end{pmatrix}.$$

Using the second method we have produced the following table:

p	$\tilde{p}$
x_1	0
x_1x_2	$4x_1x_2 + 2x_1x_4 + 2x_2x_3 + 4x_3x_4$
x_1x_3	$-2x_1^2 - 2x_1x_3 - 2x_3^2$
x_1x_4	$-2x_1x_2 - x_1x_4 - x_2x_3 - 2x_3x_4$

Since $d\mathsf{pm} = \mathsf{pm}A^{\mathsf{t}}$, where A is the Gauss-Manin connection of f in the basis ω, the fact that for all $t \in T$ and v in the tangent space of T at t, $\det(A(t)(v)) \neq 0$ (equivalently for all $t \in T$ the matrices $A_i(t)$, where $A = \frac{1}{\Delta}\sum_{i=0}^{d-1} A_i dt_i$, are $\mathbb{C}$-linear independent), implies that

pm is a local biholomorphism. This can be regarded as the infinitesimal Torelli problem in dimension zero. The global Torelli problem is whether pm is a biholomorphism between T and its image[4].

Example 1.14. Let us consider the situation in Example (1.5). For a linear differential equation $Y' = AY$, we have $\det(Y)' = det(A)Y$. We use this fact and conclude that

$$\det(\mathsf{pm}) = c \cdot \tilde{\Delta}^{\frac{-1}{2}}$$

for some constant c. The period map pm in the basis $(dx, xdx)^{\mathsf{t}}$ is a local biholomorphism. Let us denote the image of pm by $\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$. We look at the period map as a function sending $t = (t_0, t_1, t_2, t_3)$ to (x_1, x_2, x_3, x_4) and calculate the derivative of its (local) inverse F:

$$DF(x) = \frac{1}{\det(x)} \begin{pmatrix} -2t_0x_4 & -t_0x_3 - t_1x_4 & 2t_0x_2 & t_0x_1 + t_1x_2 \\ 3t_0x_3 - t_1x_4 & -2t_2x_4 & -3t_0x_1 + t_1x_2 & 2t_2x_2 \\ 2t_1x_3 & t_2x_3 - 3t_3x_4 & -2t_1x_1 & -t_2x_1 + 3t_3x_2 \\ t_2x_3 + t_3x_4 & 2t_3x_3 & -t_2x_1 - t_3x_2 & -2t_3x_1 \end{pmatrix}.$$

It is obtained from the equality $d\mathsf{pm} = \mathsf{pm}A^{\mathsf{t}}$. If $\mathsf{pm} : T \to \mathcal{L}$ is a global biholomorphism then we have four holomorphic functions $F_i, i = 0, 1, 2, 3$ on $\mathcal{L}$, where $F = \mathsf{pm}^{-1} = (F_0, F_1, F_2, F_3)$. They satisfy the differential equation obtained by the above matrix (replace t_i with F_i).

The period map sends a modular foliation $\mathcal{F}_\eta$, $\eta = \sum_{i=0}^{\mu} s_i\omega_i$, $s_i \in \mathbb{Q}$ to trivial foliations in $\mathcal{L}$ in the following sense: We define the following sub-algebra of $\mathcal{O}_{\mathcal{P}}$:

$$\mathcal{O}_\eta := \mathbb{C}[s_1x_{i,1} + s_2x_{i,2} + \cdots + s_\mu x_{i,\mu} \mid i = 1, 2, \ldots, \mu] \subset \mathcal{O}_{\mathcal{P}}.$$

The locus of points $x \in \mathcal{L}$ in which $\mathcal{O}_\eta$ is constant is a foliation in $\mathcal{L}$. Pulling back this foliation by the period map, we obtain the foliation $\mathcal{F}_\eta$ in T.

Complementary notes

1. The theory developed in this chapter was mainly for integrals over simple cycles. For instance, Theroem 1.1 and Remark 1.6 are stated only for them. Studying integrals over arbitrary topological cycles may be desired from the Galois theory point of view. For instance one may try to prove or disprove the converse of the statement in Remark 1.7: Assume that f is a polynomial over k and it is without multiple roots. Further, assume that the Galois group of f does not contain all permutations of the roots of f. Then there is non-zero elements $\omega \in \mathsf{H}$ and $\delta \in H_0(L_f, \mathbb{Z})$ such that $\int_\delta \omega = 0$.
2. We saw that the R-module H plays the role of de Rham cohomology of a zero dimensional variety. One may ask for an object representing the étale cohomology of such a variety. This may be useful, in particular when we do not assume that the characteristic of the base field is zero.
3. The leaves of a modular foliation $\mathcal{F}$, which is associated to zero dimensional integrals and is defined over $\mathbb{Q}$, are all algebraic. The following problems naturally arise: Construct the leaf space of $\mathcal{F}$ and describe its singularities and definition field. Classify the leaves of $\mathcal{F}$ in terms of their genus (the leaves corresponding to the zeros of integrals seems to have genus less than the genus of a generic leaf).

[4]The Torelli problems are stated originally for the period map who takes values in the Griffiths domain D. Our period domain $\mathcal{P}$ lives above D and for zero dimensional varieties the domain D is a one point and so our Torelli problems does not make sense for D (see Chapter 6).

Chapter 2

Modular foliations

In this chapter we define the notion of a modular foliation in a smooth variety M defined over $\mathbb{C}$ using an integrable connection ∇ in a vector bundle on M. For the main examples of the present text the variety and the connection are defined over $\mathbb{Q}$ and so we can talk about a modular foliation defined over $\mathbb{Q}$. For a complete account on connections the reader is referred to [18, 48].

There are two important classes of modular foliations: The first class is obtained by taking a quotient of a complex manifold over a group which acts discretely on it. If in the initial manifold we have an integrable foliation which is invariant under the action of the group then the quotient foliation is modular. As an example we mention a complex torus/abelian variety and its canonical foliations. Other important examples of such foliations appear in the moduli of polarized Hodge structures which we will discuss them in Chapter 6. In general, finding algebraic models for such foliations is difficult. Hilbert modular foliations (see [72] and also §2.7), the foliations induced by the Ramanujan relations (see [67]) and the canonical foliations of an abelian variety are among the few examples for which the corresponding algebraic models are well-known.

The second class of modular foliations are associated to a proper morphism $f : M \to T$ of projective varieties. They live in the parameter space T and are constant locus of integrals of differential forms in M over topological cycles in the fibers of f. Algebraic description of such foliations can be done using the Gauss-Manin connection of the fibration f. A foliation of this class has always different type of algebraic invariant varieties: the locus of parameters $t \in T$ such that the fiber over t is singular or has an algebraic cycle or a special type of morphism to another variety, are among such invariant algebraic varieties. The main objective of this text is to study this class of foliations.

2.1 Connections on vector bundles

Let M be a complex manifold, V be a locally free sheaf of rank μ on M and $D = \sum_{i=1}^{s} n_i D_i,\ n_i \in \mathbb{N}$ be a divisor in M. If there is no confusion we will also use V for the corresponding vector bundle. By $v \in V$ we mean either a section of V in some open subset of M or a germ of a section. Let

$$\nabla : V \to \Omega^1_M(D) \otimes_{\mathcal{O}_M} V$$

be a connection on V, where $\Omega^1_M(D)$ is the sheaf of meromorphic differential 1-forms η in M such that the pole order of η along $D_i,\ i = 1, 2, \ldots, s$ is less than n_i. By definition ∇

is $\mathbb{C}$-linear and satisfy the Leibniz rule:

$$\nabla(f\omega) = df \otimes \omega + f\nabla\omega,\ f \in \mathcal{O}_M,\ \omega \in V.$$

The connection ∇ induces

$$\nabla_p : \Omega_M^p(*D) \otimes_{\mathcal{O}_M} V \to \Omega_M^{p+1}(*D) \otimes_{\mathcal{O}_M} V,$$

$$\nabla_p(\alpha \otimes \omega) = d\alpha \otimes \omega + (-1)^p \alpha \wedge \nabla\omega,\ \alpha \in \Omega_M^p(*D),\ \omega \in V,$$

where $\Omega_M^p(*D)$ is the sheaf of meromorphic differential p-forms η in M with poles of arbitrary order along the support of D, i.e. $|D| = \cup_{i=1}^s D_i$. If there is no risk of confusion we will drop the subscript p of ∇_p. We say that ∇ is integrable if $\nabla \circ \nabla = 0$. Throughout the text we assume that ∇ is integrable. The set $|D|$ is also called the singular set of ∇.

A section ω of V with $\nabla\omega = 0$ is called a flat section. The integrability condition implies that the space of flat sections in a small neighborhood of $b \in M\backslash|D|$ is a $\mathbb{C}$-vector space of dimension μ. Analytic continuation of flat sections gives us the mondromy representation of ∇:

$$h : \pi_1(M\backslash|D|, b) \to \mathrm{GL}(V_b),$$

where V_b is the fiber of the vector bundle V over b (equivalently the μ-dimensional $\mathbb{C}$-vector space of the germs of flat sections around b).

It is sometimes useful to consider the case in which V is a trivial vector bundle and so it has μ global sections $\omega_i,\ i = 1, 2, \cdots, \mu$ such that in each fiber $V_x,\ x \in M$ they form a basis. We write ∇ in the basis $\omega := (\omega_1, \omega_2, \cdots, \omega_\mu)^{\mathsf{t}}$:

$$\nabla(\omega) = A \otimes \omega,\ A = [\omega_{ij}]_{1\leq i,j\leq\mu} = \begin{pmatrix} \omega_{11} & \omega_{12} & \cdots & \omega_\mu \\ \omega_{21} & \omega_{22} & \cdots & \omega_{2\mu} \\ \vdots & \vdots & \vdots & \vdots \\ \omega_{\mu 1} & \omega_{\mu 2} & \cdots & \omega_{\mu\mu} \end{pmatrix},\ \omega_{ij} \in H^0(M, \Omega_M^1(D)).$$

The matrix A is called the connection matrix of ∇. We have

$$\nabla(\nabla(\omega)) = \nabla(A \cdot \omega) = d(A) \otimes \omega - A \wedge \nabla(\omega) = (dA - A \wedge A) \otimes \omega$$

and so the integrability condition is given by:

$$dA = A \wedge A$$

or equivalently

$$d\omega_{ij} = \sum_{k=1}^{\mu} \omega_{ik} \wedge \omega_{kj},\ i, j = 1, 2, \ldots, \mu. \tag{2.1}$$

The Leibniz rule implies that for a flat section $\tilde{Y}(t) = Y(t) \cdot \omega, t \in U$ written in the basis ω, $Y(t)$ satisfies the linear multivariable differential equation:

$$dY = -Y \cdot A. \tag{2.2}$$

We may also take μ global meromorphic sections $\omega_1, \omega_2, \cdots, \omega_\mu$ of V such that they form a basis of V_x for an x in a Zariski open subset of M. This will produce unnecessary poles for the differential equation (2.2) which we call them apparent singularities. The monodromy around such singularities is identity.

Let $\tilde{\omega} = S\omega$ be another ordered set of global meromorphic sections of V with the same property as ω. Then

$$\nabla(\tilde{\omega}) = S(S^{-1}dS + A)S^{-1} \otimes \tilde{\omega},$$

where $\nabla\omega = A \otimes \omega$. This can be verified using the Leibniz rule as follows:

$$\begin{aligned} \nabla(\tilde{\omega}) &= \nabla(S\omega) = dS \otimes \omega + S\nabla\omega = dS.S^{-1} \otimes \tilde{\omega} + SAS^{-1} \otimes \tilde{\omega} \\ &= (dS.S^{-1} + SAS^{-1}) \otimes \tilde{\omega}. \end{aligned}$$

Therefore, the connection matrix in the basis $\tilde{\omega}$ is given by:

$$\tilde{A} = dS.S^{-1} + SAS^{-1}.$$

Example 2.1. Let V be the trivial bundle and let $e = \{e_i \mid i = 1, 2, \cdots, \mu\}$ be a set of trivializing sections of V. For a section $v = (f_1, f_2, \ldots, f_\mu) = \sum_{i=1}^{\mu} f_i e_i,\ f_i \in H^0(M, \mathcal{O}_M)$ of V written in the basis e, the trivial connection on V is given by:

$$\nabla(v) = (df_1, df_2, \cdots, df_\mu) = \sum_{i=1}^{\mu} df_i \otimes e_i$$

Flat sections of ∇ are constant vectors.

2.2 Operations on connections

Let $\check{V}$ be the dual vector bundle of V. There is defined a natural dual connection as follows:

$$\check{\nabla} : \check{V} \to \Omega^1_M(D) \otimes_{\mathcal{O}_M} \check{V},$$

$$\langle \check{\nabla}\delta, \omega \rangle = d\langle \delta, \omega \rangle - \langle \delta, \nabla\omega \rangle,\ \delta \in \check{V},\ \omega \in V.$$

The integrability of ∇ implies that $\check{\nabla}$ is also integrable. If $\{e_1, e_2, \ldots, e_\mu\}$ is a basis of flat sections in a neighborhood of $b \in M \backslash |D|$ then we can define its dual as follows: $\langle \delta_i, e_j \rangle = 0$ if $i \neq j$ and $= 1$ if $i = j$. We can easily check that δ_i's are flat sections. The associated monodromy for $\check{\nabla}$ with respect to this basis is just the transpose T^{t} of T. It is easy to verify that if A is the matrix form of the connection ∇ in the basis ω then $-A^{\mathsf{t}}$ is the matrix form of the dual connection $\check{\nabla}$ in the dual basis $\check{\omega}$. We can define a natural connection on $\wedge^k V = \{\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_k \mid \omega_i \in V\}$ with the pole divisor D as follows:

$$\nabla(\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_k) = \sum_{i=1}^{k} \omega_1 \wedge \omega_2 \wedge \cdots \widehat{\omega_i, \nabla\omega_i} \cdots \wedge \omega_k,$$

where $\widehat{\omega_i, \nabla\omega_i}$ means that we replace ω_i by $\nabla\omega_i$. For connections on two vector bundle V and W one can also define in a canonical way connections in the direct sum $V \oplus W$ and the tensor product $V \otimes W$ (see [18]).

Consider a group which acts on V fiberwise and linearly. This means that G acts on M and $\pi(g \cdot v) = g \cdot x$, where $\pi : V \to M$ is the vector bundle map. Moreover, the action of an element $g \in G$ as a map from V_x to $V_{g \cdot x}$ is linear. Assume that the action of G on M has no fixed or accumulation points. The quotient space $G \backslash V$ turns out to be a vector bundle over the complex manifold $G\backslash M$. We further assume that the action of each element of G is a biholomorphic mapping of M.

The group G acts also on the holomorphic sections of V in a canonical way: for $v : U \to V$ a holomorphic section of V in an open neighborhood U in M, we have $(g \cdot v)(x) := g \cdot v(x)$. We have also the canonical action of G on the sheaf $\Omega^1_M(*D) \otimes_{\mathcal{O}_M} V$:

$$g \cdot (\omega \otimes v) = g_* \omega \otimes g \cdot v,\ \omega \in \Omega^1_M(*D),\ v \in V, g \in G,$$

where $g_*\omega$ is the pull-forward of the differential form ω (we have the action of g on $\mathcal{O}_M$ given by $g \cdot f = g_* f,\ f \in \mathcal{O}_M$). A connection ∇ on V is called to be G-invariant if

$$\nabla(g \cdot v) = g \cdot \nabla(v).$$

This implies that the action of g maps flat sections of ∇ to flat sections. Finally, we have the quotient connection $G\backslash\nabla$ on the vector bundle $G\backslash V$.

2.3 Linear differential equations

We consider an integrable connection ∇ on a vector bundle V on M and a global meromorphic vector field v in M. Let ∇_v denote the composition

$$V(*) \xrightarrow{\nabla} \Omega^1_M(D) \otimes_{\mathcal{O}_M} V(*) \xrightarrow{v \otimes \mathrm{id}} V(*),$$

where $V(*)$ is the sheaf of meromorphic sections of V, and write

$$\nabla^i_v := \underbrace{\nabla_v \circ \nabla_v \circ \cdots \circ \nabla_v}_{i\text{-times}},\ \nabla^0_v = \mathrm{id},\ i = 0, 1, 2, \ldots.$$

We can iterate a global meromorphic section η of V under ∇_v and get global meromorphic sections $\nabla^i_v \eta, i = 0, 1, 2, \ldots$ of V. Since V is a vector bundle of finite rank, there exist $m \leq \mu$, μ the rank of V, and global meromorphic functions $p_0, p_1, \ldots, p_m$ on M such that

$$p_0 \eta + p_2 \nabla_v \eta + p_2 \nabla^2_v \eta + \cdots + p_m \nabla^m_v \eta = 0.$$

This is called the linear differential equation of η along the vector field v and associated to the connection V. If ∇ is the Gauss-Manin connection associated to an algebraic fibration (see Chapter 3) then it is called the Picard-Fuchs equation of η.

2.4 Modular foliations

Consider an integrable connection ∇ on a vector bundle V on M. To each global meromorphic section η of V we associate the following distribution:

$$\mathcal{F}_\eta = \{F_p \mid p \in M \backslash |D|\},\ F_p := \{v \in T_p M \mid \nabla_v(\eta) = 0\}.$$

There is a dense Zariski open subset U of M such that $\dim_{\mathbb{C}} F_p,\ p \in U$ is a fixed number. We call it the dimension of the distribution $\mathcal{F}_\eta$. The distribution $\mathcal{F}_\eta$ is integrable, i.e. for two holomorphic vector fields v_1, v_2 in some open set U' of U with $v_i(p) \in F_p,\ p \in U'$ we have $[v_1, v_2](p) \in F_p,\ p \in U'$, where $[\cdot, \cdot]$ is the Lee bracket. This follows from

$$\nabla_{[v_1, v_2]} = \nabla_{v_1} \circ \nabla_{v_2} - \nabla_{v_2} \circ \nabla_{v_1}.$$

See [18], p.11 (the reader who knows persian is also referred to [78] p. 261).

The integrability of the distribution $\mathcal{F}_\eta$ implies that there is a foliation, which we denote it again by $\mathcal{F}_\eta$, in U such that for $p \in U$ the tangent space of the foliation $\mathcal{F}_\eta$ at p is given by F_p. Geometrically the leaves of $\mathcal{F}_\eta$ are the locus L of points of M such that η is a flat section of $\nabla \mid_L$, i.e. $(\nabla \mid_L)\eta = 0$. A foliation $\mathcal{F}_\eta$ obtained in this way is called a modular foliation.

We write $\nabla \eta = [\eta_1, \eta_2, \cdots, \eta_\mu]\omega,\ \eta_i \in \Omega^1_M(*),\ i = 1, 2, \ldots, \mu$, where $\Omega^1_M(*)$ is the sheaf of meromorphic 1-forms in M and $\omega = (\omega_1, \omega_2, \ldots, \omega_\mu)^{\mathsf{t}}$ is a set of global meromorphic sections of V such that for x in a zariski open subset of M, $\omega \mid_x$ form a basis of V_x. It is left to the reader to verify that:

$$\mathcal{F}_\eta : \eta_1 = 0, \eta_2 = 0, \cdots, \eta_\mu = 0.$$

Therefore, a modular foliation extends to an singular foliation in M. The description of $\mathcal{F}_\eta$ in terms of the connection matrix can be done in the same way as in Section 1.13.

Example 2.2. Assume that all the columns of a connection matrix A are zero except the i-th one. The integrability condition is:

$$d\omega_{1i} = \omega_{1i} \wedge \omega_{ii},\ d\omega_{2i} = \omega_{2i} \wedge \omega_{ii},\ \ldots, d\omega_{ii} = 0, \ldots, d\omega_{\mu i} = \omega_{\mu i} \wedge \omega_{ii}$$

The foliations $\mathcal{F}(\omega_{ji}),\ j \neq i$ can be also obtained from a rank two connection matrix $\begin{pmatrix} \omega_{ii} & 0 \\ \omega_{ji} & 0 \end{pmatrix}$. In particular, holomorphic codimension one foliations with Godbillon-Vey sequence of length $k = 0, 1$ are modular (see [11]).

Example 2.3. (Triangular connections) For a lower triangular matrix A the integrablity condition is:

$$d\omega_{ii} = 0,\ d\omega_{i+1,i} = (\omega_{i+1,i+1} - \omega_{i,i}) \wedge \omega_{i+1,i},\ \cdots,$$

For the connection matrix

$$\begin{pmatrix} \omega_1 & 0 & 0 & \cdots & 0 \\ \omega_2 & 2\omega_1 & 0 & \cdots & 0 \\ \omega_3 & \omega_2 & 3\omega_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \omega_\mu & \omega_{\mu-1} & \omega_{\mu-2} & \cdots & \mu\omega_1 \end{pmatrix}$$

the integrablity condition reads:

$$d\omega_1 = 0,\ d\omega_i = (i-1)\omega_1 \wedge \omega_i,\ i = 2, \cdots, \mu.$$

This shows that modular foliations associated to a connection may have all possible codimensions. This can be also seen by taking the diagonal connection matrix.

In the next sections we discuss some examples of modular foliations which are already studied by many authors.

2.5 Foliations induced by closed forms

Let us consider a connection on a line bundle V. For instance, if we have a connection on a vector bundle W of rank μ then take the wedge product connection on $\wedge_{i=1}^{\mu}W$. For a global meromorphic section ω of V we can write $\nabla\omega = \omega_{11}\otimes\omega$. The integrability condition implies that $d\omega_{11} = 0$. The choice of another $\omega' = p\omega$, p being a global meromorphic function on M, will replace ω_{11} with $\omega_{11} + \frac{dp}{p}$. Now, the foliation $\mathcal{F}_{\omega_1}$ is given by $\omega_{11} = 0$. In other direction, if a foliation is given by a meromorphic closed differential form ω_{11} then it is modular in the following way: we consider the connection on the trivial line bundle whose connection matrix is the 1×1 matrix $[\omega_{11}]$. The classification of meromorphic closed 1-forms in the projective spaces is as follows:

Let $M = \mathbb{P}^n$ be the projective space of simension n and $D = -\sum_{i=1}^{s} n_i D_i$ be the pole divisor of a differential 1-form ω_{11} in $\mathbb{P}^n$. Denote the homogeneous coordinates of $\mathbb{P}^n$ by $(x_0, x_1, \ldots, x_n)$ and let D_i be given by the homogeneous polynomial $f_i(x_0, x_1, \ldots, x_n)$. Let Ω be the pull-back of the 1-form ω_{11} by the canonical projection $\mathbb{C}^{n+1} \to \mathbb{P}^n$. If $d(\omega_{11}) = 0$ then there are complex numbers $\lambda_1, \ldots, \lambda_s \in \mathbb{C}$ and a homogeneous polynomial $g(x_0, x_1, \ldots, x_n)$ such that:

1. If $n_i = 1$ then $\lambda_i \neq 0$ and if $n_i > 1$ then f_i does not divide g,

2. Ω can be written

$$\Omega = (\sum_{i=1}^{s} \lambda_i \frac{df_i}{f_i}) + d(\frac{g}{f_1^{n_1-1}\ldots f_s^{n_s-1}}), \tag{2.3}$$

3.
$$\sum_{i=1}^{s} \deg(f_i)\lambda_i = 0,\ \deg(g) = \sum_{i=1}^{s}(n_i - 1)\deg(f_i),$$

(See [14]).

Remark 2.1. The foliation $\mathcal{F}(\omega_{11})$ has the first integral

$$f_1^{\lambda_1} f_2^{\lambda_2}\cdots f_s^{\lambda_s}\exp(\frac{g}{f_1^{n_1-1}\ldots f_s^{n_s-1}}).$$

If λ_i's are rational numbers then by taking a power of the above function we can assume that λ_i's are integers and so the first integral is of the form Ae^B, where A, B are two rational functions with poles along $|D|$. Moreover, A has zeros only along $|D|$.

2.6 Foliations on abelian varieties

In this section we are going to consider holomorphic foliations in toruses/abelian varieties. For an introduction to such varieties, the reader is referred to [54] in the analytic context and to [16] in the algebraic context.

Let $G = (\mathbb{Z}^{2n}, +)$ and $e_1, e_2, \cdots, e_{2n}$ be a basis of the $\mathbb{R}$-vector space $\mathbb{C}^n$. We have the action of G on $\mathbb{C}^n$ given by:

$$(a_1, a_2, \cdots, a_{2n})\cdot z := z + \sum_{i=1}^{2n} a_i e_i,\ (a_1, a_2, \cdots, a_{2n}) \in G,\ z \in \mathbb{C}^n.$$

The torus $A := G\backslash\mathbb{C}^n = \mathbb{C}^n/\Gamma$, where $\Gamma = G \cdot 0$ and 0 is the origin of $\mathbb{C}^n$, is a complex compact manifold with the trivial tangenet bundle. It is called an abelian variety if in addition it is a projective manifold. For each linear map $f : \mathbb{C}^n \to \mathbb{C}$, the differential form df is invariant under the action of G and so it gives us a holomorphic differential form ω in A with $d\omega = 0$. Therefore, the foliation $\mathcal{F}(\omega)$ induced by $\omega = 0$ is modular. We denote by Ω^1_A the space of holomorphic differential forms in A.

A torus A has the canonical holomorphic maps

$$g_a : A \to A,\ g_a(x) = x + a,\ n_A : A \to A,\ n_A(x) = nx,\ n \in \mathbb{N}.$$

We have $g_a^*(\omega) = \omega$ and $n_A^*\omega = n\omega$ for $\omega \in \Omega^1_A$. Therefore, we have a biholomorphism $g_{b-a} : L_a \to L_b$ and a holomorphic map $L_a \to L_{na}$, where L_a denotes the leaf of $\mathcal{F}(\omega)$ through $a \in A$. In this way L_{0_A} turns out to be a complex manifold with a group structure and every leaf of $\mathcal{F}(\omega)$ is biholomorphic to L_{0_A}. Here 0_A is the zero of the group $(A, +)$.

We are interested to know when a leaf of $\mathcal{F}(\omega), \omega \in \Omega^1_A$ is an analytic subvariety of A. If A is an abelian variety then by GAGA principle an analytic subvariety of A is an algebraic subvariety of A. Form now on we work with abelian varieties. According to the above discussion if $\mathcal{F}(\omega)$ has an algebraic leaf then all the leaves of $\mathcal{F}(\omega)$ are algebraic subvarieties of A. Let us first recall some terminology related to abelian varieties.

Let A_1, A_2 be two abelian varieties of the same dimension. An isogeny between A_1 and A_2 is a surjective morphism $f : A_1 \to A_2$ of algebraic varieties with $f(0_{A_1}) = 0_{A_2}$. It is well-known that every isogeny is a group homomorphism and there is another isogeny $g : A_2 \to A_1$ such that $g \circ f = n_{A_1},\ g \circ f = n_{A_2}$. The isogeny f induces an isomorphism $f^* : \Omega^1_{A_2} \to \Omega^1_{A_1}$ of $\mathbb{C}$-vector spaces. For $A = A_1 = A_2$ simple, it turns out that $\mathrm{End}_0(A) = \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a division algebra, i.e. it is a ring, possibly non-commutative, in which every non-zero element has an inverse. An abelian variety is called simple if it does not contain a non trivial abelian subvariety. If A is an elliptic curve, i.e. $\dim A = 1$, $\mathrm{End}_0(A)$ is $\mathbb{Q}$ or a quadratic imaginary field. In the second case it is called a CM elliptic curve. Every abelian variety is isogenous to the direct product $A_1^{k_1} \times A_2^{k_2} \times \cdots \times A_s^{k_s}$ of simple, pairwise non-isogenous abelian varieties A_i and this decomposition is unique up to isogeny and permutation of the components.

Proposition 2.1. *For a holomorphic differential form on an abelian variety A, a leaf of $\mathcal{F}(\omega)$ is algebraic if and only if there is a morphism of abelian varieties $f : A \to E$ for some elliptic curve E such that ω is the pull-back of some holomorphic differential form in E.*

Proof. We prove the non-trivial part of the proposition. We assume that a leaf of $\mathcal{F}(\omega)$ is algebraic. Using g_a's we have seen that all the leaves of $\mathcal{F}(\omega)$ are algebraic and in particular, the leaf A_1 through $0 \in A$ is algebraic. It has the induced group structure and so it is an abelian subvariety of A. Poincaré reducibility theorem (see [16] p. 86 and Proposition 12.1 p. 122) implies that there is an abelian subvariety E of A such that $f : E \times A_1 \to A,\ (a, b) \mapsto a + b$ is an isogeny (here we need that A to be an abelian variety and not just a torus). Take an isogeny $g : A \to E \times A_1$ such that $f \circ g = n_A$ for some $n \in \mathbb{N}$. Since $f^*\omega$ restricted to each fiber of the projection on the first coordinate map $\pi : E \times A_1 \to E$ is zero, there is a differential form ω_1 in E such that $\pi^*(\omega_1) = f^*\omega$. The composition $\pi \circ g$ and the differential form $\frac{1}{n}\omega_1$ are the desired objects. Since $\dim(A_1) = n - 1$, E is an elliptic curve. $\square$

Let $\omega_{11}, \omega_{22}, \ldots, \omega_{nn}$ be a basis of the $\mathbb{C}$-vector space Ω^1_A. To associate all the foliations $\mathcal{F}(\omega),\ \omega := \sum_{i=1}^n t_i\omega_{ii},\ t_i \in \mathbb{C}$ to one connection we proceed as follows: In the trivial bundle $V = A \times \mathbb{C}^{n+1}$ we consider the connection:

$$\begin{pmatrix} 0 & 0 & \cdots & 0 \\ \omega_{11} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \omega_{nn} & 0 & \cdots & 0 \end{pmatrix},$$

given in the canonical sections $\omega_i,\ i = 0, 1, \ldots, n$ of V (ω_0 is a global flat section). In other words the connection is given by

$$\nabla(\omega_i) = \omega_{ii} \otimes \omega_1,\ i = 0, 1, \ldots, n+1, \omega_{00} := 0.$$

This connection is integrable and $\mathcal{F}(\omega) = \mathcal{F}_\eta$, where $\eta = \sum_{i=1}^n t_i\omega_i$. Note that η runs through all holomorphic sections of V. Let $\mathbb{P}^{n-1}(A) \cong \mathbb{P}(\Omega^1_A)$ be the space of holomorphic foliations $\mathcal{F}(\omega),\ \omega \in \Omega^1_A$.

Proposition 2.2. *There is an isomorphism $\mathbb{P}^{n-1}(A) \cong \mathbb{P}^{n-1}$ such that under this isomorphism the subspace of $\mathbb{P}^{n-1}(A)$ containing foliations with only algebraic leaves corresponds to:*

$$\mathbb{P}_1^{n_1-1}(k_1) \cup \mathbb{P}_2^{n_2-1}(k_2) \cup \cdots \cup \mathbb{P}_r^{n_r-1}(k_r),$$

where $\mathbb{P}_i^{n_i-1},\ i = 1, 2, \ldots, r$ are projective subspaces of $\mathbb{P}^{n-1}$ which do not intersect each others, $k_i \subset \mathbb{C}$ is $\mathbb{Q}$ or a quadratic imaginary field, and $\mathbb{P}_i^{n_i-1}(k_i)$ is the set of k_i-rational points of $\mathbb{P}_i^{n_i-1}$.

Note that we have

$$\sum_{i=1}^r n_i \leq n.$$

Proof. Let $P(A)$ be the subspace of $\mathbb{P}^{n-1}(A)$ containing foliations with only algebraic leaves. An isogeny $A \to B$ between two abelian varieties induces an isomorphism $\mathbb{P}^{n-1}(A) \to \mathbb{P}^{n-1}(B)$ which sends $P(A)$ to $P(B)$. Therefore, it is enough to prove the proposition for $A = A_1^{n_1} \times A_2^{n_2} \times \cdots \times A_s^{n_s}$, where A_i's are pairwise non-isogenous abelian varieties. Let us order A_i's in such a way that $A_i,\ i = 1, 2, \cdots r$ are elliptic curves and other components $A_i, r < i \leq s$ are abelian varieties of dimension bigger than 1. The fields mentioned in the proposition are $k_i := \mathrm{End}_0(A_i), i = 1, 2, \ldots, r$. It is well-known that k_i is either $\mathbb{Q}$ or a quadratic imaginary field. In the second case there is two different embedding of k_i in $\mathbb{C}$. Both have the same image in $\mathbb{C}$. We choose differential forms $\omega_i \in \Omega^1_{A_i},\ i = 1, 2, \ldots, r$ and this gives us an embedding of k_i in $\mathbb{C}$ obtained by:

$$a \mapsto \tilde{a}, \quad \text{where } a \in \mathrm{End}(A_i),\ a^*\omega_i = \tilde{a}\omega_i.$$

Define $\omega_{ij} = \pi_{ij}^*(\omega_i),\ j = 1, 2, \ldots, n_i$, where $\pi_{ij} : A_i^{n_i} \to A_i$ is the projection in the j-th coordinate. The differential forms $\omega_{ij},\ i = 1, 2, \ldots, r,\ j = 1, 2, \ldots, n_i$ form a basis of $\Omega^1_{A_1^{n_1} \times A_2^{n_2} \times \cdots \times A_r^{n_r}}$. Let $\delta_{ij} : A_i \to A$ be the map of the form $x \mapsto (\cdots, 0, x, 0, \cdots)$, where x lies in the j-th coordinate of $A_i^{n_i}$. We have

$$P(A) = P(A_1^{n_1} \times A_2^{n_2} \times \cdots \times A_r^{n_r}) = \cup_{i=1}^r P(A_i^{n_i}) = \cup_{i=1}^r \mathbb{P}^{n_i-1}(k_i).$$

Here we have identified each piece of A by its image in A through the maps δ_{ij}. The first and second equalities are obtained from the following: If $f : A \to E$, E simple, is a non-trivial morphism of abelian varieties then E is exactly isogenous to one of A_i's and the composition $A_j \overset{\delta_{js}}{\to} A \to E,\ s = 1, 2, \ldots, n_j$ is zero if $j \neq i$ and is an isogeny or zero for $j = i$. For $\mathcal{F}(\omega),\ \omega \in \Omega^1_A$ with only algebraic leaves, we have used Proposition 2.1 and obtained a first integral $f : A \to E$ and $\omega' \in \Omega^1_E$ with $f^*\omega' = \omega$, where E an elliptic curve.

Let us now prove the last equality. For a morphism $f : A_i^{n_i} \to E$, E an elliptic curve, and $\omega' \in \Omega^1_E$ we remark that $(f \circ \delta_{ij})^*\omega' = a_{ij}\omega_i$ and so $f^*(\omega) = \sum_j a_{ij}\omega_{ij}$. We have $[a_{i1} : a_{i2} : \cdots : a_{in_i}] \in \mathbb{P}^{n_i-1}(k_i)$ because

$$((f \circ \delta_{ij})^{-1} \circ (f \circ \delta_{ij'}))^*(\omega_i) = \frac{a_{ij}}{a_{ij'}}\omega_i$$

up to multiplication by a rational number (here by $(f \circ \delta_{ij})^{-1}$ we mean any isogeny $g : E \to A_i$ such that $g \circ (f \circ \delta_{ij}) = n_{A_i}$ and $(f \circ \delta_{ij}) \circ g = n_E$ for some $n \in \mathbb{N}$).

Conversely, let $\omega = \sum_j a_{ij}\omega_{ij},\ a_{ij} \in k_i$. After multiplication of ω with an integer number, there are isogenies $f_j : A_i \to A_i,\ j = 1, 2, \ldots, n_i$ such that $f_j^*(\omega_i) = a_{ij}\omega_i$. For $g : A_i^{n_i} \to A_i,\ g(x) = f_1(x) + f_2(x) + \cdots + f_{n_i}(x)$ we have $g^*(\omega_i) = \omega$.

□

Corollary 2.1. *For an abelian variety of dimension two exactly one of the following statements is true:*

1. *There is no holomorphic foliations $\mathcal{F}(\omega),\ \omega \in \Omega^1_A$ with only algebraic leaves;*
2. *There are exactly two holomorphic foliations with only algebraic leaves;*
3. *There are two foliations $\mathcal{F}(\omega_i),\ i = 1, 2$ with only algebraic leaves and all other foliations $\mathcal{F}(\omega)$ with this property are given by $\omega = \omega_1 + t\omega_2, t \in k$, where k is either $\mathbb{Q}$ or an imaginary quadratic field.*

Proof. Up to isogeny every abelian variety of dimension two is either simple or $A_1 \times A_2$ or A_1^2, where A_1 and A_2 are two non-isogenous elliptic curves. These three cases correspond to the three cases of the corollary. □

A typical example of an abelian variety with many elliptic factors is the Jacobian of the Fermat curve $x^n + y^n = 1$ (see [50]).

2.7 Hilbert modular foliations

Let F be a totally real number field of degree m over rational numbers. This means that for every embedding of F in the field of complex numbers, its image lies inside the real numbers and there are exactly m such embeddings, namely $\sigma_1, \sigma_2, \cdots, \sigma_m$. Let $\mathcal{O}_F$ be the ring of integers of F and $M := \mathbb{H} \times \mathbb{H} \times \cdots \times \mathbb{H}$ be the m-fold product of the upper half plane. The group $\mathrm{GL}(2, \mathcal{O}_F)$ acts on M as follows:

$$\gamma \cdot (z_1, z_2, \cdots, z_m) = (\sigma_1(\gamma)z_1, \sigma_2(\gamma)z_2, \cdots, \sigma_m(\gamma)z_m).$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}$. Fix a subgroup Γ of $\mathrm{GL}(2, \mathcal{O}_F)$ with finite index. A Hilbert modular form of weight $(k_1, k_2, \ldots, k_m)$ for the group Γ is an analytic function on M such

that for every $\gamma \in \Gamma$
$$\left(\prod_{i=1}^{m} \mathrm{j}(\sigma_i(\gamma), z_i)^{-k_i}\right) f(\gamma \cdot z) = f(z).$$
where $\mathrm{j}(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z) = (ad-bc)^{-\frac{1}{2}}(cz+d)$. Using modular forms one can compactify $\Gamma\backslash M$, namely $\overline{\Gamma\backslash M}$ and give an algebraic structure to $\overline{\Gamma\backslash M}$.

A Hilbert modular foliation in $\Gamma\backslash M$ is a foliation given by the constant locus of a collection of coordinates $\{z_i\}_{i\in I}$. We claim that there is a complex manifold N of dimension $3m$ and a holomorphic map $N \to M$ such that the pull-back of Hilbert modular foliations in M are modular foliations in N (in the sense of this text). Take $\tilde{\mathcal{P}}$ the set of 2-diagonal matrices of the form $\mathrm{diag}(y_1, y_2, \ldots, y_m)$, where y_i's are 2×2 matrices in
$$\mathcal{P} := \{x = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \mid \det(x) = 1,\ \mathrm{Im}(x_1\bar{x}_3) > 0\}.$$
We consider Γ as a subgroup of $\mathrm{GL}(2m, \mathcal{O}_F)$ using the embedding
$$\Gamma \hookrightarrow \mathrm{GL}(2m, \mathcal{O}_F),\ \gamma \mapsto \mathrm{diag}(\sigma_1(\gamma), \sigma_2(\gamma), \ldots, \sigma_m(\gamma)).$$
In this way, Γ acts from left on $\tilde{\mathcal{P}}$ by usual multiplication of matrices. The matrix $dX^{\mathsf{t}}X^{-\mathsf{t}}$ is invariant under the action of Γ and so gives us a matrix A of holomorphic differential 1-forms in $N := \Gamma\backslash\tilde{\mathcal{P}}$. It is easy to see that the connection associated to A is integrable. We have a canonical holomorphic map $f : N \to M$ obtained by m-fold product of the maps $\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \mapsto \frac{x_1}{x_3}$. The pull-back of a Hilbert modular foliation given by the constant locus of $\{z_i\}_{i\in I}$ is the modular foliation in N given by differential forms in $2i,\ i \in I$ rows of A.

The reader is referred to [72] for algebraization of Hilbert modular foliations in the case $m = 2$, their algebraic leaves and many other related topics.

2.8 Lins-Neto's examples

The pencils $P_i : \mathcal{F}(\omega_i + t\eta_i),\ i = 1, 2,\ t \in \mathbb{P}^1$, where
$$\omega_1 = (4x - 9x^2 + y^2)dy - 6y(1-2x)dx, \eta_1 = 2y(1-2x)dy - 3(x^2 - y^2)dx,$$
$$\omega_2 = y(x^2 - y^2)dy - 2x(y^2 - 1)dx, \eta_2 = (4x - x^3 - x^2y - 3xy^2 + y^3)dy + 2(x+y)(y^2-1)dx$$
are studied by A. Lins Neto in [55]. They satisfy
$$d\omega_i = \alpha_i \wedge \omega_i,\ i = 1, 2,$$
where
$$\alpha_i := \lambda_i \frac{dQ_i}{Q_i},\ \lambda_1 = \frac{5}{6},\ \lambda_2 = \frac{3}{4},$$
$$Q_1 = -4y^2 + 4x^3 + 12xy^2 - 9x^4 - 6x^2y^2 - y^4,\ Q_2 = (y^2 - 1)(x + 2 + y^2 - 2x)(x^2 + y^2 + 2x).$$
Consider the connection in the trivial rank 3 bundle V over $\mathbb{C}^2$ given by the connection matrix:
$$A_i := \begin{pmatrix} \alpha_i & 0 & 0 \\ \omega_i & 0 & 0 \\ \eta_i & 0 & 0 \end{pmatrix},\ i = 1, 2.$$

We have $\mathcal{F}(\omega_i+t\eta_i)=\mathcal{F}_{v_{i,t}}$, where $v_{i,t}$ is the section of V given by $x\mapsto x\times(0,1,t),\ x\in\mathbb{C}^2$. Therefore, all the elements of the pencil P_i are associated to a linear family of sections of V. Lins Neto has proved that the set

$$E_i=\{t\in\mathbb{P}^1 \mid \mathcal{F}_{v_{i,t}} \text{ has a meromorphic first integral}\}$$

is $\mathbb{Q}+\mathbb{Q}e^{2\pi i/3}$ for $i=1$ and is $\mathbb{Q}+i\mathbb{Q}$ for $i=2$. This is similar to Corollary 2.1 for abelian varieties.

Complementary notes

1. There are two main reasons to investigate the modular foliations in the context of a general integrable connection and not restrict oneself to the case of a Gauss-Manin connection. The first reason is that the works of many authors, see for instance [76, 11, 72, 55], fit into this general context. It would be of interest to interpret the results of these authors in terms of some properties of connections. The second reason is that many non-abelian integrals, where the integration takes place on non-closed cycles or topological cycles of non-algebraic objects like leaves of a foliation, are expected to be associated to certain connections similar to Gauss-Manin connections. The first example to study would be the integrals over topological cycles of a logarithmic foliation.
2. Based on the results of A. Lins Neto and Proposition 2.2 one may ask for a complete classification of modular foliations with a first integral inside a family of foliations associated to a linear family of sections of a vector bundle with an integrable connection.
3. In Remark 2.1 consider the case in which λ_i's are rational numbers and f_i, g are polynomials with algebraic coefficients. If there are two points p_1, p_2 with algebraic coordinates, outside the pole divisor of ω_{11} and lying in the same leaf of $\mathcal{F}(\omega_{11})$ then we conclude that $A(p_1)=A(p_2),\ B(p_1)=B(p_2)$ or for some $\alpha\in\mathbb{C}$ both α, e^{α} are algebraic. Does the second one happen?
4. The generalization of the arguments in Section 2.6 to foliations with arbitrary codimension is quit accessible and is left to the reader. Another way to approach foliations in abelian varieties is to consider a generic Lefschetz pencil and deform it inside foliations. In particular, the possible generalization of the result of [64, 62] for abelian surfaces may be wished.
5. The Godbillon-Vey sequence sequence of a modular foliation is not explored in this text. To start one may have a look at [11] and [29].

Chapter 3

Weighted tame polynomials over a ring

In Chapter 1 we developed basic notions related to Abelian integrals in the dimension zero. The objective of this chapter is to develop similar notions for integrals in dimension n, i.e. the integration takes place on n-dimensional homological cycles living in the fibers of a polynomial in $(n+1)$-variables. The tools of this chapter are first introduced in [65] and [63] for the family of affine varieties given by

$$f(x) - s = 0, \ s \in \mathbb{C}, \tag{3.1}$$

where $f(x) \in \mathbb{C}[x]$ is a tame polynomial in the sense of §3.3. We follow the usual conventions on rings mentioned in the Introduction. We have tried to keep as much as possible the algebraic language and meantime to explain the theorems and examples by their topological interpretations for (3.1). For this reason, we assume a basic knowledge of the topology of manifolds, their homologies and related machinery.

3.1 Homogeneous tame polynomials

Let $n \in \mathbb{N}_0$ and $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{n+1}) \in \mathbb{N}^{n+1}$. For a ring R we denote by $\mathsf{R}[x]$ the polynomial ring with coefficients in R and the variable $x := (x_1, x_2, \dots, x_{n+1})$. We consider

$$\mathsf{R}[x] := \mathsf{R}[x_1, x_2, \dots, x_{n+1}]$$

as a graded algebra with $\deg(x_i) = \alpha_i$. For $n = 0$ (resp. $n = 2$ and $n = 3$) we use the notations x (resp. x, y and x, y, z).

A polynomial $f \in \mathsf{R}[x]$ is called a homogeneous polynomial of degree d with respect to the grading α if f is a linear combination of monomials of the type

$$x^\beta := x_1^{\beta_1} x_2^{\beta_2} \cdots x_{n+1}^{\beta_{n+1}}, \ \deg(x^\beta) = \alpha \cdot \beta := \sum_{i=1}^{n+1} \alpha_i \beta_i = d.$$

For an arbitrary polynomial $f \in \mathsf{R}[x]$ one can write in a unique way $f = \sum_{i=0}^{d} f_i, \ f_d \neq 0$, where f_i is a homogeneous polynomial of degree i. The number d is called the degree of f. The Jacobian ideal of f is defined to be:

$$\mathrm{jacob}(f) := \langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \cdots, \frac{\partial f}{\partial x_{n+1}} \rangle \subset \mathsf{R}[x].$$

The Tjurina ideal is

$$\mathrm{tjurina}(f) := \mathrm{jacob}(f) + \langle f \rangle \subset \mathsf{R}[x].$$

We define also the R-modules

$$\mathsf{V}_f := \frac{\mathsf{R}[x]}{\mathrm{jacob}(f)},\ \mathsf{W}_f := \frac{\mathsf{R}[x]}{\mathrm{tjurina}(f)}.$$

These modules may be called the Milnor module and Tjurina module of f, analog to the objects with the same name in singularity theroy (see [8]).

Remark 3.1. In practice one considers V_f as an $\mathsf{R}[f]$-module. If we introduce the new parameter s and define

$$\tilde{f} := f - s \in \tilde{\mathsf{R}}[x],\ \tilde{\mathsf{R}} := \mathsf{R}[s]$$

then $\mathsf{W}_{\tilde{f}}$ as $\tilde{\mathsf{R}}$-module is isomorphic to V_f as $\mathsf{R}[f]$-module. We have introduced V_f because the main machinaries are first developed for $f-s,\ f \in \mathbb{C}[x]$ in the literature of singularities (see [65]).

Definition 3.1. A homogeneous polynomial $g \in \mathsf{R}[x]$ in the weighted ring $\mathsf{R}[x],\ \deg(x_i) = \alpha_i,\ i = 1, 2, \ldots, n+1$ has an isolated singularity at the origin if the R-module V_f is freely generated of finite rank.

In the case $\mathsf{R} = \mathbb{C}$, a homogeneous polynomial g has an isolated singularity at the origin if $Z(\frac{\partial g}{\partial x_1}, \ldots, \frac{\partial g}{\partial x_{n+1}}) = \{0\}$. This justfies the definition geometrically. If the homogeneous polynomial $g \in \mathbb{C}[x]$ is tame then the projective variety induced by $\{g = 0\}$ in $\mathbb{P}^\alpha$ is a V-manifold/quasi-smooth variety (see Steenbrink [82]). For the case $\alpha_1 = \alpha_2 = \cdots = \alpha_{n+1} = 1$ the notions of a V-manifold and smooth manifold are equivalent.

The two variable polynomial $f(x) = x^2 + y^2$ is not tame when it is considered in the ring $\mathbb{Z}[x, y]$ and it is tame in the ring $\mathbb{Z}[\frac{1}{2}][x, y]$. In a similar way $f(x, y) = t^2x^2 + y^2$ is tame in $\mathbb{Q}[t, \frac{1}{t^2}][x, y]$ but not in $\mathbb{Q}[t][x]$.

Example 3.1. Consider the case $n = 0,\ \deg(x) = 1$. For $g = x^d$ we have

$$\mathsf{V}_g = \oplus_{i=0}^{d-2} \mathsf{R} \cdot x^i \oplus \oplus_{i=d-1}^{\infty} (\mathsf{R}/d \cdot \mathsf{R}) \cdot x^i$$

and so g is tame if and only if d is invertible in R. For instance take $\mathsf{R} = \mathbb{Z}[\frac{1}{d}],\ \mathbb{Q}, \mathbb{C}$. A basis of the R-module V_g is given by $I = \{1, x, x^2, \ldots, x^{d-2}\}$. This case is already treated in Chapter 1.

Example 3.2. In the weighted ring $\mathsf{R}[x],\ \deg(x_i) = \alpha_i \in \mathbb{N}$ for a given degree $d \in \mathbb{N}$, we would like to have at least one tame polynomial of degree d. For instance, if

$$m_i := \frac{d}{\alpha_i} \in \mathbb{N},\quad i = 1, 2, \ldots, n+1$$

and all m_i's are invertible in R then the homogeneous polynomial

$$g := x_1^{m_1} + x_2^{m_2} + \cdots + x_{n+1}^{m_{n+1}}$$

is tame. A basis of the R-module V_g is given by

$$I = \{x^\beta \mid 0 \le \beta_i \le m_i - 2,\ i = 1, 2, \ldots, n+1\}.$$

For other d's we do not have yet a general method which produces a tame polynomial of degree d.

Example 3.3. For $n = 1$ and $\mathsf{R} = \mathbb{C}[x, y]$, a homogeneous polynomial has an isolated singularity at the origin if and only if in its irreducible decomposition there is no factor of multiplicity greater than one.

Throughout the present text, we assume that d is invertible in R. We use the following notations related to a homogeneous tame polynomial $g \in \mathsf{R}[x]$: We fix a basis

$$x^I := \{x^\beta \mid \beta \in I\}$$

of monomials for the R-module V_g. We also define

$$w_i := \frac{\alpha_i}{d},\ 1 \le i \le n+1,\ A_\beta := \sum_{i=1}^{n+1}(\beta_i + 1)w_i,\ \mu := \#I = \mathrm{rank}\mathsf{V}_g \tag{3.2}$$

$$\eta := (\sum_{i=1}^{n+1}(-1)^{i-1}w_i x_i \widehat{dx_i}),\ \eta_\beta := x^\beta \eta,\ \omega_\beta = x^\beta dx\ \beta \in I,$$

where

$$dx := dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{n+1},\ \widehat{dx_i} = dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_{n+1}.$$

One may call μ the Milnor number of g.[1] To make our notation simpler, we define

$$\mathbb{U}_0 := \mathrm{Spec}(\mathsf{R}),\ \mathbb{U}_1 := \mathrm{Spec}(\mathsf{R}[x])$$

and denote by $\pi : \mathbb{U}_1 \to \mathbb{U}_0$ the canonical morphism. The set of (relative) differential i-forms in $\mathbb{U}_1$ is:

$$\Omega^i_{\mathbb{U}_1/\mathbb{U}_0} := \{\sum f_{k_1,k_2,\ldots,k_i} dx_{k_1} \wedge dx_{k_2} \wedge \cdots \wedge dx_{k_i} \mid f_{k_1,k_2,\ldots,k_i} \in \mathsf{R}\}.$$

The adjective relative is used with respect to the morphism π. We define

$$\deg(dx_j) = \alpha_j,\ \deg(\omega_1 \wedge \omega_2) = \deg(\omega_1) + \deg(\omega_2),\ j = 1, 2, \ldots, n+1,\ \omega_1, \omega_2 \in \Omega^i_{\mathbb{U}_1/\mathbb{U}_0}.$$

With the above rules, $\Omega^i_{\mathbb{U}_1/\mathbb{U}_0}$ turns into a graded R-module and we can talke about homogeneous differential forms and decomposition of a differential form into homogeneous pieces. A geometric way to look at this is the following: The multiplicative group $\mathsf{R}^* = \mathsf{R} \setminus \{0\}$ acts on $\mathbb{U}_1$ by:

$$(x_1, x_2, \ldots, x_{n+1}) \to (\lambda^{\alpha_1}x_1, \lambda^{\alpha_2}x_2, \ldots, \lambda^{\alpha_{n+1}}x_{n+1}),\ \lambda \in \mathsf{R}^*.$$

We also denote the above map by $\lambda : \mathbb{U}_1 \to \mathbb{U}_1$. The polynomial form $\omega \in \Omega^i_{\mathbb{U}_1/\mathbb{U}_0}$ is weighted homogeneous of degree m if

$$\lambda^*(\omega) = \lambda^m \omega,\ \lambda \in \mathsf{R}^*.$$

For a polynomial g this means that

$$g(\lambda^{\alpha_1}x_1, \lambda^{\alpha_2}x_2, \ldots, \lambda^{\alpha_{n+1}}x_{n+1}) = \lambda^d g(x_1, x_2, \ldots, x_{n+1}),\ \forall \lambda \in \mathsf{R}^*.$$

Remark 3.2. The reader who wants to follow the present text in a geometric context may assume that $\mathsf{R} = \mathbb{C}[t_1, t_2, \ldots, t_s]$ and hence identify $\mathbb{U}_i,\ i = 0, 1$ with its geometric points, i.e.

$$\mathbb{U}_0 = \mathbb{C}^s,\ \mathbb{U}_1 = \mathbb{C}^{n+1} \times \mathbb{C}^s.$$

The map π is now the projection on the last s coordinates.

[1] J. Milnor in [60] proves that in the case $\mathsf{R} = \mathbb{C}$ there are small neighborhoods $U \subset \mathbb{C}^{n+1}$ and $S \subset \mathbb{C}$ of the origins such that $g : U \to S$ is a C^∞ fiber bundle over $S \backslash \{0\}$ whose fiber is of homotopy type of a bouquet of μ n-spheres.

3.2 De Rham Lemma

In this section we state de Rham lemma for a homogeneous tame polynomial. Originally, a similar Lemma was stated for a germ of holomorphic function $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ in [8], p.110. To make the section self sufficient we recall some facts from commutative algebra. The page numbers in the bellow paragraph refer to the Book [24].

Let $\tilde{\mathsf{R}}$ be a ring. A sequence of elements $a_1, a_2, \ldots, a_{n+1} \in \tilde{\mathsf{R}}$ is called a regular sequence if $\langle a_1, a_2, \ldots, a_{n+1}\rangle \neq \tilde{\mathsf{R}}$ and for $i = 1, 2, \ldots, n+1$, a_i is a non-zero divisor on $\frac{\tilde{\mathsf{R}}}{\langle a_1, a_2, \ldots, a_{i-1}\rangle}$ (p. 17). The dimension of a ring $\tilde{\mathsf{R}}$ is the supremum of the lengths of chains of prime ideals in $\tilde{\mathsf{R}}$ and for an ideal $I \subset \tilde{\mathsf{R}}$ we define $\dim(I) = \dim(\frac{\tilde{\mathsf{R}}}{I})$ and $\mathrm{codim}(I) = \dim(\tilde{\mathsf{R}}_I)$ (p. 225). For $I \neq \tilde{\mathsf{R}}$, the depth of the ideal I is the length of a (indeed any) maximal regular sequence in I. The ring $\tilde{\mathsf{R}}$ is called Cohen-Macaulay if the codimension and the depth of any proper ideal of $\tilde{\mathsf{R}}$ coincide (p. 452). If $\tilde{\mathsf{R}}$ is a domain, i.e. it is finitely generated over a field, then we have $\dim(I) + \mathrm{codim}(I) = \dim(\tilde{\mathsf{R}})$ (this follows from Theorem A, p. 221) but in general the equality does not hold. Let R be a Cohen-Macaulay ring. Then $\mathsf{R}[x]$ is also Cohen-Macaulay (p. 452 Proposition. 18.9). For a tame polynomial polynomial g with $I := \mathrm{jacob}(g) \subset \tilde{\mathsf{R}} := \mathsf{R}[x]$ we have:

$$\mathrm{codim}(I) := \dim \mathsf{R}[x]_I = \dim \mathsf{k}[x]_{\bar{I}} = \dim \mathsf{k}[x] - \dim \bar{I} = n + 1.$$

Here $\bar{I}$ is the Jacobian ideal of g in $\mathsf{k}[x]$, where k is the quotient field of R. In the second and last equalities we have used the fact that g is tame and hence I does no contain any non-zero element of R and $\dim \bar{I} := \dim(\frac{\mathsf{k}[x]}{\bar{I}}) = 0$. We have used also $\dim(\mathsf{k}[x]) = n + 1$ (Theorem A, p.221). We conclude that the depth of $\mathrm{jacob}(g) \subset \mathsf{R}[x]$ is $n+1$. The purpose of all what we said above is:

Proposition 3.1. *(de Rham Lemma) Let R be a Cohen-Macaulay ring and g be a homogeneous tame polynomial in $\mathsf{R}[x]$. An element $\omega \in \Omega^i_{\mathbb{U}_1/\mathbb{U}_0}, i \leq n$ is of the form $dg \wedge \eta,\ \eta \in \Omega^{i-1}_{\mathbb{U}_1/\mathbb{U}_0}$ if and only if $dg \wedge \omega = 0$. This means that the following sequnce is exact*

$$0 \to \Omega^0_{\mathbb{U}_1/\mathbb{U}_0} \xrightarrow{dg\wedge\cdot} \Omega^1_{\mathbb{U}_1/\mathbb{U}_0} \xrightarrow{dg\wedge\cdot} \cdots \xrightarrow{dg\wedge\cdot} \Omega^n_{\mathbb{U}_1/\mathbb{U}_0} \xrightarrow{dg\wedge\cdot} \Omega^{n+1}_{\mathbb{U}_1/\mathbb{U}_0}. \tag{3.3}$$

In other words

$$H^i(\Omega^\bullet_{\mathbb{U}_1/\mathbb{U}_0}, dg \wedge \cdot) = 0,\ i = 0, 1, \ldots, n.$$

Proof. We have proved the depth of $\mathrm{jacob}(g) \subset \mathsf{R}[x]$ is $n + 1$. Knowing this the above proposition follows from the main theorem of [75]. See also [24] Corollary 17.5 p. 424, Crollary 17.7 p. 426 for similar topics. $\square$

The sequence in (3.3) is also called the Koszul complex.

Proposition 3.2. *The following sequence is exact*

$$0 \xrightarrow{d} \Omega^0_{\mathbb{U}_1/\mathbb{U}_0} \xrightarrow{d} \Omega^1_{\mathbb{U}_1/\mathbb{U}_0} \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n_{\mathbb{U}_1/\mathbb{U}_0} \xrightarrow{d} \Omega^{n+1}_{\mathbb{U}_1/\mathbb{U}_0} \xrightarrow{d} 0.$$

In other words

$$H^i_{\mathrm{dR}}(\mathbb{U}_1/\mathbb{U}_0) := H^i(\Omega^\bullet_{\mathbb{U}_1/\mathbb{U}_0}, d) = 0,\ i = 1, 2, \ldots, n + 1.$$

Proof. This is [24], Exercise 16.15 c, p. 414. $\square$

Note that in the above proposition we do not need R to be Cohen-Macaulay.

Proposition 3.3. *Let* R *be a Cohen-Macaulay ring. If for* $\omega \in \Omega^i_{\mathbb{U}_1/\mathbb{U}_0}$, $1 \leq i \leq n-1$ *we have*

$$d\omega = dg \wedge \omega_1, \quad \textit{for some } \omega_1 \in \Omega^i_{\mathbb{U}_1/\mathbb{U}_0} \tag{3.4}$$

then there is an $\omega' \in \Omega^{i-1}_{\mathbb{U}_1/\mathbb{U}_0}$ *such that*

$$d\omega = dg \wedge d\omega'.$$

Proof. Since g is homogeneous, in (3.4) we can assume that

$$\deg_x(\omega_1) = \deg_x(d\omega) - d \text{ and so } \deg_x(\omega_1) < \deg_x(d\omega) \leq \deg_x(\omega).$$

We take differential of (3.4) and use Proposition 3.1. Then we have $d\omega_1 = dg \wedge \omega_2$, and again we can assume that $\deg_x(\omega_2) < \deg_x(\omega_1)$. We obtain a sequence of differential forms ω_k, $k = 0, 1, 2, 3, \ldots$, $\omega_0 = \omega$ with decreasing degrees and $d\omega_{k-1} = dg \wedge \omega_k$. Therefore, for some $k \in \mathbb{N}$ we have $\omega_k = 0$. We claim that for all $0 \leq j \leq k$ we have $d\omega_j = dg \wedge d\omega'_j$ for some $\omega'_j \in \Omega^{i-1}_{\mathbb{U}_1/\mathbb{U}_0}$. We prove our claim by decreasing induction on j. For $j = k$ it is already proved. Assume that it is true for j. Then by Proposition 3.2 we have

$$\omega_j = dg \wedge \omega'_j + d\omega'_{j-1}, \ \omega'_{j-1} \in \Omega^{i-1}_{\mathbb{U}_1/\mathbb{U}_0}.$$

Putting this in $d\omega_{j-1} = dg \wedge \omega_j$, our claim is proved for $j - 1$. □

3.3 Tame polynomials

We start this section with the definition of a tame polynomial.

Definition 3.2. A polynomial $f \in \mathsf{R}[x]$ is called a tame polynomial if there exist natural numbers $\alpha_1, \alpha_2, \ldots, \alpha_{n+1} \in \mathbb{N}$ such that the R-module V_g is freely generated R-module of finite rank (g has an isolated singularity at the origin), where $g = f_d$ is the last homogeneous piece of f in the graded algebra $\mathsf{R}[x]$, $\deg(x_i) = \alpha_i$.

In practice, we fix up a weighted ring $\mathsf{R}[x]$, $\deg(x_i) = \alpha_i \in \mathbb{N}$ and a homogeneous tame polynomial $g \in \mathsf{R}[x]$. The perturbations $g + g_1$, $\deg(g_1) < \deg(g)$ of g are tame polynomials.

Proposition 3.4. *(de Rham lemma for tame polynomials) Proposition 3.1 is valid replacing* g *with a tame polynomial* f.

Proof. If there is $\omega \in \Omega^i_{\mathbb{U}_1/\mathbb{U}_0}$, $i \leq n$ such that $df \wedge \omega = 0$ then $dg \wedge \omega' = 0$, where ω' is the last homogeneous piece of ω. We apply Proposition 3.1 and conclude that $\omega = df \wedge \omega_1 + \omega_2$ for some $\omega_1 \in \Omega^{i-1}_{\mathbb{U}_1/\mathbb{U}_0}$ and $\omega_2 \in \Omega^i_{\mathbb{U}_1/\mathbb{U}_0}$ with $\deg(\omega_2) < \deg(\omega)$ and $df \wedge \omega_2 = 0$. We repeat this argument for ω_2. Since the degree of ω_2 is decreasing, at some point we will get $\omega_2 = 0$ and then the desired form of ω. □

Recall that in §3.1 we fixed a monomial basis x^I for the R-module V_g.

Proposition 3.5. *For a tame polynomial* f*, the* R*-module* V_f *is freely generated by* x^I.

Proof. Let $f = f_0 + f_1 + f_2 + \cdots + f_{d-1} + f_d$ be the homogeneous decomposition of f in the graded ring $\mathsf{R}[x]$, $\deg(x_i) = \alpha_i$ and $g := f_d$ be the last homogeneous piece of f. Let also $F = f_0 x_0^d + f_1 x_0^{d-1} + \cdots + f_{d-1}x_0 + g$ be the homogenization of f. We claim that the set x^I generates freely the $\mathsf{R}[x_0]$-module $V := \mathsf{R}[x_0, x]/\langle \frac{\partial F}{\partial x_i} \mid i = 1, 2, \ldots, n+1\rangle$. More precisely, we prove that every element $P \in \mathsf{R}[x_0, x]$ can be written in the form

$$P = \sum_{\beta \in I} C_\beta x^\beta + R, \ R := \sum_{i=1}^{n+1} Q_i \frac{\partial F}{\partial x_i}, \tag{3.5}$$

$$\deg_x(R) \leq \deg_x(P), \ C_\beta \in \mathsf{R}[x_0], \ Q_i \in \mathsf{R}[x_0, x]. \tag{3.6}$$

Since x^I is a basis of V_g, we can write

$$P = \sum_{\beta \in I} c_\beta x^\beta + R', \ R' = \sum_{i=1}^{n+1} q_i \frac{\partial g}{\partial x_i}, \ c_\beta \in \mathsf{R}[x_0], \ q_i \in \mathsf{R}[x_0, x]. \tag{3.7}$$

We can choose q_i's so that

$$\deg_x(R') \leq \deg_x(P). \tag{3.8}$$

If this is not the case then we write the non-trivial homogeneous equation of highest degree obtained from (3.7). Note that $\frac{\partial g}{\partial x_i}$ is homogeneous. If some terms of P occur in this new equation then we have already (3.8). If not we subtract this new equation from (3.7). We repeat this until getting the first case and so the desired inequality. Now we have

$$\frac{\partial g}{\partial x_i} = \frac{\partial F}{\partial x_i} - x_0 \sum_{j=0}^{d-1} \frac{\partial f_j}{\partial x_i} x_0^{d-j-1},$$

and so

$$P = \sum_{\beta \in I} c_\beta x^\beta + R_1 - P_1, \tag{3.9}$$

where

$$R_1 := \sum_{i=1}^{n+1} q_i \frac{\partial F}{\partial x_i}, \ P_1 := x_0 (\sum_{i=1}^{n+1} \sum_{j=0}^{d-1} q_i \frac{\partial f_j}{\partial x_i} x_0^{d-j-1}).$$

From (3.8) we have

$$\deg_x(P_1) \leq \deg_x(P) - 1, \ \deg_x(R_1) \leq \deg_x(P).$$

We write again $q_i \frac{\partial f_j}{\partial x_i}$ in the form (3.7) and substitute it in (3.9). By degree conditions this process stops and at the end we get the equation (3.5) with the conditions (3.6).

Now let us prove that x^I generates the $\mathsf{R}[x_0]$-module V freely. If the elements of x^I are not $\mathsf{R}[x_0]$-independent then we have $\sum_{\beta \in I} C_\beta x^\beta = 0$ in V for some $C_\beta \in \mathsf{R}[x_0]$ or equivalently

$$\sum_{\beta \in I} C_\beta x^\beta = dF \wedge \omega \tag{3.10}$$

for some $\omega = \sum_{i=1}^{n+1} Q_i[x,x_0]\widehat{dx_i},\ Q_i \in \mathsf{R}[x,x_0]$, where d is the differnetial with respect to $x_i,\ i=1,2,\ldots,n+1$ and hence $dx_0 = 0$. Since F is homogenous in (x,x_0), we can assume that in the equality (3.10) the $\deg_{(x,x_0)}$ of the left hand side is $d + \deg_{(x,x_0)}(\omega)$. Let $\omega = \omega_0 + x_0\omega_1$ and ω_0 does not contain the variable x_0. In the equation obtained from (3.10) by putting $x_0 = 0$, the right hand side side must be zero otherwise we have a nontrivial relation between the elements of x^I in V_g. Therefore, we have $dg \wedge \omega_0 = 0$ and so by de Rham lemma (Proposition 3.1)

$$\omega_0 = dg \wedge \omega' = dF \wedge \omega' + x_0(\frac{g-F}{x_0}) \wedge \omega',$$

with $\deg_x(\omega_0) = d + \deg(\omega')$. Substituting this in ω and then ω in (3.10) we obtain a new ω with the property (3.10) and stricktly less $\deg_x$. $\square$

Proposition 3.5 implies that f and its last homogeneous piece have the same Milnor number.

Example 3.4. One of the most important class of tame polynomials are the so called hyperelliptic polynomials

$$f = y^2 + t_d x^d + t_{d-1}x^{d-1} + \cdots + t_1 x + t_0 \in \mathsf{R}[x,y],\ deg(x) = 2,\ \deg(y) = d,$$

with $g = y^2 + t_d x^d$. We assume that t_d is invertible in R. A R-basis of the V_g-module (and hence of V_f) is given by

$$I := \{1, x, x^2, \ldots, x^{d-2}\}.$$

In this example we have:

$$A_i = \frac{1}{2} + \frac{i+1}{d},\ \eta := \frac{1}{d}ydy - \frac{1}{2}ydx,$$

$$\frac{x^i dx}{y} = -2\frac{x^i dx \wedge dy}{df} = \frac{-2}{A_i}\nabla_{\frac{\partial}{\partial t_0}}(x^i\eta). \tag{3.11}$$

The last equalities will be explained in §3.9.

Definition 3.3. The polynomial

$$f = \sum_{\deg(x^\alpha)\leq d} t_\alpha x^\alpha \in \mathsf{R}[x],\ \mathsf{R} = \mathbb{Q}[\{t_\alpha \mid \deg(x^\alpha) \leq d\}]$$

is called a complete polynomial. Let $\tilde{\mathsf{R}} \subset \mathsf{R}$ be the polynomial ring generated by the coefficients of the last homogeneous piece g of f. Let also $\tilde{\mathsf{k}}$ be the field obtained by the localization of $\tilde{\mathsf{R}}$ over $\tilde{\mathsf{R}}\backslash\{0\}$. Assume that the polynomial $g \in \tilde{\mathsf{k}}[x]$ has an isolated singularity at the origin and so it has an isolated singularity at the origin as a polynomial in a localization $\tilde{\mathsf{R}}_a$ of R for some $a \in \mathsf{R}$. The variety $\{a = 0\}$ contains the locus of parameters for which g has not an isolated zero at the origin. It may contains more points. To find such an a we choose a monomial basis $x^\beta,\ \beta \in I$ of $\tilde{\mathsf{k}}[x]/\mathrm{jacob}(g)$ and write all $x_i x^\beta,\ \beta \in I,\ i = 1,2,\ldots,n+1$ as a $\tilde{\mathsf{k}}$-linear combination of x^β's and a residue in $\mathrm{jacob}(g)$. The product of the denominators of all the coefficients (in $\tilde{\mathsf{k}}$) used in the mentioned equalities is a candidate for a. The obtained a depends on the choice of the monomial basis.

Now, a complete polynomial is tame over $\mathsf{R}_{\tilde{\mathsf{R}}\backslash\{0\}}[x]$. An arbitrary tame polynomial $f \in \mathsf{R}[x]$ is a specialization of a unique complete tame polynomial, called the completion of f.

Remark 3.3. In the context of the article [9] the polynomial mapping $f : \mathbb{C}^{n+1} \to \mathbb{C}$ is tame if there is a compact neighborhood U of the critical points of f such that the norm of the Jacobian vector of f is bounded away from zero on $\mathbb{C}^n \backslash U$. It has been proved in the same article (Proposition 3.1) that f is tame if and only if the Milnor number of f is finite and the Milnor numbers of $f^w := f - (w_1 x_1 + \cdots + w_{n+1} x_{n+1})$ and f coincide for all sufficiently small $(w_1, \cdots, w_{n+1}) \in \mathbb{C}^{n+1}$. This and Proposition 3.5 imply that every tame polynomial in the sense of this article is also tame in the sense of [9]. However, the inverse may not be true (for instance take $f = x^2 + y^2 + x^2 y^2$, see [77] for other examples).

3.4 The discriminant of a polynomial

In the zero dimensional case $n = 0$ the discriminant of a polynomial is defined by means of its roots, see Chapter 1 §1.3. It seems that this definition cannot be generalized to an arbitrary dimension. But the second part of Proposition 1.2 gives us an alternative way to define the discriminant of a polynomial without looking for its roots. In this section we use this idea and we define the discriminant of a tame polynomial.

Let A be the R-linear map in V_f induced by multiplication by f. According to (3.5), V_f is freely generated by x^I and so we can talk about the matrix A_I of A in the basis x^I. For a new parameter s define

$$S(s) := \det(A - s \cdot I_{\mu \times \mu}),$$

where $I_{\mu\times\mu}$ is the identity μ times μ matrix and $\mu = \#I$. It has the property $S(f)\mathsf{V}_f = 0$. We define the discriminant of f to be

$$\Delta_f := S(0) \in \mathsf{R}.$$

Remark 3.4. According to Proposition 1.2 we have to multiply the above Δ_f with the number d^d in order to obtain the definition of discriminant in Chapter 1. Since in higher dimensional case we do not know a number like d^d, we restrict ourselves to the above definition and discard the definition in the zero dimensional case.

The discriminant has the following property

$$\Delta_f \cdot \mathsf{W}_f = 0. \tag{3.12}$$

Proposition 3.6. *Let R be a closed algebraic field. We have $\Delta_f = 0$ if and only if the affine variety $\{f = 0\} \subset \mathsf{R}^{n+1}$ is singular.*

Proof. $\Leftarrow$: If $\Delta_f \neq 0$ then A is surjective and $1 \in \mathsf{R}[x]$ can be written in the form $1 = \sum_{i=1}^{n+1} \frac{\partial f}{\partial x_i} q_i + qf$. This implies that the variety $Z := \{\frac{\partial f}{\partial x_i} = 0,\ i = 1, 2, \ldots, n+1,\ f = 0\}$ is empty.

$\Rightarrow$: If $\{f = 0\}$ is smooth then the variety Z is empty and so by Hilbert Nullstelensatz there exists $\tilde{f} \in \mathsf{R}[x]$ such that $f\tilde{f} = 1$ in V_f. This means that A is invertible and so $\Delta_f \neq 0$. $\square$

The above theorem implies that in the case of $\mathsf{R} = \mathbb{C}[t]$, the affine variety $\{\Delta_f(t) = 0\} \subset \mathbb{C}^s$ is the locus of parameters t such that the affine variety $\{f = 0\} \subset \mathbb{C}^{n+1}$ is singular.

Definition 3.4. For a tame polynomial f we say that the affine variety $\{f=0\}$ is smooth if the discriminant Δ_f of f is not zero.

Proposition 3.7. *Assume that f is a tame polynomial and $\Delta_f \neq 0$. If*

$$df \wedge \omega_2 = f\omega_1, \quad \textit{for some } \omega_2 \in \Omega^n_{\mathbb{U}_1/\mathbb{U}_0},\ \omega_1 \in \Omega^{n+1}_{\mathbb{U}_1/\mathbb{U}_0}$$

then

$$\omega_2 = f\omega_3 + df \wedge \omega_4,\ \omega_1 = df \wedge \omega_3, \quad \textit{for some } \omega_3 \in \Omega^n_{\mathbb{U}_1/\mathbb{U}_0},\ \omega_4 \in \Omega^{n-1}_{\mathbb{U}_1/\mathbb{U}_0}.$$

Proof. If ω_1 is not zero in W then the multiplication by f R-linear map in V_f has a non trivial kernel and so $\Delta_f = 0$ which contradicts the hypothesis. Now let $\omega_1 = df \wedge \omega_3$ and so $df \wedge (f\omega_3 - \omega_2) = 0$. The de Rham lemma for f (Proposition 3.4) finishes the proof. $\square$

The example bellow shows that the above proposition is not true for singular affine varieties.

Example 3.5. For a homogeneous polynomial g in the graded ring $\mathsf{R}[x]$, $\deg(x_i) = \alpha_i$ we have

$$g = \sum_{i=1}^{n+1} w_i x_i \frac{\partial g}{\partial x_i} \ \text{ equivalentely } gdx = dg \wedge \eta$$

and so the matrix A in the definition of the discriminat of g is the zero matrix. In particular, the discriminant of $g - s \in \mathsf{R}[s][x]$ is $(-s)^\mu$.

Example 3.6. Assume that $2d$ is invertible in R. For the hypergeometric polynomial $f := y^2 - p(x) \in \mathsf{R}[x,y]$, $\deg(p) = d$ we have $\mathsf{V}_f \cong \mathsf{V}_p$ and under this isomorphy the multiplication by f linear map in V_f coincide with the multiplication by p map in V_p. Therefore,

$$\Delta_f = \Delta_p.$$

In general Δ_f is not the the simplest element in R with the property (3.12). Therefore, we use

$$\mathrm{ZD}(\mathsf{W}_f) := \{a \in \mathsf{R} \mid a \cdot \mathsf{W}_f = 0\}.$$

From now on we deal with polynomials with non-zero discriminant.

3.5 The double discriminant of a tame polynomial

Let $f \in \mathsf{R}[x]$ be a tame polynomial. We consider a new parameter s and the tame polynomial $f - s \in \mathsf{R}[s][x]$. The discriminant Δ_{f-s} of $f-s$ as a polynomial in s has degree μ and its coefficients are in R. Its leading coefficient is $(-1)^\mu$ and so if μ is invertible in R then it is tame (as a polynomial in s) in $\mathsf{R}[s]$. Now, we take again the discriminant of Δ_{f-s} with respect to the parameter s and obtain

$$\check{\Delta} = \check{\Delta}_f := \Delta_{\Delta_{f-s}} \in \mathsf{R}$$

which is called the double discriminant of f. We consider a tame polynomial f as a function from $\bar{\mathsf{k}}^{n+1}$ to $\bar{\mathsf{k}}$. The set of critical values of f is defined to be $P = P_f := Z(\mathrm{jacob}(f))$ and the set of critical values of f is $C = C_f := f(P_f)$. It is easy to see that:

Proposition 3.8. *The tame polynomial f has μ distinct critical values (and hence distinct critical points) if and only of its double discriminant is not zero.*

Note that that μ is the maximum possible number for $\#C_f$.

3.6 De Rham cohomology

Let $f \in \mathsf{R}[x]$ be a tame polynomial as a in §3.3. The following quotients

$$\mathsf{H}' = \mathsf{H}'_f := \frac{\Omega^n_{\mathbb{U}_1}}{f\Omega^n_{\mathbb{U}_1} + df \wedge \Omega^{n-1}_{\mathbb{U}_1} + \pi^{-1}\Omega^1_{\mathbb{U}_0} \wedge \Omega^{n-1}_{\mathbb{U}_1} + d\Omega^{n-1}_{\mathbb{U}_1}}, \tag{3.13}$$

$$\cong \frac{\Omega^n_{\mathbb{U}_1/\mathbb{U}_0}}{f\Omega^n_{\mathbb{U}_1/\mathbb{U}_0} + df \wedge \Omega^{n-1}_{\mathbb{U}_1/\mathbb{U}_0} + d\Omega^{n-1}_{\mathbb{U}_1/\mathbb{U}_0}}$$

$$\mathsf{H}'' = \mathsf{H}''_f := \frac{\Omega^{n+1}_{\mathbb{U}_1}}{f\Omega^{n+1}_{\mathbb{U}_1} + df \wedge d\Omega^{n-1}_{\mathbb{U}_1} + \pi^{-1}\Omega^1_{\mathbb{U}_0} \wedge \Omega^n_{\mathbb{U}_1}} \cong \frac{\Omega^{n+1}_{\mathbb{U}_1/\mathbb{U}_0}}{f\Omega^{n+1}_{\mathbb{U}_1/\mathbb{U}_0} + df \wedge d\Omega^{n-1}_{\mathbb{U}_1/\mathbb{U}_0}} \tag{3.14}$$

are R-modules and play the role of de Rham cohomology of the affine variety

$$\{f = 0\} := \mathrm{Spec}(\frac{\mathsf{R}[x]}{f \cdot \mathsf{R}[x]}).$$

Here $\pi : \mathbb{U}_1 \to \mathbb{U}_0$ is the projection corresponding to $\mathsf{R} \subset \mathsf{R}[x]$. We have assumed that $n > 0$. For similar definitions in the case $n = 0$ see Chapter 1.

Remark 3.5. We will use H or $H^n_{\mathrm{dR}}(\{f = 0\})$ to denote one of the modules H' or H''. We note that for an arbitrary polynomial f such modules may not coincide with the de Rham cohomology of the affine variety $\{f = 0\}$ defined by Grothendieck, Atiyah and Hodge (see [39]). For instance, for $f = x(1 + xy) - t \in \mathsf{R}[x, y]$, $\mathsf{R} = \mathbb{C}[t]$ the differential forms $y^{k+1}dx + xy^k dy,\ k > 0$ are not zero in the corresponding H' but they are relatively exact and so zero in $H^1_{\mathrm{dR}}(\{f = 0\})$ (see [5]).

One may call H' and H'' the Brieskorn modules associated to f in analogy to the local modules introduced by Brieskorn in 1970. In fact, the classical Brieskorn modules are

$$H' = H'_f = \frac{\Omega^n_{\mathbb{U}_1/\mathbb{U}_0}}{df \wedge \Omega^{n-1}_{\mathbb{U}_1/\mathbb{U}_0} + d\Omega^{n-1}_{\mathbb{U}_1/\mathbb{U}_0}},\ H'' = H''_f := \frac{\Omega^{n+1}_{\mathbb{U}_1/\mathbb{U}_0}}{df \wedge d\Omega^{n-1}_{\mathbb{U}_1/\mathbb{U}_0}}.$$

We consider them as $\mathsf{R}[f]$-modules. In [65] we have worked with the classical ones.

Remark 3.6. The $\mathsf{R}[f]$-module H'_f is isomorphic to the $\mathsf{R}[s]$-module $\mathsf{H}'_{\tilde{f}}$, where $\tilde{f} = f - s \in \mathsf{R}[s][x]$ and s is a new parameter. A similar statement is true for the other Brieskorn module.

Remark 3.7. We have the following well-defined R-linear map

$$\mathsf{H}' \to \mathsf{H}'',\ \omega \mapsto df \wedge \omega$$

which is an inclusion by Proposition 3.7. When we write $\mathsf{H}' \subset \mathsf{H}''$ then we mean the inclusion obtained by the above map. We have

$$\frac{\mathsf{H}''}{\mathsf{H}'} = \mathsf{W}_f.$$

For $\omega \in \mathsf{H}''$ we define the Gelfand-Leray form

$$\frac{\omega}{df} := \frac{\omega'}{\Delta} \in \mathsf{H}'_\Delta, \quad \text{where } \Delta \cdot \omega = df \wedge \omega'.$$

Let us first state the main results of this section.

Theorem 3.1. *Let R be of characteristic zero and $\mathbb{Q} \subset \mathsf{R}$. If f is a tame polynomial in $\mathsf{R}[x]$ then the $\mathsf{R}[f]$-modules H'' and H' are free and $\omega_\beta, \beta \in I$ (resp. η_β, $\beta \in I$) form a basis of H'' (resp. H'). More precisely, every $\omega \in \Omega^{n+1}_{\mathbb{U}_1/\mathbb{U}_0}$ (resp. $\omega \in \Omega^{n}_{\mathbb{U}_1/\mathbb{U}_0}$) can be written*

$$\omega = \sum_{\beta \in I} p_\beta(f)\omega_\beta + df \wedge d\xi,\ p_\beta \in \mathsf{R}[f],\ \xi \in \Omega^{n-1}_{\mathbb{U}_1/\mathbb{U}_0},\ \deg(p_\beta) \leq \frac{\deg(\omega)}{d} - A_\beta \tag{3.15}$$

(resp.

$$\omega = \sum_{\beta \in I} p_\beta(f)\eta_\beta + df \wedge \xi + d\xi_1,\ p_\beta \in \mathsf{R}[t],\ \xi, \xi_1 \in \Omega^{n-1}_{\mathbb{U}_1/\mathbb{U}_0},\ \deg(p_\beta) \leq \frac{\deg(\omega)}{d} - A_\beta \tag{3.16}$$

).

We will prove the above theorem in §3.7 and §3.8.

Corollary 3.1. *Let R be of characteristic zero and $\mathbb{Q} \subset \mathsf{R}$. If f is a tame polynomial in $\mathsf{R}[x]$ then the R-modules H' and H'' are free and $\eta_\beta, \beta \in I$ (resp. ω_β, $\beta \in I$) form a basis of H' (resp. H'').*

Note that in the above corollary $\{f = 0\}$ may be singular. We call $\eta_\beta, \beta \in I$ (resp. ω_β, $\beta \in I$) the canonical basis of H' (resp. H'').

Proof. We prove the corollary for H'. The proof for H'' is similar. We consider the following canonical exact sequence

$$0 \to fH' \to H' \to \mathsf{H}' \to 0$$

Using this, Theorem 3.1 implies that H' is generated by η_β, $\beta \in I$. It remains to prove that η_β's are R-linear independent. If $a := \Sigma_{\beta \in I} r_\beta \eta_\beta = 0$, $r_\beta \in \mathsf{R}$ in H' then $a = fb$, $b \in H'$. We write b as a $\mathsf{R}[f]$-linear combination of η_β's and we obtain $r_\beta = fc_\beta(f)$ for some $c_\beta(f) \in \mathsf{R}[f]$. This implies that for all $\beta \in I$, $r_\beta = 0$. □

Theorem 3.1 is proved first for the case $\mathsf{R} = \mathbb{C}$ in [65]. In this article we have used a topological argument to prove that the forms $\omega_\beta, \beta \in I$ (resp. η_β, $\beta \in I$) are $\mathsf{R}[f]$-linear independent. It is based on the following facts: 1. η_β's generates the $\mathbb{C}[f]$-module H', 2. $\#I = \mu$ is the dimension of $H^n_{\mathrm{dR}}(\{f = c\})$ for a regular value $c \in \mathbb{C} - C$, 3. H' restricted to $\{f = 0\}$ is isomorphic to $H^n_{\mathrm{dR}}(\{f = c\})$. In the forthcomming sections we present an algebraic proof.

3.7 Proof of Theorem 3.1 for a homogeneous tame polynomial

Let $f = g$ be a homogeneous tame polynomial with an isolated singularity at origin. We explain the algorithm which writes every element of H'' of g as a $\mathsf{R}[g]$-linear combination of ω_β's. Recall that

$$dg \wedge d(P\widehat{dx_i, dx_j}) = (-1)^{i+j+\epsilon_{i,j}}\left(\frac{\partial g}{\partial x_j}\frac{\partial P}{\partial x_i} - \frac{\partial g}{\partial x_i}\frac{\partial P}{\partial x_j}\right)dx,$$

where $\epsilon_{i,j} = 0$ if $i < j$ and $= 1$ if $i > j$ and $\widehat{dx_i, dx_j}$ is dx without dx_i and dx_j (we have not changed the order of $dx_1, dx_2, \ldots$ in dx).

Proposition 3.9. *In the case $n > 0$, for a monomial $P = x^\beta$ we have*

$$\frac{\partial g}{\partial x_i} \cdot P dx = \frac{d}{d \cdot A_\beta - \alpha_i} \frac{\partial P}{\partial x_i} g dx + dg \wedge d(\sum_{j \neq i} \frac{(-1)^{i+j+1+\epsilon_{i,j}} \alpha_j}{d \cdot A_\beta - \alpha_i} x_j P \widehat{dx_i, dx_j}). \tag{3.17}$$

Proof. The proof is a straightforward calculation:

$$\sum_{j \neq i} \frac{(-1)^{i+j+1+\epsilon_{i,j}} \alpha_j}{d \cdot A_\beta - \alpha_i} dg \wedge d(x_j P \widehat{dx_i, dx_j}) =$$

$$\frac{-1}{d \cdot A_\beta - \alpha_i} \sum_{j \neq i} (\alpha_j \frac{\partial g}{\partial x_j} \frac{\partial (x_j P)}{\partial x_i} - \alpha_j \frac{\partial g}{\partial x_i} \frac{\partial (x_j P)}{\partial x_j}) dx =$$

$$\frac{-1}{d \cdot A_\beta - \alpha_i} ((d \cdot g - \alpha_i x_i \frac{\partial g}{\partial x_i}) \frac{\partial P}{\partial x_i} - P \frac{\partial g}{\partial x_i} \sum_{j \neq i} \alpha_j (\beta_j + 1)) dx =$$

$$\frac{-1}{d \cdot A_\beta - \alpha_i} (d \cdot g \frac{\partial P}{\partial x_i} - \alpha_i \beta_i P \frac{\partial g}{\partial x_i} - P \frac{\partial g}{\partial x_i} \sum_{j \neq i} \alpha_j (\beta_j + 1)) dx.$$

□

The only case in which $dA_\beta - \alpha_i = 0$ is when $n = 0$ and $P = 1$. In the case $n = 0$ for $P \neq 1$ we have

$$\frac{\partial g}{\partial x} \cdot P dx = \frac{d}{d \cdot A_\beta - \alpha} \frac{\partial P}{\partial x} g dx = \frac{d}{\alpha} x^{\beta - 1} g dx$$

and if $P = 1$ then $\frac{\partial g}{\partial x_i} \cdot P dx$ is zero in H''. Based on this observation the following works also for $n = 0$.

We use the above Proposition to write every $P dx \in \Omega^{n+1}_{\mathbb{U}_1/\mathbb{U}_0}$ in the form

$$P dx = \sum_{\beta \in I} p_\beta(g) \omega_\beta + dg \wedge d\xi, \tag{3.18}$$

$$p_\beta \in \mathsf{R}[g],\ \xi \in \Omega^{n-1}_{\mathbb{U}_1/\mathbb{U}_0},\ \deg(p_\beta(g)\omega_\beta),\ \deg(dg \wedge d\xi) \leq \deg(P dx).$$

- Input: The homogeneous tame polynomial g and $P \in \mathsf{R}[x]$ representing $[P dx] \in H''$. Output: $p_\beta, \beta \in I$ and ξ satisfying (3.18)

  We write

  $$P dx = \sum_{\beta \in I} c_\beta x^\beta \cdot dx + dg \wedge \eta,\ \deg(dg \wedge \eta) \leq \deg(P dx). \tag{3.19}$$

  Then we apply (3.17) to each monomial component $\tilde{P} \frac{\partial g}{\partial x_i}$ of $dg \wedge \eta$ and then we write each $\frac{\partial \tilde{P}}{\partial x_i} dx$ in the form (3.19). The degree of the components which make $P dx$ not to be of the form (3.18) always decreases and finally we get the desired form.

To find a similar algorithm for H' we note that if $\eta \in \Omega^n_{\mathbb{U}_1/\mathbb{U}_0}$ is written in the form

$$\eta = \sum_{\beta \in I} p_\beta(g) \eta_\beta + dg \wedge \xi + d\xi_1,\ p_\beta \in \mathsf{R}[g],\ \xi, \xi_1 \in \Omega^{n-1}_{\mathbb{U}_1/\mathbb{U}_0}, \tag{3.20}$$

where each piece in the right hand side of the above equality has degree less than $\deg(\eta)$, then

$$d\eta = \sum_{\beta\in I}(p_\beta(g)A_\beta + p'_\beta(g)g)\omega_\beta - dg\wedge d\xi \tag{3.21}$$

and the inverse of the map $\mathsf{R}[g]\to\mathsf{R}[g],\ p(g)\mapsto A_\beta.p(g)+p'(g)\cdot g$ is given by

$$\sum_{i=0}^{k} a_i g^i \to \sum_{i=1}^{k}\frac{a_i}{A_\beta+i}g^i. \tag{3.22}$$

Now let us prove that there is no $\mathsf{R}[g]$-relation between ω_β's in H''_g. This implies also that there is no $\mathsf{R}[g]$ relation between η_β's in H'_g. If such a relation exists then we take its differential and since $dg\wedge\eta_\beta = g\omega_\beta$ and $d\eta_\beta = A_\beta\omega_\beta$ we obtain a nontrivial relation in H''_g.

Since $g = dg\wedge\eta$ and x^β are R-linear independent in V_g, the existence of a non trivoal $\mathsf{R}[g]$-relation between ω_β's in H''_g implies that there is a $0\neq\omega\in H''_g$ such that $g\omega=0$ in H''_g. Therefore, we have to prove that H''_g has no torsion. Let $a\in\mathsf{R}[x]$ and

$$g\cdot a\cdot dx = dg\wedge d\omega_1,\quad \text{for some } \omega_1\in\Omega^{n-1}_{\mathbb{U}_1/\mathbb{U}_0}. \tag{3.23}$$

Since g is homogeneous, we can assume that a is also homogeneous. Now, the above equality implies that

$$dg\wedge(a\eta - d\omega_1) = 0 \overset{\text{Proposition 3.1}}{\Longrightarrow} a\eta = d\omega_1 + dg\wedge\omega_2,\quad \text{for some } \omega_2\in\Omega^{n-1}_{\mathbb{U}_1/\mathbb{U}_0}.$$

We take differential of the above equality and we conclude that

$$(\sum_{i=1}^{n+1} w_i + \frac{\deg(a)}{d})a\cdot dx = 0 \ \text{ in } H''_g.$$

Since $\mathbb{Q}\subset\mathsf{R}$, we conclude that $adx=0$ in H''_g.

Remark 3.8. The reader may have already noticed that Theorem 3.1 is not at all true if R has characteristic different form from zero. In the formulas (3.22) and (3.17) we need to divide over $d\cdot A_\beta - \alpha_i$ and $A_\beta + i$. Also, to prove that H''_g has no torsion we must be able to divide on $\sum_{i=1}^{n+1} w_i + \frac{\deg(a)}{d}$.

3.8 Proof of Theorem 3.1 for an arbitrary tame polynomial

We explain the algorithm which writes every element of H'' of f as a $\mathsf{R}[f]$-linear combination of ω_β's. We write an element $\omega\in\Omega^{n+1}_{\mathbb{U}_1/\mathbb{U}_0}, \deg(\omega)=m$ in the form

$$\omega = \sum_{\beta\in I} p_\beta(g)\omega_\beta + dg\wedge d\psi,\ p_\beta\in\mathsf{R}[g],\ \psi\in\Omega^{n-1}_{\mathbb{U}_1/\mathbb{U}_0},\ \deg(p_\beta(g)\omega_\beta)\leq m,\ \deg(d\psi)\leq m-d.$$

This is possible because g is homogeneous. Now, we write the above equality in the form

$$\omega = \sum_{\beta\in I} p_\beta(f)\omega_\beta + df\wedge d\psi + \omega',\ \text{with } \omega' = \sum_{\beta\in I}(p_\beta(g)-p_\beta(f))\omega_\beta + d(g-f)\wedge d\psi.$$

The degree of ω' is strictly less than m and so we repeat what we have done at the beginning and finally we write ω as a $\mathsf{R}[f]$-linear combination of ω_β's.

The algorithm for H' is similar and uses the fact that for $\eta \in \Omega^n_{\mathbb{U}_1/\mathbb{U}_0}$ one can write

$$\eta = \sum_{\beta \in I} p_\beta(g)\eta_\beta + dg \wedge \psi_1 + d\psi_2 \tag{3.24}$$

and each piece in the right hand side of the above equality has degree less than $\deg(\eta)$.

Let us now prove that the forms $\omega_\beta, \beta \in I$ (resp. $\eta_\beta,\ \beta \in I$) are $\mathsf{R}[f]$-linear independent. If there is a $\mathsf{R}[f]$-relation between ω_β's in H''_f, namely

$$\sum_{\beta \in I} p_\beta(f)\omega_\beta = df \wedge d\omega,\ \omega \in \Omega^{n-1}_{\mathbb{U}_1/\mathbb{U}_0}, \tag{3.25}$$

then by taking the last homogeneous piece of the relation, we obtain a nontrivial $\mathsf{R}[g]$-relations between ω_β's in H''_g or

$$dg \wedge d\omega_1 = 0,\ \omega_1 \in \Omega^{n-1}_{\mathbb{U}_1/\mathbb{U}_0},$$

where $\omega = \omega_1 + \omega_1'$ with $\deg(\omega_1') < \deg(\omega_1) = \deg(\omega)$. The first case does not happen by the proof of our theorem in the $f = g$ case (see §3.7). In the second case we use Proposition 3.10 and its Proposition 3.3 and obtain

$$d\omega_1 = dg \wedge d\omega_2,\ \omega_2 \in \Omega^{n-2}_{\mathbb{U}_1/\mathbb{U}_0},\ \deg(d\omega_1) = d + \deg(d\omega_2).$$

Now

$$df \wedge d\omega = df \wedge d(\omega_1 + \omega_1') = df \wedge (d(g - f) \wedge d\omega_1 + d\omega_1').$$

This means that we can substitute ω with another one and with less $\deg_x$. Taking ω the one with the smallest degree and with the property (3.25), we get a contradiction. In the case of H'_f the proof is similar and is left to the reader.

3.9 Gauss-Manin connection

The Tjurina module of f can be rewritten in the form

$$\mathsf{W}_f := \frac{\Omega^{n+1}_{\mathbb{U}_1}}{df \wedge \Omega^n_{\mathbb{U}_1} + f\Omega^{n+1}_{\mathbb{U}_1} + \pi^{-1}\Omega^1_{\mathbb{U}_0} \wedge \Omega^n_{\mathbb{U}_1}} \cong \frac{\Omega^{n+1}_{\mathbb{U}_1/\mathbb{U}_0}}{df \wedge \Omega^n_{\mathbb{U}_1/\mathbb{U}_0} + f\Omega^{n+1}_{\mathbb{U}_1/\mathbb{U}_0}}.$$

Looking in this way, we have the well defined differential map

$$d : \mathsf{H}' \to \mathsf{W}_f.$$

We define the Gauss-Manin connection on H' as follows:

$$\nabla : \mathsf{H}' \to \Omega^1_T \otimes_{\mathsf{R}} \mathsf{H}'$$

$$\nabla\omega = \frac{1}{\Delta}\sum_i \alpha_i \otimes \beta_i,$$

where

$$\Delta d\omega - \sum_i \alpha_i \wedge \beta_i \in f\Omega^{n+1}_{\mathbb{U}_1/\mathbb{U}_0} + df \wedge \Omega^n_{\mathbb{U}_1/\mathbb{U}_0},\ \alpha_i \in \Omega^1_{\mathbb{U}_0},\ \beta_i \in \Omega^1_{\mathbb{U}_1},$$

and Ω^1_T is the localization of $\Omega^1_{\mathbb{U}_0}$ on the multiplicative group $\{1, \Delta, \Delta^2, \dots\}$. From scheme theory point of view this is the set of differential forms defined in

$$T = \mathrm{Spec}(\mathsf{R}_\Delta) = \mathbb{U}_0 \setminus \{\Delta = 0\}.$$

When $\mathsf{R} = \mathbb{C}[t]$ we will identify T with the complex manifold $\mathbb{C}^s \backslash \{\Delta = 0\}$.

To define the Gauss-Manin connection on H'' we use the fact that $\frac{\mathsf{H}''}{\mathsf{H}'} = \mathsf{W}_f$ and define

$$\nabla : \mathsf{H}'' \to \Omega^1_T \otimes_{\mathsf{R}} \mathsf{H}'',$$

$$\nabla(\omega) = \nabla(\frac{\Delta \cdot \omega}{\Delta}) = \frac{\nabla(\Delta \cdot \omega) - d\Delta \otimes \omega}{\Delta}, \tag{3.26}$$

where $\Delta \cdot \omega = df \wedge \eta, \ \eta \in \mathsf{H}'$.

Proposition 3.10. *Let R be a localization of $\mathbb{C}[t]$ and U be an small open set in T and $\{\delta_t\}_{t \in U}, \delta_t \in H_n(\{f = 0\}, \mathbb{Z})$ be a continuous family of topological n dimensional cycles. Then*

$$d(\int_{\delta_t} \omega) = \sum \alpha_i \int_{\delta_t} \beta_i, \ \nabla\omega = \sum_i \alpha_i \otimes \beta_i, \ \alpha_i \in \Omega^1_T, \ \beta_i \in \mathsf{H}'. \tag{3.27}$$

Proof. See [1] for similar statements and their proof. The sketch of the proof is as follows: Since vanishing cycles generate the n-the cohomology of $\{f = 0\}$, we assume that δ_t is a vanishing cycle and so there exists an $n+1$-dimensional real thimble $D_t = \cup_{s \in [0,1]} \delta_{\gamma(s)}$ such that γ is a path in $\mathbb{C}^s$ connecting t to some point in $\Delta = 0$ and δ_s is the trace of δ_t when it vanishes along γ. We have

$$\int_{\delta_t} \omega = \int_{\Delta_t} d\omega = \sum_i \int_{D_t} \alpha_i \wedge \beta_i = \sum_i \int_s \alpha_i (\int_{\delta_s} \beta_i).$$

In the first equality we have used Stokes Lemma and in the second equality we have used the fact that the integral of the elements of $f\Omega^{n+1}_{\mathbb{U}_1/\mathbb{U}_0} + df \wedge \Omega^n_{\mathbb{U}_1/\mathbb{U}_0}$ on D_t is zero. □

The connection ∇ defines the operators

$$\nabla_i = \nabla : \Omega^i_T \otimes_{\mathsf{R}} \mathsf{H} \to \Omega^{i+1}_T \otimes_{\mathsf{R}} \mathsf{H}.$$

If there is no danger of confusion we will use the symbol ∇ for these operators too. The connection ∇ is an integrable connection, i.e. $\nabla \circ \nabla = 0$. It is is useful to look at the Guass-Manin connection in the following way: We have

$$\mathcal{D}_{\mathbb{U}_0} \to \mathrm{End}_{\mathbb{C}}(\mathsf{H}_\Delta), \ v \mapsto \nabla_v,$$

where $\mathcal{D}_{\mathbb{U}_0}$ is the set of vector fields in $\mathbb{U}_0$ (see §1.10) and ∇_v is the composition

$$\mathsf{H}_\Delta \xrightarrow{\nabla} \Omega^1_T \otimes_{\mathsf{R}_\Delta} \mathsf{H}_\Delta \xrightarrow{v \otimes 1} \mathsf{R}_\Delta \otimes_{\mathsf{R}_\Delta} \mathsf{H}_\Delta \cong \mathsf{H}_\Delta.$$

Note that

$$\nabla_v(r\omega) = r\nabla(\omega) + v(r) \cdot \omega, \ v \in \mathcal{D}_{\mathbb{U}_0}, \ \omega \in \mathsf{H}_\Delta, \ r \in \mathsf{R}_\Delta.$$

In particular, note that we can now iterate ∇_v, i.e. $\nabla_v^s = \nabla_v \circ \nabla_v \circ \cdots \circ \nabla_v$ s-times, and this is different from $\nabla \circ \nabla$ introduced before.

Let $\omega_1, \omega_2, \ldots, \omega_\mu$ be a basis of H and define $\omega = [\omega_1, \omega_2, \ldots, \omega_\mu]^{\mathsf{t}}$. The Gauss-Manin connection in this basis can be written in the following way:

$$\nabla\omega = A \otimes \omega, \ A \in \frac{1}{\Delta}\mathrm{Mat}^{\mu\times\mu}(\Omega^1_{\mathbb{U}_0}) \tag{3.28}$$

The integrability condition translates into $dA = A \wedge A$.

Remark 3.9. From (3.27) it follows that

$$\frac{\partial}{\partial v}\int_{\delta_t}\omega = \int_{\delta_t}\nabla_v\omega, \ \forall\omega \in \mathsf{H}, \ v \in \mathcal{D}_{\mathbb{U}_0} \tag{3.29}$$

and for any continuous family of cycles δ_t in a small neighborhood in T. For a fixed v, the operator $\nabla_v : \mathsf{H} \to \mathsf{H}_\Delta$ with the above property is unique. This follows from the fact that if $\omega \in \mathsf{H}$ restricted to all regular fibers of f is exact then ω is zero in H (a consequence of Corollary 3.1). If we want to prove an identity for the Gauss-Manin connection of a tame polynomial f over an arbitrary ring R (not necessarily functional), then we can consider the corresponding tame polynomial over a function field, use (3.29) in order to prove the equality and then conclude the same equality over the ring R.

3.10 Calculating Gauss-Manin connection

Let

$$\tilde{d} : \Omega^\bullet_{\mathbb{U}_1} \to \Omega^{\bullet+1}_{\mathbb{U}_1}$$

be the differential map with respect to variable x, i.e. $\tilde{d}r = 0$ for all $r \in \mathsf{R}$, and

$$\check{d} : \Omega^\bullet_{\mathbb{U}_1} \to \Omega^{\bullet+1}_{\mathbb{U}_1}$$

be the differential map with respect to the elements of R. It is the pull-back of the differential in $\mathbb{U}_0$. We have

$$d = \tilde{d} + \check{d},$$

where d is the total differential mapping. Let s be a new parameter and $S(s)$ be the discriminant of $f - s$. We have

$$S(f) = \sum_{i=1}^{n+1} p_i \frac{\partial f}{\partial x_i}, \ p_i \in \mathsf{k}[x]$$

or equivalently

$$S(f)dx = df \wedge \eta_f, \ \eta_f = \sum_{i=1}^{n+1}(-1)^{i-1}p_i\widehat{dx_i}. \tag{3.30}$$

To calculate ∇ of

$$\omega = \sum_{i=1}^{n+1} P_i\widehat{dx_i} \in \mathsf{H}'$$

we assume that ω has no dr, $r \in \mathsf{R}$, but the ingredient polynomials of ω may have coeficients in R. Let $\Delta = S(0)$ and

$$\tilde{d}\omega = P \cdot dx.$$

We have

$$S(f)d\omega = S(f)\tilde{d}\omega + S(f)\sum_{i=1}^{n+1} \check{d}P_i \wedge \widehat{dx_i} = \tilde{d}f \wedge (P \cdot \eta_f) + S(f)\sum_{i=1}^{n+1} \check{d}P_i \wedge \widehat{dx_i} \Rightarrow$$

$\Delta d\omega =$

$$\begin{aligned} &= (\Delta - S(f))(d\omega - \sum_{i=1}^{n+1} \check{d}P_i \wedge \widehat{dx_i}) + df \wedge (P \cdot \eta_f) + (\Delta \sum_{i=1}^{n+1} \check{d}P_i \wedge \widehat{dx_i}) - \check{d}f \wedge (P \cdot \eta_f) \\ &= (\Delta \sum_{i=1}^{n+1} \check{d}P_i \wedge \widehat{dx_i}) - \check{d}f \wedge (P \cdot \eta_f) \text{ in } \Omega^1_{\mathbb{U}_0} \otimes \mathsf{H}' \\ &= \sum_j dt_j \wedge \left(\Delta (\sum_{i=1}^{n+1} \frac{\partial P_i}{\partial t_j} \widehat{dx_i}) - \frac{\partial f}{\partial t_j} \cdot P \cdot \eta_f \right), \quad \text{in } \Omega^1_{\mathbb{U}_0} \otimes \mathsf{H}'. \end{aligned}$$

We conclude that

$$\nabla(\omega) = \frac{1}{\Delta} \left(\sum_j dt_j \otimes \left(\sum_{i=1}^{n+1} \left(\Delta \frac{\partial P_i}{\partial t_j} - (-1)^{i-1} \frac{\partial f}{\partial t_j} \cdot P \cdot p_i \right) \widehat{dx_i} \right) \right), \tag{3.31}$$

where

$$P = \sum_{i=1}^{n+1} (-1)^{i-1} \frac{\partial P_i}{\partial x_i}.$$

It is useful to define

$$\frac{\partial \omega}{\partial t_j} = \sum_{i=1}^{n+1} \frac{\partial P_i}{\partial t_j} \widehat{dx_i}.$$

Then

$$\nabla(\omega) = \frac{1}{\Delta} \left(\sum_j dt_j \otimes \left(\Delta \frac{\partial \omega}{\partial t_j} - \frac{\partial f}{\partial t_j} \cdot P \cdot \eta_f \right) \right). \tag{3.32}$$

The calculation of ∇ in H'' can be done using

$$\nabla(P \cdot dx) = \frac{df \wedge \nabla(P\eta_f) - d\Delta \otimes Pdx}{\Delta}, \ Pdx \in \mathsf{H}''$$

which is derived from (3.26). Note that we calculate $\nabla(P \cdot \eta_f)$ from (3.31). We lead to the following explicit formula

$$\nabla(P \cdot dx) = \frac{1}{\Delta} \left(\sum_j dt_j \otimes \left(\tilde{d}f \wedge \frac{\partial (P\eta_f)}{\partial t_j} - \frac{\partial f}{\partial t_j} Q_P - \frac{\partial \Delta}{\partial t_j} P \right) \right), \tag{3.33}$$

where

$$Q_P = \sum_{i=1}^{n+1} (\frac{\partial P}{\partial x_i} p_i + P \frac{\partial p_i}{\partial x_i}).$$

To be able to calculate the iterations of the Gauss-Manin connection along a vector field v in $\mathbb{U}_0$, it is useful to introduce the operators:

$$\nabla_{v,k} : \mathsf{H} \to \mathsf{H},\ k = 0, 1, 2, \ldots$$

$$\nabla_{v,k}(\omega) = \nabla_v(\frac{\omega}{\Delta^k})\Delta^{k+1} = \Delta \cdot \nabla_v(\omega) - k \cdot d\Delta(v) \cdot \omega.$$

It is easy to show by induction on k that

$$\nabla_v^k = \frac{\nabla_{v,k-1} \circ \nabla_{v,k-2} \circ \cdots \circ \nabla_{v,0}}{\Delta^k}. \tag{3.34}$$

Remark 3.10. The formulas (3.33) and (3.32) for the Gauss-Manin connection usually produce polynomials of huge size, even for simple examples. Specially when we want to iterate the Gauss-Manin connection along a vector field, the size of polynomials is so huge that even with a computer (of the time of writing this text) we get the lack of memory problem. However, if we write the result of the Gauss-Manin connection, in the canonical basis of the R-module H, and hence reduce it modulo to those differential forms which are zero in H, we get polynomials of reasonable size.

3.11 Gauss-Manin system

In this section we define the Gauss-Manin system associated to a tame polynomial. Our approach is by looking at differential forms with poles along $\{f = 0\}$ in $\mathbb{U}_1/\mathbb{U}_0$ which is a proper way when one deals with the tame polynomials in the sense of present text. We will later use the material of this section for residue of such differential forms along the pole $\{f = 0\}$.

The Gauss-Manin system for a tame polynomial f is defined to be:

$$\mathsf{M}_f = \mathsf{M} := \frac{\Omega^{n+1}_{\mathbb{U}_1/\mathbb{U}_0}[\frac{1}{f}]}{\Omega^{n+1}_{\mathbb{U}_1/\mathbb{U}_0} + d(\Omega^n_{\mathbb{U}_1/\mathbb{U}_0}[\frac{1}{f}])} \cong \frac{\Omega^{n+1}_{\mathbb{U}_1}[\frac{1}{f}]}{\Omega^{n+1}_{\mathbb{U}_1} + d(\Omega^n_{\mathbb{U}_1}[\frac{1}{f}]) + \pi^{-1}\Omega^1_{\mathbb{U}_0} \wedge \Omega^n_{\mathbb{U}_1}}.$$

It has a natural filtration given by the pole order along $\{f = 0\}$, namely

$$\mathsf{M}_i := \{[\frac{\omega}{f^i}] \in \mathsf{M} \mid \omega \in \Omega^{n+1}_{\mathbb{U}_1/\mathbb{U}_0}\},\ \mathsf{M}_1 \subset \mathsf{M}_2 \subset \cdots \subset \mathsf{M}_i \subset \cdots \subset \mathsf{M}.$$

We have well-defined canonical maps

$$\mathsf{H}'' \to \mathsf{M}_1, \omega \mapsto [\frac{\omega}{f}],$$

$$\mathsf{W} \to \mathsf{M}_i/\mathsf{M}_{i-1},\ \omega \mapsto [\frac{\omega}{f^i}].$$

Note that in M we have

$$[\frac{d\omega}{f^{i-1}}] = [\frac{(i-1)df \wedge \omega}{f^i}],\ \omega \in \Omega^n_{\mathbb{U}_1/\mathbb{U}_0}, i = 2, 3, \ldots$$

$$[\frac{df\wedge d\omega}{f^i}]=[d(\frac{df\wedge\omega}{f^i})]=0,\ \omega\in\Omega^{n-1}_{\mathbb{U}_1/\mathbb{U}_0}, i=1,2,\ldots$$

Both maps are isomorphims of R-modules. The non-trivial part is that they are injective. This follows from the following:

Proposition 3.11. *If the discriminant of a the tame polynomial f is not zero then the differential form $\frac{\omega}{f^i}$, $i\in\mathbb{N}$, $\omega\in\Omega^{n+1}_{\mathbb{U}_1/\mathbb{U}_0}$ is zero in M if and only if ω is of the form*

$$fd\omega_1-(i-1)df\wedge\omega_1+df\wedge d\omega_2+f^i\omega_3,\ \omega_1\in\Omega^{n}_{\mathbb{U}_1/\mathbb{U}_0},\ \omega_2\in\Omega^{n-1}_{\mathbb{U}_1/\mathbb{U}_0},\ \omega_3\in\Omega^{n+1}_{\mathbb{U}_1/\mathbb{U}_0}.$$

Proof. Let

$$\frac{\omega}{f^i}=d(\frac{\omega_1}{f^s})\mod\Omega^{n+1}_{\mathbb{U}_1/\mathbb{U}_0}. \tag{3.35}$$

If $s=i-1$ then ω has the desired form. If $s\geq i$ then $df\wedge\omega_1\in f\Omega^{n+1}_{\mathbb{U}_1/\mathbb{U}_0}$ and so by Proposition 3.7 we have $\omega_1=f\omega_3+df\wedge\omega_2$ and so

$$\frac{\omega}{f^i}=d(\frac{f\omega_3+df\wedge\omega_2}{f^s}),\quad \mod\Omega^{n+1}_{\mathbb{U}_1/\mathbb{U}_0}. \tag{3.36}$$

If $s=i$ then we obtain the desired form for ω. If $s>i$ we get $df\wedge d\omega_2+(s-1)df\wedge\omega_3\in f\Omega^{n+1}_{\mathbb{U}_1/\mathbb{U}_0}$ and so again by Proposition 3.7 we have $d\omega_2+(s-1)\omega_3=f\omega_4+df\wedge\omega_5$. We calculate ω_3 from this equality and substitute it in (3.36) and obtain

$$\frac{\omega}{f^i}=\frac{1}{s-1}d(\frac{f\omega_4+df\wedge\omega_5}{f^{s-1}})\mod\Omega^{n+1}_{\mathbb{U}_1/\mathbb{U}_0}.$$

We repeat this until getting the situation $s=i$. □

For any vector field in $\mathbb{U}_0$, we have a well-defined map

$$\nabla_v:\mathsf{M}\to\mathsf{M},\ \nabla=v[\frac{Pdx}{f^i}]=[\frac{\frac{\partial P}{\partial v}f-iP\frac{\partial f}{\partial v}}{f^{i+1}}dx],\ P\in\mathsf{R}[x]. \tag{3.37}$$

where $\frac{\partial P}{\partial v}$ is the differential of P with respect to the parameters in R and along the vector field v. By definition it maps M_i to M_{i+1}, $i\in\mathbb{N}$. This is also called the Gauss-Manin connection along the vector field. It is useful to identify H' by its image under $df\wedge\cdot$ in H'' and define $\mathsf{M}_0:=\mathsf{H}'$. In this case

$$\nabla_v([\frac{df\wedge\omega}{f}])=[\frac{f\frac{\partial(df\wedge\omega)}{\partial v}-\frac{\partial f}{\partial v}df\wedge\omega}{f^2}]=[\frac{\frac{\partial(df\wedge\omega)}{\partial v}+d(\frac{\partial f}{\partial v}\omega)}{f}]$$

and so ∇_v maps M_0 to M_1. To see the relation of the Gauss-Manin connection of this section with the Gauss-Manin connection of §3.9 we need the following proposition:

Proposition 3.12. *The multiplication by Δ in M maps M_i to M_{i-1} for all $i\in\mathbb{N}$.*

Proof. The multiplication by Δ in W is zero and so for a given $\frac{\omega}{f^i}$ we can write

$$\Delta\frac{\omega}{f^i}=\frac{f\omega_1+df\wedge\omega_2}{f^i}=\frac{\omega_1}{f^{i-1}}+\frac{1}{i-1}(\frac{d\omega_2}{f^{i-1}}-d(\frac{\omega_2}{f^{i-1}}))$$

which is equal to $\frac{\omega_1+\frac{1}{i-1}d\omega_2}{f^{i-1}}$ in M. □

Now, it is easy to see that $\Delta\cdot\nabla_v:\mathsf{H}\to\mathsf{H},\ \mathsf{H}=\mathsf{H}',\mathsf{H}''$ of this section and §3.9 coincide.

3.12 The case of a homogeneous polynomial

Bellow for simplicity we use d to denote the differential operator with respect to the variables $x_1, x_2, \ldots, x_{n+1}$. Let us consider a homogenous polynomial g in the graded ring $\mathsf{R}[x],\ \deg(x_i) = \alpha_i$. We have the equalities

$$g = \sum_{i=1}^{n+1} w_i x_i \frac{\partial g}{\partial x_i} \text{ equivalentely } g dx = dg \wedge \eta,$$

$$g\omega_\beta = dg \wedge \eta_\beta,\ d\eta = (w \cdot 1)dx,\ d\eta_\beta = A_\beta \omega_\beta.$$

The discriminant of the polynomial g is zero. We define $f := g - t \in \mathsf{R}[t][x]$ which is tame and its discriminant is $(-t)^\mu$. The above qualities imply that

$$\nabla_{\frac{\partial}{\partial t}} \eta_\beta = \frac{A_\beta}{t}\eta_\beta,\ \nabla_{\frac{\partial}{\partial t}}(\omega_\beta) = \frac{(A_\beta - 1)}{t}\omega_\beta.$$

This implies that in the case $\mathsf{R} = \mathbb{C}$ we have

$$\frac{\partial}{\partial t}\int_{\delta_t} \eta_\beta = \frac{A_\beta}{t}\int_{\delta_t} \eta_\beta.$$

Therefore there exists a constant number C depending only on η_β and δ_t such that $\int_{\delta_t} \eta_\beta = Ct^{A_\beta}$. One can take a branch of t^{A_β} so that $C = \int_{\delta_1} \eta_\beta$.

We have

$$\frac{t\omega_\beta}{f^k} = \frac{-f\omega_\beta + dg \wedge \eta_\beta}{f^k} = (-1 + \frac{A_\beta}{k-1})\frac{\omega_\beta}{f^{k-1}} \text{ in } \mathsf{M}$$

and so

$$\frac{\omega_\beta}{f^k} = \frac{1}{t^{k-1}}(-1 + \frac{A_\beta}{k-1})(-1 + \frac{A_\beta}{k-2}) \cdots (-1 + \frac{A_\beta}{1})\frac{\omega_\beta}{f} \text{ in } \mathsf{M}. \tag{3.38}$$

3.13 $\mathsf{R}[\theta]$ structure of H''

In this section we consider the $\mathsf{R}[s]$-modules H'' and H', where $s\omega := f\omega$. We have the following well-defined map:

$$\theta : H'' \to H',\ \theta\omega = \eta, \text{ where } \omega = d\eta.$$

We have used the fact that $H^n_{\mathrm{dR}}(\mathbb{U}_1/\mathbb{U}_0) = 0$ (see Proposition 3.2). It is well-defined because:

$$df \wedge d\eta_1 = d\eta_2 \Rightarrow \eta_2 = df \wedge \eta_1 + d\eta_3, \text{ for some } \eta_3 \in \Omega^{n-1}_{\mathbb{U}_1/\mathbb{U}_0}.$$

Using the inclusion $H' \to H''$, $\omega \mapsto df \wedge \omega$, both H' and H'' are now $\mathsf{R}[s,\theta]$-modules. The relation between $\mathsf{R}[s]$ and $\mathsf{R}[\theta]$ structures is given by:

Proposition 3.13. *We have:*

$$\theta \cdot s = s \cdot \theta - \theta \cdot \theta$$

and for $n \in \mathbb{N}$

$$\theta^n s = s\theta^n - n\theta^{n+1}.$$

Proof. The map $d : H' \to H''$ satisfies

$$d \cdot s = s \cdot d + df,$$

where s stands for the mapping $\omega \mapsto s\omega$ and df stands for the mapping $\omega \mapsto df \wedge \omega,\ \omega \in H'$. Composing the both sides of the above equality by θ we get the first statement. The second statement is proved by induction. □

For a homogeneous polynomial with an isolated singularity at the origin we have $d\eta_\beta = A_\beta \omega_\beta$ and so

$$\theta\omega_\beta = \frac{s}{A_\beta}\omega_\beta.$$

Remark 3.11. The action of θ on H'' is inverse to to the action of the Gauss-Manin connection with respect to the parameter s in $f - s = 0$ (we have composed the Gauss-Manin connection with $\frac{\partial}{\partial s}$). This arises the following question: Is it possible to construct similar structures for H' and H''?

Complementary notes

1. The calculation of the Gauss-Manin connection in this chapter is done for smooth affine hypersurfaces. Its generalization for arbitrary codimension varieties seems to be quit accessible. In this direction, the tools worked out for complete intersection singularities (see [30, 32, 74, 75]) may be carried into the context of complete intersection affine varieties.
2. We have postponed the definition of a mixed Hodge structure in H to Chapter 5, where we have assume that $\mathsf{R} = \mathbb{Q}[t]$. The original definition of a mixed Hodge structure on $H^n_{\mathrm{dR}}(\{f = 0\})$ is done by Deligne using the hypercohomology of the sheaf of meromorphic forms in the compactification of $\{f = 0\}$. Can one give a reasonable algorithm for calculating such a mixed Hodge structure for an arbitrary R?
3. For a given tame polynomial f, the deformation $f^w = f - (w_1x_1 + \cdots + w_{n+1}x_{n+1})$ has μ distinct critical values for all w in some Zariski open subset of $\mathbb{C}^{n+1}$. This follows from [60], Appendix B, see also [9] Proposition 2.2. For practical reasons one may also want to see whether f^w has μ distinct critical values or not.

Chapter 4

Topology of tame polynomials

It was S. Lefschetz who for the first time studied systematically the topology of smooth projective varieties. Later, his theorems were translated into the language of modern Algebraic Geometry, using Hodge theory, sheaf theory and spectral sequences. "But none of these very elegant methods yields Lefschetz's full geometric insight, e.g. they do not show us the famous vanishing cycles"(K. Lamotke). A direction in which Lefschetz's topological ideas were developed was in the study of the topology of hypersurface singularities. The objective of this chapter is to study the topology of the fibers of tame polynomials following the local context [1] and the global context [52]. To make this chapter self sufficient, we have put many well-known materials from the mentioned references. By a tame polynomial $f \in \mathbb{C}[x]$ in this chapter we mean that the polynomial $f - t \in \mathsf{R}[x]$, $\mathsf{R} = \mathbb{C}[t]$ is tame in the sense of Chapter 3. We denote by C the set of critical values of f and by μ the Milnor number of $f - t$.

4.1 Vanishing cycles and orientation

We consider in $\mathbb{C}$ the canonical orientation $\frac{1}{-2\sqrt{-1}} dx \wedge d\bar{x} = d(\mathrm{Re}(x)) \wedge d(\mathrm{Im}(x))$. This corresponds to the anti-clockwise direction in the complex plane. In this way, every complex manifold carries an orientation obtained by the orientation of $\mathbb{C}$, which we call it the canonical orientation. For a complex manifold of dimension n and an holomorphic nowhere vanishing differential n-form ω on it, the orientation obtained from $\frac{1}{(-2\sqrt{-1})^n}\omega \wedge \overline{\omega}$ differs from the canonical one by $(-1)^{\frac{n(n-1)}{2}}$ (as an exercise compare the orientation $\mathrm{Re}(\omega) \wedge \mathrm{Im}(\omega)$ with the canonical one. Assume that the complex manifold is $(\mathbb{C}^n, 0)$ and $\omega = dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n$). For instance for a tame polynomial f, the Gelfand-Leray form $\frac{dx}{df}$ in each regular fiber of f is such an n-form. Holomorphic maps between complex manifolds preserve the canonical orientation. For a zero dimensional manifold an orientation is just a map which associates ± 1 to each point of the manifold.

Let $f = x_1^2 + x_2^2 + \cdots + x_{n+1}^2$. For a real positive number t, the n-th homology of the complex manifold $L_t := \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{C}^{n+1} \mid f(x) = t\}$ is generated by the so called vanishing cycle

$$\delta_t = \mathbb{S}_n(t) := L_t \cap \mathbb{R}^{n+1}.$$

It vanishes along the vanishing path γ which connects t to 0 in the real line. The (Lefschetz) thimble

$$\Delta_t := \cup_{0 \le s \le t} \delta_s = \{x \in \mathbb{R}^{n+1} \mid f(x) \le t\}$$

df

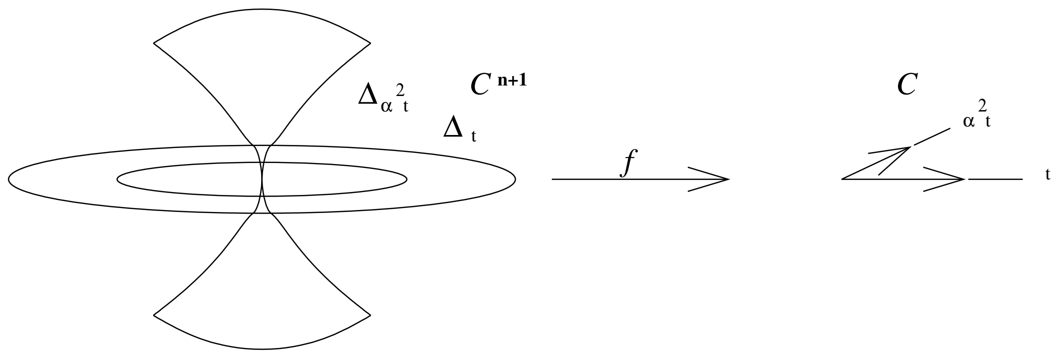

Figure 4.1: Intersection of thimbles

is a real $(n+1)$-dimensional manifold which generates the relative $(n+1)$-th homology $H_{n+1}(\mathbb{C}^{n+1}, L_t, \mathbb{Z})$. We consider for $\mathbb{S}_n(t)$ the orientation η such that $\eta \wedge \mathrm{Re}(df)$ is $\mathrm{Re}(dx_1) \wedge \mathrm{Re}(dx_2) \wedge \cdots \wedge \mathrm{Re}(dx_{n+1})$, which is an orientation for Δ_t. Let α be a complex number near to 1 with $\mathrm{Im}(\alpha) > 0, |\alpha| = 1$ and

$$h : L_t \to L_{\alpha^2 t},\ x \mapsto \alpha \cdot x.$$

The oriented cycle $h_* \delta_t$ is obtained by the mondromy of δ_t along the shortest path which connects t to $\alpha^2 t$. Now the orientation of Δ_t wedge with the orientation of $h_* \Delta_t$ is:

$$\begin{aligned}
&= \mathrm{Re}(dx_1) \wedge \mathrm{Re}(dx_2) \wedge \cdots \wedge \mathrm{Re}(dx_{n+1}) \wedge \mathrm{Re}(\alpha^{-1} dx_1) \wedge \mathrm{Re}(\alpha^{-1} dx_2) \wedge \cdots \wedge \mathrm{Re}(\alpha^{-1} dx_{n+1}) \\
&= (-1)^{\frac{n^2+n}{2}} \mathrm{Im}(\alpha)^{n+1} \mathrm{Re}(dx_1) \wedge \mathrm{Im}(dx_1) \wedge \mathrm{Re}(dx_2) \wedge \mathrm{Im}(dx_2) \wedge \cdots \wedge \mathrm{Re}(dx_{n+1}) \wedge \mathrm{Im}(dx_{n+1}) \\
&= (-1)^{\frac{n^2+n}{2}} \text{ the canonical orienatation of } \mathbb{C}^{n+1}.
\end{aligned}$$

This does not depend on the orientation η that we chose for δ_t. The assumption $\mathrm{Im}(\alpha) > 0$ is equivalent to the fact that $\mathrm{Re}(dt) \wedge h_* \mathrm{Re}(dt)$ is the canonical orientation of $\mathbb{C}$. We conclude that:

Proposition 4.1. *The orientation of* $(\mathbb{C}^{n+1}, 0)$ *obtained by the intersection of two thimbles is* $(-1)^{\frac{n(n+1)}{2}}$ *times the orientation of* $(\mathbb{C}, 0)$ *obtained by the intersection of their vanishing paths.*

See Figure 4.1.

4.2 Picard-Lefschetz theory of tame polynomials

Let us consider a tame polynomial in the ring $\mathsf{R}[x]$, where R is a localization of the polynomial ring $\mathbb{C}[t] := \mathbb{C}[t_1, t_2, \ldots, t_s]$ over some multiplicative subgroup of $\mathbb{C}[t]$. Recall that g denotes the last homogeneous piece of f and

$$\mathbb{U}_0 := \mathrm{Spec}(\mathsf{R}),\ \mathbb{U}_1 := \mathrm{Spec}(\mathsf{R}[x]),\ T := \mathbb{U}_0 \backslash \{\Delta_f = 0\}.$$

Let $\mathbb{N}_{n+1} = \{1, 2, \ldots, n+1\}$, $S = \{i \in \mathbb{N}_{n+1} \mid \alpha_i = 1\}$ and $S^c = \mathbb{N}_{n+1} \backslash S$.

Definition 4.1. The homogeneous polynomial g has a strongly isolated singularity at the origin if g has an isolated singularity at the origin and for all $R \subset \{1, 2, 3 \ldots, n+1\}$ with $S \subset R$, g restricted to $\cap_{i \in R}\{x_i = 0\}$ has also an isolated singularity at the origin.

If $\alpha_1 = \alpha_2 = \cdots = \alpha_{n+1} = 1$ then the condition 'strongly isolated' is the same as 'isolated'. The Picard-Lefschetz theory of tame polynomials is based on the following statement:

Theorem 4.1. *If the last homogeneous piece of a tame polynomial f is either independent of any parameter in R or it has a strongly isolated singularity at the origin then then the projection $\pi : \mathbb{U}_1 \to \mathbb{U}_0$ is a locally trivial C^∞ fibration over T.*

Proof. We give only a sketch of the proof. First, assume that the last homogeneous piece of f, namely g, has a strongly isolated singularity at the origin. Let us add the new variable x_0 to $\mathsf{R}[x]$ and consider the homogenization $F(x_0, x) \in \mathsf{R}[x_0, x]$ of f. Let F_t be the specialization of F in $t \in T$. Define

$$\bar{\mathbb{U}}_1 := \{([x_0 : x], t) \in \mathbb{P}^{1,\alpha} \times T \mid F_t(x_0, x) = 0\},$$

where $\mathbb{P}^{1,\alpha}$ is the weighted projective space of type $(1, \alpha) = (1, \alpha_1, \alpha_2, \ldots, \alpha_{n+1})$. Let $\bar{\pi} : \bar{\mathbb{U}}_1 \to \mathbb{U}_0$ be the projection in $\mathbb{U}_0$. If all the weights α_i are equal to 1 then $D := \bar{\mathbb{U}}_1 \backslash \mathbb{U}_1$ is a smooth submanifold of $\bar{\mathbb{U}}_1$ and $\bar{\pi}$ and $\bar{\pi} \mid_D$ are proper regular (i.e. the derivative is surjective). For this case one can use directly Ehresmann's fibration theorem (see [23, 52]). For arbitrary weights we use the generalization of Ehresmann's theorem for stratified varieties. In $\mathbb{P}^{1,\alpha}$ we consider the following stratification

$$(\mathbb{P}^{1,\alpha} \backslash \mathbb{P}^{\alpha}) \cup (\mathbb{P}^{\alpha} \backslash \mathbb{P}^{S^c}) \cup \cup_{I \subset S^c} (\mathbb{P}^I \backslash \mathbb{P}^{<I}),$$

where for a subset I of $\mathbb{N}_{n+1}$, $\mathbb{P}^R$ denotes the sub projective space of the weighted projective space $\mathbb{P}^\alpha$ given by $\{x_i = 0 \mid i \in \mathbb{N}_{n+1} \backslash I\}$ and $\mathbb{P}^{<I} := \cup_{J \subset I,\ J \neq I} \mathbb{P}^J$. Now in T consider the one piece stratification and in $\mathbb{P}^{1,\alpha} \times T$ the product stratification. This gives us a stratification of $\bar{\mathbb{U}}_1$. The morphism $\bar{\pi}$ is proper and the fact that g has a strongly isolated singularity at the origin implies that $\bar{\pi}$ restricted to each strata is regular. We use Verdier Theorem ([83], Theorem 4.14, Remark 4.15) and obtain the local trivilaization of π on a small neighborhood of $t \in T$ and compatible with the stratification of $\bar{\mathbb{U}}_1$. This yields to a local trivialization of π around t. If g is independent of any parameter in R then $\bar{\mathbb{U}}_1 \backslash \mathbb{U}_1 = G \times \mathbb{U}_0$, where G is the variety induced in $\{g = 0\}$ in $\mathbb{P}^\alpha$. We choose an arbitrary stratification G and the product stratification in $G \times \mathbb{U}_0$ and apply again Verdier Theorem. □

The hypothesis of Theorem 4.1 is not the best one. For instance, the homogeneous polynomial $g = x^3 + tzy + tz^2$ in the ring $\mathsf{R}[x, y, z]$, $\mathsf{R} = \mathbb{C}[t, s, \frac{1}{t}]$, $\deg(x) = 2$, $\deg(y) = \deg(z) = 3$ depends on the parameter t and $g(x, y, 0)$ has not an isolated singularity at the origin. However, π is a C^∞ locally trivial fibration over T. I do not know any theorem describing explicitly the atypical values of the morphism π. Such theorems must be based either on a precise desingularization of $\bar{\mathbb{U}}_1$ and Ehresmann's theorem or various types of stratifications depending on the polynomial g. For more information in this direction the reader is referred to the works of J. Mather, R. Thom and J. L. Verdier around 1970 (see [57] and the references there). Theorem 4.1 (in the general context of morphism of algebraic varieties) is also known as the second theorem of isotopy (see [83] Remark 4.15).

Let λ be a path in $\mathbb{U}_0$ connecting b_0 to b_1 and defined up to homotopy. Theorem 4.1 gives us a unique map $h_\lambda : L_{b_0} \to L_{b_1}$ defined up to homotopy. In particular, for $b := b_0 = b_1$ we have the action of $\pi_1(\mathbb{U}_0, b)$ on the homology group $H_n(L_b, \mathbb{Z})$. The image of $\pi_1(\mathbb{U}_0, b)$ in $\mathrm{Aut}(H_n(L_b, \mathbb{Z}))$ is called the monodromy group.

4.3 Distinguished set of vanishing cycles

First, let us recall some definitions from local theory of vanishing cycles. Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a holomorphic function with an isolated critical point at $0 \in \mathbb{C}^{n+1}$. We take convenient neighborhoods U of $0 \in \mathbb{C}^{n+1}$ and V of $0 \in \mathbb{C}$ such that $f : U \to V$ is a C^∞ fiber bundle over $V \backslash \{0\}$. Let $t_i \in V \backslash \{0\},\ i = 1, 2, \cdots, s$ (not necessarily distinct) and λ_i be a path which connects 0 to t_i in V. We assume that λ_i's do not intersect each other except at their start/end points and at 0 they intersect each other transversally. We also assume that the embedded oriented sphere $\delta_i \subset f^{-1}(t_i)$ vanishes along λ_i. The sphere δ_i is called a vanishing cycle and is defined up to homotopy.

Definition 4.2. *The ordered set of vanishing cycles $\delta_1, \delta_2, \cdots, \delta_s$ is called distinguished if*

1. $(\lambda_1, \lambda_2, \cdots, \lambda_s)$ *near* 0 *is the clockwise direction;*
2. *for a versal deformation $\tilde{f}$ of f with μ distinguished critical values, where μ is the Milnor number of f, the deformed paths $\tilde{\lambda}_i$ do not intersect each other except possibly at their end points t_i's.*

Historically, one is interested to the full distinguished set of vanishing cycles, i.e. the one with μ elements and with $b := t_1 = t_2 = \cdots = t_\mu$. From now on by a distinguished set of vanishing cycles we mean the full one. It is well-known that a full distinguished set of vanishing cycles form a basis of $H_n(f^{-1}(b), \mathbb{Z})$ (see [1]).

Example 4.1. For $f := x^d$ the point $0 \in \mathbb{C}$ is the only critical value of f. Let $\lambda(s) = s,\ 0 \leq s \leq 1$. The set

$$\delta_i := [\zeta_d^{i+1}] - [\zeta_d^i],\ i = 0, \ldots, d-2$$

is a distinguished set of vanishing cycles for $H_0(\{f = 1\}, \mathbb{Z})$. The vanishing takes place along λ (see [1] Theorem 2.15).

Let $f \in \mathbb{C}[x]$ be a tame polynomial. We fix a regular value $b \in \mathbb{C} \backslash C$ of f and consider a system of paths $\lambda_i,\ i = 1, 2, \ldots, \mu$ connecting the points of C to the point b. Again, we assume that λ_i's do not intersect each other except at their start/end points and at the points of C they intersect each other transversally. We call λ_i's a distinguished set of paths. In a similar way as in Definition 4.2 we define a distinguished set of vanishing cycles $\delta_i \subset f^{-1}(b),\ i = 1, 2, \ldots, \mu$ (defined up to homotopy). For each singularity p of f we use a separate versal deformation which is defined in a neighborhood of p. If the completion of f has a non zero double discriminant then we can deform f and obtain another tame polynomial $\tilde{f}$ with the same Milnor number in a such a way that f and $\tilde{f}$

have C^∞ isomorphic regular fibers and $\tilde{f}$ has distinct μ critical values. In this case we can use $\tilde{f}$ for the definition of a distinguished set of vanishing cycles.

Fix an embedded sphere in $f^{-1}(b)$ representing the vanishing cycle δ_i. For simplicity we denote it again by δ_i.

Theorem 4.2. *For a tame polynomial $f \in \mathbb{C}[x]$ and a regular value b of f, the complex manifold $f^{-1}(b)$ has the homotopy type of $\cup_{i=1}^{\mu}\delta_i$. In particular, a distinguished set of vanishing cycles generates $H_n(f^{-1}(b), \mathbb{Z})$.*

Proof. The proof of this theorem is a well-known argument in Picard-Lefschetz theory, see for instance [52] §5, [9] Theorem 1.2, [61] Theorem 2.2.1 and [21]. We have reproduced this argument in the proof of Theorem 4.4 □

In the literature the union $\cup_{i=1}^{\mu}\delta_i$ is known as the bouquet of μ spheres.

Theorem 4.3. *If the tame polynomial $f \in \mathbb{C}[x]$ has μ distinct critical values and the discriminant of its completion is irreducible then for two vanishing cycles δ_0, δ_1 in a regular fiber of f, there is a homotopy class $\gamma \in \pi_1(\mathbb{C}\backslash C, b)$ such that $h_\gamma(\delta_0) = \pm\delta_1$, where C is the set of critical values of f.*

Similar theorems are stated in [52] 7.3.5 for generic Lefschetz pencils, in [61] Theorem 2.3.2, Corollary 3.1.2 for generic pencils of type $\frac{F^p}{G^q}$ in $\mathbb{P}^n$ and in [1] Theorem 3.4 for a versal deformation of a singularity. Note that in the above theorem we are still talking about the homotopy classes of vanishing cycles. I believe that the discriminant of complete tame polynomials is always irreducible. This can be checked easily for $\alpha_1 = \alpha_2 = \cdots = \alpha_{n+1} = 1$ and many particular cases of weights.

Proof. Let $F \in \mathsf{R}[x]$, $\mathsf{R} = \mathbb{C}[t]$ be the completion of f and $\Delta_0 := \{t \in \mathbb{U}_0 \mid \Delta_F(t) = 0\}$. We consider $f - s,\ s \in \mathbb{C}$ as a line G_{c_0} in $\mathbb{U}_0$ which intersects Δ_0 transversally in μ points. If there is no confusion we denote by b the point in $\mathbb{U}_0$ corresponding to $f - b$. Let D be the locus of points $t \in \Delta_0$ such that the line G_t through b and t intersects Δ_0 at μ distinct points. Let also δ_0 and δ_1 vanish along the paths λ_0 and λ_1 which connect b to $c_0, c_1 \in G_{c_0} \cap \Delta_0$, respectively. Since the set D is a proper algebraic subset of Δ_0 and Δ_0 is an irreducible variety and $c_0, c_1 \in \Delta_0\backslash D$, there is a path w in $\Delta_0\backslash D$ from c_0 to c_1. After a blow up at the point b and using the Ehresmann's theorem, we conclude that: There is an isotopy

$$H : [0,1] \times G_{c_0} \to \cup_{t\in[0,1]} G_{w(t)}$$

such that

1. $H(0,\cdot)$ is the identity map;

2. for all $a \in [0,1]$, $H(a,.)$ is a C^∞ isomorphism between G_{c_0} and $G_{w(a)}$ which sends points of Δ_0 to Δ_0;

3. For all $a \in [0,1]$, $H(a,b) = b$ and $H(a,c_0) = w(a)$

Let $\lambda'_a = H(a,\lambda_0)$. In each line $G_{w(a)}$ the cycle δ_0 vanishes along the path λ'_a in the unique critical point of $\{F_{w(s)} = 0\}$. Therefore δ_0 vanishes along λ'_1 in $c_1 = w(1)$. Consider λ_1 and λ'_1 as the paths which start from b and end in a point b_1 near c_1 and put $\lambda = \lambda'_1 - \lambda_1$. By uniqueness (up to sign) of the Lefschetz vanishing cycle along a fixed path we can see that the path λ is the desired path. □

Let $f \in \mathbb{C}[x]$ be a tame polynomial and λ be a path in $\mathbb{C}$ which connects a regular value $b \in \mathbb{C}\backslash C$ to a point $c \in C$ and do not cross C except at the mentioned point c. To λ one can associate an element in $\tilde{\lambda} \in \pi_1(\mathbb{C}\backslash\mathbb{C}, b)$ as follows: The path $\tilde{\lambda}$ starts from b goes along λ until a point near c, turns around c anti clockwise and returns to b along λ. By the monodromy along the path and around c we mean the monodromy associated to $\tilde{\lambda}$. The associated monodromy is given by the Picard-Lefschetz formula/mapping:

$$a \mapsto a + \sum_{\delta} (-1)^{\frac{(n+1)(n+2)}{2}} \langle a, \delta \rangle, \tag{4.1}$$

where δ runs through a basis of distinguished vanishing cycles which vanish in the critical points of the fiber $f^{-1}(c)$. The above mappings keep the intersection form $\langle \cdot, \cdot \rangle$ invariant, i.e.

$$\left\langle a + (-1)^{\frac{(n+1)(n+2)}{2}} \langle a, \delta \rangle \delta, b + (-1)^{\frac{(n+1)(n+2)}{2}} \langle b, \delta \rangle \delta \right\rangle = \langle a, b \rangle, \ \forall a, b \in H_n(\{f = 0\}, \mathbb{Z}).$$

This follows from (4.3) and the fact that $\langle \cdot, \cdot \rangle$ is $(-1)^n$-symmetric. I do not know whether in general a $\langle \cdot, \cdot \rangle$-preserving map from $H_n(\{f = 0\}, \mathbb{Z})$ to itself is a composition of some Picard-Lefschetz mappings. The positive answer to this question may change our point of view on the moduli of polarized Hodge structures (see §6.16).

Definition 4.3. A cycle $\delta \in H_n(\{f = 0\}, \mathbb{Z})$, f a tame polynomial, is called the cycle at infinity if its intersection with all other cycles in $H_n(\{f = 0\}, \mathbb{Z})$ (including itself) is zero.

4.4 Direct sum of polynomials

Let $f \in \mathbb{C}[x]$ and $g \in \mathbb{C}[y]$ be two polynomials in variables $x := (x_1, x_2, \ldots, x_{n+1})$ and $y := (y_1, y_2 \ldots, y_{m+1})$ respectively. In this section we study the topology of the variety

$$X := \{(x, y) \in \mathbb{C}^{n+1} \times \mathbb{C}^{m+1} \mid f(x) = g(y)\}$$

in terms of the topology of the fibrations $f : \mathbb{C}^{n+1} \to \mathbb{C}$ and $g : \mathbb{C}^{m+1} \to \mathbb{C}$. We start with a definition.

Definition 4.4. The join $X * Y$ of two topological spaces X and Y is the quotient space of the direct product $X \times I \times Y$, where $I = [0, 1]$, by the equivalence relation:

$$(x, 0, y_1) \sim (x, 0, y_2) \ \forall \ y_1, y_2 \in Y, \ x \in X,$$

$$(x_1, 1, y) \sim (x_2, 1, y) \ \forall \ x_1, x_2 \in X, \ y \in Y.$$

Let X and Y be compact oriented real manifolds and $\pi : X * Y \to I$ be the projection on the second coordinate. The real manifold $X * Y \backslash \pi^{-1}(\{0, 1\})$ has a canonical orientation obtained by the wedge product of the orientations of X, I and Y. Does $X * Y$ have a structure of a real oriented manifold? It does not seem to me that the answer is positive for arbitrary X and Y. In the present text we only need the following proposition which gives partially a positive answer to our question. Let

$$\mathbb{S}_n := \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1\}$$

be the n-dimensional sphere with the orientation $\frac{dx}{d(x_1^2 + x_2^2 + \cdots + x_{n+1}^2)}$.

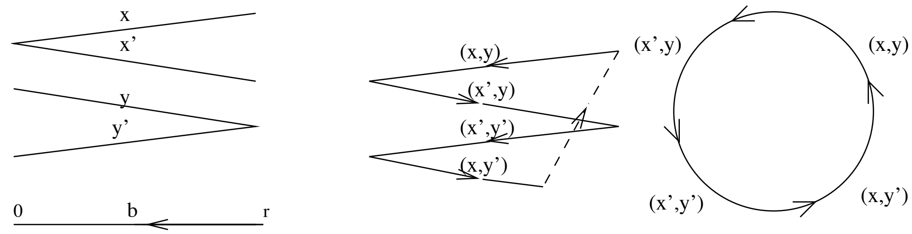

Figure 4.2: Join of zero dimensional cycles

Proposition 4.2. *We have*

$$\mathbb{S}_n * \mathbb{S}_m \overset{C^0}{\cong} \mathbb{S}_{n+m+1}, \ n, m \in \mathbb{N}_0,$$

which is an isomorphisim of oriented manifolds outside $\pi^{-1}(\{0,1\})$.

Proof. For the proof of the above diffeomorphism we write $\mathbb{S}_{n+m+1}$ as the set of all $(x,y) \in \mathbb{R}^{n+m+2}$ such that

$$x_1^2 + \cdots + x_{n+1}^2 = 1 - (y_1^2 + \cdots + y_{m+1}^2)$$

Now, let t be the above number and let it varies from 0 to 1. We have the following isomorphism of topological spaces:

$$\mathbb{S}_{n+m+1} \to \mathbb{S}_n * \mathbb{S}_m, \ (x,y) \mapsto \begin{cases} (\frac{x}{\sqrt{t}}, t, \frac{y}{\sqrt{1-t}}) & t \neq 0,1 \\ (0,0,y) & t = 0 \\ (x,1,0) & t = 1 \end{cases}$$

The Figure (4.2) shows a geometric construction of $\mathbb{S}_0 \times \mathbb{S}_0$. The proof of the statement about orientations is left to the reader. □

Let $f \in \mathbb{C}[x]$ and $g \in \mathbb{C}[y]$ be two tame polynomials in $n+1$, repectively $m+1$, variables. Let also C_1 (resp. C_2) denotes the set of critical values of f (resp. g). We assume that $C_1 \cap C_2 = \emptyset$, which implies that the variety X is smooth. Fix a regular value $b \in \mathbb{C}\backslash(C_1 \cup C_2)$ of both f and g. Let $\delta_{1b} \in H_n(f^{-1}(b), \mathbb{Z})$ and $\delta_{2b} \in H_m(g^{-1}(b), \mathbb{Z})$ be two vanishing cycles and $t_s, \ s \in [0,1]$ be a path in $\mathbb{C}$ such that it starts from a point in C_1, crosses b and ends in a point of C_2 and never crosses $C_1 \cup C_2$ except at the mentioned cases. We assume that δ_{1b} vanishes along $t_.^{-1}$ when s tends to 0 and δ_{2b} vanishes along $t_.$ when s tends to 1. Now

$$\delta_{1b} * \delta_{2b} \cong \delta_{1b} *_{t_.} \delta_{2b} := \cup_{s \in [0,1]} \delta_{1t_s} \times \delta_{2t_s} \in H_{n+m+1}(X, \mathbb{Z})$$

is an oriented cycle. Note that its orientation changes when we change the direction of the path $t_.$. We call the triple $(t_s, \delta_1, \delta_2) = (t_s, \delta_{1t_.}, \delta_{2t_.})$ an admissible triple.

Let $b \in \mathbb{C}\backslash(C_1 \cup C_2)$. We take a system of distinguished paths $\lambda_c \ c \in C_1 \cup C_2$, where λ_c starts from b and ends at c. Let $\delta_1^1, \delta_1^2, \cdots, \delta_1^{\mu} \in H_n(f^{-1}(b), \mathbb{Z})$ and $\delta_2^1, \delta_2^2, \cdots, \delta_2^{\mu'} \in H_m(g^{-1}(b), \mathbb{Z})$ be the corresponding distinguished basis of vanishing cycles. Note that many vanishing cycles may vanish along a path in one singularity.

Theorem 4.4. *The* $\mathbb{Z}$*-module* $H_{n+m+1}(X, \mathbb{Z})$ *is freely generated by*

$$\gamma := \delta_1^i * \delta_2^j, \ i = 1, 2, \ldots, \mu, \ j = 1, 2, \cdots, \mu',$$

where we have taken the admissible triples $(\lambda_{c_j}\lambda_{c_i}^{-1}, \delta_1^i, \delta_2^j), \ c_i \in C_1, \ c_j \in C_2$.

Proof. The proof which we present for this theorem is similar a well-known argument in Picard-Lefschetz theory, see for instance [52] or Theorem 2.2.1 of [61]. The homologies bellow are with $\mathbb{Z}$ coefficients.

The fibration $\pi : X \to \mathbb{C},\ (x,y) \mapsto f(x) = f(y)$ is toplogically trivial over $\mathbb{C}\backslash(C_1 \cup C_2)$. Let $Y = f^{-1}(b) \times g^{-1}(b)$. We have

(4.2)

$$0 = H_{n+m+1}(Y) \to H_{n+m+1}(X) \to H_{n+m+1}(X,Y) \xrightarrow{\partial} H_{n+m}(Y) \to H_{n+m}(X) \to \cdots$$

We take small open disks D_c around each point $c \in C_1 \cup C_2$. Let b_c be a point near c in D_c and $X_c = \pi^{-1}(\lambda_c \cup D_c)$. We have

$$H_{n+m}(Y) \cong H_n(f^{-1}(b)) \otimes_{\mathbb{Z}} H_n(g^{-1}(b))$$

and

$$\begin{aligned} H_{n+m+1}(X,Y) &\cong \oplus_{c\in C_1\cup C_2} H_{n+m+1}(X_c, Y) \\ &\cong \oplus_{c\in C_1\cup C_2} H_{n+m+1}(X_c, Y_{b_c}) \\ &\cong \oplus_{c\in C_1} H_{n+1}(f^{-1}(D_c), f^{-1}(b_c)) \oplus \oplus_{c\in C_2} H_{m+1}(g^{-1}(D_c), g^{-1}(b_c)). \end{aligned}$$

We look $H_{n+m+1}(X)$ as the kernel of the boundary map ∂ in (4.2). Let us take two cycles δ_1 and δ_2 form the pieces of the last direct sum in the above equation and assume that $\partial\delta = 0$, where $\delta = \delta_1 - \delta_2$. If δ_1 and δ_2 belongs to different classes, according to $c \in C_1$ or $c \in C_2$, then δ is the join of two vanishing cycles. Otherweise, $\delta = 0$ in $H_{n+m+1}(X, \mathbb{Z})$. □

It is sometimes useful to take $g = b' - g'$, where b' is a fixed complex number and g' is a tame polynomial . The set of critical values of g' is denoted by C_2' and hence the set of critical values of g is $C_2 = b' - C_2'$. We define $t = F(x,y) := f(x) + g'(y)$ and so $X = F^{-1}(b')$. The set of critical values of F is $C_1 + C_2'$ and the assumption that $C_1 \cap (b' - C_2')$ is empty implies that b' is a regular value of F. Let $(t_s, \delta_{1b}, \delta_{2b})$ be an admissible triple and t_s starts from c_1 and ends in $b' - c_2'$.

Proposition 4.3. *The topological cycle $\delta_{1b} * \delta_{2b}$ is a vanishing cycle along the path $t_. + c_2$ with respect to the fibration $F = t$.*

Proof. See Figure 4.3. □

Remark 4.1. Let $b \in \mathbb{C}\backslash(C_1 \cup C_2)$. We take a system of distinguished paths $\lambda_c\ c \in C_1 \cup C_2$, where λ_c starts from b and ends at c (see Figure 4.3). If the points of the set C_1 (resp. C_2) are enough near (resp. far from) each other then the collection of translations given in Proposition 4.3 gives us a system of paths, which is distinguished after performing a proper homotopy, starting from the points of $C_1 + C_2'$ and ending in b'. This together with Theorem 4.2 gives an alternative proof to Theorem 4.4.

Example 4.2. Let us assume that all the critical values of f and $g' = b' - g$ are real. Moreover, assume that f (resp. g) has non-degenerated critical points with distinct images. For instance, in the case $n = m = 0$ take

$$f := (x-1)(x-2)\cdots(x-m_1),\ g' := (x-m_1-1)(x+2)\cdots(x-m_1-m_2).$$

Take $b' \in \mathbb{C}$ with $\mathrm{Im}(b') > 0$. We take direct segment of lines which connects the points of C_1 to the points of $b' - C_2'$. The set of joint cycles constructed in this way, is a basis of vanishing cycles associated the direct segment of paths which connect b' to the points of $C_1 + C_2'$ (see Figure 4.3, B).

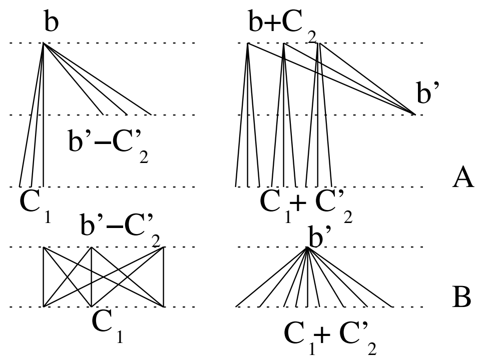

Figure 4.3: A system of distinguished paths

Example 4.3. Using the machinery introduced in this section, we can find a distinguished basis of vanishing cycles for $H_n(\{g=1\},\mathbb{Z})$, where $g=x_1^{m_1}+x_2^{m_2}+\cdots+x_{n+1}^{m_{n+1}}$, $2\leq m_i\in\mathbb{N}$ is discussed in Example 3.2. Let

$$\Gamma := \{(t_1,t_2,\ldots,t_{n+1})\in\mathbb{R}^{n+1}\mid t_i\geq 0, \sum_{i=1}^{n+1} t_i = 1\}.$$

For $i=1,2,\ldots,n+1$ we take the distinguished set of vanishing cycles δ_{i,β_i}, $\beta_i = 0,1,\ldots,m_i-2$ given in Example 4.1 and define the joint cycles

$$\delta_\beta = \delta_{m_1,\beta_1} * \delta_{m_2,\beta_2} * \cdots * \delta_{m_{n+1},\beta_{n+1}} :=$$

$$\cup_{t\in\Gamma}\delta_{m_1,\beta_1,t_1}\times\delta_{m_2,\beta_2,t_2}\times\cdots\times\delta_{m_{n+1},\beta_{n+1},t_{n+1}}\in H_n(\{g=1\},\mathbb{Z}),\ \beta\in I,$$

where $I := \{(\beta_1,\ldots,\beta_{n+1})\mid 0\leq\beta_i\leq m_i-2\}$. They are ordered lexicographically and form a distinguished set of vanishing cycles in $H_n(\{g=1\},\mathbb{Z})$. Another description of δ_β's is as follows: For $\beta\in I$ and $a_i=0,1$, where $i=1,2,\ldots,n+1$, let

$$\Gamma_{\beta,a}:\Gamma\to\{g=1\},\ \Gamma_{\beta,a}(t) = (t_1^{\frac{1}{m_1}}\zeta_{m_1}^{\beta_1+a_1}, t_2^{\frac{1}{m_2}}\zeta_{m_2}^{\beta_2+a_2},\ldots,t_{n+1}^{\frac{1}{m_{n+1}}}\zeta_{m_{n+1}}^{\beta_{n+1}+a_{n+1}}),$$

where for a positive number r and a natural number s, $r^{\frac{1}{s}}$ is the unique positive s-th root of r. We have

$$\delta_\beta := \sum_a (-1)^{\sum_{i=1}^{n+1}(1-a_i)}\Gamma_{\beta,a}.$$

4.5 Calculation of the Intersection form

Let us consider two tame polynomials $f,g\in\mathbb{C}[x]$. A critical value c of f is called non-degenerated if the fiber $f^{-1}(c)$ contains only one singularity and the Mlinor number of that singularity is one. Around such a singularity f can be written in the form $X_1^2+X_2^2+\cdots+X_{n+1}^2+c$ for certain local coordinate functions X_i.

For two oriented paths $t_., t'_.$ in $\mathbb{C}$ which intersect each other at b transversally the notation $t_.\times_b^+ t'_.$ means that $t_.$ intersects $t'_.$ in the positive direction, i.e. $dt_.\wedge dt'_.$ is the canonical orientation of $\mathbb{C}$. In a similar way we define $t_.\times_b^- t'_.$ (see Figure 4.4).

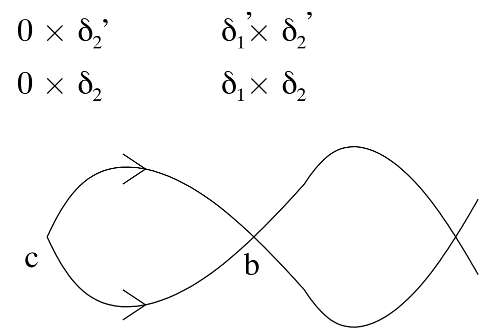

Figure 4.4: Two paths in $\mathbb{C}$

Theorem 4.5. *Let $(t_{\cdot}, \delta_1, \delta_2)$ and $(t'_{\cdot}, \delta'_1, \delta'_2)$ be two admissible triples. Assume that $t_{\cdot}$ and $t'_{\cdot}$ intersect each other transversally in their common points and the start/end critical points of $t_{\cdot}$ and $t'_{\cdot}$ are non-degenerated. Then*

$$\langle \delta_1 * \delta_2, \delta'_1 * \delta'_2 \rangle = (-1)^{nm+n+m} \sum_b \epsilon_1(b) \langle \delta_{1b}, \delta'_{1b} \rangle \langle \delta_{2b}, \delta'_{2b} \rangle$$

where b runs through all intersection points of $t_{\cdot}$ and $t'_{\cdot}$,

$$\epsilon_1(b) = \begin{cases} 1 & t_{\cdot} \times_b^+ t'_{\cdot} \text{ and } b \text{ is not a start/end point} \\ -1 & t_{\cdot} \times_b^- t'_{\cdot} \text{ and } b \text{ is not a start/end point} \\ (-1)^{\frac{n(n-1)}{2}} & t_{\cdot} \times_b^+ t'_{\cdot} \text{ and } b \text{ is a start point} \\ (-1)^{\frac{n(n+1)}{2}+1} & t_{\cdot} \times_b^- t'_{\cdot} \text{ and } b \text{ is a start point} \\ (-1)^{\frac{m(m-1)}{2}} & t_{\cdot} \times_b^+ t'_{\cdot} \text{ and } b \text{ is an end point} \\ (-1)^{\frac{m(m+1)}{2}+1} & t_{\cdot} \times_b^- t'_{\cdot} \text{ and } b \text{ is an end point} \end{cases},$$

and by $\langle 0, 0 \rangle$ we mean 1.

Proof. Let $t_{\cdot}$ intersect $t'_{\cdot}$ transversally at a point b. Let also a_1, a_2, a'_1, a'_2 be the orientation elements of the cycles $\delta_1, \delta_2, \delta'_1, \delta'_2$ and a and a' be the orientation element of $t_{\cdot}$ and $t'_{\cdot}$. We consider two cases:

1. b is not the end/start point of neither $t_{\cdot}$ nor $t'_{\cdot}$: Assume that the cycles δ_1 and δ'_1 (resp. δ_2 and δ'_2) intersect each other at p_1 (resp. p_2) transversally. The cycles $\gamma = \delta_1 * \delta_2$ and $\gamma' = \delta'_1 * \delta_2$ intersect each other transversally at (p_1, p_2). The orientation element of the whole space X obtained by the intersection of γ and γ' is:

$$a_1 \wedge a \wedge a_2 \wedge a'_1 \wedge a' \wedge a'_2 = (-1)^{nm+n+m} (a_1 \wedge a'_1) \wedge (a \wedge a') \wedge (a_2 \wedge a'_2)$$

   This is $(-1)^{nm+n+m}$ times the canonical orientation of X.

2. $b = c$ is, for instance, the start point of both $t_{\cdot}$ and $t'_{\cdot}$ and δ_1, δ'_1 vanish in the point $p_1 \in \mathbb{C}^{n+1}$ when t tends to c. Assume that the cycles δ_2 and δ'_2 intersect each other transversally at p_2. By assumption, p_1 is a non-degenerated critical point of f and so both cycles γ, γ' are smooth around (p_1, p_2) and intersect each other transversally at (p_1, p_2). The orientation element of the whole space X obtained by the intersection of γ and γ' is:

$$(a_1 \wedge a) \wedge a_2 \wedge (a'_1 \wedge a') \wedge a'_2 = (-1)^{(n+1)m} (a_1 \wedge a) \wedge (a'_1 \wedge a') \wedge a_2 \wedge a'_2.$$

Note that $a_1 \wedge a$ has meaning and is the orientation of the thimble formed by the vanishing of δ_1 at p_1. According to Proposition 4.1, $(a_1 \wedge a) \wedge (a_1' \wedge a')$ is the canonical orientation of $\mathbb{C}^{n+1}$ multiplied with ϵ, where $\epsilon = (-1)^{\frac{n(n+1)}{2}}$ if $t_{.} \times_b^+ t'_{.}$ and $= (-1)^{\frac{n(n+1)}{2}+n+1}$ otherwise.

□

Remark 4.2. One can use Theorem 4.5 to calculate the intersection matrix of $H_n((f+g')^{-1}(b'), \mathbb{Z})$ in the basis given by Theorem 4.4. This calculation in the local case is done by A. M. Gabrielov (see [1] Theorem 2.11). To state Gabrielov's result in the context of this text take f and g two tame polynomials such that the set C_1 can be separated from C_2 by a real line in $\mathbb{C}$. Then take b a point in that line. The advantage of our calculation is that it works in the global context and the vanishing cycles are constructed explicitly.

Remark 4.3. In Theorem 4.5 we may discard the assumption on the critical points in the following way: In the case in which δ_1 and δ_1' (resp. δ_2 and δ_2') vanish on the same critical point, we assum that they are distinguished (see Definition 4.2). Note that if two vanishing cycles vanish along transversal paths in the same singularity then the corresponding thimbles are not necessarily transversal to each other, except when the singularity is non-degenerated.

Proposition 4.4. *The self intersection of a vanishing cycle of dimension n is given by*

$$(-1)^{\frac{n(n-1)}{2}}(1+(-1)^n). \tag{4.3}$$

Proof. By Proposition 4.3 a joint cycle of two vanishing cycle is also a vanishing cycle. We apply Theorem 4.5 in the case $\delta_1 = \delta_1'$ and $\delta_2 = \delta_2'$ and conclude that the self intersection a_n of a vanishing cycle of dimension n satisfies

$$a_{n+m+1} = (-1)^{nm+n+m}((-1)^{\frac{n(n-1)}{2}} a_m + (-1)^{\frac{m(m+1)}{2}+1} a_n), \ a_0 = 2, \quad n, m \in \mathbb{N}_0.$$

It is easy to see that (4.3) is the only function with the above property. □

Example 4.4. (Stabilization) We take $g = y_1^2 + y_2^2 + \cdots + y_{m+1}^2$ and f an arbitrary tame polynomial. Let $\delta_1, \delta_2, \cdots, \delta_\mu$ be a distinguished set of vanishing cycles in $H_n(f^{-1}(0), \mathbb{Z})$ and δ be the vanishing cycle in $H_n(f^{-1}(0), \mathbb{Z})$ (up to multiplication by ± 1 it is unique). The intersection form in the basis $\tilde{\delta}_i = \delta_i * \delta$ is given by

$$\langle \tilde{\delta}_i, \tilde{\delta}_j \rangle = (-1)^{nm+n+m+\frac{m(m-1)}{2}} \langle \delta_i, \delta_j \rangle, \ i > j,$$

$$\langle \tilde{\delta}_i, \tilde{\delta}_j \rangle = (-1)^{nm+n+m+\frac{m(m+1)}{2}+1} \langle \delta_i, \delta_j \rangle, \ i < j,$$

(see [1] Theorem 2.14). Now let us assume that $m = 0$ and $n = 1$. Choose δ_i's as in Example 4.1. In this basis the intersection matrix is:

$$\Psi_0 = \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}.$$

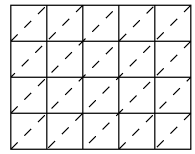

Figure 4.5: Dynkin diagram of $x^5 + y^4$

The intersection matrix in the basis $\tilde{\delta}_i,\ i = 1, 2, \cdots, \mu$ is of the form:

$$\tilde{\Psi}_0 = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 0 \end{pmatrix}.$$

As an exercise, construct a symplectic basis of the Riemann surface X using the basis $\tilde{\delta}_i,\ i = 1, 2, \ldots, \mu$ and its intersection matrix.

Example 4.5. We consider f and $g = b' - g'$, where f and g' are two homogeneous tame polynomials. The point $0 \in \mathbb{C}$ (resp. $b' \in \mathbb{C}$) is the only critical value of f (resp. g) and so, upto homotopy, there is only one path connecting 0 to b'. We choose the stright piece of line $t_s = sb',\ 0 \leq s \leq 1$ as the path for our admissible triples. For a point b between 0 and b' in $t.$ we choose a distinguished set of vanishing cycles $\delta_i,\ i = 1, 2 \ldots, \mu_1$ (resp. $\gamma_j,\ j = 1, 2 \ldots, \mu_2$) of f (resp. g) in the fiber $f^{-1}(b)$ (resp. $g^{-1}(b)$). By Theorem 4.4, the cycles

$$\delta_i * \gamma_j,\ i = 1, 2, \ldots, \mu_1,\ j = 2, \ldots, \mu_2$$

generate $H_1(\{f + g = b'\}, \mathbb{Z})$. The intersection matrix in this basis is given by

$$\langle \delta_i * \gamma_j, \delta_{i'} * \gamma_{j'} \rangle = \begin{cases} \operatorname{sgn}(j' - j)^{n+1}(-1)^{(n+1)(m+1) + \frac{n(n+1)}{2}} \langle \gamma_j, \gamma_{j'} \rangle & \text{if } i' = i \ \&\ j' \neq j \\ \operatorname{sgn}(i' - i)^{m+1}(-1)^{(n+1)(m+1) + \frac{m(m+1)}{2}} \langle \delta_i, \delta_{i'} \rangle & \text{if } j' = j \ \&\ i' \neq i \\ \operatorname{sgn}(i' - i)(-1)^{(n+1)(m+1)} \langle \delta_i, \delta_{i'} \rangle \langle \gamma_j, \gamma_{j'} \rangle & \text{if } (i' - i)(j' - j) > 0 \\ 0 & \text{if } (i' - i)(j' - j) < 0 \end{cases}$$

Example 4.6. In the case $f := x^{m_1}$ and $g := b' - y^{m_2}$,

$$\delta_i := [\zeta_{m_1}^{i+1} b^{\frac{1}{m_1}}] - [\zeta_{m_1}^{i} b^{\frac{1}{m_1}}],\ i = 0, \ldots, m_1 - 2$$

(resp.

$$\gamma_j := [\zeta_{m_2}^{j+1} (b' - b)^{\frac{1}{m_2}}] - [\zeta_{m_1}^{j} (b' - b)^{\frac{1}{m_2}}],\ j = 0, \ldots, m_2 - 2)$$

is a distinguished set of vanishing cycles for $H_0(\{f = b\}, \mathbb{Z})$ (resp. $H_0(\{g = b\}, \mathbb{Z})$), where we have fixed a value of $b^{\frac{1}{m_1}}$ and $b^{\frac{1}{m_2}}$. See Figure (4.2) for a tentative picture of the join cycle $\delta_i * \gamma_j$ with $\delta_i = x - y$ and $\gamma_j = x' - y'$. The upper triangle of intersection matrix in this basis is given by:

$$\langle \delta_i * \gamma_j, \delta_{i'} * \gamma_{j'} \rangle = \begin{cases} 1 & \text{if } (i' = i \ \&\ j' = j + 1) \vee (i' = i + 1 \ \&\ j' = j) \\ -1 & \text{if } (i' = i \ \&\ j' = j - 1) \vee (i' = i + 1 \ \&\ j' = j + 1) \\ 0 & \text{otherwise} \end{cases}.$$

This shows that Figure 4.5 is the associated Dynkin diagram.

Example 4.7. The calculation of the Dynkin diagram of tame polynomials of the type $g = x^{m_1} + x^{m_2} + \cdots + x^{m_{n+1}}$ is done first by F. Pham (see [1] p. 66). It follows from Example 4.5 and by induction on n that the intersection map in the basis $\delta_\beta,\ \beta \in I$ of Example 4.3 is given by:

$$\text{(4.4)}\quad \langle \delta_\beta, \delta_{\beta'} \rangle = (-1)^{\frac{n(n+1)}{2}} (-1)^{\Sigma_{k=1}^{n+1} \beta'_k - \beta_k},\ \beta = (\beta_1, \beta_2, \ldots, \beta_{n+1}), \beta' = (\beta'_1, \beta'_2, \ldots, \beta'_{n+1})$$

for $\beta_k \leq \beta'_k \leq \beta_k + 1,\ k = 1, 2, \ldots, n+1, \beta \neq \beta'$, and

$$\langle \delta_\beta, \delta_\beta \rangle = (-1)^{\frac{n(n-1)}{2}} (1 + (-1)^n),\ \beta \in I.$$

In the remaining cases, except those arising from the previous ones by a permutation, we have $\langle \delta_\beta, \delta_{\beta'} \rangle = 0$.

4.6 Integration over joint cycles

First, let us introduce some notations. For two $n \times n$ matrices A, B, the matrix $A * B$ is the coordinate wise product of A and B. We have used the lexicographical order in $A * B$. By $\mathrm{diag}(a_\beta)$ we mean the diagonal matrix formed by a_β's. For a $n \times n$ diagonal matrix C we have $(A \cdot C) * B = (A * B) \cdot \tilde{C}$, where $\tilde{C}$ is a $n^2 \times n^2$ diagonal matrix obtained from C in the following way: each entry of C is repeated in $\tilde{C}$ n-times.

Let f and g be two tame polynomials in $n+1$ and $m+1$ variables respectively. In the previous sections we studied the topology of the variety $X = \{f - g = 0\}$ in terms of vanishing cycles of f and g. From this section on, we discuss the integration of differential polynomial forms on such vanishing cycles.

Proposition 4.5. *Let ω_1 (resp. ω_2) be an $(n+1)$-form (resp. $(m+1)$-form) in $\mathbb{C}^{n+1}$ (resp. $\mathbb{C}^{m+1}$). Let also $(t_., \delta_{1b}, \delta_{2b})$ be an admissible triple and $I_1(t) = \int_{\delta_{1t}} \frac{\omega_1}{df}$ and $I_2(t) = \int_{\delta_{2t}} \frac{\omega_2}{dg}$. Then*

$$\int_{\delta_{1b} *_{t_.} \delta_{2b}} \frac{\omega_1 \wedge \omega_2}{d(f-g)} = \int_{t_.} I_1(t) I_2(t) dt.$$

Proof. We have

$$\omega_1 \wedge \omega_2 = df \wedge \frac{\omega_1}{df} \wedge dg \wedge \frac{\omega_2}{dg} = d(f-g) \wedge \frac{\omega_1}{df} \wedge dg \wedge \frac{\omega_2}{dg}$$

and so restricted to X we have

$$\frac{\omega_1 \wedge \omega_2}{d(f-g)} = \frac{\omega_1}{df} \wedge dt \wedge \frac{\omega_2}{dg},$$

where t is the holomorphic function on X defined by $t(x, y) := f(x) = g(y)$. Now, the proposition follows by integration in parts. □

Consider a tame polynomial $f \in \mathbb{C}[x]$ with the regular value $0 \in \mathbb{C}$. The period matrix of f with respect to the basis $\omega = (\omega_{\beta'},\ \beta' \in I)^{\mathsf{t}}$ of $H^n_{\mathrm{dR}}(\{f = 0\})$ and $\delta = (\delta_\beta,\ \beta \in I)^{\mathsf{t}}$ of $H_n(\{f = 0\}, \mathbb{Z})$ is given by

$$\mathsf{pm}(\{f = 0\}) = [\int_{\delta_\beta} \omega_{\beta'}]_{\beta, \beta' \in I}.$$

Let us consider the situation in Example 4.5. Let

$$\mathsf{pm}(\{f=1\}) := [a_{\beta_1\beta_1'}] = [\int_{\delta_{\beta_1}} \omega_{\beta_1'}],\ \mathsf{pm}(\{g=1\}) := [b_{\beta_2\beta_2'}] = [\int_{\delta_{\beta_2}} \omega_{\beta_2'}].$$

We have

$$\begin{aligned}\int_{\delta_{\beta_1}*\delta_{\beta_2}} \frac{x^{\beta_1'}y^{\beta_2'}dx\wedge dy}{d(f+g)} &= a_{\beta_1\beta_1'}b_{\beta_2,\beta_2'}{b'}^{A_{(\beta_1,\beta_2)}-1}\int_0^1 s^{A_{\beta_1}-1}(1-s)^{A_{\beta_2}-1}ds \\ &= a_{\beta_1\beta_1'}b_{\beta_2,\beta_2'}{b'}^{A_{(\beta_1,\beta_2)}-1}B(A_{\beta_1},A_{\beta_2}).\end{aligned}$$

We have used the following identity/definition

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 s^{a-1}(1-s)^{b-1}ds,\ a,b,\in\mathbb{Q}.$$

We conclude that

$$\text{(4.5)}\quad \mathsf{pm}(\{f+g=b'\}) = \mathsf{pm}(\{f=1\}) * \mathsf{pm}(\{g=1\})\cdot \mathrm{diag}({b'}^{A_{(\beta_1,\beta_2)}-1}B(A_{\beta_1},\ A_{\beta_2})).$$

Proposition 4.6. *We have*

$$\mathsf{pm}(\{x^{m_1}+x^{m_2}+\cdots+x^{m_{n+1}}=b'\}) =$$

$$\mathsf{pm}_{m_1} * \mathsf{pm}_{m_2} * \cdots * \mathsf{pm}_{m_{n+1}} \cdot \mathrm{diag}({b'}^{A_\beta - 1}\frac{\Pi_{i=1}^{n+1}\Gamma(\frac{\beta_i+1}{d})}{\Gamma(\sum_{i=1}^{n+1}\frac{\beta_i+1}{d})}),$$

where we have used the canonical basis $\omega_\beta \in \mathsf{H}''$, $\beta\in I$ in Example 3.2 and δ_β, $\beta\in I$ in Example 4.3 for the definition of period matrix.

Proof. By the equality 1.9 in §1.7 we know that

$$\mathsf{pm}(\{x^m=1\}) = \mathsf{pm}_m$$

Successive uses of 4.5 will give us the desired equality of the proposition. □

Example 4.8. We have

$$\mathsf{pm}(\{y^2-1=\}) = \frac{1}{2}[(-1)-1] = [1],\ \mathsf{pm}(\{x^n-1=0\}) = \mathsf{pm}_n$$

Therefore

$$\mathsf{pm}(y^2+x^n=1) = -\mathsf{pm}_n\cdot \mathrm{diag}(B(\frac{1}{2},\frac{1}{n}),B(\frac{1}{2},\frac{2}{n}),\cdots,B(\frac{1}{2},\frac{n-1}{n})).$$

4.7 Reduction of integrals

Let us consider the notations in Example 4.5. This time assume that f is an arbitrary tame polynomial but g is still a homogeneous tame polynomial. Let

$$\mathsf{pm}(\{g=1\},\frac{y^{\beta_2'}dy}{dg}) := [b_{\beta_2\beta_2'}]_{1\times\mu_2} = [\int_{\delta_{\beta_2}}\frac{y^{\beta_2'}dy}{dg}]_{1\times\mu_2},$$

$$A_{\beta_2'} := \sum_{i=1}^{m+1} \frac{\beta_{2,i}' + 1}{m_i} = \frac{\beta_3' + 1}{m}, \; m = [m_1, m_2, \ldots, m_{n+1}]$$

and

$$\mathsf{pm}(\{z^m = 1\}, \frac{z^{\beta_3'} dz}{dz^m}) = [b_{\beta_3,\beta_3'}] = [\int_{\delta_{\beta_3}} \frac{z^{\beta_3'} dz}{dz^m}].$$

Note that for $(\beta_3' + 1) \mid m$ the above matrix is zero. We have:

$$\int_{\delta_{\beta_1} * \delta_{\beta_2}} \frac{x^{\beta_1'} y^{\beta_2'} dx \wedge dy}{d(f+g)} = b_{\beta_2,\beta_2'} \int_{t.} t^{A_{\beta_2'}-1} I_1(t) dt = b_{\beta_2,\beta_2'} \int_{t.} t^{\frac{\beta_3'+1}{m}-1} I_1(t) dt.$$

We consider two cases

1. $A_{\beta_2} \notin \mathbb{N}$: We continue the above equalities

$$= \frac{b_{\beta_2,\beta_2'}}{b_{\beta_3,\beta_3'}} \int_{\delta_{\beta_1} * \delta_{\beta_3}} \frac{x^{\beta_1'} z^{\beta_3'} dx \wedge dz}{d(f + z^q)}$$

2. $A_{\beta_2} \in \mathbb{N}$:

$$= b_{\beta_2,\beta_2'} \int_{t.} t^{A_\beta - 1} I_1(t) dt = b_{\beta_2,\beta_2'} \int_{\Delta} f^{A_\beta - 1} x^{\beta_1'} dx = \int_{\delta_{\beta_1,0}} \theta(\frac{f^{A_\beta - 1} x^{\beta_2'} dx}{df})$$

   where $\Delta_0 := \cup_{s \in [0,1]} \delta_{\beta_1, t_s} \in H_{n+1}(\mathbb{C}^{n+1}, f^{-1}(t), \mathbb{Z})$, $\delta_1 \in H_n(\{f = 0\}, \mathbb{Z})$ and θ is defined in section 3.13.

Complementary notes

1. The main theorems of this Chapter are stated for homologies. For further development of the content of the present text in the direction of iterated integrals (see [69] and the references there), we need to work with homotopy groups. Since vanishing cycles are defined up to homotopy, it is quit possible to rewrite the contents of this Chapter using homotopy groups.
2. The topology of an arbitrary polynomial is much more complicated than the topology of a tame polynomial. The reader is referred to [9] for the study of the topology of a class of polynomials including the tame polynomials. This article does not contain information on the intersection of cycles.

Chapter 5

Weighted tame polynomials over $\mathbb{Q}[t]$

This chapter is the continuation of Chapter 3. In order to have the geometric intuition, we assume that R is a localization of the ring $\mathbb{Q}[t]$, where $t = (t_1, t_2, \ldots, t_s)$ is a multi parameter. As before, f is a tame polynomial in $\mathsf{R}[x]$ and we will freely use the notations related to f. For a fixed value c of t, we denote by f_c the polynomial obtained by replacing c instead of t in f. We denote by L_c the affine hypersurface $\{f_c = 0\} \subset \mathbb{C}^{n+1}$. By a topological cycle $\delta \in H_n(\{f = 0\}, \mathbb{Z})$ we mean a continuous family of cycles $\{\delta_t\}_{t \in V}$, $\delta_t \in H_n(\{f_t = 0\}, \mathbb{Z})$, where V is a small neighborhood in $T := \mathbb{U}_0 \backslash \{\Delta = 0\}$. Our objective is to analyze the modular foliations associated to the Gauss-Manin connection in §3.9. We introduce two filtrations on H which reflect the mixed Hodge structure of the regular fibers of f and state their relations with modular foliations. We also introduce the notion of a Hodge locus and show that it is invariant under certain modular foliations.

5.1 Period map

Let $\omega = (\omega_1, \omega_2, \ldots, \omega_\mu)^{\mathsf{t}}$ be a basis of H. In this basis we can write the matrix of the Gauss-Manin connection ∇:

$$\nabla\omega = A \otimes \omega, \;\; A \in \mathrm{Mat}^{\mu\times\mu}(\Omega^1_T).$$

A fundamental matrix of solutions for the linear differential equation

$$d\mathsf{pm} = \mathsf{pm} \cdot A^{\mathsf{t}}$$

(with pm a $\mu \times \mu$ unknown matrix) is given by the period matrix

$$\mathsf{pm}(t) = [\int_\delta \omega^{\mathsf{t}}] = \begin{pmatrix} \int_{\delta_1}\omega_1 & \int_{\delta_1}\omega_2 & \cdots & \int_{\delta_1}\omega_\mu \\ \int_{\delta_2}\omega_1 & \int_{\delta_2}\omega_2 & \cdots & \int_{\delta_2}\omega_\mu \\ \vdots & \vdots & \vdots & \vdots \\ \int_{\delta_\mu}\omega_1 & \int_{\delta_\mu}\omega_2 & \cdots & \int_{\delta_\mu}\omega_\mu \end{pmatrix},$$

where $\delta = (\delta_1, \delta_2, \ldots, \delta_\mu)^{\mathsf{t}}$ is a basis of the $\mathbb{Z}$-module $H_0(\{f = 0\}, \mathbb{Z})$. This follows from Proposition 3.10. The matrix pm is called the period matrix of f (in the basis δ and ω). By Theorem 4.2 we know that δ can be chosen as a distinguished set of vanishing cycles.

Let us assume that in R there is only one parameter, namely $t = t_1$. It follows that for a fixed $\tilde{\omega} \in \mathsf{H}$ the integrals

$$\int_\delta \tilde{\omega},\ \delta \in H_n(\{f=0\},\mathbb{Z}) \tag{5.1}$$

spans the solution space of a linear differential equation:

$$p_0(t)y^{(m)} + p_1(t)y^{(m-1)} + \cdots + p_{m-1}(t)y' + p_m(t)y = 0, \tag{5.2}$$

$$p_i \in \mathbb{Q}[t],\ i = 0,1,\ldots,m,\ p_0(t) \neq 0,$$

where y' means derivation with respect to the parameter t. The linear differential equations arising in this way are usually called Picard-Fuchs equations. The number m is called the order of the differential equation (5.2) and it is less than or equal μ. If $m = \mu$ then the integrals (5.1) form a basis of the solution space of (5.2).

5.2 Modular foliations

The definition of a modular foliation $\mathcal{F}_\eta,\ \eta \in \mathsf{H}$ in $\mathbb{U}_0$ is done in a similar way as in Definition 1.1 in §1.13. It is the constant locus of integrals $\int_\delta \eta,\ \delta \in H_n(\{f=0\},\mathbb{Z})$ and its algebraic description using the Gauss-Manin connection matrix is done as in (1.18).

Even in some simple examples, such as a hypergeometric polynomial of type $(5,2)$ with 5 parameters, the calculation of the Guass-Manin connection leads to huge polynomials in parameters. This means that the differential forms defining the modular foliations are usually huge (both the size of coefficients and the number of monomials). However, one can find vector fields tangent to $\mathcal{F}_\eta$ which are short enough to write them in a mathematical paper (see §5.4).

Modular foliations may be of dimension zero in the parameter space of tame polynomials with few parameters, for instance one parameter. If

$$\omega_{i,1} \wedge \omega_{i,2} \wedge \cdots \wedge \omega_{i,\mu} \neq 0,\ \forall i = 1,2,\ldots,\mu,$$

where $A = [\omega_{ij}]_{i,j\in I}$ is the Gauss-Manin connection matrix, then the leaves of $\mathcal{F}_{\omega_i}$ are complex manifolds of dimension bigger than zero. In order to obtain non-trivial modular foliations, we may consider complete tame polynomials which have all the possible parameters and differential forms which depend on many parameters. However, so much parameters may not be necessary. The choice of proper parameters will be done example by example. In general the number of parameters of a complete tame polynomial is less than its Milnor number (see 5.6) and so if the differential form η does not depend on any parameter then the modular foliation $\mathcal{F}_\eta$ can be zero dimensional (we do not have yet a precise theorem on this).

5.3 Modular foliations associated to the Weierstrass family

In this section we briefly review the modular foliations associated to the Weierstrass family of elliptic curves. For more information on this topic the reader is referred to [67].

We consider the polynomial

$$f = y^2 - 4t_0(x-t_1)^3 + t_2(x-t_1) + t_3,\ t \in \mathbb{C}^4 \tag{5.3}$$

which is tame with $\mathsf{R} = \mathbb{Q}[\frac{1}{t_0}, t]$, $\deg(x) = 2, \deg(y) = 3$. For f we have the following data:

$$\Delta = \frac{1}{27t_0}(27t_0t_3^2 - t_2^3)$$

$$\nabla\omega = \frac{1}{27t_0\Delta}(\sum_{i=0}^{3} A_i dt_i)\omega, \quad \text{where } \omega = (\frac{dx}{y}, \frac{xdx}{y})^{\mathsf{t}} \text{ and}$$

$$A_0 = \begin{pmatrix} 3/2t_0t_1t_2t_3 - 9t_0t_3^2 + 1/4t_2^3 & -3/2t_0t_2t_3 \\ 3/2t_0t_1^2t_2t_3 + 9t_0t_1t_3^2 - 1/2t_1t_2^3 + 1/8t_2^2t_3 & -3/2t_0t_1t_2t_3 - 18t_0t_3^2 + 3/4t_2^3 \end{pmatrix} \tag{5.4}$$

$$A_1 = \begin{pmatrix} 0 & 0 \\ 27t_0^2t_3^2 - t_0t_2^3 & 0 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} -9/2t_0^2t_1t_3 + 1/4t_0t_2^2 & 9/2t_0^2t_3 \\ -9/2t_0^2t_1^2t_3 + 1/2t_0t_1t_2^2 - 3/8t_0t_2t_3 & 9/2t_0^2t_1t_3 - 1/4t_0t_2^2 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 3t_0^2t_1t_2 - 9/2t_0^2t_3 & -3t_0^2t_2 \\ 3t_0^2t_1^2t_2 - 9t_0^2t_1t_3 + 1/4t_0t_2^2 & -3t_0^2t_1t_2 + 9/2t_0^2t_3 \end{pmatrix}.$$

The period domain is defined to be

$$\mathcal{P} := \left\{ \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \in \mathrm{GL}(2, \mathbb{C}) \mid \mathrm{Im}(x_1\overline{x_3}) > 0 \right\}. \tag{5.5}$$

It lives over the Poincaré upper half plane/Griffiths domain $D = \mathbb{H} := \{x+iy \mid Im(y) > 0\}$ (the mapping $x \mapsto \frac{x_1}{x_2}$). The algebraic group

$$G_0 = \left\{ \begin{pmatrix} k_1 & k_3 \\ 0 & k_2 \end{pmatrix} \mid \ k_3 \in \mathbb{C}, k_1, k_2 \in \mathbb{C}^* \right\} \tag{5.6}$$

acts on $\mathcal{P}$ from the right.

We have the period map

$$\mathsf{pm} : T \to \mathrm{SL}(2, \mathbb{Z})\backslash\mathcal{P}, \ t \mapsto \left[\frac{1}{\sqrt{2\pi i}} \begin{pmatrix} \int_{\delta_1} \frac{dx}{y} & \int_{\delta_1} \frac{xdx}{y} \\ \int_{\delta_2} \frac{dx}{y} & \int_{\delta_2} \frac{xdx}{y} \end{pmatrix} \right]$$

where $\sqrt{i} = e^{\frac{2\pi i}{4}}$ and

$$T := \mathbb{C}^4 \backslash \{t = (t_0, t_1, t_2, t_3) \in \mathbb{C}^4 \mid t_0(27t_0t_3^2 - t_2^3) = 0\}$$

and (δ_1, δ_2) is a basis of the $\mathbb{Z}$-module $H_1(\{f = 0\}, \mathbb{Z})$ such that the intersection matrix in this basis is $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. It is not difficult to show that the period map is a biholomorphy and so $\mathrm{SL}(2, \mathbb{Z})\backslash\mathcal{P}$ has the canonical structure T of an algebraic quasi-affine variety such that the action of G_0 from the right is algebraic. The action of G_0 on T is given by:

$$t \bullet g := (t_0k_1^{-1}k_2^{-1}, t_1k_1^{-1}k_2 + k_3k_1^{-1}, t_2k_1^{-3}k_2, t_3k_1^{-4}k_2^2)$$

$$t = (t_0, t_1, t_2, t_3) \in \mathbb{C}^4, g = \begin{pmatrix} k_1 & k_3 \\ 0 & k_2 \end{pmatrix} \in G_0. \tag{5.7}$$

We denote by

$$F = (F_0, F_1, F_2, F_3) : \mathcal{P} \to T$$

the inverse of the period map. Taking F of (5.7) we have:

$$F_0(xg) = F_0(x)k_1^{-1}k_2^{-1},$$

$$F_1(xg) = F_1(x)k_1^{-1}k_2 + k_3k_1^{-1}, \tag{5.8}$$

$$F_2(xg) = F_2(x)k_1^{-3}k_2,\ F_3(xg) = F_3(x)k_1^{-4}k_2^2,\ \forall x \in \mathcal{P},\ g \in G_0.$$

It satisfies the differential equation

$$x.A(F(x))^{\mathsf{t}} = I.$$

We consider pm as a map sending the vector (t_0, t_1, t_2, t_3) to (x_1, x_2, x_3, x_4). Its derivative at t is a 4×4 matrix whose i-th column constitutes of the first and second row of $\frac{1}{\Delta} x A_i^{\mathsf{t}}$. Therefore, we have

$$(dF)_x = (d\mathsf{pm})_t^{-1} =$$

$$\det(x)^{-1} \begin{pmatrix} -F_0x_4 & F_0x_3 & F_0x_2 & -F_0x_1 \\ \frac{1}{12F_0}(12F_0F_1^2x_3 - 12F_0F_1x_4 - F_2x_3) & -F_1x_3 + x_4 & \frac{1}{12F_0}(-12F_0F_1^2x_1 + 12F_0F_1x_2 + F_2x_1) & F_1x_1 - x_2 \\ 4F_1F_2x_3 - 3F_2x_4 - 6F_3x_3 & -F_2x_3 & -4F_1F_2x_1 + 3F_2x_2 + 6F_3x_1 & F_2x_1 \\ \frac{1}{3F_0}(18F_0F_1F_3x_3 - 12F_0F_3x_4 - F_2^2x_3) & -2F_3x_3 & \frac{1}{3F_0}(-18F_0F_1F_3x_1 + 12F_0F_3x_2 + F_2^2x_1) & 2F_3x_1 \end{pmatrix}.$$

Restricting the first column of the above equality to $\begin{pmatrix} z & -1 \\ 1 & 0 \end{pmatrix}$, $z \in \mathbb{H}$, one conclude that $(F_1(z), F_2(z), F_3(z))$ satisfy the following ordinary differential equation:

$$\begin{cases} \dot{t}_1 = t_1^2 - \frac{1}{12}t_2 \\ \dot{t}_2 = 4t_1t_2 - 6t_3 \\ \dot{t}_3 = 6t_1t_3 - \frac{1}{3}t_2^2 \end{cases} \tag{5.9}$$

With the notations of §5.2, the holomorphic foliation induced by the above differential equation in $\mathbb{C}^3$ is $\mathcal{F}_{\frac{xdx}{y}}$.

5.4 Modular foliations associated to genus two curves

We consider the following family of hyperelliptic genus two curves

$$f := y^2 - x^5 + t_1x^4 + t_2x^3 + t_3x^2 + t_4x + t_5 = 0,\ t \in T := \mathbb{C}^5 \backslash \{\Delta = 0\},$$

where Δ is the discriminant of f. We calculate the Gauss-Manin connection matrix in the basis

$$\omega = (dx \wedge dy, xdx \wedge dy, x^2dx \wedge dy, x^3dx \wedge dy)^{\mathsf{t}}$$

using the algorithms developed in Chapter 3 (note that $\frac{x^idx}{y} = -2\frac{x^idx\wedge dy}{df}$). But the ingredient polynomials have so big size that they do not fit to a mathematical paper. However, the vector field X_i tangent to the foliation $\mathcal{F}_{\frac{x^{i-1}dx}{y}}, i = 1, 2, 3, 4$ has not a huge size: For $\mathcal{F}_{\frac{dx}{y}}$ we have:

$$X_1 = -5\frac{\partial}{\partial t_1} + 4t_1\frac{\partial}{\partial t_2} + 3t_2\frac{\partial}{\partial t_3} + 2t_3\frac{\partial}{\partial t_4} + t_4\frac{\partial}{\partial t_5}.$$

The solution of X_1 passing through a is given by the coefficients of

$$y^2 - (x+z)^5 + a_1(x+z)^4 + a_2(x+z)^3 + a_3(x+z)^2 + a_4(x+z) + a_5$$

and so all solutions of X_1 are algebraic. This is natural because $\frac{dx}{y}$ is invariant under $(x,y)\mapsto(x+b,y),\ b\in\mathbb{C}$. We have also

$$\mathcal{F}_{\frac{xdx}{y}}:\ X_2=-3t_4\frac{\partial}{\partial t_1}+(2t_1t_4-10t_5)\frac{\partial}{\partial t_2}+(8t_1t_5+t_2t_4)\frac{\partial}{\partial t_3}+6t_2t_5\frac{\partial}{\partial t_4}+(4t_3t_5-t_4^2)\frac{\partial}{\partial t_5}$$

$$\mathcal{F}_{\frac{x^2dx}{y}}:\ X_3=(-4t_3t_5+t_4^2)\frac{\partial}{\partial t_1}+(-12t_4t_5)\frac{\partial}{\partial t_2}+(8t_1t_4t_5-4t_2t_3t_5+t_2t_4^2-40t_5^2)\frac{\partial}{\partial t_3}+$$
$$(32t_1t_5^2+4t_2t_4t_5-8t_3^2t_5+2t_3t_4^2)\frac{\partial}{\partial t_4}+(24t_2t_5^2-12t_3t_4t_5+3t_4^3)\frac{\partial}{\partial t_5}$$

$$\mathcal{F}_{\frac{x^3dx}{y}}:\ X_4=(8t_2t_5^2-4t_3t_4t_5+t_4^3)\frac{\partial}{\partial t_1}+(-16t_1t_2t_5^2+8t_1t_3t_4t_5-2t_1t_4^3-8t_3t_5^2+2t_4^2t_5)\frac{\partial}{\partial t_2}+$$
$$(-24t_2^2t_5^2+12t_2t_3t_4t_5-3t_2t_4^3-24t_4t_5^2)\frac{\partial}{\partial t_3}+$$
$$(16t_1t_4t_5^2-40t_2t_3t_5^2+2t_2t_4^2t_5+16t_3^2t_4t_5-4t_3t_4^3-80t_5^3)\frac{\partial}{\partial t_4}+$$
$$(64t_1t_5^3-32t_2t_4t_5^2-16t_3^2t_5^2+24t_3t_4^2t_5-5t_4^4)\frac{\partial}{\partial t_5}.$$

5.5 Reconstructing the period matrix

We have seen that the period matrix $X=\mathsf{pm}^{\mathsf{t}}$ satisfies the differential equation $dX=A^{\mathsf{t}}\cdot X$. Fix a point $t_0\in T$ and let γ be a path in T which connects t_0 to $t\in T$. The analytic continuation of the flat section throught $I_{\mu\times\mu}:=X(t_0)^{-1}X(t_0)$ and along γ is $X(t_0)^{-1}X(t)$. This gives us the equality

$$\mathsf{pm}(t)=(I_{\mu\times\mu}-\int_\gamma A+\int_\gamma AA-\int_\gamma AAA+\cdots)\mathsf{pm}(t_0),\tag{5.10}$$

where we have used iterated integrals (for further details see [40]). Note that the above series is convergent and the sum is homotopy invariant but its pieces are not homotopy invariants. The equality (5.10) implies that if we know the value of period matrix for just one point t_0 then we can construct the period matrix of other points of T using the Gauss-Manin connection. The calculation of period matrix for examples of tame polynomials in $\mathbb{C}[x]$ is done in §4.6.

5.6 Deformation of hypersurfaces

For a given smooth hypersurface M of degree d in $\mathbb{P}^{n+1}$ is there any deformation of M which is not embedded in $\mathbb{P}^{n+1}$? We need the answer of this question because it would be essential to us to know that the fibers of a tame polynomial f form the most effective family of affine hypersurfaces. The answer to our question is given by Kodaira-Spencer Theorem which we are going to explain it in this section. For the proof and more information on deformation of complex manifolds the reader is referred to [51], Chapter 5.

Let M be a complex manifold and $M_t, t\in B:=(\mathbb{C}^s,0),\ M_0=M$ be a deformation of M_0 which is topologically trivial over B. We say that the parameter space B is effective if the Kodaira-Spencer map

$$\rho_0:T_0B\to H^1(M,\Theta)$$

is injective, where Θ is the sheaf of vector fields on M. It is called complete if any other family which contain M is obtained from $M_t, t \in B$ in a canonical way (see [51], p. 228).

Theorem 5.1. *If ρ_0 is surjective at 0 then $M_t, t \in B$ is complete.*

Let $m = \dim_{\mathbb{C}} H^1(M, \Theta)$. If one finds an effective deformation of M with $m = \dim B$ then ρ_0 is surjective and so by the above theorem it is complete.

Let us now M be a smooth hypersurface of degree d in the projective space $\mathbb{P}^{n+1}$. Let T be the projectivization of the coefficient space of smooth hypersurfaces in $\mathbb{P}^{n+1}$. In the definition of M one has already $\dim T = \binom{n+1+d}{d} - 1$ parameters, from which only

$$m := \binom{n+1+d}{d} - (n+2)^2$$

are not obtained by linear transformations of $\mathbb{P}^{n+1}$.

Theorem 5.2. *Assume that $n \geq 2$, $d \geq 3$ and $(n,d) \neq (2,4)$. There exists a m-dimensional smooth subvariety of T through the parameter of M such that the Kodaira-Spencer map is injective and so the corresponding deformation is complete.*

For the proof see [51] p. 234. Let us now discuss the exceptional cases. For $(n,d) = (2,4)$ we have 19 effective parameter but $\dim H^1(M, \Theta) = 20$. The difference comes from a non algebraic deformation of M (see [51] p. 247). In this case M is a $K3$ surface. For $n = 1$, we are talking about the deformation theory of a Riemann surface. According to Riemann's well-known formula, the complex structure of a Riemann surface of genus $g \geq 2$ depends on $3g - 3$ parameters which is again $\dim H^1(M, \Theta)$ ([51] p. 226).

5.7 Mixed Hodge Structure in H

Let $c \in \mathbb{U}_0 \backslash \{\Delta = 0\}$ ($\{f_c = 0\}$ is smooth). The mixed Hodge structure $(W_\bullet, F^\bullet)$ of $H^n(L_c, \mathbb{C})$ is defined by Deligne in [19] using the hypercohomology interpretation of the cohomology of L_c and the sheaf of meromorphic forms with logarithmic poles along the compactification divisor of L_c (see Appendix A). It is natural to define the mixed Hodge structure of H as follows.

Definition 5.1. *The piece $\mathsf{W}_m\mathsf{H}$, $m \in \mathbb{Z}$ (resp. $\mathsf{F}^k\mathsf{H}$, $k \in \mathbb{Z}$) consists of elements $\psi \in \mathsf{H}$ such that the restriction of ψ on all L_c, $c \in \mathbb{U}_0 \backslash \{\Delta = 0\}$ belongs to $W_m H^n(L_c, \mathbb{C})$ (resp. $F^k H^n(L_c, \mathbb{C})$).*

Since in our situation H is a freely generated R-module of finite rank, the pieces of the mixed Hodge structure of H are also freely generated. Their rank is equal to to the dimensions of the mixed Hodge structure of a regular fiber of f. This fact is not trivial and will be clear in §5.17.

Note that we do no know yet whether $\mathrm{Gr}_{\mathsf{F}}^k \mathrm{Gr}_m^{\mathsf{W}} \mathsf{H}$, $k, m \in \mathbb{Z}$ are freely generated R-modules (see (A.3) for the notations). In the same way we define the mixed Hodge structure of the localization of H over multiplicative subgroups of R.

Definition 5.2. *A set $B = \cup_{m,k \in \mathbb{Z}} B_m^k \subset \mathsf{H}$ is a basis of H compatible with the mixed Hodge structure if it is a basis of the R-module H and moreover each B_m^k form a basis of $\mathrm{Gr}_F^k \mathrm{Gr}_m^W \mathsf{H}$.*

Theorem 5.3. *For a weighted homogeneous polynomial $g \in \mathsf{R}[x]$ with an isolated singularity at the origin, the set*

$$B = \cup_{k=1}^{n} B_{n+1}^{k} \cup \cup_{k=0}^{n} B_{n}^{k}$$

with

$$B_{n+1}^{k} = \{\eta_\beta \mid A_\beta = n-k+1\}, B_n^k = \{\eta_\beta \mid n-k < A_\beta < n-k+1\},$$

is a basis of the R-module H' associated to $g - s \in \mathsf{R}[s][x]$ compatible with the mixed Hodge structure. The same is true for H'' replacing η_β with ω_β.

This is a consequence of a theorem of Steenbrink in [82]. We postpone the proof to the section §5.17 in which we have stated a similar Theorem for an arbitrary tame polynomial with non-zero discriminant.

5.8 Hodge numbers

For polynomials $f \in \mathsf{R}[x]$ satisfying the hypothesis of Theorem 4.1, the dimensions of the pieces of the mixed Hodge structure of a regular fiber of f are constants depending only on f and not the parameter (see for instance [84] Proposition 9.20). We call such constants the Hodge numbers.

We are going to consider the weighted polynomial ring $\mathbb{C}[x]$ with $\deg(x_i) = \alpha_i \in \mathbb{N}$. For a given degree $d \in \mathbb{N}$, we would like to have at least one homogeneous polynomial $g \in \mathbb{C}[x]$ with an isolated singularity at the origin and of degree d. For instance for $\alpha_1, \alpha_2, \ldots, \alpha_{n+1} \mid d$ we have the polynomial

$$g = x_1^{m_1} + x_2^{m_2} + \cdots + x_{n+1}^{m_{n+1}},\ m_i := \frac{d}{\alpha_i}.$$

For other d's we do not have yet a general method which produces a tame polynomial of degree d. The vector space $\mathsf{V}_g = \mathbb{C}[x]/\mathrm{jacob}(f)$ has the following basis of monomials

$$x^\beta,\ \beta \in I := \{\beta \in \mathbb{Z}^{n+1} \mid 0 \le \beta_i \le m_i - 2\}$$

and $\mu = \#I = \Pi_{i=1}^{n+1}(m_i - 1)$. In this case

$$A_\beta = \sum_{i=1}^{n+1} \frac{(\beta_i + 1)}{m_i}.$$

Proposition 5.1. *For the affine variety*

$$V = V(m_1, \ldots, m_{n+1}) := \{g = 1\} \subset \mathbb{C}^{n+1}$$

we have

$$h_0^{k-1,n-k} := \dim_{\mathbb{C}}(\mathrm{Gr}_F^{n+1-k}\mathrm{Gr}_{n+1}^W V) = \#\{\beta \in I \mid A_\beta = k\},$$

$$h_0^{k-1,n-k+1} := \dim(\mathrm{Gr}_F^{n+1-k}\mathrm{Gr}_n^W V) = \#\{\beta \in I \mid k-1 < A_\beta < k\}.$$

The above proposition follows from Theorem 5.3. For $\beta \in I$ we have $A_\beta = A_{m-\beta-2}$, where $m - \beta - 2 := (m_1 - \beta_1 - 2, m_2 - \beta_2 - 2, \cdots)$. We have the symmetric sequence of numbers $(h_0^{k-1,n-k},\ k = 1, 2, \ldots, n)$ and $(h_0^{k-1,n-k+1},\ k = 1, 2, \ldots, n+1)$ which correspond to the classical Hodge numbers of the primitive cohomologies of the weighted projective varieties:

$$V_\infty = V_\infty(m_1, \dots, m_{n+1}) := \{g = 0\} \subset \mathbb{P}^{\alpha_1, \alpha_2, \dots, \alpha_{n+1}},$$

$$\bar{V} = V \cup V_\infty \subset \mathbb{P}^{1, \alpha_1, \dots, \alpha_{n+1}}$$

respectively. Here are some tables of Hodge numbers of weighted hypersurfaces obtained by Proposition 5.1.

$n = 2, \alpha_0 = \alpha_1 = \alpha_2 = \alpha_3 = 1$

d	1	2	3	4	5	6	7	8	9	10
$h_0^{0,2}$	0	0	0	1	4	10	20	35	56	84
$h_0^{1,1}$	0	1	6	19	44	85	146	231	344	489
$h_0^{2,0}$	0	0	0	1	4	10	20	35	56	84

$n = 3, \alpha_0 = \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1$

d	1	2	3	4	5	6	7	8	9	10
$h_0^{0,3}$	0	0	0	0	1	5	15	35	70	126
$h_0^{1,2}$	0	0	5	30	101	255	540	1015	1750	2826
$h_0^{2,1}$	0	0	5	30	101	255	540	1015	1750	2826
$h_0^{3,0}$	0	0	0	0	1	5	15	35	70	126

$n = 4, \alpha_0 = \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = 1$

d	1	2	3	4	5	6	7	8	9	10
$h_0^{0,4}$	0	0	0	0	0	1	6	21	56	126
$h_0^{1,3}$	0	0	1	21	120	426	1161	2667	5432	10116
$h_0^{2,2}$	0	1	20	141	580	1751	4332	9331	18152	32661
$h_0^{3,1}$	0	0	1	21	120	426	1161	2667	5432	10116
$h_0^{4,0}$	0	0	0	0	0	1	6	21	56	126

$n = 2, \alpha_0 = \alpha_1 = \alpha_2 = 1,\ \alpha_3 = 3$

d	3	6	9	12	15	18	21	24	27	30
$h_0^{0,2}$	0	1	11	39	94	185	321	511	764	1089
$h_0^{1,1}$	0	19	92	255	544	995	1644	2527	3680	5139
$h_0^{2,0}$	0	1	11	39	94	185	321	511	764	1089

Riemann surfaces: Let us consider the case $n = 1$ and let $\alpha_i := \frac{m_i}{(m_1, m_2)},\ i = 1, 2$

Proposition 5.2. *The variety V_∞ has*

$$\#\{(\beta_1, \beta_2) \in \mathbb{Z}^2 \mid \frac{\beta_1 + 1}{m_1} + \frac{\beta_2 + 1}{m_2} = 1,\ 0 \le \beta_i \le m_i - 2,\ i = 1, 2\} + 1 = (m_1, m_2)$$

points and the genus of $\bar{V}$ is

$$g(\bar{V}) = \frac{(m_1 - 1)(m_2 - 1) - (m_1, m_2) + 1}{2}$$

Proof. A point of V_∞ can be written in the form

$$[1 : \zeta_{m_2}^i] = [(\zeta_{\alpha_1}^j)^{\alpha_1} : \zeta_{\alpha_1}^{j\alpha_2}\zeta_{m_2}^i] = [1 : \zeta_{m_2}^{i+jm_1}].$$

Therefore, the number of points of V_∞ is $\#(\mathbb{Z}_{m_2}/m_1\mathbb{Z}_{m_2}) = (m_1, m_2)$. According to Proposition 5.1 the number of β's such that $A_\beta = 1$ is the dimension of the 0-th primitive cohomology of V_∞ which is the number of points of V_∞ mines one. □

The only cases in which $\overline{V(m_1, m_2)}$ is an elliptic curve are

$$\{m_1, m_2\} = \{2, 4\}, \{2, 3\}, \{3, 3\}.$$

The Riemann surface $V(d, 2)$ belongs to the family of Hypergeometric Riemann surfaces. Its genus is

$$g(V(d,2)) = \begin{cases} \frac{d-1}{2} & \text{if } d \text{ is odd} \\ \frac{d-2}{2} & \text{if } d \text{ is even} \end{cases}.$$

Hypersurfaces of type $1, h, 1$**:** We want to find all the hypersufaces $\bar{V}$ with the first Hodge number equal to 1. This means that we have to find all $m := (m_1, m_2, \cdots, m_{n+1})$ such that

$$1 - \frac{1}{m_{n+1}} \leq \frac{1}{m_1} + \frac{1}{m_2} + \cdots + \frac{1}{m_{n+1}} < 1,\ 2 \leq m_1 \leq m_2 \leq \cdots \leq m_{n+1}. \tag{5.11}$$

Note that the above conditions imply that $m_1 \leq n + 2 \leq m_{n+1}$. We have

$$1 - \frac{1}{m_k} \leq 1 - \frac{1}{m_{n+1}} \leq \frac{1}{m_1} + \frac{1}{m_2} + \cdots + \frac{1}{m_{n+1}} \leq \frac{1}{m_1} + \frac{1}{m_2} + \cdots + \frac{1}{m_{k-1}} + \frac{n+2-k}{m_k}$$

and so

$$m_k \leq (n + 3 - k)(1 - \frac{1}{m_1} - \frac{1}{m_2} - \cdots - \frac{1}{m_{k-1}})^{-1} \leq (n + 3 - k) \cdot p_k,$$

where p_k is defined by induction as follows: $p_1 = 1,\ p_2 = 2$ and p_{k+1} is the maximum of $(1 - \frac{1}{m_1} - \frac{1}{m_2} - \cdots - \frac{1}{m_{k-1}})^{-1}$ for $m_i \leq (n + 3 - i)p_i$ with $\frac{1}{m_1} + \frac{1}{m_2} + \cdots + \frac{1}{m_{k-1}} < 1$. All these imply that for a fixed n the number of such m's is finite. By some computer calculations one expects that we have always $n \leq 3$.

Hodge numbers $1, 10, 1$**:** The regular fibers of the following homogeneous tame polynomials have the Hodge numbers $1, 10, 1$:

$$g = x^7 + y^3 + z^2,\ x^5 + y^4 + z^2, x^4 + y^3 + z^3$$

For these examples there is no cycle at infinity and so the intersection form is non-degenerate(see §5.10).

5.9 Versal deformation vs. tame polynomial

We may consider homogeneous polynomial g with an isolated singularity at the origin as a holomorphic map from $(\mathbb{C}^{n+1}, 0)$ to $(\mathbb{C}, 0)$ and hence consider its versal deformation

$$f(x) = g + \sum_{\beta \in I} t_\beta x^\beta \in \mathsf{R}[x],\ \mathsf{R} := \mathbb{C}[t_\beta \mid \beta \in I],$$

where $\{x^\beta \mid \beta \in I\}$ form a monomial basis of the vector space $\mathbb{C}[x]/\mathrm{jacob}(g)$. In general, the degree of f is bigger than the degree of g and so f may not be a tame polynomial in our context. From topological point of view, the middle cohomology of a generic fiber of f has dimension bigger than the dimension of the middle cohomology of a regular fiber of g and some new singularities and hence vanishing cycles appear after deforming g in the above way. Let us analyze a versal deformation in a special case:

Let $g := x_1^{m_1} + x_2^{m_2} + \cdots + x_{n+1}^{m_{n+1}}$. In this case $I = \{\beta \in \mathbb{Z}^{n+1} \mid 0 \le \beta_i \le m_i - 2\}$. The Milnor number of f is $\mu := \#I = (m_1 - 1)(m_2 - 1)\cdots(m_{n+1} - 1)$ and $A_\beta = \sum_{i=1}^{n+1} \frac{\beta_i + 1}{m_i}$. The versal deformation of g has degree less than $d := [m_1, m_2, \cdots, m_{n+1}]$ if and only if for all $x^\beta \in I$

$$\sum_{i=1}^{n+1} (m_i - 2)\frac{[m_1, m_2, \cdots, m_{n+1}]}{m_i} \le [m_1, m_2, \cdots, m_{n+1}].$$

$$\Leftrightarrow \sum_{i=1}^{n+1} \frac{1}{m_i} \ge \frac{n}{2} \tag{5.12}$$

The equality may happen only for $x_1^{m_1-2} x_2^{m_2-2} \cdots x_{n+1}^{m_{n+1}-2}$. We have

$$\frac{n}{2} \le \sum_{i=1}^{n+1} \frac{1}{m_i} \le A_\beta \le (n+1) - \sum_{i=1}^{n+1} \frac{1}{m_i} \le \frac{n}{2} + 1.$$

It is an easy exercise to verify that (5.12) happens if and only if $m := (m_1, m_2, \ldots, m_{n+1})$ belongs to:

$$(2, 2, \ldots, 2, 2, a),\ a \ge 2,\ (2, 2, \ldots, 2, 3, b), b = 3, 4, 5 \tag{5.13}$$

$$(2, 2, \ldots, 2, 3, 6),\ (2, 2, \ldots, 3, 3, 3), (2, 2, \ldots, 2, 4, 4). \tag{5.14}$$

or their permutation. The equality in (5.12) happens only in the cases (5.14). We conclude that:

Proposition 5.3. *The versal deformation of $g = x_1^{m_1} + x_2^{m_2} + \cdots + x_{n+1}^{m_{n+1}}$, $m_1 \le m_2 \le \cdots \le m_{n+1}$ does not increase the degree of g if and only if m is in the list (5.13) or (5.14). In this case we have*

1. *For $n = 2k$ even, the list of Hodge numbers of the variety $\{g = 0\} \subset \mathbb{P}^\alpha$ (resp. $\{g = 1\} \subset \mathbb{P}^{1,\alpha}$) is of the form $\cdots 0, 1, 1, 0 \cdots$ (resp. $\cdots 0, \mu - 2, 0 \cdots$) if m belongs to the list (5.14) and it is of the form $\cdots 0, 0, 0 \cdots$ (resp. $\cdots, 0, \mu, 0, \cdots$) otherwise,*
2. *For $n = 2k - 1$ odd, the list of Hodge numbers of the variety $\{g = 0\} \subset \mathbb{P}^\alpha$ (resp. $\{g = 1\} \subset \mathbb{P}^{1,\alpha}$) is of the form $\cdots 0, \mu - 2h, 0 \cdots$ (resp. $\cdots 0, h, h, 0 \cdots$),*

where

$$\mu := (m_1 - 1)\cdots(m_{n+1} - 1),\ \mu - 2h := \{\beta \in \mathbb{Z}^{n+1} \mid 0 \le \beta_i \le m_i - 2,\ A_\beta = k\}.$$

5.10 Hodge/Lefschetz cycles and cycles at infinity

Let f be a tame polynomial over $\mathbb{C}$ with $n+1$ variables. We further assume that f has a non-zero discriminant Δ_f and so $\{f=0\}$ is a smooth variety.

Definition 5.3. For an $\omega \in \mathsf{H}$, a cycle $\delta \in H_n(\{f=0\}, \mathbb{Z})$ is called an ω-cycles if $\int_\delta \omega = 0$. It is called a cycle at infinity if

$$\int_\delta \omega = 0, \ \forall \omega \in \mathsf{W}_n,$$

where $(\mathsf{W}_\bullet, \mathsf{F}^\bullet)$ is the mixed Hodge structure of H. Let n be an even number. A cycle $\delta \in H_n(\{f=0\}, \mathbb{Z})$ is called a Hodge cycle if

$$\int_\delta \omega = 0, \ \forall \omega \in \mathsf{F}^{\frac{n}{2}+1} \cap \mathsf{W}_n.$$

For $n=2$ we will also call δ the Lefschetz cycle. By definition the cycles at infinity are Hodge cycles. We say that two Hodge cycles δ_1, δ_2 are equivalent if $\delta_1 - \delta_2$ is a cycle at infinity. We denote by $[\delta_1]$ the equivalent class of the Hodge cycle δ_1.

Let M be a compactification of $\{f=0\}$ and $i : H_n(\{f=0\}, \mathbb{Z}) \to H_n(M, \mathbb{Z})$ be the map induced by the inclusion $\{f=0\} \subset M$. It is a classical fact that the kernel of i is the set of cycles at infinity and a cycle $\delta \in H_n(\{f=0\}, \mathbb{Z})$ is a cycle at infinity if and only if $< \delta, \delta' >= 0$ for all $\delta' \in H_n(\{f=0\}, \mathbb{Z})$. For a Hodge cycle δ, the cycle $i(\delta)$ is Hodge in the classical sense.

Let n be an even natural number and $Z = \sum_{i=1}^s r_i Z_i$, where $Z_i, \ i = 1, 2, \ldots, s$ is a subvariety of M of complex dimension $\frac{n}{2}$ and $r_i \in \mathbb{Z}$. Using a resolution map $\tilde{Z}_i \to M$, where $\tilde{Z}_i$ is a complex manifold, one can define an element $\sum_{i=1}^s r_i [Z_i] \in H_n(M, \mathbb{Z})$ which is called an algebraic cycle (see [6]). The assertion of the Hodge conjecture is that if we consider the rational homologies then a Hodge cycle $\delta \in H_n(\{f=0\}, \mathbb{Q})$ is an algebraic cycle, i.e. there exist subvarieties $Z_i \subset M$ of dimension $\frac{n}{2}$ and rational numbers r_i such that $i(\delta) = \sum r_i [Z_i]$. The difficulty of this conjecture lies in constructing varieties just with their homological information.

By our definition of a Hodge cycle we do not lose anything as it is explained in the following remark.

Remark 5.1. Let M be a hypersurface of even dimension n in the projective space $\mathbb{P}^{n+1}$. By first Lefschetz theorem $H_m(M, \mathbb{Z}) \cong H_m(\mathbb{P}^{n+1}, \mathbb{Z})$, $m < n$ and so the only interesting Hodge cycles are in $H_n(M, \mathbb{Z})$. Let P^n be a hyperplane in general position with respect to M. The intersection $N := P^n \cap M$ is a submanifold of M and is a smooth hypersurface in P^n. Let $\delta \in H_n(M, \mathbb{Z})$. There is an algebraic cycle $[Z] \in H_n(M, \mathbb{Z})$ and integer numbers a, b such that $< \delta - \frac{a}{b}[Z], [N] >= 0$ and so $b\delta - a[Z]$ is in the image of i. The proof of this fact goes as follows: Let $P^{\frac{n}{2}+1}$ be a sub-projective space of $\mathbb{P}^{n+1}$ such that $P^{\frac{n}{2}+1}$ and $P^{\frac{n}{2}} := P^{\frac{n}{2}+1} \cap P^n$ are in general position with respect to M. Put $Z = P^{\frac{n}{2}+1} \cap M$. By Lefschetz first theorem $H_{n-2}(M) \cong H_{n-2}(\mathbb{P}^{n+1}) \cong \mathbb{Z}$. If $a :=< \delta, [N] >$ and $b :=< [Z], [N] >$ then $< \delta - \frac{a}{b}[Z], [N] >= 0$ (b is the degree of $M \cap P^{\frac{n}{2}+1} \cap P^n$ in $P^{\frac{n}{2}+1} \cap P^n$ and so it is not zero).

It is remarkable to mention that:

Proposition 5.4. *Let f be a tame polynomial over $\mathbb{C}$ with a non-zero discriminant and $\delta \in H_n(\{f=0\},\mathbb{Z})$. If $\int_\delta \mathsf{W}_n \cap \mathsf{F}^{n-[\frac{n}{2}]} = 0$ then $\int_\delta \mathsf{W}_n = 0$ and so δ is the cycle at infinity.*

Proof. Let M be the compactification of $\{f=0\}$. The elements of $\mathsf{W}_n \cap \mathsf{F}^{n-[\frac{n}{2}]}$ induce elements in $H^n_{\rm dR}(M)$ which are represented by C^∞ $(n-p,p)$-differential forms with $p = 0,1,\ldots,n-[\frac{n}{2}]$. Since $\int_\delta \bar{\omega} = \overline{\int_\delta \omega}$, we conclude that the integration of all elements of $H^n_{\rm dR}(M)$ over δ is zero. □

Example 5.1. Let us consider the tame polynomial $f = x^2+y^4+z^5-1$ over $\mathbb{Q}$. For this example there is no non-zero cycle at infinity and the Hodge numbers of $\overline{L_f}$ is $1,10,1$. According to Theorem 5.3, $\mathsf{F}^2\mathsf{H}''$ is a one dimensional $\mathbb{Q}$-vector space generated by $\omega_0 := dx\wedge dy \wedge dz$. We have a distinguished set of vanishing cycles

$$\delta_{2,0} * \delta_{4,\beta_2} * \delta_{5,\beta_3},\ \beta_2 = 0,1,2,\ \beta_2 = 0,1,2,3.$$

(see Example 4.3) and the period matrix of ω_0 in the above basis is given by

$$\mathsf{pm}(\omega_0) = \frac{1}{20}[-1]_{1\times 1} * [i-1,-1-i,-i+1]_{1\times 3} * [\zeta_5 - 1, \zeta_5^2-\zeta_5,\ldots,\zeta_5^5-\zeta_5^4]_{1\times 5}.$$

Therefore,

$$\delta_{2,0} * (\delta_{4,0}+\delta_{4,2}) * \delta_{5,i},\ i=1,2,\ldots,4 \tag{5.15}$$

are Lefschetz cycles. It is easy to verify that the entries of $\mathsf{pm}(\omega_0)$ generate a $\mathbb{Q}$-vector space of dimension 8 and $[\mathbb{Q}(\zeta_5,i):\mathbb{Q}] = 8$. This implies that the the $\mathbb{Q}$-vector space of Lefschetz cycles is generated by (5.15). Consider the morphism

$$s : \{x^2+y^4+z^5=1\} \to \{x^2+y^2+z^5=1\},\ (x,y,z)\mapsto (x,y^2,z).$$

The compactification of the target variety has Hodge numbers $0,4,0$. It does not have also non zero cycles at infinity. Therefore, all the cycles in $H_2(\{x^2+y^2+z^5=1\},\mathbb{Q})$ are Lefschetz. For $i=0,1,2,3$ we have

$$\mathsf{pm}(yz^i\omega_0, x^2+y^4+z^5=1) = \frac{1}{12}[-1] * [-2,2,-2]_{1\times 3} * [\zeta_5^i - 1,\ldots,\zeta_5^{4i}-\zeta_5^{3i}]_{1\times 4}$$

and so the cycles

$$\delta_{2,0} * (\delta_{4,0}+\delta_{4,1}) * \delta_{5,j},\ \delta_{2,0} * (\delta_{4,1}+\delta_{4,2}) * \delta_{5,j},\ j=1,2,\ldots,4 \tag{5.16}$$

are in the kernel of s_*, where s_* is the map induced by s in the 2-th homology. The cycles (5.15) and (5.16) form a basis of $H_2(\{f=0\},\mathbb{Q})$. We conclude that the cycles (5.16) form a basis of the kernel of s_* and s induces an isomorphism in the corresponding $\mathbb{Q}$-vector space of Lefschetz cycles.

5.11 An example

We are going to analyze the Hodge cycles of $g = x_1^{m_1} + x_2^{m_2} + \cdots + x_{n+1}^{m_{n+1}} - 1$ in more details. We use Theorem 5.3 and Proposition 4.6 and obtain an arithmetic interpretation of Hodge cycles which does not involve any topological argument. This is explained in the next paragraphs.

Hodge cycles: For each natural number m let

$$I_m := \{0, 1, 2, \dots, m-2\},$$

$$\int_\beta \beta' := \zeta_m^{(\beta+1)(\beta'+1)} - \zeta_m^{\beta(\beta'+1)},\ \beta, \beta' \in I_m$$

and

$$\mathsf{pm}_m(\beta) := [\int_0 \beta, \int_1 \beta, \dots, \int_{m-2} \beta]^{\mathsf{t}} =$$

$$[\zeta_m^{\beta+1} - 1, \zeta_m^{2(\beta+1)} - \zeta_m^{(\beta+1)}, \cdots, \zeta_m^{(m-1)(\beta+1)} - \zeta_m^{(m-2)(\beta+1)}]^{\mathsf{t}},\ \beta \in I_m.$$

For a collection M of $n+1$ natural numbers, which may has repetitions and all of them are greater than one, let

$$I_M = \prod_{m \in M} I_m.$$

We denote the elements of I_M by $\beta = (\beta_1, \beta_2, \cdots, \beta_{n+1})$. In this text I_m's are considered to be disjoint. We define

$$\int_\beta \beta' = \prod_{i=1}^{n+1} \int_{\beta_i} \beta_i',\ \beta, \beta' \in I_M,$$

$$\mathsf{pm}(\beta) = \mathsf{pm}(\beta_1) * \mathsf{pm}(\beta_2) * \cdots * \mathsf{pm}(\beta_m),\ \beta = (\beta_1, \beta_2, \dots, \beta_{n+1}) \in I_M.$$

By $\mathbb{Z}$-linearity we define $\int_\delta \omega,\ \omega, \delta \in \mathbb{Z}[I_M]$. For $\beta \in \mathbb{Z}[I_M]$ a β-cycle is an element $\delta \in \mathbb{Z}[I_M]$ such that $\int_\delta \beta = 0$. A β-cycle written in the canonical basis of $\mathbb{Z}[I_M]$ is a $1 \times \mu$ matrix δ with coefficients in $\mathbb{Z}$ such that $\delta \cdot \mathsf{pm}(\beta) = 0$. An element $\delta \in \mathbb{Z}[I_M]$ which is a β-cycle for all

$$\beta \in I_M, A_\beta \notin \mathbb{Z},\ A_\beta < \frac{n}{2}$$

is called a Hodge cycle.

Proposition 5.5. *Let $\beta \in I_M$. For natural numbers $m_1, m_2, \dots, m_{n+1}$ such hat*

$$[\mathbb{Q}(\zeta_{m_1}, \zeta_{m_2}, \dots, \zeta_{m_{n+1}}), \mathbb{Q}] = (m_1 - 1)(m_2 - 1) \cdots (m_{n+1} - 1) \tag{5.17}$$

there does not exist a non zero β-cycle. In particular, there does not exist a non zero Hodge cycle, and also, there does not exist a cycle at infinity and so

$$\forall \beta' \in I_M, A_{\beta'} \notin \mathbb{N}.$$

The condition (5.17) is satisfied if and only if all m_i's are prime numbers.

Proof. Let $k_i = \mathbb{Q}(\zeta_{m_1}, \zeta_{m_2}, \dots, \zeta_{m_i}),\ i = 1, 2, \dots, n+1$. Since

$$[k_{n+1}, \mathbb{Q}] = [k_{n+1} : k_n] \cdots [k_2 : k_1][k_1 : \mathbb{Q}],\ [k_i, k_{i-1}] \leq m_i - 1,$$

the condition (5.17) implies that $[k_i, k_{i-1}] = m_i - 1$ and so m_i is a prime number. If all m_i's are prime the condition (5.17) is trivially true.

For the proof of the first statement of the theorem, we have to prove that the entries of $\mathsf{pm}(\beta)$ form a $\mathbb{Q}$-basis of $\mathbb{Q}(\zeta_{m_1}, \zeta_{m_2}, \dots, \zeta_{m_{n+1}})$. This statement can be easily proved by induction on n (since m_i's are prime, we can assume that $\beta = (0, 0, \dots, 0)$). □

The intersection map: In the freely generated $\mathbb{Z}$-module $\mathbb{Z}[I_M]$ we consider the bilinear form $\langle , \rangle$ which satisfies

$$\langle \beta, \beta' \rangle = (-1)^n \langle \beta', \beta \rangle, \beta, \beta \in \mathbb{Z}[I_M],$$

$$\langle (\beta_1, \beta_2, \dots, \beta_{n+1}), (\beta_1', \beta_2', \dots, \beta_{n+1}') \rangle = (-1)^{\frac{n(n+1)}{2}} (-1)^{\Sigma_{k=1}^{n+1} \beta_k' - \beta_k}$$

for $\beta_k \leq \beta_k' \leq \beta_k + 1, \ k = 1, 2, \dots, n+1, \ \beta \neq \beta'$, and

$$\langle \beta, \beta \rangle = (-1)^{\frac{n(n-1)}{2}} (1 + (-1)^n), \ \beta \in I_M.$$

In the remaining cases, except those arising from the previous ones by a permutation, we have $\langle \beta, \beta' \rangle = 0$. This bilinear map corresponds to the intersection map of $H_n(\{g = 1\}, \mathbb{Z})$, see Example 4.7.

The cycles at infinity: Using the geometric interpretation of cycles at infinity, one can see that a cycle $\delta \in \mathbb{Z}[I]$ is a cycle at infinity if and only if it is β-cycle for all $\beta \in I$ with $A_\beta \notin \mathbb{N}$, i.e.

$$\int_\delta \beta = 0, \ \forall (\beta \in I, \ A_\beta \notin \mathbb{N}).$$

Equivalently, a cycle $\delta \in \mathbb{Z}[I_M]$ is a cycle at infinity if

$$\langle \delta, \beta \rangle = 0, \ \forall \beta \in I_M$$

Joint cycles: For any two disjoint subset $M = M_1 \cup M_2$ of M, we have a canonical map

$$I_{M_1} \times I_{M_2} \to I_M, (\delta_1, \delta_2) \mapsto \delta_1 * \delta_2,$$

which extends to $\mathbb{Z}[I_{M_1}] \times \mathbb{Z}[I_{M_2}] \to \mathbb{Z}_{I_M}$. A cycle $\delta \in \mathbb{Z}[I_M]$ is called a joint cycle if it has the following property: There exists a decomposition $M = M_1 \cup M_2$ into disjoint non empty sets such that $\delta = \delta_1 * \delta_2, \ \delta_i \in \mathbb{Z}[I_{M_i}], \ i = 1, 2$. By the definition, if $M = M_1 \cup M_2$ and $\delta_1 \in \mathbb{Z}[I_{M_1}]$ is a β_1 cycle then for all $\beta_2 \in I_{M_2}$ and $\delta_2 \in \mathbb{Z}[I_{M_2}]$, $\delta_1 * \delta_2$ is a (β_1, β_2)-cycle.

5.12 Two conjectures

In this section we state two consequences of the Hodge conjecture. For the fact that these conjectures are followed by the Hodge conjecture the reader is referred to the Deligne's lecture [20].

Let f be a tame polynomial over a localization of $\mathbb{Q}[t]$ and $c \in T$. The polynomial f_c is tame over $\mathbb{C}$ and we have a well-defined restriction map $\mathsf{H}_f \to \mathsf{H}_{f_c}$. Let k be a subfield of $\mathbb{C}$ containing the coordinates of c. In fact f_c is tame over k. We denote by $\mathsf{H}_{f_c}(k)$ the corresponding de Rham cohomology. If the Hodge conjecture is true then we have:

Conjecture 5.1. Let c be a $\bar{\mathbb{Q}}$-rational point of $\mathbb{U}_0$ ($c \in \mathbb{U}_0(\bar{\mathbb{Q}})$). For a Hodge cycle $\delta \in H_n(L_c, \mathbb{Q})$ and a differential form $\omega \in \mathsf{W}_n \mathsf{H}_{f_c}(\bar{\mathbb{Q}})$ we have:

$$\int_\delta \omega \in (2\pi i)^{\frac{n}{2}} \bar{\mathbb{Q}}.$$

(see [20], Proposition 1.5). Now consider a field homomorphism $\sigma : \mathbb{C} \to \mathbb{C}$ ($\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$). σ acts on $\mathbb{U}_0$ and we have a well-defined one to one map $\mathsf{H}_{f_c} \to \mathsf{H}_{f_{\sigma(c)}}$, which is obtained by acting σ on the coefficients of differential forms in $\mathbb{C}^{n+1}$ and is denoted again by σ.

Definition 5.4. *A cycle $\delta \in H_n(L_c, \mathbb{Z})$ is called absolute if for all $\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ there is a cycle $\delta' \in H_n(L_{\sigma(c)}, \mathbb{Z})$ such that*

$$\int_\delta \omega = \int_{\delta'} \sigma(\omega), \ \forall \omega \in \mathsf{W}_n \mathsf{H}_{f_c}.$$

If such a δ' exists then it is unique (up to cycles at infinity) and we write $\sigma(\delta) := \delta'$

If the Hodge conjecture is true then we have:

Conjecture 5.2. *Every Hodge cycle is absolute.*

5.13 The loci of Hodge cycles

We consider a tame polynomial f defined over R, a localization of $\mathbb{Q}[t]$. We take a basis $\omega_1, \omega_2, \ldots, \omega_k$ of the freely generated R-module $\mathsf{F}^{\frac{n}{2}+1} \cap \mathsf{W}_n$. Take also a Hodge cycle $\delta_{t_0} \in H_n(\{f_{t_0} = 0\}, \mathbb{Z})$, where f_{t_0} is the specialization of f in $t = t_0$ and $t_0 \in T$ is a regular parameter. By definition we have

$$\int_{\delta_{t_0}} \omega_i = 0, \ i = 1, 2, \ldots, k.$$

For t near to t_0, denote by $\delta_t \in H_n(\{f_t = 0\}, \mathbb{Z})$ the cycle obtained by the monodromy of δ_{t_0}. The variety

$$X_{t_0} := \{t \in (\mathbb{U}_0, t_0) \mid \int_{\delta_t} \omega_i = 0, \ i = 1, 2, \ldots, k\}$$

is called the (local) loci of Hodge cycles. It is a germ of an analytic variety, possibly reducible, defined around a small neighborhood of t_0.

Theorem 5.4. *There exists an algebraic set $Y_{t_0} \subset \mathbb{U}_0$ such that Y_{t_0} in a small neighborhood of t_0 in $\mathbb{U}_0$ coincide with X_{t_0}.*

For a proof of the above theorem see [13]. Again, we will call Y_{t_0} the loci of Hodge cycles through t_0. The importance of the loci of Hodge cycles from the point of view of the present text is described in the following proposition:

Proposition 5.6. *The loci of Hodge cycles through t_0 is invariant by the foliation:*

$$\mathcal{F}_{\mathrm{Hodge}} := \cap_{i=1}^k \mathcal{F}_{\omega_i},$$

where $\omega_i, \ i = 1, 2, \ldots, k$ generate the R-module $\mathsf{F}^{\frac{n}{2}+1} \cap \mathsf{W}_n$.

Proof. Let $\delta_{t_0} \in H_n(L_{t_0}, \mathbb{Z})$ be a Hodge cycle. By definition, on a leaf L_i of $\mathcal{F}_{\omega_i}, \ i = 1, 2, \ldots, k$, the integral $\int_{\delta_t} \omega_i$ is constant and so $(L_i, t_0) \subset \{t \in (\mathbb{U}_0, t_0) \mid \int_{\delta_t} \omega_i = 0\}$. Taking intersection for all $i = 1, 2, \ldots, k$ we get the statement of the proposition. □

It is natural to look for a classification of tame polynomials for which $\mathcal{F}_{\mathrm{Hodge}}$ has a transcendent leaf. We will just discuss the Example 5.1:

Example 5.2. We consider the homogeneous polynomial $g = x^2 + y^4 + z^5$ of degree 20 in $\mathbb{Q}[x, y, z]$, $\deg(x) = 10$, $\deg(y) = 5$, $\deg(z) = 4$. The set of monomials of degree less than 20 is:

$$(1), x, (y), (y^2), y^3, (z), (z^2), (z^3), z^4, xy, xz, xz^2, (yz), (yz^2), (yz^3), (y^2z), (y^2z^2), y^3z$$

(those in parenthesis and y^2z^3 form a basis of V_g). We consider the tame polynomial $f = g + \sum_m t_m m$ over $\mathbb{Q}[\cup_m\{t_m\}]$, where m runs through the above monomials. We want to study the modular foliation $\mathcal{F}_{\omega_0}$, where $\omega_0 = dx \wedge dy \wedge dz$. In Example 5.1, (5.15) we have found a basis δ_i, $i = 1, 2, 3, 4$ of the $\mathbb{Q}$-vectors space of Lefschetz cycles in $H_2(\{g = 1\}, \mathbb{Q})$. Let

$$I_i(t) = \int_{\delta_{i,t}} \omega_0,\ t \in (\mathbb{U}_0, [g-1]),\ i = 1, 2, 3, 4.$$

In Theorem 5.5 part 2 we have proved that ω_0 form a R_Δ-basis of $\mathsf{F}^2\mathsf{H}_\Delta$. Therefore, for all $r \in \mathbb{Q}$, $i, j = 1, 2, 3, 4$ the local variety $\{I_i - rI_j = 0\}$ is a part of an algebraic variety in $\mathbb{U}_0$. This implies that $\frac{I_i}{I_j}$ belong to the algebraic closure of k, where k is the localization of R over $\mathsf{R}\backslash\{0\}$. Proposition 5.6 implies that

$$\frac{I_i}{I_j},\ i, j = 1, 2, 3, 4,\ i \neq j$$

belongs to the closure of the first integral field of $\mathcal{F}_{\omega_0}$, i.e. they are constant along the leaves of $\mathcal{F}_{\omega_0}$.

Note that ω_0 is invariant under $s : \mathbb{C}^3 \to \mathbb{C}^3, s(x, y, z) = (x + a, y + b, z + c)$ for all $a, b, c \in \mathbb{C}$.

Example 5.3. Let us consider $g = x_1^d + x_2^d + \cdots + x_{n+1}^d$ with $d > n + 1$ and the tame polynomial $f = g - \sum_\alpha t_\alpha x^\alpha$, where α runs through $\deg(x^\alpha) < d$, $x^\alpha \neq x_i^{d-1}$, $i = 1, 2, \ldots, n + 1$. If the differential forms

$$\frac{\partial}{\partial t_\alpha}\frac{dx}{f} = \frac{x^\alpha dx}{f^2}$$

are R-independent in M then the modular foliation $\mathcal{F}_\omega$ is trivial, i.e. its leaves are points. Note that $\omega_\alpha := x^\alpha dx$'s are R-linearly independent.

5.14 $\mathbb{G}_m$ action

Recall the notion of a complete tame polynomial f in Definition 3.3. For $t_0 \in \mathbb{U}_0$ we define $[f_{t_0}] := t_0$. Let $f = f_0 + f_1 + \cdots + f_{d-1} + f_d$, $g := f_d$, where f_i is a homogeneous polynomial of degree i in x. Let $\mathbb{G}_m$ be the the multiplicative group $(\mathbb{C}, \cdot)$. It acts on $\mathbb{C}^{n+1}$ by

$$(x_1, x_2, \ldots, x_{n+1}) \to (\lambda^{\alpha_1}x_1, \lambda^{\alpha_2}x_2, \ldots, \lambda^{\alpha_{n+1}}x_{n+1})$$

and on $\mathbb{U}_0$ by

$$[\lambda \bullet f] := [\frac{f(\lambda^{\alpha_1}x_1, \lambda^{\alpha_2}x_2, \ldots, \lambda^{\alpha_{n+1}}x_{n+1})}{\lambda^d}] = [\lambda^{-d}f_0 + \lambda^{-d+1}f_1 + \cdots \lambda^{-1}f_d + g],$$

where $\alpha_i = \deg(x_i)$. The action of $\lambda \in \mathbb{G}_m$ gives us an isomorphy $\tilde{\lambda} : \{\lambda \bullet f = 0\} \to \{f = 0\}$ and hence hence it leaves

$$\Sigma_0 := \{t \in \mathbb{U}_0 \mid \Delta_f = 0\}$$

invariant. We have

$$\tilde{\lambda}^{-1}(\omega_\beta) = \lambda^{d \cdot A_\beta} \omega_\beta,$$

where $\omega_\beta,\ \beta \in I$ is the canonical basis of H and so the modular foliations $\mathcal{F}_{\omega_\beta},\ \beta \in I$ are invariant under the action of $\mathbb{G}_m$.

Fix a natural number $1 < s \leq d$. The action of $e^{\frac{2\pi i}{s}} \in \mathbb{G}_m$ on $\mathbb{U}_0$ has the fixed points

$$\Sigma_s := \{[g + \sum_{s|(d-i)} f_i]\}.$$

A loci of Hodge cycles in $\mathbb{U}_0$ contains the point $[g] \in \Sigma_0 \subset \mathbb{U}_0$. A loci of Hodge cycles containing $[g-1]$ contains the one dimesnional variety $[g-t],\ t \in \mathbb{C}$.

5.15 Residue map

Let us be given a closed submanifold N of real codimension c in a manifold M. The Leray (or Thom-Gysin) isomorphism is

$$\tau : H_{k-c}(N, \mathbb{Z}) \tilde{\to} H_k(M, M-N, \mathbb{Z})$$

holding for any k, with the convention that $H_s(N) = 0$ for $s < 0$. Roughly speaking, given $\delta \in H_{k-c}(N)$, its image by this isomorphism is obtained by thickening a cycle representing δ, each point of it growing into a closed c-disk transverse to N in M (see for instance [15] p. 537). Let N be a connected codimension one submanifold of the complex manifold M of dimension n. Writing the long exact sequence of the pair $(M, M-N)$ and using τ we obtain:

$$\cdots \to H_{n+1}(M, \mathbb{Z}) \to H_{n-1}(N, \mathbb{Z}) \xrightarrow{\sigma} H_n(M-N, \mathbb{Z}) \xrightarrow{i} H_n(M, \mathbb{Z}) \to \cdots \tag{5.18}$$

where σ is the composition of the boundary operator with τ and i is induced by inclusion. Let $\omega \in H^n(M-N, \mathbb{C}) := \check{H}_n(M-N, \mathbb{Z}) \otimes \mathbb{C}$, where $\check{H}_n(M-N, \mathbb{Z})$ is the dual of $H_n(M-N, \mathbb{Z})$. The composition $\omega \circ \sigma : H_{n-1}(N, \mathbb{Z}) \to \mathbb{C}$ defines a linear map and its complexification is an element in $H^{n-1}(N, \mathbb{C})$. It is denoted by $\mathrm{Resi}_N(\omega)$ and called the residue of ω in N. We consider the case in which ω in the n-th de Rham cohomology of $M-N$ is represented by a meromorphic C^∞ differential form ω' in M with poles of order at most one along N. Let $f_\alpha = 0$ be the defining equation of N in a neighborhood U_α of a point $p \in N$ in M and write $\omega' = \omega_\alpha \wedge \frac{df}{f}$. For two such neighborhoods U_α and U_β with non empty intersection we have $\omega_\alpha = \omega_\beta$ restricted to N. Therefore, we get a $(n-1)$-form on N which in the de Rham cohomology of N represents $\mathrm{Resi}_N \omega$ (see [35] for details). This is called the Poincaré residue.

Let us be given a tame polynomial f, $c \in T := \mathbb{U}_0 \backslash \{\Delta = 0\}$ and $\omega \in \Omega^{n+1}_{\mathbb{U}_1/\mathbb{U}_0}$. We can associate to $\frac{\omega}{f^k},\ k \in \mathbb{N}$ its residue in L_c which is going to be an element of $H^n(L_c, \mathbb{C})$ (we first substitute t with c in $\frac{\omega}{f}$ and then take the residue as it is explained in the previous paragraph). This gives us a global section $\mathrm{Resi}[\frac{\omega}{f^k}]$ of the n-th cohomology bundle of the fibration f over T. It is represented by the element $[\frac{\omega}{f^k}] \in \mathsf{M}$, where M is defined in §3.11.

Having Proposition 3.12 in mind, we regard $\mathrm{Resi}(\frac{\omega}{f^k})$ as an element in the localization of H over $\{1, \Delta, \Delta^2, \dots\}$. In the case $k = 1$, $\mathrm{Resi}(\frac{\omega}{f}) = [\omega] \in \mathsf{H}''$.

If v is a vector field in $\mathbb{U}_0$ then we have

$$v \int_\delta \mathrm{Resi}(\frac{\omega}{f^k}) = v \int_{\sigma(\delta)} \frac{\omega}{f^k} = \int_{\sigma(\delta)} \nabla_v([\frac{\omega}{f^k}]) = \int_\delta \mathrm{Resi}(\nabla_v([\frac{\omega}{f^k}]))$$

and so

$$\mathrm{Resi}(\nabla_v([\frac{\omega}{f^k}]) = \nabla_v(\mathrm{Resi}([\frac{\omega}{f^k}]).$$

5.16 The numbers $d_\beta,\ \beta \in I$

In this section on we consider the homogenization

$$F = F(x_0, x) = x_0^d f(\frac{x_1}{x_0^{\alpha_1}}, \frac{x_2}{x_0^{\alpha_2}}, \dots, \frac{x_{n+1}}{x_0^{\alpha_{n+1}}})$$

of a tame polynomial f over R with the new variable x_0. Let $\mathsf{V}_F := \mathsf{R}[x_0, x]/\mathrm{jacob}(F)$. The R-module V_F may not be freely generated but $\mathsf{V}_F \otimes_{\mathsf{R}} \mathsf{k}$ is a k-vector space with a basis

$$x^\beta x_0^{\beta_0},\ \beta \in I,\ 0 \le \beta_0 < d_\beta \tag{5.19}$$

for some $d_\beta \in \mathbb{N}$ (see for instance [65] Lemma 5 for a proof). This implies that there exists $\mathsf{a} \in \mathsf{R}$ such that $\mathsf{V}_F \otimes_{\mathsf{R}} \mathsf{R}_{\mathsf{a}}$ is freely generated by (5.19). In this section we give an algorithm which gives us such d_β's and a.

From the proof of Proposition 3.5 it follows that x^I generates freely the $\mathsf{R}[x_0]$-module

$$V := \frac{\mathsf{R}[x_0, x]}{\langle \frac{\partial F}{\partial x_i} \mid i = 1, 2, \dots, n+1 \rangle}.$$

Let

$$A_F : V \to V,\ A_F(G) = \frac{\partial F}{\partial x_0} G,\ G \in V.$$

Proposition 5.7. *The matrix of A_F in the basis x^I is of the form $d \cdot [x_0^{K_{\beta,\beta'}} c_{\beta,\beta'}]$, where*

$$K_{\beta,\beta'} := \min\{0, d - 1 + \deg(x^\beta) - \deg(x^{\beta'})\}$$

and $A_f := [c_{\beta,\beta'}]$ is the multiplication by f in V_f. In particular, if $A_{\beta'} - A_\beta \ge 1$ then $c_{\beta,\beta'} = 0$ and

$$\det(A_F) = \Delta_f x_0^{(d-1)\mu}.$$

Proof. Since the polynomial F is weighted homogeneous, we have $\sum_{i=0}^{n+1} \alpha_i x_i \frac{\partial F}{\partial x_i} = d \cdot F$ and so $x_0 \frac{\partial F}{\partial x_0} = d.F$ in V (note that $\alpha_0 = 1$ by definition). Let

$$F.x^\beta = \sum_{\beta' \in I} x^{\beta'} c_{\beta,\beta'}(x_0) + \sum_{i=1}^{n+1} \frac{\partial F}{\partial x_i} q_i,\ c_{\beta,\beta'}(x_0) \in \mathsf{R}[x_0],\ q_i \in \mathsf{R}[x_0, x]. \tag{5.20}$$

Since the left hand side is homogeneous of degree $d + \deg(x^\beta)$ we can assume that the pieces of the right hand side are also homogeneous of the same degree. This can be done by taking an arbitrary equation (5.20) and subtracting the unnecessary parts. □

Let
$$V_F = \frac{\mathsf{R}[x_0, x]}{\text{jacob}(F)} = \frac{V}{< \frac{\partial F}{\partial x_0} q \mid q \in V >} = \frac{\mathsf{R}[x_0]^\mu}{A_F \mathsf{R}[x_0]^\mu}.$$

Here $\mathsf{R}[x_0]^\mu$ is the set of $\mu \times 1$ matrices with entries in $\mathsf{R}[x_0]$. In order to find a basis of the form (5.19), we introduce a kind of Gaussian elimination in A_F and simplify it. For this reason we introduce the operation $GE(\beta_1, \beta_2, \beta_3)$. Here, β_1, β_2(resp. β_3) denote the rows (resp. column) of A_F. For $\beta \in I$ let $(A_F)_\beta$ be the β-th row of A_F.

- Input: A_F and $\beta_1, \beta_2, \beta_3 \in I$ with $A_{\beta_1} \leq A_{\beta_2}$. Output: a matrix A'_F.

  We replace $(A_F)_{\beta_2}$ with
  $$-\frac{(A_F)_{\beta_2,\beta_3}}{(A_F)_{\beta_1,\beta_2}} * (A_F)_{\beta_1} + (A_F)_{\beta_2}$$
  and take a the c_{β_1,β_2} in $(A_F)_{\beta_1,\beta_2} = c_{\beta_1,\beta_2} \cdot x_0^{K_{\beta_1,\beta_2}}$. Since for all $\beta_4 \in I$ we have
  $$K_{\beta_2,\beta_3} + K_{\beta_1,\beta_4} = K_{\beta_1,\beta_3} + K_{\beta_2,\beta_4}$$
  the obtained matrix A'_F is of the form $[x_0^{K_{\beta,\beta'}} c'_{\beta,\beta'}]$, $c'_{\beta_2,\beta_3} = 0$ and $A'_F.\mathsf{R}_\mathsf{a}[x_0]^\mu = A_F \mathsf{R}_\mathsf{a}[x_0]^\mu$.

We give an example of algorithm which calculates d_β's.

- Input: A_F and an ideal $\mathsf{b} \subset \mathsf{R}$. Output: $d_\beta, \beta \in I$ and $\mathsf{a} \notin \mathsf{R} \backslash \mathsf{b}$ such that $V_F \otimes_\mathsf{R} \mathsf{R}_\mathsf{a}$ is freely generated as an R_a-modules by (5.19).

  We identify I with $\{1, 2, \ldots, \mu\}$ and assume that
  $$\beta_1 \leq \beta_2 \Rightarrow A_{\beta_1} \geq A_{\beta_2}.$$
  The algorithm has μ steps indexed by $\beta = \mu, \mu - 1, \ldots, 1$. In $\beta = \mu$ we have $A(\beta) = A_F$ and $\mathsf{a} = 1$. In the step β we find the first β_1 such that the coefficient c of $x_0^{K_{\beta,\beta_1}}$ in $A(\beta)_{\beta,\beta_1}$ does not belong to b and put $d_{\beta_1} = d - 1 + \deg(x^\beta) - \deg(x^{\beta_1})$ and a to be the old value of a times c. For $\beta_2 = \beta - 1, \ldots, 1$ we make $GE(\beta, \beta_2, \beta_1)$ and then go to the step $\beta - 1$.

Note that
$$A_\beta < n + 1, \; d_\beta < d(n + 2 - A_\beta), \; \sum_{\beta \in I} d_\beta = \mu(d - 1).$$

The first one is already in Theorem 5.3. The second inequality is obtained by applying the first inequality associated to F:
$$A_{(d_\beta - 1, \beta)} = A_\beta + \frac{d_\beta - 1 + 1}{d} < n + 2.$$

The Milnor number of F is $\sum_{\beta \in I} d_\beta$ and equals to the Milnor number of $g - x_0^d$ which is $\mu(d - 1)$.

Proposition 5.8. *For a tame polynomial f we have:*

1. *Let* $\mathsf{b} \subset \mathsf{R}$ *be an ideal. There exist a map* $\beta \in I \to d_\beta \in \mathbb{N} \cup \{0\}$ *and* $\mathsf{a} \in \mathsf{R}$ *such that* $\mathsf{a} \notin \mathsf{b}$ *and* $\mathsf{R}_\mathsf{a}[x, x_0]/\mathrm{jacob}(F)$ *as an* R_a*-module is freely generated by*
$$x^\beta x_0^k, \ 0 \le k \le d_\beta - 1.$$
2. *If* $f(0)$ *is not a root of a monic polynomial in one variable and coefficients in* R *then for some* $\mathsf{a} \in \mathsf{R}$ *the* R_a*-module* $\mathsf{R}_\mathsf{a}[x, x_0]/\mathrm{jacob}(F)$ *is freely generated by*
$$x^\beta x_0^k, \ 0 \le k \le d - 2.$$

Proof. The first part is already proved. Proof of the second part. By various use of the operation GE on A_F we get $(A_F)_{\beta,\mu} = 0, \ \beta \in I \backslash \{\mu\}$. We repeat this for $\mu - 1$ and obtain $(A_F)_{\beta,\mu-1} = 0, \beta \in I \backslash \{\mu, \mu - 1\}$. After μ-times we get a lower triangular matrix. We always divide on a rational function in $f(0)$ with coefficients in R such that the leading coefficient of its numerator is one. By our hypothesis, division by zero does not occur. □

5.17 Mixed Hodge structure of M

Since the inclusion $\mathsf{H} \to \mathsf{M}$ induces an isomorphism of R-modules $\mathsf{M}_\Delta \cong \mathsf{H}_\Delta$, we can define the mixed Hodge structure of M in a similar way as we did it in Definition 5.1. Recall that $\{x^\beta \mid \beta \in I\}$ is a monomial basis of V_g.

Proposition 5.9. *Every element of* M *can be written as an* R*-linear sum of the elements*

$$\frac{\omega_\beta}{f^k}, \ \beta \in I, \ 1 \le k \le n+1, \ A_\beta \le k. \tag{5.21}$$

Proof. Let us be given an element $\frac{\omega}{f^k}$ in M. According to Corollary 3.1, we write $\omega = \sum_{\beta \in I} a_\beta \omega_\beta + df \wedge d\omega_2 + f\omega_1$ and so
$$\frac{\omega}{f^k} = \sum_{\beta \in I} a_\beta \frac{\omega_\beta}{f^k} + \frac{\omega_1}{f^{k-1}} \text{ in } \mathsf{M}.$$

We repeat this argument for ω_1. At the end we get $\frac{\omega}{f^k}$ as a linear combination of $\frac{\omega_\beta}{f^i}, \ \beta \in I, \ k \in \mathbb{N}$. An alternative way is to say that ω can be written as an $\mathsf{R}[f]$-linear combinations of $\omega_\beta, \ \beta \in I$ modulo $df \wedge d\Omega^{n-1}_{\mathbb{U}_1/\mathbb{U}_0}$ (see Theorem 3.1). We define the degree of $\frac{\omega}{f^k}, \ k \in \mathbb{N}, \omega \in \Omega^{n+1}_{\mathbb{U}_1/\mathbb{U}_0}$ to be $\deg(\omega) - \deg(f^k)$ and the degree conditions (3.15) implies that the we have used only $\frac{\omega_\beta}{f^i}$ with $\deg(\frac{\omega_\beta}{f^i}) \le \deg(\frac{\omega}{f^k})$.

Now, we have to get rid of elements of type $\frac{\omega_\beta}{f^k}, \ A_\beta > k$. Given such an element, in M we have:
$$\frac{\omega_\beta}{f^k} = \frac{1}{A_\beta}\frac{d\eta_\beta}{f^k} = \frac{k}{A_\beta}\frac{df \wedge \eta_\beta}{f^{k+1}} = \frac{k}{A_\beta}\frac{f\omega_\beta + (g-f)\omega_\beta + d(f-g)\wedge\eta_\beta}{f^{k+1}}$$
and so
$$\frac{\omega_\beta}{f^k} = \frac{k}{A_\beta - k}\frac{(g-f)\omega_\beta + d(f-g)\wedge\eta_\beta}{f^{k+1}}. \tag{5.22}$$

The degree of the right hand side of (5.22) is less than $d(A_\beta - k)$, which is the degree of the left hand side. We write the right hand side in terms of $\frac{\omega_{\beta'}}{f^s}, \ \beta' \in I, s \in \mathbb{N}$ and repeat (5.22) for these new terms. Since each time the degree of the new elements $\frac{\omega'_\beta}{f^s}$ decrease, at some pont we get the desired form for $\frac{\omega_\beta}{f^k}$. □

Recall the notations in Appendix A.5.

Proposition 5.10. *For a monomial x^β with $A_\beta = k \in \mathbb{N}$, the meromorphic form $\frac{x^\beta dx}{f^k}$ has a pole of order one at infinity and its Poincaré residue at infinity is $\frac{X^\beta \eta_\alpha}{g^k}$.*

Proof. Let us write the above form in the homogeneous coordinates (A.8). We use $d(\frac{X_i}{X_0^{\alpha_i}}) = X_0^{-\alpha_i} dX_i - \alpha_i X_i X_0^{-\alpha_i - 1} dX_0$ and

$$\begin{aligned}
\frac{x^\beta dx}{f^k} &= \frac{(\frac{X_1}{X_0^{\alpha_1}})^{\beta_1} \cdots (\frac{X_{n+1}}{X_0^{\alpha_{n+1}}})^{\beta_{n+1}} d(\frac{X_1}{X_0^{\alpha_1}}) \wedge \cdots \wedge d(\frac{X_{n+1}}{X_0^{\alpha_{n+1}}})}{f(\frac{X_1}{X_0^{\alpha_1}}, \cdots, \frac{X_{n+1}}{X_0^{\alpha_{n+1}}})^k} \\
&= \frac{X^\beta \eta_{(1,\alpha)}}{X_0^{(\sum_{i=1}^{n+1} \beta_i \alpha_i) + (\sum_{i=1}^{n+1} \alpha_i) + 1 - kd} (X_0 \tilde{F} - g(X_1, X_2, \cdots, X_{n+1}))^k} \\
&= \frac{X^\beta \eta_{(1,\alpha)}}{X_0 (X_0 \tilde{F} - g(X_1, X_2, \cdots, X_{n+1}))^k} \\
&= \frac{dX_0}{X_0} \wedge \frac{X^\beta \eta_\alpha}{(X_0 \tilde{F} - g)^k}
\end{aligned}$$

The last equality is up to forms without pole at $X_0 = 0$. The restriction of $\frac{X^\beta \eta_\alpha}{(X_0 \tilde{F} - g)^k}$ to $X_0 = 0$ gives us the desired form. $\square$

Theorem 5.5. *For a tame polynomial f with a non zero discriminant Δ we have:*

1. *The elements*
$$\frac{\omega_\beta}{f^k}, \ \beta \in I, A_\beta = n + 1 - i,$$
form a R-basis of $\mathrm{Gr}^i_{\mathsf{F}} \mathrm{Gr}^{\mathsf{W}}_{n+1} \mathsf{M}$.

2. *The elements*
$$\frac{\omega_\beta}{f^k}, \ \beta \in I, k \in \mathbb{N}, A_\beta \leq k \leq n + 1 - i \tag{5.23}$$
generate $\mathsf{F}^i \mathsf{M}_\Delta \cap \mathsf{W}_n \mathsf{M}_\Delta$ as an R_Δ-module.

3. *If $f(0)$ is not a root of a monic polynomial in one variable and coefficients in R then the elements*
$$\frac{\omega_\beta}{f^k}, \ -[-A_\beta] = k, \ A_\beta < k,$$
form a R_{a}-basis of $\mathrm{Gr}^{n+1-k}_{\mathsf{F}} \mathrm{Gr}^{\mathsf{W}}_n \mathsf{M}_a$ for some $\mathsf{a} \in \mathsf{R}$.

4. *Let $\mathsf{b} \subset \mathsf{R}$ be an ideal. There exist a map $\beta \in I \to d_\beta \in \mathbb{N} \cup \{0\}$ and $\mathsf{a} \in \mathsf{R}$ such that $\mathsf{a} \notin \mathsf{b}$ and*
$$\frac{\omega_\beta}{f^k}, \ A_\beta + \frac{1}{d} \leq k \leq A_\beta + \frac{d_\beta}{d}, \tag{5.24}$$
form a R_{a}-basis of $\mathrm{Gr}^{n+1-k}_{\mathsf{F}} \mathrm{Gr}^{\mathsf{W}}_n \mathsf{M}_{\mathsf{a}}$.

Proof. Theorem A.3 and Proposition 5.10 imply that for all $c \in T$ the residue of the forms

$$\frac{x^\beta dx}{f^k},\ A_\beta = k \tag{5.25}$$

in L_c form a basis of $\mathrm{Gr}_F^{n+1-k}\mathrm{Gr}_{n+1}^W H^n(L_c, \mathbb{C})$. This proves the first part of the theorem.

For the proof of the next statements we note that every $\frac{\omega_\beta}{f^k},\ A_\beta < k$ in the homogeneous coordinates $(X_0, X_1, \ldots, X_{n+1})$ is of the form

$$\frac{X_0^{\beta_0} X^\beta \eta_{(1,\alpha)}}{F^k},$$

where β_0 is defined through the equality $A_\beta + \frac{\beta_0+1}{d} = k$ (see §A.5).

We write an element of $\mathsf{F}^i\mathsf{M} \cap \mathsf{W}_n\mathsf{M}$ in terms of (5.21) and according to the first part there will not be terms of the type $\frac{\omega_\beta}{f^k},\ A_\beta = k$. Now, the second statement follows form Theorem A.3 immediately.

The third and fourth statements follows from Theorem A.3 and Proposition 5.8. □

Proof of Theorem 5.3: This follows form Theorem 5.5 and (3.38).

5.18 Griffiths transversality

In the free module H we have introduced the mixed Hodge structure and the Gauss-Manin connection. It is natural to ask whether there is any relation between these two concepts or not. The answer is given by the following theorem:

Theorem 5.6. *Let $(\mathsf{W}_\bullet, \mathsf{F}^\bullet)$ be the mixed Hodge structure of H. The Gauss-Manin connection on H satisfies:*

1. *Griffiths transversality:*

$$\nabla(\mathsf{F}^i) \subset \Omega_T^1 \otimes_{\mathsf{R}} \mathsf{F}^{i-1},\ i = 1, 2, \ldots, n.$$

2. *No residue at infinity: We have*

$$\nabla(\mathsf{W}_n) \subset \Omega_T^1 \otimes_{\mathsf{R}} \mathsf{W}_n.$$

3. *Residue killer: For a tame polynomial f of degree d defined over a polynomial parameter ring $\mathsf{R} = \mathbb{C}[t]$, $\omega \in \mathsf{H}$ and a polynomial vector field $v \in \mathcal{D}_{\mathbb{U}_0}$ such that $\deg(\frac{\partial f}{\partial v}) < d$ there exists $k \in \mathbb{N}$ such that $\nabla_v^k \omega \in \mathsf{W}_n$.*

Proof. Griffiths transversality has been proved in [34, 33] for Hodge structures. For a recent text see also [84]. The proof for mixed Hodge structures is similar and can be found in [87, 88]. The proof in the context of tame polynomials is as follows:

It is enough to prove the theorem for the Gauss-Manin connection ∇_v along a vector field $v \in \mathcal{D}_{\mathbb{U}_0}$. We have to prove that ∇_v maps $\mathsf{F}^i\mathsf{M}$ to $\mathsf{F}^{i-1}\mathsf{M}$. By Leibniz rule, it is enough to take an element ω in the set (5.23) and prove that $\nabla_v\omega$ is in $\mathsf{F}^{i-1}\mathsf{M}$. This follows from Theorem 5.5, part 1,2 and (3.37).

The second part of the theorem follows form (3.37) and the fact that $\frac{\omega_\beta}{f^k},\ A_\beta < k$ generate $\mathsf{W}_n\mathsf{M}$

For the third part of the theorem we present two proof, one is algebraic and the other is topological: We write an element of M as a R-linear combination of (5.21). By the hypothesis, the coefficients are polynomials in the parameters t. The condition $\deg(\frac{\partial f}{\partial v}) < d$ implies that for all ω in the set (5.21) we have $\nabla_v \omega \in \mathsf{W}_n\mathsf{M}$. These two imply the third statement. For the topological proof we make the assumption that the last homogeneous part of g does not depend on any parameter. This implies in particular that for any vector field $v \in \mathcal{D}_{\mathbb{U}_0}$ we have $\deg(\frac{\partial f}{\partial v}) < d$. The monodromy group does not change a cycle at infinity $\delta \in H_n(\{f = 0\}, \mathbb{Z})$. Considering $\delta = \{\delta_t\}_{t\in T}$ as a family of cycles in the fibers of f, the tameness condition implies that δ extends also to the singular fibers of f and so $\int_\delta \omega$ is a one valued function in $\mathbb{U}_0 = \mathbb{C}^s$ and hence a polynomial in the parameters. □

Corollary 5.1. *The codimension of a modular foliation $\mathcal{F}_\omega,\ \omega \in \mathsf{F}^i$ is at most the rank of the (free) R-module F^{i+1}.*

Proof. Let us choose a basis $\omega_i,\ i = 1, 2, \ldots, s$ of the free R-module F^{i+1}. According to Griffiths transversality for $\omega \in \mathsf{F}^i$ we can write $\nabla\omega = \sum_{i=1}^s \eta_i \otimes \omega_i$ and so $\mathcal{F}_\omega$ is given by $\eta_i = 0,\ i = 1, 2, \ldots, s$. □

Recall that the first integral field of a foliation $\mathcal{F}(V)$, V a k-sub vector space of meromorphic differential forms in $\mathbb{U}_0$, is defined to be the set

$$\{f \in \mathsf{k} \mid df \in V\},$$

where k is the localization of R over $\mathsf{R}\backslash\{0\}$. It is not hard to see that the above set is indeed a field. Using the argument of the proof of the second part of Theorem 6.14 we can prove that:

Proposition 5.11. *The modular foliation $\mathcal{F}_\omega, \omega \in \mathsf{H}\backslash\mathsf{W}_n$ has a non trivial first integral field.*

Proof. The integration of ω over a cycle at infinity is a nontrivial element of the first integral field of the foliation $\mathcal{F}_\omega$. □

Complementary notes

1. Concerning the notion of a versal deformation discussed in §5.9, it seems to be reasonable to expect that for a homogeneous polynomial with an isolated singularity at the origin $0 \in \mathbb{C}^{n+1}$ there is an affine embedding $\mathbb{C}^{n+1} \subset \mathbb{C}^N$ and a kind of versal deformation of g in $\mathbb{C}^N$ such that the topological structure of the global fibers do not change. To investigate these kind of problems, one has to introduce the notion of tame ideals corresponding to codimension bigger than one varieties. The first examples to investigate are $x^5 + y^5$ and $x^3 + y^7$.
2. A hypersurface of degree d in $\mathbb{P}^{n+1}$ depends on $\binom{d+n+1}{n+1} = \frac{1}{(n+1)!}d^{n+1} + \cdots$ parameters and its cohomology is of dimension $(d-1)^{n+1} - (d-1)^n + \cdots$. For n fixed and d tending to ∞ the first number is smaller than the second one and this gives the impression that modular foliations of differential forms which do not depend on any parameter, are trivial foliations, i.e. their leaves are points. It would be essential to the purpose of the present text to classify all the cases in which we have non trivial foliations (see Example 5.3). However, note that for any tame polynomial we can construct differential forms which depends on some parameters and the corresponding foliations are non trivial (see Example 1.12).
3. Let $\omega_1, \omega_2 \in W_n\mathsf{H}$. Calculate $\int_{\overline{L_t}} [\omega_1] \wedge [\omega_2] \in \mathsf{R}_\Delta$. Now let $\omega_1, \omega_2 \in \mathsf{H}\backslash W_n\mathsf{H}$. Calulate $\int_{\overline{L_t}\backslash L_t} [\mathrm{Resi}(\omega_1)] \wedge [\mathrm{Resi}(\omega_2)] \in \mathsf{R}_\Delta$. For similar calculations see [1] p.454 or [20]. Here $[\cdot]$ denotes classes in the corresponding de Rham cohomologies.
4. Having Proposition 5.6 in mind, one may formulate a premature conjecture that every codimension k algebraic set invariant by $\mathcal{F}_{\mathrm{Hodge}}$ is a locus of Hodge cycles.

5. Having the discussion of §5.3 in mind, we expect that a solution of of the vector fields $X_i,\ i=2,3,4$ given in §5.4 is parameterized by some one dimensional subspace of the Siegel domain $\mathbb{H}_2$. To find such an space we have to explicitly determine the image of the period map. It is an analytic subset of 4×4 matrices and has dimension between 3 and 15. Its dimension is bigger than 3 because the moduli of genus two curves is of dimension three and the hyperelliptic family contains all genus two curves(up to biholomorphism). Its dimension is less than or equal 15 because of the property $P1$ in §6.7.
6. Under what kind of conditions on a family of varieties and a differential form, all the leaves of the corresponding modular foliation are algebraic? Can one describe all algebraic varieties invariant by a modular foliation? For instance, for the modular foliations of §5.4 one may speculate that X_i's have at least one transcendent solution. One may also speculate that all algebraic varieties invariant by some X_i is the locus of parameters such that some geometric phenomenon of a fixed topological type, like contraction to an elliptic curve or having a special automorphism, occurs. Such spaces in the Siegel domain $\mathbb{H}_2$ are studied by G. Humbert in [43] and carry his name.
7. In the case $n=2$, Lefschetz $(1,1)$ theorem (see for instance [84]) implies that every Lefschetz cycle is algebraic. The proof is not constructive and it would be interesting to give a proof in which one constructs algebraic cycles with prescribed homology classes. A simple example to look is $f=x^2+y^2+z^N,\ N\geq 2$ in which all the the two dimensional cycles are Lefschetz.

Chapter 6

Moduli of Polarized Hodge Structures

In this chapter we construct an analytic variety U and an action of an algebraic group G_0 on U from the right such that U/G_0 is the moduli space of ploarized Hodge structures of a fixed type. The space U lives over the so called Griffiths domain (see [36]) and has the advantage that it carries certain modular foliations such that their pull-back by period maps are the geometric modular foliations constructed in Chapter 5. Our hope is that U has a canonical structure of an algebraic variety such that the action of G_0 is algebraic and the corresponding modular foliations are of geometric origin. If this is the case then one may look to the action of G_0 on U from the point of view of geometric invariant theory, see [70]. Since we know partial compactifications of U/G_0 (see [46]) in the analytic context, the algeraic version would be also of interest. The objective of this chapter in the case of Hodge structures of type $h^{10} = h^{01} = 1$ is already realized in [66, 67]. In this case $U = \mathrm{SL}(2,\mathbb{Z})\backslash\mathcal{P}$, where

$$\mathcal{P} := \{\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \in \mathrm{GL}(2,\mathbb{C}) \mid \mathrm{Im}(x_1\overline{x_3}) > 0\}.$$

The algebraic group

$$G_0 = \{\begin{pmatrix} k_1 & k_3 \\ 0 & k_2 \end{pmatrix} \mid \ k_1, k_2, k_3 \in \mathbb{C},\ k_1k_2 \neq 0\}$$

acts from right on U by the usual multiplication of matrices. In §5.3 we saw that

$$U \overset{\text{bihol.}}{\cong} \{(t_0, t_1, t_2, t_3) \in \mathbb{C}^4 \mid t_0(27t_0t_3^2 - t_2^3) \neq 0\}$$

and under the above biholomorphism the action of G_0 is given by

$$t \bullet g = (t_0k_1^{-1}k_2^{-1}, t_1k_1^{-1}k_2 + k_3k_1^{-1}, t_2k_1^{-3}k_2, t_3k_1^{-4}k_2^2),$$

$$t = (t_0, t_1, t_2, t_3) \in \mathbb{C}^4,\ g = \begin{pmatrix} k_1 & k_3 \\ 0 & k_2 \end{pmatrix} \in G_0.$$

The complex manifold $\mathcal{P}$ contains copies of $\mathbb{C}^*$ and so it is not a Hermitian symmetric domain (see [42],[59]). This rules out the direct use of Baily-Borel theorem (see [2]) on $\mathcal{P}$.

6.1 The space of polarized lattices

We fix a $\mathbb{C}$-vector space V_0 of dimension h, a natural number $m \in \mathbb{N}$ and a $h \times h$ integer valued matrix Ψ_0 such that the associated bilinear form

$$\mathbb{Z}^h \times \mathbb{Z}^h \to \mathbb{Z},\ (a,b) \to a\Psi_0 b^t$$

is non-degenerate, symmetric if m is even and skew if m is odd. Note that in the case of $\mathbb{Z}$-modules by non-degenerate we mean that the associated morphism

$$\mathbb{Z}^h \to (\mathbb{Z}^h)^\vee,\ a \to (b \to a^t \Psi_0 b)$$

is a an isomorphism, where $\vee$ means the dual of a $\mathbb{Z}$-module.

A lattice $V_\mathbb{Z}$ in V_0 is a $\mathbb{Z}$-module generated by a basis of V_0. A polarized lattice $(V_\mathbb{Z}, \psi_\mathbb{Z})$ of type Ψ_0 is a lattice $V_\mathbb{Z}$ together with a bilinear map $\psi_\mathbb{Z} : V_\mathbb{Z} \times V_\mathbb{Z} \to \mathbb{Z}$ such that in a $\mathbb{Z}$-basis of $V_\mathbb{Z}$, $\psi_\mathbb{Z}$ has the form Ψ_0.

Let $\mathcal{L}$ be the space of polarized lattices of type Ψ_0 in V_0. Usually, we denote an element of $\mathcal{L}$ by $x, y, \ldots$ and the associated lattice (resp. bilinear form) by $V_\mathbb{Z}(x), V_\mathbb{Z}(y), \ldots$ (resp. $\psi_\mathbb{Z}(x), \psi_\mathbb{Z}(y), \ldots$). Let R be any subring of $\mathbb{C}$. For instance, R can be $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}$, $\mathbb{Z}[\sqrt{2}]$ and etc.. We define

$$V_R(x) := V_\mathbb{Z}(x) \otimes_\mathbb{Z} R \text{ and } \psi_R(x) : V_R(x) \times V_R(x) \to R \text{ the induced map.}$$

Conjugation with respect to $x \in \mathcal{L}$ of an element $a = \sum a_i \delta_i \in V_0$, where $V_\mathbb{Z}(x) = \sum \mathbb{Z}\delta_i$, is defined by

$$\overline{a}^x = \sum \overline{a}_i \delta_i,$$

where $\overline{s},\ s \in \mathbb{C}$ is the usual conjugation of complex numbers.

6.2 Polarized lattices and automorphism groups

We fix $x_0 \in \mathcal{L}$ and define Γ_R to be a subgroup of $P := \mathrm{Aut}(V_0)$ containing all $p \in P$ such that p induces an element in $\mathrm{Aut}(V_R(x_0), \psi_R(x_0))$, i.e. it induces an R-linear map $V_R(x_0) \to V_R(x_0)$ with

$$\psi_R(x_0)(pa, pb) = \psi_R(x_0)(a, b),\ \forall a, b \in V_R.$$

We will mainly make use of $\Gamma_\mathbb{Z}$. We define the action of P on $\mathcal{L}$ from right. For $p \in P$ and $x \in \mathcal{L}$, xp is defined by:

$$V_\mathbb{Z}(xp) := p^{-1}(V_\mathbb{Z}(x)),\ \psi_\mathbb{Z}(xp)(v_1, v_2) := \psi_\mathbb{Z}(pv_1, pv_2),\ \forall v_1, v_2 \in V_\mathbb{Z}(xp).$$

By definition we have

$$\psi_\mathbb{C}(xp)(v_1, v_2) = \psi_\mathbb{C}(pv_1, pv_2),\ \forall v_1, v_2 \in V_0.$$

Note that the group P acts on V_0 from left in a natural way:

$$pv = p(v),\ p \in P,\ v \in V_0.$$

Proposition 6.1. *For all $v \in V_0$, $x \in \mathcal{L}$ and $p \in P$, we have*

$$\overline{v}^{xp} = p^{-1}\overline{pv}^x.$$

Proof. Take a $\mathbb{Z}$-basis $\delta_i, i \in I$ of $V_{\mathbb{Z}}(x)$. Then $p^{-1}(\delta_i),\ i \in I$ is a $\mathbb{Z}$-basis of $V_{\mathbb{Z}}(xp)$ and we write

$$v = \sum_{i\in I} a_i p^{-1}(\delta_i) \text{ or equivalently } p(v) = \sum_{i\in I} a_i\delta_i,\ a_i \in \mathbb{C}.$$

Now

$$\overline{v}^{xp} = \sum_{i\in I} \overline{a_i} p^{-1}(\delta_i) = p^{-1}(\sum_{i\in I} \overline{a_i}\delta_i) = p^{-1}\overline{pv}^{x}$$

□

Proposition 6.2. *We have:*

1. *$\Gamma_{\mathbb{Z}}$ is a discrete subgroup of P;*

2. *The canonical map*

$$\alpha : \Gamma_{\mathbb{Z}}\backslash P \to \mathcal{L},\ \alpha(p) = x_0 p \tag{6.1}$$

   is well-defined and isomorphism;

3. *$\mathcal{L}$ has a canonical structure of a complex manifold.*

Proof. 1. The set $V_{\mathbb{Z}}(x_0)$ is a discerte subset of V_0. 2. We have

$$\Gamma_{\mathbb{Z}} = \{p \in P \mid x_0 p = x_0\}.$$

3. The action of $\Gamma_{\mathbb{Z}}$ on P has no fixed points and no accumulation points. □

From now on for $p \in P$ we define

$$V_{\mathbb{Z}}(p) := V_{\mathbb{Z}}(\alpha(p)), \psi_{\mathbb{Z}}(p) := \psi_{\mathbb{Z}}(\alpha(p)),\ \overline{v}^{p} = \overline{v}^{\alpha(p)}$$

and so on.

6.3 Poincaré dual

In this section we explain the notion of Poincaré dual in the context of current chapter. Let $(V_{\mathbb{Z}}(x), \psi_{\mathbb{Z}}(x))$ be a polarized lattice and $\delta \in V_{\mathbb{Z}}(x)^{\vee}$, where $\vee$ means the dual of a $\mathbb{Z}$-module. We will use the notation

$$\int_{\delta} \omega := \delta(\omega),\ \forall \omega \in V_0.$$

The Poincaré dual $\delta^{\mathsf{pd}} \in V_{\mathbb{Z}}(x)$ is the unique element with the property:

$$\int_{\delta} \omega = \psi_{\mathbb{Z}}(x)(\delta^{\mathsf{pd}}, \omega),\ \forall \omega \in V_{\mathbb{Z}}(x).$$

It exists and is unique because $\psi_{\mathbb{Z}}$ is non-degenerate. Using the Poincaré duality one defines the dual polarization:

$$\psi_{\mathbb{Z}}(x)^{\vee}(\delta_i, \delta_j) := \psi_{\mathbb{Z}}(x)(\delta_i^{\mathsf{pd}}, \delta_j^{\mathsf{pd}}),\ \delta_i, \delta_j \in V_{\mathbb{Z}}(x)^{\vee}.$$

Proposition 6.3. *We have:*

$$(A^\vee\delta)^{\mathsf{pd}} = A^{-1}\delta^{\mathsf{pd}}, \ \forall A \in \Gamma_{\mathbb{Z}}, \ \delta \in V_{\mathbb{Z}}(x_0)^\vee,$$

where $A^\vee : V_{\mathbb{Z}}(x_0)^\vee \to V_{\mathbb{Z}}(x_0)^\vee$ *is the induced dual map.*

Proof. The proposition follows from:

$$\int_{A^\vee\delta}\omega = \int_\delta A\omega = \psi_{\mathbb{Z}}(x_0)(\delta^{\mathsf{pd}}, A\omega) = \psi_{\mathbb{Z}}(x_0)(A^{-1}\delta^{\mathsf{pd}}, \omega).$$

□

We define

$$\Gamma_{\mathbb{Z}}^\vee := \mathrm{Aut}(V_{\mathbb{Z}}(x_0)^\vee, \psi_{\mathbb{Z}}(x_0)^\vee).$$

It follows from the above proposition that

$$\Gamma_{\mathbb{Z}} \to \Gamma_{\mathbb{Z}}^\vee, \ A \mapsto A^\vee$$

is an isomorphism of groups.

6.4 Period matrix

Sometimes it is convenient to have explicite coordinate functions on P. In this section we explain such functions.

Let $I = \{1, 2, \ldots, h\}$ and $\omega = (\omega_i)_{i\in I}$ be a $\mathbb{C}$-basis of V_0. In this chapter a basis of V_0 is written as a $h \times 1$ matrix of elements of V_0. We take a $\mathbb{Z}$-basis δ_{x_0} of $V_{\mathbb{Z}}(x_0)^\vee$ such that the matrix of $\psi_{\mathbb{Z}}(x_0)$ in the basis δ_{x_0} is Ψ_0. For an arbitrary lattice $V_{\mathbb{Z}}(x)$ with $p \in P, \alpha(p) = x,$, where α is the map in (6.1), we obtain a $\mathbb{Z}$-basis $\delta = \delta_x := p^\vee(\delta_{x_0})$, where $p^\vee : V_{\mathbb{Z}}(x_0)^\vee \to V_{\mathbb{Z}}(x)^\vee$ is the map induced in the dual spaces. We define the period matrix in the following way:

$$\mathsf{pm} = \mathsf{pm}(x) = [\int_\delta \omega^{\mathsf{t}}]_{h\times h} := \begin{pmatrix} \int_{\delta_1}\omega_1 & \int_{\delta_1}\omega_2 & \cdots & \int_{\delta_1}\omega_h \\ \int_{\delta_2}\omega_1 & \int_{\delta_2}\omega_2 & \cdots & \int_{\delta_2}\omega_h \\ \vdots & \vdots & \vdots & \vdots \\ \int_{\delta_h}\omega_1 & \int_{\delta_h}\omega_2 & \cdots & \int_{\delta_h}\omega_h \end{pmatrix}.$$

We identify P with the space of the above matrices, which is $\mathrm{GL}(h, \mathbb{C})$. We write an element A of $\Gamma_{\mathbb{Z}}$ in the basis δ_{x_0}, and redefine $\Gamma_{\mathbb{Z}}$:

$$\Gamma_{\mathbb{Z}} := \{A \in \mathrm{GL}(h, \mathbb{Z}) \mid A\Psi_0 A^{\mathsf{t}} = \Psi_0\}.$$

The action of $\Gamma_{\mathbb{Z}}$ (resp. P) on P from left(resp. right) is the usual multiplication of matrices.

Instead of the period matrix it is usefull to use the matrix

$$\mathsf{q} = \mathsf{q}(x), \quad \text{where } \delta^{\mathsf{pd}} = \mathsf{q}\omega$$

Then we have:

$$(\delta^{\mathsf{pd}})^{\mathsf{t}} = \omega^{\mathsf{t}}\mathsf{q}^{\mathsf{t}} \Longrightarrow \Psi_0 = \mathsf{pm}\cdot\mathsf{q}^{\mathsf{t}}.$$

Again, the action of $\Gamma_{\mathbb{Z}}$ (resp. P) on P from left(resp. right) is the usual multiplication of matrices. If we identify V_0 with $\mathbb{C}^h$ through the basis ω then q is a matrix whose rows are the entries of δ.

6.5 A canonical connection on $\mathcal{L}$

Recall the terminology related to connections in Chapter 2. We consider the trivial bundle $\mathcal{H} = V_0 \times \mathcal{L}$ on $\mathcal{L}$. On $\mathcal{H}$ we have a well-defined connection

$$\nabla : \mathcal{H} \to \Omega^1_{\mathcal{L}} \times \mathcal{H}$$

such that a flat section s of ∇ in an small open set $U \subset \mathcal{L}$ satisfies $s(x) \in V_{\mathbb{Z}}(x)$. Let $\omega = \{\omega_i\}_{i=1}^h$ be a basis of V_0. We can consider ω_i as a global section of $\mathcal{H}$ and so we have

$$\nabla\omega = A \otimes \omega,\ A = \begin{pmatrix} \omega_{11} & \omega_{12} & \cdots & \omega_{1h} \\ \omega_{21} & \omega_{22} & \cdots & \omega_{2h} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_{h1} & \omega_{h2} & \cdots & \omega_{hh} \end{pmatrix},\ \omega_{ij} \in H^0(\mathcal{L}, \Omega^1_{\mathcal{L}}). \tag{6.2}$$

The connection ∇ is integrable and so $dA = A \wedge A$:

$$d\omega_{ij} = \sum_{k=1}^{h} \omega_{ik} \wedge \omega_{kj},\ i, j = 1, 2, \ldots, h. \tag{6.3}$$

Similar formula as in(6.3) appears in the discussion of frames in Hermitian Geometry (See P.A. Griffiths article [37]). Let δ be a basis of flat sections. Write $\delta = \mathsf{q}\omega$. We have

$$\omega = \mathsf{q}^{-1}\delta \Rightarrow \nabla(\omega) = d(\mathsf{q}^{-1})\mathsf{q}\omega \Rightarrow$$

$$A = d\mathsf{q}^{-1} \cdot \mathsf{q} = d(\mathsf{pm}^{\mathsf{t}} \cdot \Psi_0^{-\mathsf{t}}) \cdot (\Psi_0^{\mathsf{t}} \cdot \mathsf{pm}^{-\mathsf{t}}) = d(\mathsf{pm}^{\mathsf{t}}) \cdot \mathsf{pm}^{-\mathsf{t}}.$$

We have used the equality $\Psi_0 = \mathsf{pm} \cdot \mathsf{q}^{\mathsf{t}}$.

6.6 Some functions on $\mathcal{L}$

First of all recall that if δ and ω be two basis of V_0, $\delta = p\omega$ for some $p \in \mathrm{GL}(2, \mathbb{C})$ and a linear form on V_0 in the basis δ (resp. ω) has the matrix form A (resp. B) then $pBp^t = A$.

For two vectors $\omega_1, \omega_2 \in V_0$ one can define the following holomorphic function on $\mathcal{L}$

$$f_{\omega_1,\omega_2}(x) = \psi_{\mathbb{C}}(x)(\omega_1, \omega_2). \tag{6.4}$$

To obtain all such possible holomorphic functions we first choose a basis ω of V_0 and for $x \in \mathcal{L}$ we write $\delta_x = \mathsf{q} \cdot \omega$. Then

$$\mathsf{pm}^t \Psi_0^{-\mathsf{t}} \mathsf{pm} = (\mathsf{q}^{-1})^t \Psi_0 \mathsf{q}^{-1} = [f_{\omega_i,\omega_j}]_{i,j \in I} \tag{6.5}$$

(we have used the identitity $\Psi_0 = \mathsf{q} \cdot \mathsf{pm}^{\mathsf{t}}$). Other functions as in (6.4) are $\mathbb{C}$-linear combination of the entries of the above matrix. It is remarkable that the matrix $F = [f_{\omega_i,\omega_j}]_{i,j \in I}$ satisfies the differential equation:

$$dF = A \cdot F + F \cdot A^{\mathsf{t}}, \tag{6.6}$$

where A is the connection matrix. It is easy to check that every solution of the above differential equation is of the form $\mathsf{pm}^t \cdot C \cdot \mathsf{pm}$ for some constant $h \times h$ matrix C with entries in $\mathbb{C}$ (if F is a solution of (6.6) then $F \cdot \mathsf{pm}^{-1}$ is a solution of $dY = A \cdot Y$).

We fix an isomorphism of $\mathbb{C}$-vectorspaces $o : \wedge^h V_0 \cong \mathbb{C}$. It is called an orientation. Now, we have the determinant map

$$\det : (V_0)^h \to \mathbb{C},\ \det(\omega_1, \omega_2, \ldots, \omega_h) := o(\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_h).$$

Using that one can define

$$\det{}^2 : \mathcal{L} \to \mathbb{C},\ \det{}^2(x) := \det(\delta_1, \delta_2, \cdots, \delta_h)^2 = \det(\mathsf{q})^2 = \frac{\det(\Psi_0)^2}{\det(\mathsf{pm})^2},$$

where $\delta := (\delta_i)_{i\in I}$ is a $\mathbb{Z}$-basis of $V_{\mathbb{Z}}(x)$ for which the bilinear form $\psi_{\mathbb{Z}}(x)$ has the form Ψ_0. Taking another basis will contribute $\det(A),\ A \in \Gamma_{\mathbb{Z}}$, which is $+1$ or -1, to the det function and so $\det^2$ is a well-defined function. In case $\det(A) = 1$ for all $A \in \Gamma_{\mathbb{Z}}$, we can define the det function in $\mathcal{L}$.

We have a plenty of non holomorphic functions on $\mathcal{L}$. For two elements $\omega_1, \omega_2 \in V_0$ we define

$$f_{\omega_1,\bar{\omega}_2} : \mathcal{L} \to \mathbb{C},\ f_{\omega_1,\bar{\omega}_2}(x) = \psi_{\mathbb{C}}(x)(\omega_1, \overline{\omega_2}^x).$$

Let $\omega_i, i = 1, 2, \ldots$ be as before. We write

$$\delta = \overline{\mathsf{q}}^\epsilon \overline{\omega}^x$$

The entries of the bellow matrix gives us a set which spans the $\mathbb{R}$-vector space of real functions obtained in the above way:

$$\mathsf{pm}^t \Psi_0^{-\mathsf{t}} \overline{\mathsf{pm}} = (\mathsf{q}^{-1})^t \Psi_0 \overline{\mathsf{q}}^{-1} = [f_{\omega_i,\bar{\omega}_j}]_{i,j\in I} \tag{6.7}$$

The matrix $G = [f_{\omega_i,\bar{\omega}_j}]_{i,j\in I}$ satisfies the differential equation:

$$dG = A \cdot G + G \cdot \overline{A}^{\mathsf{t}}, \tag{6.8}$$

where A is the connection matrix.

For $\omega \in V_0,\ x \in \mathcal{L}$ and $\epsilon \in \{0, 1\}$ define $\overline{\overline{\omega}}^{x,\epsilon} = \omega$ if $\epsilon = 0$ and $= \overline{\omega}^x$ otherwise. Let $\epsilon : I \to \{0, 1\}$ be a function and $\omega = (\omega_i)_{i\in I}$ be h elements in V_0. We have the following complex valued analytic function on $\mathcal{L}$:

$$f^\epsilon_\omega : \mathcal{L} \to \mathbb{C},\ f^\epsilon_\omega(x) = \det(\overline{\overline{\omega}}^{x,\epsilon}).$$

6.7 Hodge filtrations

We fix Hodge numbers

$$h^{i,m-i} \in \mathbb{N} \cup \{0\},\ h^i := \sum_{j=i}^{m} h^{j,m-j},\ i = 0, 1, \ldots, m,\ h^0 = h$$

and a filtration

$$F_0^\bullet : \{0\} = F_0^{m+1} \subset F_0^m \subset \cdots \subset F_0^1 \subset F_0^0 = V_0,\ dim(F_0^i) = h^i \tag{6.9}$$

on V_0. We define

$$H^{i,m-i}(x) := F_0^i \cap \overline{F_0^{m-i}}^x$$

and the following properties for $x \in \mathcal{L}$:

1. $\psi_{\mathbb{C}}(x)(F_0^i, F_0^j) = 0,\ \forall i,j,\ i+j > m;$
2. $V_0 = \oplus_i H^{i,m-i}(x);$
3. $(-1)^{i+\frac{m}{2}}\psi_{\mathbb{C}}(x)(v, \overline{v}^x) > 0,\ \forall v \in H^{i,m-i}(x),\ v \neq 0.$

Throughout the text we call these properties P1, P2 and P3.

Proposition 6.4. *Fix a polarized lattice $x \in \mathcal{L}$.*

1. P1 implies that

$$\psi_{\mathbb{C}}(H^{i,m-i}(x), H^{j,m-j}(x)) = 0 \text{ except for } i+j=m.$$

2. $\sum_i H^{i,m-i}(x) = \oplus_i H^{i,m-i}(x)$ if and only if

$$F_0^i \cap \overline{F_0^j}^x = 0,\ \forall\ i+j > m. \tag{6.10}$$

Proof. 1. We have $\psi_{\mathbb{C}}(H^{i,m-i}(x), H^{j,m-j}(x)) = \psi_{\mathbb{C}}(F_0^i \cap \overline{F_0^{m-i}}^x, F_0^j \cap \overline{F_0^{m-j}}^x) = 0$. Because if $i+j > m$ then $\psi_{\mathbb{C}}(F_0^i, F_0^j) = 0$ and if $i+j < m$ then $\psi_{\mathbb{C}}(\overline{F_0^i}^x, \overline{F_0^j}^x) = 0$.

2. If $a_{m-k,k} + \cdots + a_{0,m} = 0,\ a_{i,m-i} \in H^{i,m-i}(x)$ then

$$-a_{m-k,k} = a_{m-k-1,k+1} + \cdots + a_{0,m} \in F_0^{m-k} \cap \overline{F_0^{k+1}}^x \Rightarrow a_{k,m-k} = 0.$$

The proof in other direction is a consequence of

$$F_0^i \cap \overline{F_0^j}^x = H^{i,m-i}(x) \cap H^{m-j,j}(x),\ i+j > m.$$

□

6.8 The analytic variety U

Define

$$\mathcal{K} := \{x \in \mathcal{L} \mid x \text{ satisfies P1 }\},$$

$$U := \{x \in \mathcal{L} \mid x \text{ satisfies P1,P2, P3 }\}.$$

We also define

$$\tilde{\mathcal{K}} := \{x \in P \mid x \text{ satisfies P1 }\},$$

$$\mathcal{P} := \{x \in P \mid x \text{ satisfies P1,P2, P3 }\}.$$

Definition 6.1. *A basis $\omega_i, i = 1,2,\ldots,h$ of V is compatible with the filtration $F_0^\bullet$ if $\omega_i,\ i = 1,2,\ldots,h^i$ is a basis of F_0^i for all i.*

Proposition 6.5. *The set $\mathcal{K}$ is an analytic subset of $\mathcal{L}$ and U is an open subset of $\mathcal{K}$.*

Proof. Take a basis $\omega_i,\ i=1,2,\ldots,h$ of V_0 compatible with the Hodge filtration. The property P1 is given by

$$f_{\omega_r,\omega_s}(x)=0,\ r\leq h^i,\ s\leq h^j,\ i+j>m$$

and so $\mathcal{K}$ is an analytic subset of $\mathcal{L}$.

Now choose a basis $\delta=(\delta_i)_{i\in I}$ of $V_{\mathbb{Z}}(x)$ and write $\delta=p\omega$ as before. Using ω we may assume that $V_0=\mathbb{C}^h$ and δ constitutes of the rows of p. We have

$$\omega=p^{-1}\delta\Longrightarrow\overline{\omega}^x=\overline{p}^{-1}\delta=\overline{p}^{-1}p\omega$$

Therefore, the rows of $\overline{p}^{-1}p$ are complex conjugate of the the entries of ω. Now it is easy to verify that if the property (6.10), $\dim(H^{i,m-i}(x))=h^{i,m-i}$ and P3 are valid for one x then they are valid for all points in a small neighborhood of x (for P3 we may first restirct $\psi_{\mathbb{C}}$ to the sphere of radius 1 and center $0\in\mathbb{C}^h$). □

6.9 Moduli of polarized Hodge structures

We fix

$$G_0:=\{p\in P\mid p(F_0^\bullet)=F_0^\bullet\}$$

and let G_0 to act from right on P.

Proposition 6.6. *The properties P1, P2 and P3 are invariant under the action of G_0.*

Proof. Let $x\in\mathcal{L},\ g\in G_0$ and $\omega\in V_0$. We have

$$\begin{aligned}H^{i,m-i}(xg)&=F_0^i\cap\overline{F_0^{m-i}}^{xg}=F_0^i\cap g^{-1}\overline{g(F_0^{m-i})}^x=F_0^i\cap g^{-1}(\overline{F_0^{m-i}}^x)\\&=g^{-1}(F_0^i\cap\overline{F_0^{m-i}}^x)=g^{-1}(H^{i,m-i}(x))\end{aligned}$$

and

$$\psi_{\mathbb{C}}(xg)(\omega,\overline{\omega}^{xg})=\psi_{\mathbb{C}}(x)(g\omega,gg^{-1}\overline{g\omega}^x)=\psi_{\mathbb{C}}(x)(g\omega,\overline{g\omega}^x).$$

These equalities prove the proposition. □

The space U/G_0 is called the moduli of polarized Hodge structures.

6.10 The classical approach to the moduli of polarized Hodge structures

In this section we give the classical approach to the moduli of polarized Hodge structures due to P. Griffiths. The reader is referred to [47, 46] for more developments in this direction.

Let us fix the $\mathbb{C}$-vector space V_0 and the Hodge numbers as in §6.7. Let also F be the space of filtrations (6.9) in V_0. In fact, F has a natural structure of a compact smooth projective variety. We fix again the polarized lattice $(V_{\mathbb{Z}}(x_0),\psi_{\mathbb{Z}}(x_0))$ and define the Griffiths domain:

$$D:=\{F^\bullet\in \mathrm{F}\mid (V_{\mathbb{Z}}(x_0),\psi_{\mathbb{Z}}(x_0),F^\bullet)\text{ satisfies P1, P2 and P3 }\}.$$

The group $\Gamma_{\mathbb{Z}}$ acts on V_0 from right in the usual way and this gives us an action of $\Gamma_{\mathbb{Z}}$ on D. The space $\Gamma_{\mathbb{Z}}\backslash D$ is the moduli of polarized Hodge structure in the Griffiths sense.

Proposition 6.7. *There is a canonical isomorphism*

$$\beta : U/G_0 \xrightarrow{\sim} \Gamma_{\mathbb{Z}}\backslash D.$$

Proof. We take $x \in U$ and an isomorphism $\gamma : (V_{\mathbb{Z}}(x), \psi_{\mathbb{Z}}(x)) \xrightarrow{\sim} (V_{\mathbb{Z}}(x_0), \psi_{\mathbb{Z}}(x_0))$. The Hodge filtration $F_0^\bullet$ under this isomorphism gives us a Hodge filtration on V_0 with respect to the lattice $V_{\mathbb{Z}}(x_0)$ and so it gives us a point $\beta(x) \in D$. Different choices of γ leads us to the action of $\Gamma_{\mathbb{Z}}$ on $\beta(x)$. Therefore, we have a well-defined map

$$\beta : U \to \Gamma_{\mathbb{Z}}\backslash D.$$

Since $G_0 = \mathrm{Aut}(V_0, F_0^\bullet)$, β induces the desired isomorphism. □

The Griffiths domain is the moduli of polarized Hodge structures of a fixed type and with a $\mathbb{Z}$-basis in which the polarization has a fixed matrix form. Our domian U is the moduli of polarized Hodge structures of a fixed type and with a $\mathbb{C}$-basis compatible with Hodge filtration. Since cohomolgy with integer coefficients is not defined in algebraic geometry over an arbitrary field but de Rham cohomology and its Hodge filtration is defined, the Griffiths domain does not seem to have an algebraic counterpart but U corresponds to the moduli of smooth projective varieties $X/\mathbb{C}$ with certain differential forms on X and certain topological invariants fixed. This arises the hope that U has a natural algebraic structure.

We may define the space

$$DU := \{(F^\bullet, x) \in \mathrm{F} \times \mathcal{L} \mid (F^\bullet, x) \text{ satisfies P1, P2 and P3 }\}.$$

The canonical projection $\pi_D : DU \to \mathrm{F}$ (resp.$\pi_U : DU \to \mathcal{L}$) is a holomorphic fiber bundle with fibers biholomorphic to U (resp. D). If DU has a canonical structure of an algebraic variety and $\pi_D : DU \to \mathrm{F}$ is a morphism of algebraic varieties then U has a canonical structure of an algebraic variety. Therefore, to find an algebraic structure for DU is as much difficult as for U.

6.11 On biholomorphisim group of $\mathcal{P}$

We would like to investigate the group of biholomorphic mappings of $\mathcal{P}$. We have seen that G_0 acts from right on $\mathcal{P}$.

Let $P_{\mathbb{R}}$ be a subgroup of $P := \mathrm{Aut}(V_0)$ containing all $p \in P$ such that p induces an isomorphism of $V_{\mathbb{R}}(x_0)$. For all $A \in P_{\mathbb{R}},\ \omega \in\ V_0,\ p \in P$ we have

$$\overline{A\omega}^{x_0} = A\overline{\omega}^{x_0} \tag{6.11}$$

and also

$$\overline{\omega}^p = p^{-1}(\overline{p\omega}^{x_0}) \tag{6.12}$$

because

$$\overline{\omega}^p = \overline{\omega}^{\alpha(p)} = \overline{\omega}^{\alpha(1)p} = p^{-1}(\overline{p\omega}^{\alpha(1)}) = p^{-1}(\overline{p\omega}^{x_0}).$$

In particular (6.11) and (6.12) imply that

$$\overline{\omega}^{Ap} = \overline{\omega}^p,\ A \in P_{\mathbb{R}},\ p \in P. \tag{6.13}$$

Proposition 6.8. *The properties P1, P2 and P3 are invariant under the action of $P_{\mathbb{R}}$ from left on P.*

Proof. We use the equalities (6.11), (6.12) and (6.13) and we have

$$H^{i,m-i}(Ap) = F_0^i \cap \overline{F_0^{m-i}}^{Ap} = H^{i,m-i}(p)$$

and

$$\begin{aligned}\psi_{\mathbb{C}}(Ap)(\omega, \overline{\omega}^{Ap}) &= \psi_{\mathbb{C}}(\alpha(Ap))(\omega, p^{-1}\overline{p\omega}^{x_0}) \\ &= \psi_{\mathbb{C}}(\alpha(1))(Ap\omega, App^{-1}\overline{p\omega}^{x_0}) = \psi_{\mathbb{C}}(x_0)(Ap\omega, \overline{Ap\omega}^{x_0}).\end{aligned}$$

for $A \in P_{\mathbb{R}}$ and $p \in P$. These equalities prove the proposition. □

The Propositions 6.8 and 6.6 imply that the biholomorphism group of $\tilde{\mathcal{P}}$ contains the algebraic group G_0 and the real group $P_{\mathbb{R}}$.

6.12 Period map

Let $f : X \to S$ be a regular and proper map of two smooth verieties over $\mathbb{C}$, $H = \cup_{t\in S} H^m(X_t, \mathbb{C})$ the cohomology bundle and

$$V_f := H^0(S, H)$$

the space of global sections of H. We assume that there is a filtration $F_f^\bullet$ of V_f such that this filtration restricted to $H^m(X_t, \mathbb{C})$ gives us the Hodge filtration. For examples of such algebraic families see Chapter 5. We fix also a basis $\{\omega_i\}_{i=1}^h$ (resp. $\{\omega_{0,i}\}_{i=1}^h$) of V_f (resp. V_0) compatible with $F_f^\bullet$ (resp. $F_0^\bullet$).

We identify $(H^m(X_t, \mathbb{C}), F^\bullet, \{\omega_i\}_{i=1}^h)$ with $(V_0, F_0^\bullet, \{\omega_{0,i}\}_{i=1}^h)$, sending ω_i to $\omega_{0,i}$. Then under this identification, $(H^m(X_t, \mathbb{Z}), < \cdot, \cdot >)$ is mapped to a polarized lattice in V_0 and so we have a point in U. The obtained map

$$\mathsf{pm} : S \to U$$

is called the period map. Usually, we take $V_0 = V_f$ and $F_0 = F_f$ and we do not mention the choice of bases. Each $(H^m(X_t, \mathbb{C}), F^\bullet)$ is canonically isomorphic to $(V_0, F_0^\bullet)$ and under this canonical isomorphism $(H^m(X_t, \mathbb{Z}), < \cdot, \cdot >)$ is mapped to a polarized lattice in V_0.

The period map satisfy the so called Griffiths transversality:

$$\mathsf{pm}^{-1}(\omega_{ij}) = 0, \ i \le h^{m-x}, \ j \ge h^{m-x-1}, \ x = 0, 1, \dots, m-1. \tag{6.14}$$

6.13 The Siegel upper half plane $m = 1, h^{10} = h^{01} = g$

In this section we consider the case in which the weight m is 1 and the polarization matrix is:

$$\Psi_0 = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix},$$

where I_g is the $g \times g$ identity matrix. It satisfies $\Psi_0^{\mathsf{t}} = \Psi_0^{-1}$. In this case $g := h^{10} = h^{01}$ and $h = 2g$. We take a basis $\omega = \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$ of V_0 with $o(\omega_1 \wedge \omega_2) = 1$ and a basis δ of $V_{\mathbb{Z}}^{\vee}$ and write the associated period matrix in the form:

$$\mathsf{pm}(x) = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix},$$

where $x_i, i = 1, \ldots, 4$ are $g \times g$ matrices. We have

$$\Gamma_{\mathbb{Z}} = \mathrm{S}p(2g, \mathbb{Z}) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(N, \mathbb{Z}) \mid a^{\mathsf{t}}c = c^{\mathsf{t}}a,\ b^t d = d^t b,\ a^t d - c^t b = 1\},$$

$$G_0 = \{\begin{pmatrix} k_1 & k_3 \\ 0 & k_2 \end{pmatrix} \in \mathrm{GL}(2g, \mathbb{C}) \mid \det(k_1)\det(k_4) \neq 0\}.$$

The matrices (6.5) and (6.7) have the form:

$$\begin{pmatrix} f_{\omega_1,\omega_1} & f_{\omega_1,\omega_2} \\ f_{\omega_2,\omega_1} & f_{\omega_2,\omega_2} \end{pmatrix} = \begin{pmatrix} x_1^{\mathsf{t}} & x_3^{\mathsf{t}} \\ x_2^{\mathsf{t}} & x_4^{\mathsf{t}} \end{pmatrix} \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = \begin{pmatrix} -x_3^{\mathsf{t}}x_1 + x_1^{\mathsf{t}}x_3 & -x_3^{\mathsf{t}}x_2 + x_1^{\mathsf{t}}x_4 \\ -x_4^{\mathsf{t}}x_1 + x_2^{\mathsf{t}}x_3 & -x_4^{\mathsf{t}}x_2 + x_2^{\mathsf{t}}x_4 \end{pmatrix}$$

respectively

$$\begin{pmatrix} f_{\omega_1,\bar{\omega}_1} & f_{\omega_1,\bar{\omega}_2} \\ f_{\omega_2,\bar{\omega}_1} & f_{\omega_2,\bar{\omega}_2} \end{pmatrix} = \begin{pmatrix} x_1^{\mathsf{t}} & x_3^{\mathsf{t}} \\ x_2^{\mathsf{t}} & x_4^{\mathsf{t}} \end{pmatrix} \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix} \begin{pmatrix} \bar{x}_1 & \bar{x}_2 \\ \bar{x}_3 & \bar{x}_4 \end{pmatrix} = \begin{pmatrix} -x_3^{\mathsf{t}}\bar{x}_1 + x_1^{\mathsf{t}}\bar{x}_3 & -x_3^{\mathsf{t}}\bar{x}_2 + x_1^{\mathsf{t}}\bar{x}_4 \\ -x_4^{\mathsf{t}}\bar{x}_1 + x_2^{\mathsf{t}}\bar{x}_3 & -x_4^{\mathsf{t}}\bar{x}_2 + x_2^{\mathsf{t}}\bar{x}_4 \end{pmatrix}.$$

The properties P1 and P3 imply that $x_3^{\mathsf{t}}x_1 = x_1^{\mathsf{t}}x_3$ and $-\sqrt{-1}(-x_3^{\mathsf{t}}\bar{x}_1 + x_1^{\mathsf{t}}\bar{x}_3)$ is a positive matrix. The property P2 implies that x_1 and x_2 have non zero determinant and so $x := x_1 x_2^{-1}$ is well-defined invertible matrix which satisfies the famous Riemann relations:

$$x^{\mathsf{t}} = x,\ \mathrm{Im}(x) \text{ is a positive matrix.}$$

The set of matrices $x \in \mathrm{Mat}^{g\times g}(\mathbb{C})$ with the above properties is called the Siegel upper half plane and is denoted by $\mathbb{H}_g$. The arithmetic group $\Gamma_{\mathbb{Z}}$ on $\mathbb{H}_g$ is given by:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x = (ax + b)(cx + d)^{-1},\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_{\mathbb{Z}},\ x \in \mathbb{H}_g.$$

The morphism

$$U/G_0 \to \Gamma_{\mathbb{Z}} \backslash \mathbb{H}_g,$$

is given by

$$\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \to x_1 x_3^{-1}.$$

Now let us calculate the entries of the connection matrix $A = [\omega_{ij}]_{i,j \in I}$.

$$A = d(\mathsf{pm}^{\mathsf{t}}) \cdot \mathsf{pm}^{-\mathsf{t}} = \begin{pmatrix} dx_1^{\mathsf{t}} & dx_3^{\mathsf{t}} \\ dx_2^{\mathsf{t}} & dx_4^{\mathsf{t}} \end{pmatrix} \begin{pmatrix} x_1^{\mathsf{t}} & x_3^{\mathsf{t}} \\ x_2^{\mathsf{t}} & x_4^{\mathsf{t}} \end{pmatrix}^{-1}.$$

One may use the formula

$$\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}^{-1} = \begin{pmatrix} x_2^{-1}x_4(x_1x_2^{-1}x_4 - x_3)^{-1} & -(x_1x_2^{-1}x_4 - x_3)^{-1} \\ -x_1^{-1}x_3(x_4 - x_2x_1^{-1}x_3)^{-1} & (x_4 - x_2x_1^{-1}x_3)^{-1} \end{pmatrix}$$

and obtain explicit expression for A. For instance in the case $g = 1$ we have:

$$A = \frac{1}{\det(x)} \begin{pmatrix} x_4dx_1 - x_2dx_3 & x_1x_3 - x_3dx_1 \\ x_4dx_2 - x_2dx_4 & x_1dx_4 - x_3dx_2 \end{pmatrix}$$

See the books [49, 25, 56] for more information on Siegel modular forms.

6.14 Modular foliations in U

In §6.5 we defined the connection ∇ on $\mathcal{L}$ and determined its matrix A. We restrict ∇, A and so on to U and, if there is no danger of confusion, we use the same notations for the the new ones. We have now modular foliations in U associated to a global section $\sum \omega_i p_i,\ p_i \in \mathcal{O}(U)$ of $\mathcal{H}$ and the connection ∇. In particular, the modular foliation $\mathcal{F}_{\omega_i}$ is the locus of points $x \in U$ such the i-th column of pm is constant.

6.15 Loci of Hodge cycles

In this section we assume that m is even. A cycle $\delta \in V_{\mathbb{Z}}(x_0), x_0 \in U$ is called a Hodge cycle if

$$\int_{\delta} F_0^{\frac{m}{2}+1} = 0.$$

Fix a Hodge cycle $\delta_{x_0} \in V_{\mathbb{Z}}(x_0)$. The loci of Hodge cycles through x_0 is given by

$$\{x \in (U, x) \mid \int_{\delta_x} F_0^{\frac{m}{2}+1} = 0\},$$

where $\delta_x = p^{-1}(x_0)$ and p is in a small neighborhood of the identity automorphism in $\mathcal{P}$. It is an analytic subset of (U, x_0) and it may not be irreducible. It is too premature claim, to say that a loci of of Hodge cycles is a part of a global analytic subvariety of U (similar to Theorem 5.4). However, it seems to me that the following statement is true: Let $\mathsf{pm} : T \to U$ be an analytic map form an algebraic variety variety T to the period domain U which satisfies the Griffiths transversality (6.14). Then the pull-back of any local loci of Hodge cycles by pm is a part of an algebraic subvariety of T. The possible proof must be reconstructed from the arguments in [13].

6.16 Vanishing cycles and Picard-Lefschetz formula

For a family of n-dimensional hypersurfeces studied in the previous chapters, we have the notion of a vanishing cycle and we know that not every cycle in the n-th homology is a vanishing cycle. Therefore it would be reasonable to translate this notion into the moduli of polarized Hodge structures. In this section we want to do this.

We take a finite set $\delta_{x_0} = \{\delta_{1,x_0}, \delta_{2,x_0}, \ldots, \delta_{\mu,x_0}\}$ which generates $V_{\mathbb{Z}}(x_0)^\vee$ as $\mathbb{Z}$-module. The reason why we do not assume that it form a basis $V_{\mathbb{Z}}(x_0)$ will be clarified in §6.17. This corresponds to a distinguished set of vanishing cycles in the geometric context of Chapter 4. We define each element of δ_{x_0} to be a vanishing cycle. For an arbitrary lattice $V_{\mathbb{Z}}(x)$ with $p \in P, \alpha(p) = x$, we obtain a $\mathbb{Z}$-basis $\delta = \delta_x := p^\vee(\delta_{x_0})$, where $p^\vee : V_{\mathbb{Z}}(x_0)^\vee \to V_{\mathbb{Z}}(x)^\vee$ is the map induced in the dual spaces. Again, each entry of δ_x is defined to be a vanishing cycle. Following the property (4.3) of geometric vanishing cycles, it is natural to assume that the dual polarization $\langle \cdot, \cdot \rangle$ in $V_{\mathbb{Z}}(x_0)$ satisfies

$$\langle \delta_i, \delta_i \rangle = (-1)^{\frac{m(m-1)}{2}}(1 + (-1)^m),\ i = 1, 2, \ldots, h. \tag{6.15}$$

To each vanishing cycle $\delta_i \in V_{\mathbb{Z}}(x)$ we define the Picard-Lefschez mapping:

$$\mathsf{pl}_i(x) : V_{\mathbb{Z}}(x_0)^\vee \to V_{\mathbb{Z}}(x_0)^\vee,\ a \mapsto a + (-1)^{\frac{(m+1)(m+2)}{2}} \langle a, \delta_i \rangle \delta_i.$$

Note that $\mathsf{pl}_i(x_0) \in \Gamma_{\mathbb{Z}}$. Let $\Gamma_{\mathbb{Z}}^{\mathsf{pl}}$ be the subgroup of $\Gamma_{\mathbb{Z}}$ generated by $\mathsf{pl}_i,\ i = 1, 2, \dots, \mu$. If in the geometric context of Chapter 4, a $\langle \cdot, \cdot \rangle$-preserving map from $H_n(\{f = 0\}, \mathbb{Z})$ to itself, where f is tame polynomial over $\mathbb{C}$ with non-zero discriminant, is a composition of some Picard-Lefschetz mappings, then it would be reasonable to assume that

$$\Gamma_{\mathbb{Z}}^{\mathsf{pl}} = \Gamma_{\mathbb{Z}}.$$

6.17 Moduli of mixed Hodge structures

The reader may have noticed that in Chapters 3 and 5 we have worked with the mixed Hodge structures of affine varieties associated to tame polynomials and in this Chapter we have worked only with pure Hodge structures. In this section we sketch the construction of the period domain associated to such mixed Hodge structures. Since the construction is similar to the case of usual Hodge structures, we left the details to the reader.

Let us fix a $\mathbb{C}$-vector space V_0 of dimension μ and the Hodge numbers $h_0^{k-1,n-k},\ k = 1, 2, \dots, n, h^{k-1,n-k+1},\ k = 1, 2, \dots, n+1$ in §5.8. We fix also a lattice $V_{\mathbb{Z}}(x_0) \subset V_0$, a weight filtration of $\mathbb{Z}$-modules,

$$0 = W_{n-1,\mathbb{Z}}(x_0) \subset W_{n,\mathbb{Z}}(x_0) \subset W_{n+1,\mathbb{Z}}(x_0) = V_{\mathbb{Z}}(x_0),\ \mathrm{rank} W_{n,\mathbb{Z}}(x_0) = \sum_{k=1}^{n+1} h_0^{k-1,n-k+1}$$

(it has just one non-trivial piece) and for $m = n, n+1$ a $(-1)^m$-symmetric non-degenerated bilinear form $\psi_{m,\mathbb{Z}}(x_0)$ on $\mathrm{Gr}_m^W V_{\mathbb{Z}}(x_0)$. We also consider a decreasing filtration

$$0 = F_0^{n+1} \subset F_0^n \subset \cdots \subset F_0^1 \subset F_0^0 = V_0$$

with

$$\dim_{\mathbb{C}} \mathrm{Gr}_F^{n+1-k} \mathrm{Gr}_{n+1}^W V_0 := h_0^{k-1,n-k},\ \dim_{\mathbb{C}} \mathrm{Gr}_F^{n+1-k} \mathrm{Gr}_n^W V_0 := h_0^{k-1,n-k+1}$$

and call it the Hodge filtration. All the above data is called a polarized mixed Hodge structure if the induced structures in $\mathrm{Gr}_m V_0,\ m = n, n+1$ form a polarized Hodge structure. The Hodge structure induced in $\mathrm{Gr}_{n+1} V_0$ is called the Hodge structure at infinity.

Now, if we fix the filtration $F_0^\bullet$, the matrices of polarizations $\psi_{m,\mathbb{Z}},\ m = n, n+1$ and consider the space of all lattices $V_{\mathbb{Z}}(x_0)$ which result into a mixed Hodge structure as above then we obtain a complex space similar to U in §6.8. Contrary to this construction, if we fix the lattice $V_{\mathbb{Z}}(x_0)$ and let the Hodge filtration varies we obtain a complex space similar to the Griffiths domain in §6.10.

For a tame polynomial f over a function field and with the last homogeneous polynomial g, in Theorem 5.5 we constructed a basis of a localization of H compatible with the mixed Hodge structure. As in 6.12 we can define the period map in this case. The variation of a parameter in g results into the variation of the corresponding Hodge structures at infinity. In the definition of D or U corresponding to mixed Hodge structures, it may be useful to fix the Hodge structure at infinity (both the lattice and the Hodge filtration). In this we case have to assume that g does not depend on any parameter.

Having Theorem 4.2 in mind and the definition of vanishing cycles in 6.16, we may take a $\mathbb{Z}$-basis of $V_{\mathbb{Z}}(x_0)$ and call its elements as vanishing cycle.

6.18 Isogeny of Hodge structures

Isogeny: A morphism of $\mathbb{Z}$-modules

$$f_{\mathbb{Z}} : V_{\mathbb{Z}}(x) \to V_{\mathbb{Z}}(y),\ x, y \in U \tag{6.16}$$

such that $f_{\mathbb{C}} : V_0 \to V_0$ respects the Hodge filtration(i.e. $f_{\mathbb{C}}(F_0^i) \subset F_0^i,\ i = 0, 1, 2, \dots, m+1$) is called a morphism of Hodge structures. It is called a strong isogeny if in addition it satisfies

$$\psi_{\mathbb{Z}}(y)(f_{\mathbb{Z}}(a), f_{\mathbb{Z}}(b)) = c \cdot \psi_{\mathbb{Z}}(x)(a, b),\ \forall a, b \in V_{\mathbb{Z}}(x) \tag{6.17}$$

for some non-zero constant $c \in \mathbb{Q}$ depending only on $f_{\mathbb{Z}}$. The constant c will be called the semi exponent of the strong isogeny $f_{\mathbb{Z}}$. The condition (6.17) and the fact that $\psi_{\mathbb{Z}}(x)$ is non-degenerate imply that $f_{\mathbb{Z}}$ is injective. Since $V_{\mathbb{C}}(x)$ and $V_{\mathbb{C}}(y)$ have the same dimension, we conclude that $f_{\mathbb{C}}$ is an isomorphism of $\mathbb{C}$-vector spaces. It is not necessary for $f_{\mathbb{Z}}$ to be surjective. We say that $f_{\mathbb{Z}}$ is an isogeny if instead of the condition (6.17) we use the weaker condition that $f_{\mathbb{Z}}$ is injective (and hence $f_{\mathbb{C}}$ is an isomorphism of $\mathbb{C}$-vector spaces).

In the case of $m = 1$, the definition of an isogeny is equivalent to the classical definition of an isogeny between toruses: $f_{\mathbb{Z}}$ is an isogeny if the induced map on the Jacobian varieties $F_0^1/V_{\mathbb{Z}}(x) \to F_0^1/V_{\mathbb{Z}}(y)$ is an isogeny (see [54] and its references). [1] We use the fact that $V_{\mathbb{Z}}(y)/f_{\mathbb{Z}}(V_{\mathbb{Z}}(x))$ is a finite abelian group and so it is of the form $(\mathbb{Z}/n_1\mathbb{Z}) \times \cdots \times (\mathbb{Z}/n_k\mathbb{Z})$. The notion of a strong isogeny does no seem to be treated in the literature. We will need it in §6.19 in order to introduce the notion of Hecke operators. In the case of Hodge structures arising from elliptic curves (see 6.13, the case $g = 1$) the notions of an isogeny and a strong isogeny are equivalent.

Degree and exponent of an isogeny: The degree of an isogeny $f_{\mathbb{Z}}$ is the cardinality of the finite group $V_{\mathbb{Z}}(y)/f_{\mathbb{Z}}(V_{\mathbb{Z}}(x))$ and its exponent is the exponent of the group G, i.e. the smallest natural number n such that $nG = 0$.

The first example of a strong isogeny is multiplication by an integer mapping:

$$k_{V_{\mathbb{Z}}(x)} : V_{\mathbb{Z}}(x) \to V_{\mathbb{Z}}(x),\ \delta \mapsto k\delta,\ k \in \mathbb{Z}.$$

Since for a Hodge structure $V_{\mathbb{Z}}(x)$ and an integer k we have $V_{\mathbb{Z}}(x)/kV_{\mathbb{Z}}(x) = (\mathbb{Z}/k\mathbb{Z})^{\operatorname{rank}(V_{\mathbb{Z}}(x))}$, the degree of $k_{V_{\mathbb{Z}}(x)}$ is $k^{\operatorname{rank}(V_{\mathbb{Z}}(x))}$ and its exponent is k.

Inverse of an isogeny: For any strong isogeny $f_{\mathbb{Z}}$ with exponent k, there exists another isogeny

$$g_{\mathbb{Z}} : V_{\mathbb{Z}}(y) \to V_{\mathbb{Z}}(x)$$

such that

$$f_{\mathbb{Z}} \circ g_{\mathbb{Z}} = k_{V_{\mathbb{Z}}(y)},\ g_{\mathbb{Z}} \circ f_{\mathbb{Z}} = k_{V_{\mathbb{Z}}(x)}. \tag{6.18}$$

To construct $g_{\mathbb{Z}}$ we note that $k \circ f_{\mathbb{C}}^{-1} : V_0 \to V_0$ is a $\mathbb{C}$-isomorphism which sends $V_{\mathbb{Z}}(y)$ into $V_{\mathbb{Z}}(x)$. Its restriction to $V_{\mathbb{Z}}(y)$ is our $g_{\mathbb{Z}}$.

[1] Every element $a \in V_{\mathbb{Z}}(x)$ can be written in a unique way as $b + \overline{b}^x$ and the map $a \mapsto b$ gives an inclusion $V_{\mathbb{Z}}(x) \hookrightarrow F_0^1$.

Isogeny in a coordinate system: Let $f_{\mathbb{Z}}$ be an strong isogeny as in (6.16). Let us take δ_x^{pd}, δ_y^{pd} and ω with

$$\delta_x^{\mathsf{pd}} = \mathsf{q}_x\omega,\ \delta_y^{\mathsf{pd}} = \mathsf{q}_y\omega,$$

as in Section 6.4. We write $f_{\mathbb{Z}}$ in the integral basis: $f_{\mathbb{Z}}(\delta_x^{\mathsf{pd}}) = A\delta_y^{\mathsf{pd}}$, where A is a matrix with integer valued entries. We write also $f_{\mathbb{C}}$ in the complex basis ω: $f_{\mathbb{C}}(\omega) = g\omega,\ g \in G_0$. All these imply that

$$A\mathsf{q}_y = \mathsf{q}_x g, A \in \mathrm{Mat}(h,\mathbb{Z}),\ g \in G_0 \tag{6.19}$$

The condition (6.17) translates into:

$$A\Psi_0 A^{\mathsf{t}} = c\Psi_0 \tag{6.20}$$

The exponent of $f_{\mathbb{Z}}$ divides the absolute value of the determinant of A. To see this one uses the equality $f_{\mathbb{Z}}(\delta_x^{\mathsf{pd}}) = A\delta_y^{\mathsf{pd}}$ and the formula of the inverse of A, in which A^{-1} is equal to $1/\det(A)$ times a matrix in $\mathrm{Mat}^{h\times h}(\mathbb{Z})$. The exponent of $f_{\mathbb{Z}}$ may not be equal to $\det(A)$, for instance take $f_{\mathbb{Z}}$ to be the multiplication by an integer number. The degree of $f_{\mathbb{Z}}$ is $\#(\mathbb{Z}^h/A\mathbb{Z}^h)$. The equality (6.20) implies that $|\det(A)| = |c|^{\frac{h}{2}}$. This equality implies that c is an integer and it is a square if h is odd. It also shows that except for $h = 2$ case the determinant of an strong isogeny cannot be an arbitrary number.

Endomorphism algebra: Let $V_{\mathbb{Z}}(x)$ be a Hodge structure. We define $\mathrm{End}_{\mathbb{Z}}(x)$ to be the the set of all morphism of Hodge structures from $V_{\mathbb{Z}}(x)$ to itself. Multiplication by an integer k gives an element of $\mathrm{End}_{\mathbb{Z}}(x)$. Therefore, we have a natural embedding $\mathbb{Z} \hookrightarrow \mathrm{End}_{\mathbb{Z}}(x)$. Now, $\mathrm{End}_{\mathbb{Z}}(x)$ is a freely generated $\mathbb{Z}$-module of rank less than $\mathrm{rank}(V_{\mathbb{Z}}(x))^2$(this number is the rank of the $\mathbb{Z}$-module of all $\mathbb{Z}$-linear self mappings of $V_{\mathbb{Z}}(x)$). According to discussion in (6.18), in the non-commutative ring

$$\mathrm{End}_{\mathbb{Q}}(x) := \mathrm{End}_{\mathbb{Z}}(x) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

the isogenies are invertible.

6.19 Hecke operators

Let us take a natural $p \in \mathbb{N}$ and assume that p is a square if h is odd. This implies that $p^{\frac{h}{2}}$ is a natural number. The p-th Hecke operator is defined as follows:

$$T_p : \mathcal{O}(U) \to \mathcal{O}(U),\ T_p f(y) = \sum_x f(x),\ y \in U,$$

where x runs through those elements in U such that the identity map in V_0 induces a strong isogeny $f_{\mathbb{Z}} : V_{\mathbb{Z}}(x) \to V_{\mathbb{Z}}(y)$ with the semi exponent p. The fact that the above sum is finite is equivalent to say that the quotient $\Gamma_{\mathbb{Z}}\backslash\Gamma_{\mathbb{Z}}^p$ is a finite set, where

$$\Gamma_{\mathbb{Z}}^p := \{A \in \mathrm{Mat}^{h\times h}(\mathbb{Z}) \mid A\Psi_0 A^{\mathsf{t}} = p\Psi_0\}.$$

The proof or disproof of the second statement does not seem to be a difficult problem.

Let us prove the equivalency assertion. We take the matrix forms of an isogeny $f_{\mathbb{Z}}$: $V_{\mathbb{Z}}(x) \to V_{\mathbb{Z}}(y)$ as we have done it in the paragraph around (6.20) and (6.19). In our case

$f_{\mathbb{Z}}$ is the identity map, $V_{\mathbb{Z}}(x) \subset V_{\mathbb{Z}}(y)$, $\delta_x^{\mathsf{pd}} = A\delta_y^{\mathsf{pd}}$, g is the identity matrix, $\mathsf{q}_x = A\mathsf{q}_y$ and $|c| = p$. Let $[A_1], [A_2], \dots, [A_s]$ be the classes of equivalencies of $\Gamma_{\mathbb{Z}}^p$ under the action of $\Gamma_{\mathbb{Z}}$ form the left. Using the map $\alpha : \mathcal{P} \to U,\ \mathsf{q}_x \mapsto x$, we lift the definition of the p-th Hecke operator to $\mathcal{P}$ and it turns out that:

$$T_p f(\mathsf{q}_x) = \sum_{i=1}^{s} f(A_i \mathsf{q}_x),$$

where f is a left $\Gamma_{\mathbb{Z}}$-invariant function on $\mathcal{P}$. The corresponding sum of $T_p f$ is also left $\Gamma_{\mathbb{Z}}$-invariant and does not depend on the choice of A_i in its class $[A_i]$ or q_x in $[\mathsf{q}_x] \in U$. This is because for $B \in \Gamma_{\mathbb{Z}}$ each $A_i B,\ i = 1, 2, \dots, s$ can be written as $B_i A_{\sigma_i},\ B_i \in \Gamma_{\mathbb{Z}}$ and σ is just a permutation of i's.

Let us consider the case of Hodge structures arising from elliptic curves (§6.13, the case $g = 1$). The notion of Hecke operators in this section arises from the same notion which acts on the space of modular forms on the Poincaré upper half plane (see for instance [53, 66]). It seems to me that this is the only well-studied case in the literature. In this case for a natural number p we have $\Gamma_{\mathbb{Z}}^p = \mathrm{Mat}_p(2, \mathbb{Z})$ and one usually choose the following representatives

$$\begin{pmatrix} \frac{p}{d} & b \\ 0 & d \end{pmatrix},\ d \mid p, 0 \leq b \leq d - 1$$

for $\mathrm{SL}(2, \mathbb{Z}) \backslash \Gamma_{\mathbb{Z}}^p$.

Complementary notes

1. In the best case we expect that U has a natural algebraic structure such that the action of G_0 becomes algebraic and every period map is a morphism of algebraic varieties. This is along the same ideas of P. Griffiths in [36]. To realize this hope we need a generalization of Baily-Borel Theorem (see [2]) on the unique algebraic structure of quotients of symmetric Hermitian domains by discrete arithmetic groups. To begin with, we must verify when for a holomorphic function f in $\mathcal{P}$ the associated Poincaré type series

$$\tilde{f}(x) := \sum_{A \in \Gamma_{\mathbb{Z}}} f(Ax)$$

   converges. In [2], Theorem 4.5, one can find a argument, due to Harish-Chandra, on the convergence of Poincaré series in the case of Hermitian symmetric domains.

2. The global Torelli problem says that the period map is one to one. For a brief description of this problem see [3]. Since the period domain constructed in this text lives over the Griffiths domain, it is quit possible that for some examples the period domain introduced in this text is one to one but the period map with its image in the Griffiths domain is not. It would be interesting to investigate the differences in details.

3. The constriction of modular/automorphic forms associated to a moduli of Hodge structures is a difficult problem which needs to be treated example by example and involves many other problems such as the Torelli problem. It is well-known that the Torelli problem for K3 surfaces is true and so the inverse of the period map exists for this case. However, the construction of such an inverse map in terms of certain automorphic functions is not known.

Appendix A

Mixed Hodge structure of affine varieties

Our main examples of modular foliations in Chapters 3 and 5 are associated to a family of affine hypersufaces and polynomial differential forms in $\mathbb{C}^{n+1}$. Such differential forms have poles at infinity and the corresponding pole order gives us the first numerical invariant to distinguish between differential forms and hence the corresponding modular foliations. Another way to distinguish between differential forms is by looking at their classes in the de Rham cohomology and its Hodge filtration. It is believed that there exists a close relation between the mentioned concepts and the testimonies to this belief are P. Griffiths theorem on the Hodge filtration of the complement of a smooth hypersurface (see §A.6) and some calculations related to Riemann surfaces.

A.1 Hypercohomology

Let us be given a complex of analytic sheaves on a complex manifold M:

$$\mathcal{S}^\bullet : \mathcal{S}^0 \xrightarrow{d} \mathcal{S}^1 \xrightarrow{d} \mathcal{S}^2 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{S}^n \xrightarrow{d} \cdots, \quad d \circ d = 0. \tag{A.1}$$

We call $\mathcal{S}^\bullet$ a differential complex. We are going to define the hypercohomologies $\mathbb{H}^m(M, \mathcal{S}^\bullet)$, $m = 0, 1, 2, \dots$ of $\mathcal{S}^\bullet$. Let $\mathcal{U} = \{U_i\}_{i\in I}$ be a Stein covering of M, i.e. each U_i is Stein (and so the intersection of finitely many of them is also Stein, see for instance [12] Proposition 1.5). Consider the double complex

$$\begin{array}{ccccccccccc}
\uparrow & & \uparrow & & \uparrow & & & & \uparrow & & \\
\mathcal{S}^0_n & \to & \mathcal{S}^1_n & \to & \mathcal{S}^2_n & \to & \cdots & \to & \mathcal{S}^n_n & \to \\
\uparrow & & \uparrow & & \uparrow & & & & \uparrow & & \\
\mathcal{S}^0_{n-1} & \to & \mathcal{S}^1_{n-1} & \to & \mathcal{S}^2_{n-1} & \to & \cdots & \to & \mathcal{S}^n_{n-1} & \to \\
\uparrow & & \uparrow & & \uparrow & & & & \uparrow & & \\
\vdots & & \vdots & & \vdots & & & & \vdots & & \\
\uparrow & & \uparrow & & \uparrow & & & & \uparrow & & \\
\mathcal{S}^0_2 & \to & \mathcal{S}^1_2 & \to & \mathcal{S}^2_2 & \to & \cdots & \to & \mathcal{S}^n_2 & \to \\
\uparrow & & \uparrow & & \uparrow & & & & \uparrow & & \\
\mathcal{S}^0_1 & \to & \mathcal{S}^1_1 & \to & \mathcal{S}^2_1 & \to & \cdots & \to & \mathcal{S}^n_1 & \to \\
\uparrow & & \uparrow & & \uparrow & & & & \uparrow & & \\
\mathcal{S}^0_0 & \to & \mathcal{S}^1_0 & \to & \mathcal{S}^2_0 & \to & \cdots & \to & \mathcal{S}^n_0 & \to
\end{array} \tag{A.2}$$

Here $\mathcal{S}^i_j$ is the disjoint union of global sections of $\mathcal{S}^i$ in the open sets $\cap_{i\in I_1}U_i$, $I_1 \subset I,\ \#I_1 = j$. The horizontal arrows are usual differential operator d of $\mathcal{S}^i$'s and vertical arrows are differential operators δ in the sense of Cech cohomology. The k-th piece of the total chain of (A.2) is

$$\mathcal{L}^k := \oplus_{i=0}^k \mathcal{S}^i_{k-i}$$

with the the differential operator $d' = d + (-1)^k\delta : \mathcal{L}^k \to \mathcal{L}^{k+1}$. The hypercohomology $\mathbb{H}^m(M, \mathcal{S}^\bullet)$ is the total cohomology of the double complex (A.2), i.e.

$$\mathbb{H}^m(M, \mathcal{S}^\bullet) := H^m(\mathcal{L}^\bullet, d').$$

This definition is independent of the choice of $\mathcal{U}$. For an analytic sheaf $\mathcal{S}$ on M let $\Gamma\mathcal{S}$ denote the global sections of $\mathcal{S}$. For any differential complex $\mathcal{S}^\bullet$ and an open set U in M we have the cohomologies:

$$H^k(\mathcal{S}^\bullet(U), d) := \frac{\ker(\mathcal{S}^k(U) \to \mathcal{S}^{k+1}(U))}{\mathrm{Im}(\mathcal{S}^{k-1}(U) \to \mathcal{S}^k(U))},\ k = 0, 1, 2, \ldots.$$

Proposition A.1. *If $H^\mu(M, \mathcal{S}^i) = 0,\ \mu > 0,\ i \geq 0$ then*

$$\mathbb{H}^m(M, \mathcal{S}^\bullet) \cong H^m(\Gamma\mathcal{S}^\bullet, d).$$

Proof. This is an immediate consequence of the definition of the hypercohomology. The hypothesis implies that the vertical arrows in A.2 are exact. Every elememnt in $\mathcal{L}^k$ is reduced to an element in $\mathcal{S}^k_0$ whose δ is zero and so corresponds to a global section of $\mathcal{S}^k$. □

A.2 Logarithmic differential forms and mixed Hodge structures of affine varieties

Let M be a projective smooth variety and $D = D_1 + D_2 + \cdots + D_s$ be a normal crossing divisor in M, i.e. to each point $p \in M$ there are holomorphic coordinates $z_1, z_2, \ldots, z_n$ around p such that $D = \{z_1 = 0\} + \cdots + \{z_i = 0\}$ for some i depending on p. Let also $\Omega^i_M(\log D)$ be the sheaf of meromorphic i-forms ω in M with logarithmic poles along D, i.e. ω and $d\omega$ have poles of order at most one along D. This is equivalent to the fact that around each point $p \in M$ the sheaf $\Omega^k_M(\log D)$ is generated by k-times wedge products of $\frac{dz_1}{z_1}, \frac{dz_2}{z_2}, \cdots, \frac{dz_i}{z_i}, dz_{i+1}, \cdots, dz_n$, where i is as above. Let $\mathcal{A}^\bullet$ be the complex of C^∞ differential forms in $M\backslash D$ and $j : M\backslash D \to M$ be the inclusion. A map between two differential complexes, $p : \mathcal{S}^\bullet_1 \to \mathcal{S}^\bullet_2$ is called to be a quasi-isomorphism if the induced maps

$$H^k(\mathcal{S}_{1,x}, d) \to H^k(\mathcal{S}_{2,x}, d),\ k = 0, 1, \ldots,\ x \in M$$

are isomorphisms, where $\mathcal{S}^k_x$ is the stalk of $\mathcal{S}^k$ over $x \in M$.

Proposition A.2. *The canonical map*

$$\Omega^\bullet(\log D) \to j_*\mathcal{A}^\bullet$$

is a quasi isomorphism.

This proposition is proved [35] and [18]. See also [84] Proposition 8.18. The above proposition implies that we have an isomorphism

$$H^k(M\backslash D,\mathbb{C})\cong \mathbb{H}^k(M,\Omega_M^\bullet),\ k=0,1,2,\dots.$$

The Hodge filtration on $H^m(M\backslash D,\mathbb{C})$ is given by

$$F^kH^m(M\backslash D,\mathbb{C}):=\mathrm{Im}(\mathbb{H}^m(F^k\Omega_M^\bullet(\log(D))\to \mathbb{H}^m(\Omega_M^\bullet(\log(D))))$$

where for a differential complex $\mathcal{S}^\bullet$

$$F^k\mathcal{S}^\bullet:=\mathcal{S}^{\bullet\geq k}:\underbrace{0\to 0\to\cdots\to 0}_{k \text{ times } 0}\to\mathcal{S}^k\to\mathcal{S}^{k+1}\to\cdots\to\mathcal{S}^n\to\cdots$$

is the bete filtration. By definition we have $F^0/F^1\cong H^n(M,\mathcal{O}_M)$. The weight filtration on $H^m(M\backslash D,\mathbb{C})$ is given by

$$W_kH^m(M\backslash D,\mathbb{C}):=\mathrm{Im}(\mathbb{H}^m(P_k\Omega_M^\bullet(\log(D))\to \mathbb{H}^m(\Omega_M^\bullet(\log(D)))),\ k\in\mathbb{Z},$$

where $P_k\Omega_M^\bullet(\log(D))$ is the Deligne pole order filtration: A logarithmic differential m-form ω is in $P_k\Omega_M^m(\log(D))$ if in local coordinates, there does not appear a wedge product of more than k'-times $\frac{dz_i}{z_i}, k'>k$ in ω. The Hodge filtration induces a filtration on $Gr_a^W:=W_a/W_{a-1}$ and we set

$$Gr_F^bGr_a^W:=F^bGr_a^W/F^{b+1}Gr_a^W=\frac{(F^b\cap W_a)+W_{a-1}}{(F^{b+1}\cap W_a)+W_{a-1}},a,b\in\mathbb{Z} \tag{A.3}$$

In the next sections we will introduce two other filtrations which are called again pole order filtrations and have completely distinct nature.

A.3 Pole order filtration

For an analytic sheaf $\mathcal{S}$ on M we denote by $\mathcal{S}(*D)$ the sheaf of meromorphic sections of $\mathcal{S}$ with poles of arbitrary order along D. For $k=(k_1,k_2,\dots,k_s)\in\mathbb{N}_0^s$ we denote by $\mathcal{S}(kD)$ the sheaf of meromorphic sections of $\mathcal{S}$ with poles of order at most k_i along $D_i,\ i=1,2,\dots,s$. For $\mathcal{S}(*D)$ we have also the pole filtration:

$$P^k\mathcal{S}^\bullet(*D):\underbrace{0\to 0\to\cdots\to 0}_{k \text{ times } 0}\to\mathcal{S}_0^k\to\mathcal{S}_1^{k+1}\to\cdots\to\mathcal{S}_{p-k}^p\to\cdots,$$

where

$$\mathcal{S}_{p-k}^p:=\cup_{|n|\leq p-k}\mathcal{S}^j((n+1)D) \text{ if } p\geq k,$$
$$(n+1)=(n_1+1,n_2+1,\dots,n_s+1),\ |n|=n_1+n_2+\cdots+n_s.$$

Let $(\mathcal{S}_1^\bullet,F)$ and $(\mathcal{S}_2^\bullet,F)$ be two filtered differential complexes and $p:(\mathcal{S}_1^\bullet,F)\to(\mathcal{S}_2^\bullet,P)$ a map between them, i.e. we have a collection of maps $p_k:F^k\mathcal{S}_1\to P^k\mathcal{S}_2$ such that the following diagram commutes

$$\begin{array}{ccc} F^{k+1}\mathcal{S}_1^\bullet & \to & P^{k+1}\mathcal{S}_2^\bullet \\ \downarrow & & \downarrow \\ F^k\mathcal{S}_1^\bullet & \to & P^k\mathcal{S}_2^\bullet \end{array},\ k=0,1,\dots$$

The map p is called a quasi-isomorphism of filtered complexes if $p_k,\ k=0,1,\dots$ are quasi-isomorphism.

Theorem A.1. *The inclusion*

$$(\Omega^\bullet_M(\log D), F) \subset (\Omega^\bullet_M(*D), P) \tag{A.4}$$

is a quasi-isomorphisms of filtered differential complexes.

Proof. Note that this is a local statement and so we can suppose that $M = (\mathbb{C}^n, 0),\ D = \{z_1 = 0\} + \{z_2 = 0\} + \cdots \{z_s = 0\}$. The proof can be found in [18] Proposition 3.13, [19] Proposition 3.1.8. See also [85] Proposition 8.18. According to [17] p.647, Deligne has inspired the above theorem from the work of Griffiths . □

The above proposition implies that the Hodge filtration on $H^m(M - D, \mathbb{C})$ is also given by

$$F^i H^m(M\backslash D, \mathbb{C}) = \mathrm{Im}(\mathbb{H}^m(P^i\Omega^\bullet_M(*D)) \to \mathbb{H}^m(\Omega^\bullet_M(*D))).$$

A.4 Another pole order filtration

In this section we assume that D is a positive divisor, i.e. the associated line bundle is positive. From this what we need is the following: For any coherent analytic sheaf $\mathcal{S}$ on M we have

$$H^k(M, \mathcal{S}(*D)) = 0,\ k = 1, 2, \ldots \tag{A.5}$$

We do not assume that D is a normal crossing divisor.

Theorem A.2. *(Atiyah-Hodge-Grothendieck) If D is positive then*

$$H^k(M\backslash D, \mathbb{C}) \cong H^k(\Gamma\Omega^\bullet_M(*D), d),\ k = 0, 1, 2, \ldots \tag{A.6}$$

Proof. The proof follows from Proposition A.1 and (A.5). □

From now on assume that D is irreducible. To each cohomology class $\alpha \in H^m(M\backslash D, \mathbb{C})$ we can associate $P(\alpha) \in \mathbb{N}$ which is the minimum number k such that there exists a meromorphic m-form in M with poles of order k along D and represents α in the isomorphism (A.6). We have

$$P(\alpha + \beta) \le max\{P(\alpha), P(\beta)\},\ P(k\alpha) = P(\alpha),\ \alpha, \beta \in H^m(M\backslash D, \mathbb{C}),\ k \in \mathbb{C}\backslash\{0\}.$$

Using the above facts for a $\mathbb{C}$-basis of $H^m(M\backslash D, \mathbb{C})$, we can find a number h such that for all $\alpha \in H^m(M\backslash D, \mathbb{C})$ we have $P(\alpha) \le h$. We take the minimum number h with the mentioned property. Now we have the filtration

$$H_0 \subset H_1 \subset \cdots \subset H_{h-1} \subset H_h = H^m(M\backslash D, \mathbb{C})$$

$$H_i = \{\alpha \in H^m(M\backslash D, \mathbb{C}) \mid P(\alpha) \le i\},\ i = 0, 1, \ldots, h.$$

We call it the new pole order filtration.

A.5 Weighted projective spaces

In this section we recall some terminology on weighted projective spaces. For more information on weighted projective spaces the reader is referred to [22, 82].

Let n be a natural number and $\alpha := (\alpha_1, \alpha_2, \ldots, \alpha_{n+1})$ be a vector of natural numbers whose greatest common divisor is one. The multiplicative group $\mathbb{C}^*$ acts on $\mathbb{C}^{n+1}$ in the following way:

$$(X_1, X_2, \ldots, X_{n+1}) \to (\lambda^{\alpha_1} X_1, \lambda^{\alpha_2} X_2, \ldots, \lambda^{\alpha_{n+1}} X_{n+1}),\ \lambda \in \mathbb{C}^*.$$

We also denote the above map by λ. The quotient space

$$\mathbb{P}^\alpha := \mathbb{C}^{n+1}/\mathbb{C}^*$$

is called the projective space of weight α. If $\alpha_1 = \alpha_2 = \cdots = \alpha_{n+1} = 1$ then $\mathbb{P}^\alpha$ is the usual projective space $\mathbb{P}^n$ (Since n is a natural number, $\mathbb{P}^n$ will not mean a zero dimensional weighted projective space). One can give another interpretation of $\mathbb{P}^\alpha$ as follow: Let $G_{\alpha_i} := \{e^{\frac{2\pi\sqrt{-1}m}{\alpha_i}} \mid m \in \mathbb{Z}\}$. The group $\Pi_{i=1}^{n+1} G_{\alpha_i}$ acts discretely on the usual projective space $\mathbb{P}^n$ as follows:

$$(\epsilon_1, \epsilon_2, \ldots, \epsilon_{n+1}), [X_1 : X_2 : \cdots : X_{n+1}] \to [\epsilon_1 X_1 : \epsilon_2 X_2 : \cdots : \epsilon_{n+1} X_{n+1}].$$

The quotient space $\mathbb{P}^n/\Pi_{i=1}^{n+1} G_{\alpha_i}$ is canonically isomorphic to $\mathbb{P}^\alpha$. This canonical isomorphism is given by

$$[X_1 : X_2 : \cdots : X_{n+1}] \in \mathbb{P}^n/\Pi_{i=1}^{n+1} G_{\alpha_i} \to [X_1^{\alpha_1} : X_2^{\alpha_2} : \cdots : X_{n+1}^{\alpha_{n+1}}] \in \mathbb{P}^\alpha.$$

Let d be a natural number. The polynomial (resp. the polynomial form) ω in $\mathbb{C}^{n+1}$ is weighted homogeneous of degree d if

$$\lambda^*(\omega) = \lambda^d \omega,\ \lambda \in \mathbb{C}^*.$$

For a polynomial g this means that

$$g(\lambda^{\alpha_1} X_1, \lambda^{\alpha_2} X_2, \ldots, \lambda^{\alpha_{n+1}} X_{n+1}) = \lambda^d g(X_1, X_2, \ldots, X_{n+1}),\ \forall \lambda \in \mathbb{C}^*.$$

Let g be an irreducible polynomial of (weighted) degree d. The set $g = 0$ induces a hypersurface D in $\mathbb{P}^\alpha$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{n+1})$. If g has an isolated singularity at $0 \in \mathbb{C}^{n+1}$ then Steenbrink has proved that D is a V-manifold/quasi-smooth variety. A V-manifold may be singular but it has many common features with smooth varieties (see [82, 22]).

For a polynomial form ω of degree dk, $k \in \mathbb{N}$ in $\mathbb{C}^{n+1}$ we have $\lambda^* \frac{\omega}{g^k} = \frac{\omega}{g^k}$ for all $\lambda \in \mathbb{C}^*$. Therefore, $\frac{\omega}{g^k}$ induce a meromorphic form on $\mathbb{P}^\alpha$ with poles of order k along D. If there is no confusion we denote it again by $\frac{\omega}{g^k}$. The polynomial form

$$\eta_\alpha = \sum_{i=1}^{n+1} (-1)^{i-1} \alpha_i X_i \widehat{dX_i}, \tag{A.7}$$

where $\widehat{dX_i} = dX_1 \wedge \cdots \wedge dX_{i-1} \wedge dX_{i+1} \wedge \cdots \wedge dX_{n+1}$, is of degree $\sum_{i=1}^{n+1} \alpha_i$.

Let $\mathbb{P}^{(1,\alpha)} = \{[X_0 : X_1 : \cdots : X_{n+1}] \mid (X_0, X_1, \cdots, X_{n+1}) \in \mathbb{C}^{n+2}\}$ be the projective space of weight $(1, \alpha)$, $\alpha = (\alpha_1, \ldots, \alpha_{n+1})$. One can consider $\mathbb{P}^{(1,\alpha)}$ as a compactification of $\mathbb{C}^{n+1} = \{(x_1, x_2, \ldots, x_{n+1})\}$ by putting

$$x_i = \frac{X_i}{X_0^{\alpha_i}},\ i = 1, 2, \cdots, n+1 \tag{A.8}$$

The projective space at infinity $\mathbb{P}^{\alpha}_{\infty} = \mathbb{P}^{(1,\alpha)} - \mathbb{C}^{n+1}$ is of weight $\alpha := (\alpha_1, \alpha_2, \ldots, \alpha_{n+1})$.

Let f be a tame polynomial of degree d in §3 and g be its last quasi-homogeneous part. We take the homogenization $F = X_0^d f(\frac{X_1}{X_0^{\alpha_1}}, \frac{X_2}{X_0^{\alpha_2}}, \ldots, \frac{X_{n+1}}{X_0^{\alpha_{n+1}}})$ of f and so we can regard $\{f = 0\}$ as an affine subvariety in $\{F = 0\} \subset \mathbb{P}^{(1,\alpha)}$.

A.6 Complement of hypersurfaces

This section is dedicated to a classic theorem of Griffiths in [35]. Its generalization for quasi-homogeneous spaces is due to Steenbrink in [82].

Theorem A.3. *Let $g(X_1, X_2, \cdots, X_{n+1})$ be a weighted homogeneous polynomial of degree d, weight $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{n+1})$ and with an isolated singularity at $0 \in \mathbb{C}^{n+1}$(and so $D = \{g = 0\}$ is a V-manifold). We have*

$$H^n(\mathbb{P}^{\alpha} - D, \mathbb{C}) \cong \frac{H^0(\mathbb{P}^{\alpha}, \Omega^n(*D))}{dH^0(\mathbb{P}^{\alpha}, \Omega^{n-1}(*D))}$$

and under the above isomorphism

$$Gr_F^{n+1-k} Gr_{n+1}^W H^{n+1}(\mathbb{P}^{\alpha} - D, \mathbb{C}) := F^{n-k+1}/F^{n-k+2} \cong \frac{H^0(\mathbb{P}^{\alpha}, \Omega^n(kD))}{dH^0(\mathbb{P}^{\alpha}, \Omega^{n-1}((k-1)D)) + H^0(\mathbb{P}^{\alpha}, \Omega^n((k-1)D))} \tag{A.9}$$

where $0 = F^{n+1} \subset F^n \subset \cdots \subset F^1 \subset F^0 = H^n(\mathbb{P}^{\alpha} - D, \mathbb{C})$ is the Hodge filtration of $H^n(\mathbb{P}^{\alpha} - D, \mathbb{C})$. Let $\{X^{\beta} \mid \beta \in I\}$ be a basis of monomials for the Milnor vector space

$$\mathbb{C}[X_1, X_2, \cdots, X_{n+1}] / < \frac{\partial g}{\partial X_i} \mid i = 1, 2, \ldots, n+1 >$$

A basis of (A.9) is given by

$$\frac{X^{\beta}\eta_{\alpha}}{g^k},\ \beta \in I,\ A_{\beta} = k \tag{A.10}$$

where

$$\eta_{\alpha} = \sum_{i=1}^{n+1} (-1)^{i-1} \alpha_i X_i \widehat{dX_i}$$

In the situation of the above theorem $F^0 = F^1$. The essential ingredient in the proof is Bott's vanishing theorem for quasi-homogeneous spaces and Proposition A.1.

Complementary notes

1. As the reader may have noticed, Theorem A.3 implies that for the complement of smooth hypersurfaces the pole order filtrations in §A.3 and §A.4 are the same up to reindexing the pieces. In this point the following question arises: Can one find the pieces of the Hodge filtration of $H^m(M\backslash, \mathbb{C})$ inside the pieces of the new pole order filtration? Using Riemann-Roch Theorem one can find also positive answers to this question for Riemann surfaces. However, the question in general, as far as I know, is open.

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