Painlevé VI equations with algebraic solutions and family of curves [1]

Hossein Movasati, Stefan Reiter
Instituto de Matemática Pura e Aplicada, IMPA,
Estrada Dona Castorina, 110,
22460-320, Rio de Janeiro, RJ, Brazil,
E-mail: hossein@impa.br, reiter@impa.br

Abstract

In families of Painlevé VI differential equations having common algebraic solutions we classify all the members which come from geometry, i.e. the corresponding linear differential equations which are Picard-Fuchs associated to families of algebraic varieties. In our case, we have one family with zero dimensional fibers and all others are families of curves. We use the classification of families of elliptic curves with four singular fibers done by Herfurtner in 1992 and generalize the results of Doran in 2001 and Ben Hamed and Gavrilov in 2005.

1 Introduction

Along the solutions of the sixth Painlevé differential equation written in the vector field form

$$(1) \qquad PVI_\theta: \ \frac{\partial K}{\partial \mu}\frac{\partial}{\partial \lambda} - \frac{\partial K}{\partial \lambda}\frac{\partial}{\partial \mu} + \frac{\partial}{\partial t}$$

in $\mathbb{C}^3$ with coordinates (λ, μ, t), where

$$t(t-1)K = \lambda(\lambda-1)(\lambda-t)\mu^2 - (\theta_2(\lambda-1)(\lambda-t)+\theta_3\lambda(\lambda-t)+(\theta_1-1)\lambda(\lambda-1))\mu + \kappa(\lambda-t),$$

$$\kappa = \frac{1}{4}((\sum_{i=1}^{3}\theta_i - 1)^2 - \theta_4^2),$$

and $\theta = (\theta_1, \theta_2, \theta_3, \theta_4)$ is a fixed multi-parameter, the linear differential equation

$$(2) \qquad y'' + p_1(z)y' + p_2(z)y = 0$$

$$p_1(z) := \frac{1-\theta_1}{z-t} + \frac{1-\theta_2}{z} + \frac{1-\theta_3}{z-1} - \frac{1}{z-\lambda},$$

$$p_2(z) := \frac{\kappa}{z(z-1)} - \frac{t(t-1)K}{z(z-1)(z-t)} + \frac{\lambda(\lambda-1)\mu}{z(z-1)(z-\lambda)}$$

is isomonodromic, i.e. its monodromy group representation is constant. Recently, there have been works on algebraic solutions of (1) using linear differential equations coming from geometry, i.e. those who are Picard-Fuchs equations associated to families of varieties. Linear equations (2) with finite monodromy come automatically from geometry and this is the origin of many algebraic solutions known until now (see [3] and the references therein). Doran in [6] took 5 deformable families of elliptic curves with non-constant j

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invariant along the deformation parameter and with exactly four singular fibers, which were already classified by Herfurtner in [10], and obtained algebraic solutions for (1) (see Table 1, column 1 and 2, row 3-6 for $a = c = \frac{1}{2}$). In general one can obtain such algebraic solutions by taking pull-backs of the Gauss hypergeometric equation (see [12] and the references therein). Ben Hamed and Gavrilov in [1] took zero dimensional families of three points varieties, constructed directly from the Herfurtner list, and they obtained the same algebraic solutions in the (t, λ)-space but for different parameters θ. They have also noticed that in the parameter space of Painlevé equations (1), the points obtained by them and Doran lie in families with algebraic solutions whose projections in the (t, λ) space is independent of the parameter of the family. Then by a straightforward calculation they showed that up to the Okamoto transformations corresponding to the Möbius transformation of $\mathbb{P}^1$, such families of Painlevé equations are given by the first and second column of Table 1. The main result of this article is to classify the members of such families which come from geometry (third column of Table 1).

Table 1: Algebraic solutions of families of the sixth Painlevé equation

Algebraic solution	$(\theta_1, \theta_2, \theta_3, \theta_4)$	family of algebraic varieties
$\{\lambda = t = 0\} \cup \{\lambda = t = 1\}$	$(0, 1 - c,$ $c - a - b,$ $b - a)$	$y = x^{1-a}(1 - x)^b(z - x)^{1+a-c}, \ \frac{dx}{y}$
$\lambda = (\frac{-a+1}{a+2c-3})b, \ t = b^2$ $\mu = \frac{-a-2c+3}{2b}$	$(\frac{1}{2}, a - 1,$ $\frac{1}{2}, -(a + 2c - 3))$	zero dimensional varieties
$\lambda = -b, \ t = b^2$ $\mu = \frac{-a-2c+2}{2b}$	$(c - \frac{1}{2}, a + c - 1,$ $c - \frac{1}{2}, a + c - 1)$	$y = (4x^2 - g_2x + g_3)^c(x + g_2/4)^a, \ \frac{dx}{y}$ $g_2 = 4(\tilde{z}^2 + \tilde{z})$ $g_3 = -12\tilde{b}\tilde{z}^3 - 8\tilde{z}^4 + 8\tilde{z}^3 - 8\tilde{z}^2$ $\tilde{b} = \frac{3}{4}(b + \frac{1}{b}) + \frac{1}{2}, \tilde{z} = \frac{-1}{b}z$
$\lambda = \frac{-2b-1}{b^2}, \ t = \frac{2b+1}{b^4+2b^3}$ $\mu = \frac{(-2a+3)b^2(b+2)}{2(b+1)^2}$	$(a - \frac{1}{2}, 3(a - \frac{1}{2}),$ $a - \frac{1}{2}, a - \frac{1}{2})$	$y = (4x^3 - g_2x - g_3)^a, \ \frac{dx}{y}$ $g_2 = 12\tilde{z}^2(\tilde{z}^2 + 2\tilde{b}\tilde{z} + 1)$ $g_3 = 4\tilde{z}^3(2\tilde{z}^3 + 3(\tilde{b}^2 + 1)\tilde{z}^2 + 6\tilde{b}\tilde{z} + 2)$ $\tilde{b} = \frac{2}{3}(b + \frac{1}{b}) - \frac{1}{3}, \ \tilde{z} = -\frac{b^2+2b}{3}z$
$\lambda = \frac{b^3+b^2+3b+3}{b^3+b^2-5b+3}, \ t = \frac{b^4-6b^2-8b-3}{b^4-6b^2+8b-3}$ $\mu = \frac{(3a-2)(b-1)^2(b+3)}{24(b+1)}$	$(a - \frac{1}{2}, \frac{1}{2},$ $a - \frac{1}{2}, a - \frac{1}{2})$	$y = (4x^3 - g_2x - g_3)^a, \ \frac{dx}{y}$ $g_2 = 3\tilde{z}^3(\tilde{z} + \tilde{b}), \ g_3 = \tilde{z}^5(\tilde{z} + 1)$ $\tilde{b} = \frac{2}{3}\frac{b^2-3}{b^2+3} + \frac{1}{3}, \tilde{z} = -\frac{b^3-3b^2+3b-1}{b^3-3b^2+3b-9}z$
$\lambda = \frac{-2b^2-4}{b^4-6b^2}, \ t = \frac{-12b^2+8}{b^6-6b^4}$ $\mu = \frac{(-3a+2)b^2(b^2+2)(b^2-6)}{12(b^2-2)^2}$	$(a - \frac{1}{2}, \frac{1}{3},$ $a - \frac{1}{2}, 2a - 1)$	$y = (4x^3 - g_2x - g_3)^a, \ \frac{dx}{y}$ $g_2 = 3\tilde{z}^3(\tilde{z} + 2\tilde{b})$ $g_3 = \tilde{z}^4(\tilde{z}^2 + 3\tilde{b}\tilde{z} + 1)$ $\tilde{b} = \frac{1}{4}(b + \frac{2}{b}), \ \tilde{z} = -\frac{2b^3}{3b^2-2}z$

Theorem 1. *In the columns 1 and 2 of Table 1 the corresponding linear differential equations (2) come from geometry if and only the exponent parameters a and c in column 2 (and b in the first row) are rational numbers. The corresponding family of algebraic varieties and differential form are listed in column 3. Moreover, the second family (together with its algebraic solution) is Okamoto equivalent to the third family and the fourth family is Okamoto equivalent to the fifth family. Both Okamoto transformations are equivalent to the middle convolution of the corresponding Fuchsian systems.*

In the first row of Table 1, the corresponding linear differential equation (2) is the Gauss hypergeometric equation and the geometric interpretation is classical and it has nice applications in the theory of the special values of Gauss hypergeometric functions (see [14]). Note that in this case the projection of the corresponding algebraic curve in the (λ, μ, t) space into the (λ, t) space is just the zero dimensional variety $\{(0, 0), (1, 1)\}$.

In all other cases it is a curve in the (λ, t) space. In the literature one finds mainly the equations of such curves. Note also that the parametrization of the algebraic solutions in column 1 is different from the parametrization of the solutions of the vector field (1).

Let us put the parameters a and c in column 2 and rows $3, 4, 5, 6$ equal to $\frac{1}{2}$. We obtain families of elliptic curves $y^2 = 4x^3 - g_2(z, b)x - g_3(z, b)$ with exactly four singular fibers and with j-invariant depending on an extra parameter b. These are exactly four families of the five Herfurtner families. The missing family in the Herfurtner list is the one given by $g_2 = 3(z - 1)(z - b^2)^3$, $g_3 = (z - 1)(z - b^2)^4(z + b)$. The corresponding family of curves $y = (4x^3 - g_2 x - g_3)^{\tilde{a}}$ gives us the Painlevé equation and its algebraic solution in Table 1, row 3 by setting $c = \frac{5}{6}$, $a = \tilde{a} - \frac{1}{3}$. Note that in this case we find two apparently different geometric interpretations for the same Painlevé equation. Note also that in the mentioned five families of elliptic curves just for the family used in row 3 the polynomial $4x^3 - g_2(z, b)x - g_3(z, b)$ is reducible in x. In row 2 for a and c rational numbers we have shown that the monodromy group of the linear equation (2) is a dihedral group and so it is finite. Also the other families are related via the middle convolution to (third order) differential equations whose monodromy groups are finite imprimitive reflection groups for rational parameters. In [2] Boalch started with third order differential equations to obtain (via a construction equivalent to the middle convolution) algebraic solutions and the parameters of the corresponding Painlevé VI differential equations. Recently Cantat and Loray showed in [4, Prop. 5.4] that any algebraic solution of a Painlevé VI differential equation having degree 2, 3, or 4 belongs (up to Okamoto transformation) to one of the families in Table 1.

Using Singular (see [8]), for each linear equation (2) we have calculated the corresponding Fuchsian system in the Schlesinger form and the Okamoto transformation of the algebraic solutions in Table 1 corresponding to exchanging the role of the first and second coordinates of the system. The details of the calculation is explained for the example in row 3 of Table 1 and for the others the reader is referred to the home-page of the first author.

Let us explain the content of each section. In §2 we introduce systems of linear differential equations in two and three variables. Pulling these back we get Fuchsian systems with four singularities. In §3 we recall some well-known facts about linear differential equations and in §4 we explain how to calculate Fuchsian systems in Schlesinger form associated to families of curves in Table 1, column 3. Such calculations are explained for the third row of Table 1 in §5. In §6 we recall some basic facts about the middle convolution and in §7 we show that the algebraic solution in row 2 of Table 1 can be obtained via the middle convolution of the Fuchsian system corresponding to the algebraic solution in row 3 of Table 1. In §8 we show that the Fuchsian system associated to the Painlevé equation in row 2 of Table 1 has finite monodromy if $a, c \in \mathbb{Q}$ and so it comes from variation of zero dimensional varieties. Further we point out how all the other Picard-Fuchs differential equations are related via the middle convolution to those with finite monodromy. Finally, in §9 we show that the middle convolution relates the results in Table 1 row 4 and row 5.

2 Linear systems in two variables

For $a, b, c \in \mathbb{C}$ fixed, we consider the following family of transcendent curves:

(3)
$$E : y = f(x),$$

$$f(x) := (t_1 - t_3)^{\frac{1}{2}(1-a-c)}(t_1 - t_2)^{\frac{1}{2}(1-a-b)}(t_2 - t_3)^{\frac{1}{2}(1-b-c)}(x - t_1)^a(x - t_2)^b(x - t_3)^c$$

Here $t = (t_1, t_2, t_3)$ is a parameter in

$$T := \{t \in \mathbb{C}^3 \mid (t_1 - t_2)(t_2 - t_3)(t_3 - t_1) \neq 0\}.$$

We distinguish three, not necessarily closed, paths in E. In the x-plane let $\tilde{\delta}_i$, $i = 1, 2, 3$, be the straight path connecting t_{i+1} to t_{i-1}, $i = 1, 2, 3$ (by definition $t_4 := t_1$ and $t_0 = t_3$). There are many paths in E which are mapped to $\tilde{\delta}_i$ under the projection on x. We choose one of them and call it δ_i. We can make our choices so that $\delta_1 + \delta_2 + \delta_3$ is a limit of a closed and homotopic-to-zero path in E. For instance, we can take the path $\tilde{\delta}_i$'s in such a way that the triangle formed by them has almost zero area. Now, we have the integral

(4)
$$\int_\delta \frac{p(x)dx}{y} = \int_{\tilde{\delta}} \frac{p(x)dx}{f(x)}, \quad p \in \mathbb{C}[x],$$

where δ is one of the paths explained above. By a linear change in the variable x such integrals can be written in terms of the Gauss hypergeometric function (see [11]). Another way to study the integrals (4) is by using Pochhammer cycles. For simplicity we explain it for the pairs (t_1, t_2). The Pochhammer cycle associated to the points $t_1, t_2 \in \mathbb{C}$ and the path $\tilde{\delta}_3$ is the commutator

$$\tilde{\alpha}_3 = [\gamma_1, \gamma_2] = \gamma_1^{-1} \cdot \gamma_2^{-1} \cdot \gamma_1 \cdot \gamma_2,$$

where γ_1 is a loop along $\tilde{\delta}_3$ starting and ending at some point in the middle of $\tilde{\delta}_1$ which encircles t_1 once anti-clockwise, and γ_2 is a similar loop with respect to t_2. It is easy to see that the cycle $\tilde{\alpha}_3$ lifts up to a closed path α_3 in E_t and if $a, b \notin \mathbb{Z}$ then

$$\int_{\alpha_3} \frac{p(x)dx}{y} = (1 - e^{-2\pi i a})(1 - e^{-2\pi i b}) \int_{\tilde{\alpha}_3} \frac{p(x)}{f(x)} dx.$$

(see [11], Proposition 3.3.7).

For a fixed $\mathsf{a} \in T$, the period map is given by:

(5)
$$\mathsf{pm} : (T, \mathsf{a}) \to \mathrm{GL}(2, \mathbb{C}), \ t \mapsto \begin{pmatrix} \int_{\delta_1} \frac{dx}{y} & \int_{\delta_2} \frac{dx}{y} \\ \int_{\delta_1} \frac{xdx}{y} & \int_{\delta_2} \frac{xdx}{y} \end{pmatrix},$$

where (T, a) means a small neighborhood of a in T. The map pm can be extended along any path in T with the starting point a. The period map pm is a fundamental system for the linear differential equation $dY = AY$ in $\mathbb{C}^3$, where

(6)
$$A =$$

$$\frac{1}{(t_1 - t_2)(t_1 - t_3)} \begin{pmatrix} \frac{1}{2}(b + c - 2)t_1 + \frac{1}{2}(a + c - 1)t_2 + \frac{1}{2}(a + b - 1)t_3 & -a - b - c + 2 \\ at_2 t_3 + (b-1)t_1 t_3 + (c-1)t_1 t_2 & -\frac{1}{2}(b + c - 2)t_1 - \frac{1}{2}(a + c - 1)t_2 - \frac{1}{2}(a + b - 1)t_3 \end{pmatrix} dt_1$$

$$+ (\cdots)dt_2 + (\cdots)dt_3$$

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The matrix coefficient of dt_2 (resp dt_3) is obtained by permutation of t_1 with t_2 and a with b (resp. t_1 with t_3 and a with c) in the matrix coefficient of dt_1 written above. Now, for the multi-valued function

$$y = (27t_3^2 - t_2^3)^{\frac{1}{2}(\frac{1}{2}-a)}(4x^3 - t_2x - t_3)^a$$

we have the system

(7)
$$A = \frac{1}{27t_3^2 - t_2^3}\left(\begin{pmatrix} \frac{1}{4}t_2^2 & -27at_3 + 18t_3 \\ -\frac{9}{4}at_2t_3 + \frac{3}{4}t_2t_3 & -\frac{1}{4}t_2^2 \end{pmatrix}dt_2 + \right.$$
$$\left. \begin{pmatrix} -\frac{9}{2}t_3 & 18at_2 - 12t_2 \\ \frac{3}{2}at_2^2 - \frac{1}{2}t_2^2 & \frac{9}{2}t_3 \end{pmatrix}dt_3\right)$$

and for

$$y = (t_2^2 + 2t_3)^{\frac{1}{2}(1-a-c)}(t_2^2 - 16t_3)^{\frac{1}{2}(\frac{1}{2}-c)}(4x^2 - t_2x + t_3)^c(x + \frac{1}{4}t_2)^a$$

we have

(8)
$$A = \frac{1}{(t_2^2 - 16t_3)(t_2^2 + 2t_3)}$$

$$\left(\begin{pmatrix} (6at_2t_3 - 6ct_2t_3 - \frac{1}{2}t_2^3 + 5t_2t_3) & (-48at_3 - 96ct_3 + 96t_3) \\ (12at_3^2 - 3ct_2^2t_3 + t_2^2t_3 - 4t_3^2) & (-6at_2t_3 + 6ct_2t_3 + \frac{1}{2}t_2^3 - 5t_2t_3) \end{pmatrix}dt_2 + \right.$$
$$\left. \begin{pmatrix} (-3at_2^2 + 3ct_2^2 + t_2^2 + 8t_3) & (24at_2 + 48ct_2 - 48t_2) \\ (-6at_2t_3 + \frac{3}{2}ct_2^3 - \frac{1}{2}t_2^3 + 2t_2t_3) & (3at_2^2 - 3ct_2^2 - t_2^2 - 8t_3) \end{pmatrix}dt_3\right)$$

The calculation of the matrix A for the elliptic case $a = b = c = \frac{1}{2}$ is classical and goes back to Griffiths [9]. In fact the matrix A in (6) can be also calculated from the well-known Fuchsian system for hypergeometric functions, i.e. for the case $t_1 = 0$, $t_2 = 1$, $t_3 = t$. The algorithms for calculating such a matrix are explained in [13] and the implementation of such algorithms in a computer can be found in the first author's home-page.

3 Review of Fuchsian differential equations

Here we collect some basic facts about Fuchsian differential equations which can all be found in [11]. Let $D = \frac{d}{dz}$ and

(9)
$$D^2y + p_1(z)Dy + p_2(z)y = 0$$

be a Fuchsian differential equation with regular singularities at $t_1, \ldots, t_m \in \mathbb{C}$ and $t_{m+1} = \infty$. Then by [11, Prop. 4.2]

$$p_1(z) = \sum_{i=1}^m \frac{a_i}{z - t_i}, \quad p_2(z) = \sum_{i=1}^m \frac{b_i}{(z - t_i)^2} + \sum_{i=1}^m \frac{c_i}{(z - t_i)}, \quad a_i, b_i, c_i \in \mathbb{C}$$

where $p_2(z)\prod_{i=1}^m(z - t_i)^2$ is a polynomial in $\mathbb{C}[z]$ of degree at most $2(m-1)$, i.e. $\sum c_i = 0$. The following table of the regular singular points $t_1, \ldots, t_{m+1}$ and the exponents s_i^1, s_i^2 at t_i is called the Riemann scheme of the differential equation

$$\begin{pmatrix} t_1 & \cdots & t_{m+1} \\ s_1^1 & \cdots & s_{m+1}^1 \\ s_1^2 & \cdots & s_{m+1}^2 \end{pmatrix},$$

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where the sum of all exponents satisfies the Fuchs relation $\sum_{i=1}^{m+1}\sum_{j=1}^{2} s_j^i = m - 1$. The exponents at the singularity t_i, $i = 1, \ldots, m$, are the roots of $s(s-1) + a_i s + b_i = 0$ (see [11, p. 170]) and at t_{m+1} the exponents satisfy

$$s(s+1) - (\sum_{i=1}^{m} a_i)s + (\sum_{i=1}^{m} b_i + (\sum_{i=1}^{m} c_i t_i)) = 0.$$

The second order Fuchsian differential equation (9) can be transformed into SL-form by substituting y by fy, where $0 \neq f$ satisfies $Df = -\frac{1}{2}p_1(z)f$. Then by [11, p. 166]

$$(10) \qquad D^2 y = p(z)y, \quad p(z) = -p_2(z) + \tfrac{1}{4}p_1(z)^2 + \tfrac{1}{2}Dp_1(z).$$

From a two dimensional system $DY = QY$, $Q = (q_{ij})$ (not necessarily a Fuchsian system), we obtain the second order differential equation (9) for the first coordinate y_1 of $Y = (y_1, y_2)^{\mathrm{tr}}$ (see [11, Lemma 6.1.1]) with

$$(11) \qquad p_1(z) = -D\log(q_{12}(z)) - \mathrm{Tr}(Q), \quad p_2(z) = \det(Q(z)) - Dq_{11} + q_{11}D\log q_{12}.$$

If λ is a zero of q_{12} of order r and $\lambda \notin \{t_1, \ldots, t_m\}$ then $z = \lambda$ is an apparent singular point with the exponents 0 and $r + 1$ (see [11, Lemma 6.1.2]).

4 The algorithm

In this section we explain how to obtain the algebraic solutions of the Painlevé VI differential equation starting from the families of curves in Table 1 column 3, row 3, 4, 5, 6, which are constructed directly from the Herfurtner list of families of elliptic curves (see [10]). For the convenience of the reader we have listed the five families which we need:

Table 2: List of deformable families of elliptic curves $y^2 = 4x^3 - g_2 x - g_3$ with four singular fibers and with non constant j invariant along the deformation

name	deformation
1	$g_2 = 3(z-1)(z-b^2)^3, \; g_3 = (z-1)(z-b^2)^4(z+b)$
2	$g_2 = 12z^2(z^2 + bz + 1), \; g_3 = 4z^3(2z^3 + 3bz^2 + 3bz + 2)$
3	$g_2 = 12z^2(z^2 + 2bz + 1), \; g_3 = 4z^3(2z^3 + 3(b^2+1)z^2 + 6bz + 2)$
4	$g_2 = 3z^3(z+b), \; g_3 = z^5(z+1)$
5	$g_2 = 3z^3(z+2b), \; g_3 = z^4(z^2 + 3bz + 1)$

Let us consider $f(x) = 4x^3 - g_2 x - g_3$, where g_2, g_3 correspond to one of the five families of elliptic curves with parameter b in the Herfurtner list. Since the roots of the discriminant $\Delta = 27g_2^3 - g_3^2$ of f, which will be the singular points of the corresponding differential equation for $\int_\delta \frac{\mathrm{d}x}{y}$, are not all rational functions in b we substitute b by a suitable rational function in b such that the transformed roots are rational in b. Such substitutions are done by the equalities $\tilde{b} = \cdots$ in Table 1 column 3. In the next step we check whether the polynomial $f(x)$ factorizes over $\mathbb{Q}(b, z)$. It turns out that this only happens for second family. In that case we have

$$(12) \qquad f(x) = (4x^2 - g_2 x + g_3)(x + \frac{g_2}{4}).$$

Substituting g_2 and g_3 in the list 2 (resp. 1, 3, 4, 5) of Table 2 for t_2 and t_3 in (8) (resp. (7)) we obtain a system $DY = AY$. We can now compute the second order differential

equation satisfied by $\int_\delta \frac{dx}{y}$, where δ is a Pochhammer cycle, using the formula (11) and (10). It always turns out that (9) has four singularities at $t_1, t_2 = 0, t_3$ and $t_4 = \infty$ and 1 apparent singularity at λ with exponents 0 and 2. Hence, the SL-form can be written as follows ([11, p. 173])

$$p(z) = \sum_{i=1}^{3} \frac{a_i}{(z - t_i)^2} + \frac{a_4}{z(z - t_3)} + \frac{t_1(t_1 - t_3)/t_3 \cdot L}{z(z - t_1)(z - t_3)} + \frac{3}{4} \frac{1}{(z - \lambda)^2} - \frac{\lambda(\lambda - t_3)/t_3 \cdot \nu}{z(z - t_3)(z - \lambda)},$$

where

$$a_i = \frac{1}{4}(\theta_i^2 - 1), \ i = 1, 2, 3, \qquad a_4 = -\tfrac{1}{4}(\textstyle\sum_{i=1}^{3} \theta_i^2 - \theta_4^2 - 1) - \tfrac{1}{2},$$

$$L = t_3((p(z) - \frac{a_1}{(z - t_1)^2})(z - t_1))_{z=t_1}, \quad \nu = -t_3((p(z) - \tfrac{3}{4(z-\lambda)^2})(z - \lambda))_{z=\lambda}.$$

The corresponding Riemann-scheme is

$$\begin{pmatrix} t_1 & t_2 & t_3 & \infty & \lambda \\ \frac{1-\theta_1}{2} & \frac{1-\theta_2}{2} & \frac{1-\theta_3}{2} & \frac{-1-\theta_4}{2} & -\frac{1}{2} \\ \frac{1+\theta_1}{2} & \frac{1+\theta_2}{2} & \frac{1+\theta_3}{2} & \frac{-1+\theta_4}{2} & \frac{3}{2} \end{pmatrix}.$$

The system

(13) $DY = QY, \quad Q = \sum_{k=1}^{3} \frac{Q_k}{z - t_k}, \ Q_k = Q_k(t) = (q_{ij}^k) \in \mathrm{Mat}_n(\mathbb{C}), \ k = 1, 2, 3$

of Schlesinger type with the 4 regular singularities at $t_1, t_2 = 0, t_3, t_4 = \infty$ can now be determined: We can assume that θ_i and 0 are the eigenvalues of Q_i and that (if $\theta_4 \neq 1$)

$$-\sum_{i=1}^{3} Q_i(t) = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha + \theta_4 - 1 \end{pmatrix}, \quad 2\alpha + \sum_{i=1}^{4} \theta_i = 1.$$

Further we normalize the singularities via the Möbius transformation $z \mapsto z \cdot t_3$ to $t = \frac{t_1}{t_3}, 0, 1$ and ∞. (Note that also the apparent singularity is transformed to $\frac{\lambda}{t_3}$ which we will also denote again by λ.) Then by [11, Prop. 6.3.1] the matrices $Q_i, i = 1, 2, 3$, can be expressed as follows:

(14) $Q_i = \begin{pmatrix} M_i(W_i - W) & -M_i \\ -(W_i - W)(M_i(W - W_i) + \theta_i) & \theta_i - M_i(W_i - W) \end{pmatrix}, \ i = 1, \dots, 3,$

	M_i	W_i	
$i = 1$	$\frac{\lambda - t}{t(t-1)}$	$\lambda(\lambda - 1)(\mu + \frac{\alpha}{\lambda})$	$(\theta_4 - 1)W = \sum_{i=1}^{3} W_i(M_iW_i - \theta_i),$
$i = 2$	$\frac{\lambda}{t}$	$(\lambda - t)(\lambda - 1)(\mu + \frac{\alpha}{\lambda}) - \frac{t\alpha}{\lambda}$	$\mu = \nu - \frac{1}{2}\sum_{i=1}^{3} \frac{1-\theta_i}{\lambda - t_i},$
$i = 3$	$\frac{\lambda - 1}{1 - t}$	$\lambda(\lambda - t)(\mu + \frac{\alpha}{\lambda})$	$\alpha = -\frac{1}{2}(\sum_{i=1}^{4} \theta_i - 1).$

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Note that a zero of $Q_{1,2}$ has order $r = 1$, which is the apparent singularity λ for the first coordinate. If we compute the differential equation for the second coordinate or equivalently if we compute the differential equation for the first coordinate of the transformed system

$$DY = TQT^{-1}Y, \quad T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

then $(\theta_1, \ldots, \theta_4)$ is transformed to $(\theta_1, \ldots, \theta_3, 2 - \theta_4)$.

5 An example

We demonstrate how the algorithm in §4 yields the results for the third row of of Table 1. We start with the second family in Table 2. The roots of the discriminant of f are

$$0, \omega_1, \omega_2, \quad \omega_{1,2} = -\tfrac{1}{3}(2b - 1 \pm 2\sqrt{b^2 - b - 2})$$

Solving the diophantic equation $\alpha^2 - \alpha - 2 = \beta^2$, we substitute b by $\frac{3}{4}(b + \frac{1}{b}) + \frac{1}{2}$ in g_2 and g_3. Thus the new roots are

$$t_1 = -b, \quad t_2 = 0, \quad t_3 = -\frac{1}{b}.$$

Since $f(x) = 4x^3 - g_2 x - g_3$ factorizes we get (12) with the new coefficients

$$g_2 := 4(z^2 + z), \quad g_3 := \frac{-9b^2 z^3 - 8bz^4 + 2bz^3 - 8bz^2 - 9z^3}{b},$$

Substituting g_2 and g_3 for t_2 and t_3 in (8) we obtain a system which does not fit in our paper and we do not write it here. Computing the SL-form (10) of the differential equation for the first coordinate of the system we obtain the following parameters: (note that we have normalized the singularities via the Möbius transformation $z \mapsto z \cdot t_3$)

$$(t_1, t_2, t_3) = (t, 0, 1), \quad (\theta_1, \theta_2, \theta_3, \theta_4) = (c - \tfrac{1}{2}, a + c - 1, c - \tfrac{1}{2}, a + c - 1),$$

$$\lambda = -b, \quad t = b^2, \quad \mu = \frac{-a - 2c + 2}{2b}, \quad \nu = -\frac{3}{4b}.$$

The system (13) using (14) reads:

$$q_{11}^1 = \frac{(ba + a + 2bc - 3b - 1)(a + 2c - 2)}{4b(a + c - 2)},$$

$$q_{12}^1 = \frac{1}{b(b - 1)},$$

$$q_{21}^1 = \frac{(b - 1)(ba + a - 2b + 2c - 2)(-ba - a - 2bc + 3b + 1)(a + 2c - 2)(a - 1)}{16b(a + c - 2)^2},$$

$$q_{22}^1 = -q_{11}^1 + (c - \frac{1}{2})$$

$$q_{11}^2 = \frac{(-b^2 + 2b - 1)(a + 2c - 2)(a - 1)}{4b(a + c - 2)},$$

$$q_{12}^2 = \frac{1}{b},$$

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$$q_{21}^2 = \frac{(b-1)^2((2-a-2c)(a-1)(b^2+1) + (-2a^2 - 4ac + 6a - 4c^2 + 8c - 4)b)(a+2c-2)(a-1)}{16b(a+c-2)^2},$$

$$q_{22}^2 = -q_{11}^2 + (a - c - 1),$$

$$q_{11}^3 = \frac{(ba + a - b + 2c - 3)(a + 2c - 2)}{4(a + c - 2)},$$

$$q_{12}^3 = -\frac{1}{b-1},$$

$$q_{21}^3 = \frac{(b-1)(ba + a - b + 2c - 3)(ba + a + 2bc - 2b - 2)(a + 2c - 2)(a - 1)}{16(a+c-2)^2},$$

$$q_{22}^3 = -q_{11}^3 + \left(c - \frac{1}{2}\right)$$

and

$$-(Q_1 + Q_2 + Q_3) = \begin{pmatrix} -(a + 2c - 2) & 0 \\ 0 & -c \end{pmatrix}.$$

The apparent singularity of the second coordinate y_2 is the zero of Q_{21}:

$$\tilde{\lambda} = -b + \frac{(-2a - 2c + 4)(b+1)^2 b}{(a-2)(a+2c-3)(b+1)^2 + 2(a^2 + 2ca - 5a + 2c^2 - 6c + 6)b}.$$

6 Middle convolution

For the convenience of the reader we give here a short review of the middle convolution for Fuchsian systems (see [5]).

Let $f(z) = (f_1(z), \ldots, f_n(z))^{\mathrm{tr}}$ be a solution of the Fuchsian system

$$DY = AY = \sum_{i=1}^r \frac{A_i}{z - t_i} Y, \quad A_i \in \mathrm{Mat}_n(\mathbb{C}),$$

and $[t_i, y] = \gamma_{t_i}^{-1} \gamma_y^{-1} \gamma_{t_i} \gamma_y$ be a Pochhammer cycle around t_i and y. Then the Euler transform of $f(z)$ with respect to $\mu \in \mathbb{C}$ and $[t_i, y]$

$$\begin{pmatrix} \int_{[t_i, y]} f(z)(y - z)^\mu \frac{dz}{z - t_1} \\ \vdots \\ \int_{[t_i, y]} f(z)(y - z)^\mu \frac{dz}{z - t_r} \end{pmatrix}, \quad i = 1, \ldots, r,$$

is a solution of the (Okubo-) system

$$(yI_{nr} - T)DY = BY := \left(\begin{pmatrix} A_1 & \cdots & A_r \\ & \vdots & \\ A_1 & \cdots & A_r \end{pmatrix} + \mu\, I_{nr} \right) Y, \quad T = \mathrm{diag}(t_1 I_n, \ldots, t_r I_n)$$

$$\Leftrightarrow DY = \sum_{i=1}^r \frac{B_i}{y - t_i} Y, \quad B_i \in \mathrm{Mat}_{nr}(\mathbb{C}),$$

where I_n is the identity $n \times n$ matrix. In general this system is not irreducible and has the following two $\langle B_1, \ldots, B_r \rangle$-invariant subspaces:

$$\mathfrak{k} = \oplus_{i=1}^{r} \ker(A_i), \quad \mathfrak{l} = \ker(B) = \langle (v, \ldots, v)^{\mathrm{tr}} \mid v \in \ker(\sum_{i=1}^{r} A_i + \mu I_n) \rangle.$$

Factoring out this subspace we obtain a Fuchsian system in dimension m

$$m = nr - \sum_{i=1}^{r} \dim(\ker A_i) - \dim(\ker(\sum_{i=1}^{r} A_i + \mu I_n)).$$

Let M_i be the monodromy of the system $DY = AY$ at t_i. Then, if the system is irreducible and

$$\mathrm{rk}(A_i) = \mathrm{rk}(M_i - I_n), \quad i = 1, \ldots, r$$

$$\mathrm{rk}(\sum_{i=1}^{r} A_i + \mu I_n) = \mathrm{rk}(M_1 \cdots M_r \lambda - I_n), \quad \lambda = e^{2\pi i \mu}$$

then the factor system is again irreducible and it is called the *middle convolution* of $DY = AY$ with μ. Thus if we start with a two dimensional system in Schlesinger form (13), where the parameters are $(\theta_1, \ldots, \theta_4)$ then there are in general two possibilities to get again a new two dimensional system (13) via the middle convolution. We can apply the middle convolution either with $\mu = \alpha$ or with $\mu = \alpha + \theta_4 - 1$ and diagonalize the residue matrix at ∞ to

$$\begin{pmatrix} \tilde{\alpha} & 0 \\ 0 & \tilde{\alpha} + \tilde{\theta}_4 - 1 \end{pmatrix}.$$

This changes the parameters as follows:

$$(\tilde{\theta}_1, \ldots, \tilde{\theta}_4) = (\theta_1 + \mu, \ldots, \theta_3 + \mu, \theta_4 - \mu + 2\alpha), \quad \tilde{\alpha} = -\mu.$$

Note that it is possible to write down also the transformation for the apparent singularity (see [7, Section 5]) which is also known as an Okamoto transformation.

7 Application of the middle convolution I

In this section we show that the algebraic solution in row 2 of Table 1 can be obtained via the middle convolution of the Schlesinger system corresponding to the algebraic solution in row 3 of Table 1.

We continue with our example by determining the middle convolution of $DY = QY$ with $\mu = -c$. Hence, for the (Okubo-) system $(zI_6 - T)DY = BY \iff DY = \sum_{i=1}^{3} \frac{B_i}{y - t_i} Y$, $B_i \in Mat_6(\mathbb{C})$, we get

$$B = \begin{pmatrix} Q_1 - c \cdot I_2 & Q_2 & Q_3 \\ Q_1 & Q_2 - c \cdot I_2 & Q_3 \\ Q_1 & Q_2 & Q_3 - c \cdot I_2 \end{pmatrix}$$

and $T = \mathrm{diag}(t, t, 0, 0, 1, 1)$. The $\langle B_1, B_2, B_3 \rangle$ invariant subspace has dimension 4, since

$$\ker Q_i = \langle \begin{pmatrix} q_{1,2}^i \\ -q_{1,1}^i \end{pmatrix} \rangle \qquad \mathfrak{l} = \langle (0, 1, 0, 1, 0, 1)^{\mathrm{tr}} \rangle.$$

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In order to get a 2 dimensional factor system we set

$$
S = \begin{pmatrix}
q^1_{1,2} & 0 & 0 & 0 & 0 & 0 \\
-q^1_{1,1} & 0 & 0 & 1 & 0 & 0 \\
0 & q^2_{1,2} & 0 & 0 & 1 & 0 \\
0 & -q^2_{1,1} & 0 & 1 & 0 & 0 \\
0 & 0 & q^3_{1,2} & 0 & 0 & 1 \\
0 & 0 & -q^3_{1,1} & 1 & 0 & 0
\end{pmatrix}.
$$

Thus

$$
D(SY) = \sum_{i=1}^{3} \frac{SB_iS^{-1}}{z - t_i}(SY) = \begin{pmatrix}
* & 0 & 0 & 0 & * \\
0 & * & 0 & 0 & * \\
0 & 0 & * & 0 & * \\
0 & 0 & 0 & 0 & * \\
0 & 0 & 0 & 0 & \tilde{A}
\end{pmatrix}(SY)
$$

and we get the 2 dimensional factor system $DY = \tilde{A}Y = \sum_{i=1}^{3} \frac{\tilde{A}_i}{z-t_i}Y$. Transforming this system into Schlesinger form via

$$
Y \mapsto \tilde{S}Y, \quad \tilde{S} = \begin{pmatrix}
-(\sum_{i=1}^{3}\tilde{A}_i + c \cdot I_2)_{1,2} & -(\sum_{i=1}^{3}\tilde{A}_i - c \cdot I_2 - a \cdot I_2 + 2 \cdot I_2)_{1,2} \\
(\sum_{i=1}^{3}\tilde{A}_i + c \cdot I_2)_{1,1} & (\sum_{i=1}^{3}\tilde{A}_i - c \cdot I_2 - a \cdot I_2 + 2 \cdot I_2)_{1,1}
\end{pmatrix},
$$

where the columns of $\tilde{S}$ consist of eigenvectors of $\sum_{i=1}^{3}\tilde{A}_i$ with respect to the eigenvalues $-c$ and $c + a - 2$, we obtain finally

(15) $$DY = AY, \ A = \frac{A_1}{z - b^2} + \frac{A_2}{z} + \frac{A_3}{z - 1}$$

$$
A_1 = \frac{1}{4b}\begin{pmatrix}
-ab - a - 2bc + b + 1 & ab + a + 2bc - 3b - 1 \\
-ab - a - 2bc + b + 1 & ab + a + 2bc - 3b - 1
\end{pmatrix},
$$

$$
A_2 = \frac{1}{4b}\begin{pmatrix}
ab^2 + 2ab + a - b^2 - 2b - 1 & ab^2 - a - b^2 + 1 \\
-ab^2 + a + b^2 - 1 & -ab^2 + 2ab - a + b^2 - 2b + 1
\end{pmatrix},
$$

$$
A_3 = \frac{1}{4}\begin{pmatrix}
-ab - a + b - 2c + 1 & -ab - a + b - 2c + 3 \\
ab + a - b + 2c - 1 & ab + a - b + 2c - 3
\end{pmatrix}
$$

and

$$
-(A_1 + A_2 + A_3) = \begin{pmatrix}
c & 0 \\
0 & -(a + c - 2)
\end{pmatrix}.
$$

Computing the entries of A we get

$$
A_{11} = \frac{-4cz^2 + ((1 - a + 2c)(b^2 + 1) + 2(1 - a)b)z + (a - 1)(b + 1)^2 b}{4z(z - 1)(z - b^2)},
$$

$$
A_{12} = \frac{(b^2 - 1)((a + 2c - 3)z + b(a - 1))}{4z(z - 1)(z - b^2)},
$$

$$
A_{21} = \frac{(b^2 - 1)((1 - a - 2c)z + (b - ab))}{4z(z - 1)(z - b^2)},
$$

$$
A_{22} = \frac{4(a + c - 2)z^2 + ((5 - 3a - 2c)(b^2 + 1) + 2(a - 1)b)z + (1 - a)(b - 1)^2 b}{4z(z - 1)(z - b^2)}.
$$

Thus the parameters are

$$(\theta_1, \ldots, \theta_4) = (-1/2, a - 1, -1/2, -(a + 2c - 3)).$$

The apparent singularity λ_1 for the first, resp. λ_2 for the second coordinate is

$$\lambda_1 = (\frac{-a+1}{a+2c-3})b, \qquad \lambda_2 = (\frac{-a+1}{a+2c-1})b.$$

Since $t = b^2$ we obtain the relation

$$\theta_4^2 \, \lambda_1^2 = t \, \theta_2^2, \qquad (2 - \theta_4)^2 \, \lambda_2^2 = t \, \theta_2^2.$$

Computing the SL-form for the first coordinate (after transforming $Y \mapsto (z-1)^{1/2}(z - t)^{1/2}Y$ which changes the parameters $(\theta_1, \ldots, \theta_4)$ (but not the apparent singularities) to $(1/2, a - 1, 1/2, -(a + 2c - 3))$ we get a simple formula for μ_1

$$\mu_1 = \frac{-a - 2c + 3}{2b}.$$

8 Monodromy

In this section we show that (15) arises also as a pullback of hypergeometric differential equations. By determining the monodromy group it turns out that this monodromy group is finite and therefore by a well known result of Klein the claim follows. Further we indicate that all the other families of Picard-Fuchs equations are related to those with finite monodromy.

Proposition 1. *The projective monodromy group of (15) is (up to conjugation) contained in the orthogonal group* $\mathrm{GO}_2(\mathbb{C})$. *Moreover, if the parameters a and c are rational numbers, i.e. if (15) is a Picard-Fuchs equation, then the monodromy group is even finite, i.e. a dihedral group.*

Proof. To prove this statement let M_t, M_0, M_1 denote the monodromy at $t, 0$ and 1. If the parameter b tends to 1 we see that the system becomes reducible and monodromy group is abelian:

$$A \to \begin{pmatrix} \frac{-a-cz+1}{z^2-z} & 0 \\ 0 & \frac{a+c-2}{z-1} \end{pmatrix}.$$

Hence $M_t M_1$ and M_0 commute and are diagonal matrices. Since M_t and M_1 are reflections, the group generated by them is an orthogonal group. Hence M_t and M_1 normalize also M_0 and the claim follows. $\square$

Applying the Möbius transformation $z \mapsto \frac{1}{z}$ that permutes the residue matrices Q_i of the Schlesinger system corresponding to row 5 (resp. row 6), column 2 in Table 1 and scaling the new Q_2 we get the following pairs of eigenvalues for $Q_1, Q_2, Q_3, -(Q_1+Q_2+Q_3)$

$$(a - \frac{1}{2}, 0), \ (a - \frac{3}{2}, 0), \ (a - \frac{1}{2}, 0), \ (-\frac{3a-3}{2}, -\frac{3a-2}{2})$$

resp.

$$(a - \frac{1}{2}, 0), \ (-2a + 2, 0), \ (a - \frac{1}{2}, 0), \ (-\frac{1}{3}, -\frac{2}{3}).$$

The middle convolution with $\mu = -(a - 1)$ yields a three dimensional Fuchsian system with the following triples of eigenvalues of the residue matrices

$$(\frac{1}{2}, 0, 0),\ (-\frac{1}{2}, 0, 0),\ (\frac{1}{2}, 0, 0),\ (-\frac{a-1}{2}, -\frac{a}{2}, a - 1)$$

resp.

$$(\frac{1}{2}, 0, 0),\ (-3a + 3, 0, 0),\ (\frac{1}{2}, 0, 0),\ (-\frac{1}{3} + a - 1, -\frac{2}{3} + a - 1, a - 1).$$

Using the explicit construction for the middle convolution and similar arguments as in the above Proposition, one easily sees that the monodromy groups of these third order differential equations are finite imprimitive reflection groups contained in $T \rtimes S_3$, where T denotes the group of diagonal matrices. In the next section we show that in Table 1 row 4 and row 5 are also related via the middle convolution. Hence all the families of Picard-Fuchs equations corresponding to Table 1 are related to those with finite monodromy. Since the corresponding Picard-Fuchs differential equation of row 1 in Table 1 is the hypergeometric one it is well known that it is obtained via the convolution of a one dimensional differential equation with finite monodromy.

9 Application of the middle convolution II

As in the previous example we show that the middle convolution relates the results in Table 1 row 4 and row 5.

The system (13) in Schlesinger form corresponding to the differential equation satisfied by $\int_\delta \frac{dx}{y}$, where y is from row 4 in Table 1, reads:

$$q_{1,1}^1 = \frac{(18a^2b^2 + 18a^2b + 18a^2 - 3ab^3 - 30ab^2 - 48ab - 36a + 2b^3 + 12b^2 + 24b + 16)}{18ab + 18a - 27b - 27}$$

$$q_{1,2}^1 = \frac{-b^4 - 2b^3}{b^2 - 1}$$

$$q_{2,1}^1 = \frac{q_{11}^1 q_{22}^1}{q_{12}^1}$$

$$q_{2,2}^1 = (a - \frac{1}{2}) - q_{1,1}^1$$

$$q_{1,1}^2 = \frac{(3a - 2)(b^2 + (-6a + 7)b + 1)(b - 1)^2}{18ab^2 - 27b^2}$$

$$q_{1,2}^2 = b^2 + 2b$$

$$q_{2,1}^2 = \frac{q_{11}^2 q_{22}^2}{q_{12}^2}$$

$$q_{2,2}^2 = (3a - \frac{3}{2}) - q_{1,1}^2$$

$$q_{1,1}^3 = \frac{(18a^2b^3 + 18a^2b^2 + 18a^2b - 36ab^3 - 48ab^2 - 30ab - 3a + 16b^3 + 24b^2 + 12b + 2)}{18ab^3 + 18ab^2 - 27b^3 - 27b^2}$$

$$q_{1,2}^3 = \frac{b^2 + 2b}{b^2 - 1}$$

$$q_{2,1}^3 = \frac{q_{11}^3 q_{22}^3}{q_{12}^3}$$

$$q_{2,2}^3 = (a - \frac{1}{2}) - q_{1,1}^3$$

$$-(Q_1 + Q_2 + Q_3) \quad = \quad \begin{pmatrix} -3a+2 & 0 \\ 0 & -2a+\frac{1}{2} \end{pmatrix}.$$

Applying the middle convolution to $DY = QY$ with $\mu = -(3a-2)$ and transforming the 2 dimensional factor system into Schlesinger form with singularities at $t, 0, 1$ and ∞ we get the system (13) with

$$q^1_{1,1} = \frac{(3a-2)(b+(-6a+2))(b+2)^2}{9(4a-1)(b+1)}$$

$$q^1_{1,2} = \frac{b+2}{(72a-18)(b+1)}$$

$$q^1_{2,1} = \frac{(3a-2)((6a-4)b^3+(-36a^2+60a-24)b^2+(24a-21)b-5)(-b+(6a-2))(b+2)}{9(4a-1)(b+1)}$$

$$q^1_{2,2} = (-2a+\frac{3}{2}) - q^1_{1,1}$$

$$q^2_{1,1} = \frac{(-3a+2)(b^2+(-6a+4)b+1)(b^2+b+1)}{9(4a-1)b^2}$$

$$q^2_{1,2} = \frac{-(b^2+b+1)}{18(4a-1)b^2}$$

$$q^2_{2,1} = \frac{((4-6a)b^4+(36a^2-54a+20)b^3+(36a^2-96a+33)b^2+(36a^2-54a+20)b+4-6a)}{9(4a-1)b^2}.$$
$$(3a-2)(-b^2+(6a-4)b-1)$$

$$q^2_{2,2} = \frac{1}{2} - q^2_{1,1}$$

$$q^3_{1,1} = \frac{(3a-2)((-6a+2)b+1)(2b+1)^2}{9(4a-1)b^2(b+1)}$$

$$q^3_{1,2} = \frac{2b+1}{18(4a-1)b^2(b+1)}$$

$$q^3_{2,1} = \frac{(3a-2)(-5b^3+(24a-21)b^2+(-36a^2+60a-24)b+(6a-4))((6a-2)b-1)(2b+1)}{9(4a-1)b^2(b+1)}$$

$$q^3_{2,2} = (-2a+\frac{3}{2}) - q^3_{1,1}$$

$$-(Q_1+Q_2+Q_3) = \begin{pmatrix} 3a-2 & 0 \\ 0 & a-\frac{3}{2} \end{pmatrix} = \begin{pmatrix} 3a-2 & 0 \\ 0 & 3a-2+(-2a+\frac{3}{2})-1 \end{pmatrix}.$$

We obtain the parameters $(\theta_1, \ldots, \theta_4) = (-2a+\frac{3}{2}, \frac{1}{2}, -2a+\frac{3}{2}, -2a+\frac{3}{2})$. The apparent singularity for the first coordinate is

$$\lambda \quad = \quad \frac{b^2+b+1}{b^3+2b^2}.$$

Using that $t = \frac{2b+1}{b^4+2b^3}$ we see that λ and t satisfy

$$\lambda^4 - 2t\lambda^3 - 2\lambda^3 + 6t\lambda^2 - 2t^2\lambda - 2t\lambda + t^3 - t^2 + t = 0.$$

Since this is also the relation for

$$t = \frac{b^4 - 6b^2 - 8b - 3}{b^4 - 6b^2 + 8b - 3}, \qquad \lambda = \frac{b^3 + b^2 + 3b + 3}{b^3 + b^2 - 5b + 3}$$

from Table 1 row 5 the claim follows.

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