Heun equations coming from geometry [1]

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Abstract

We give a list of Heun equations which are Picard-Fuchs associated to families of algebraic varieties. Our list is based on the classification of families of elliptic curves with four singular fibers done by Herfurtner. We also show that pull-backs of hypergeometric functions by rational Belyi functions with restricted ramification data give rise to Heun equations.

1 Introduction

For a linear differential equation which depends on some parameters, it is a natural question to ask for which values of the parameters the specialized differential equation comes from geometry. We say that the linear differential equation comes from geometry if there is a proper family of algebraic varieties $X \to \mathbb{P}^1$ over $\mathbb{C}$ and a differential form $\omega \in H^i_{\mathrm{dR}}(X/\mathbb{P}^1)$ such that the periods $\int_{\delta_z} \omega$, where $\delta_z \in H_i(X_z, \mathbb{Z})$ is a continuous family of cycles, span the solutions space of the linear differential equation. Such linear differential equations are also called Picard-Fuchs equations (for further details see [1], Chapter II, §1).

If a linear differential equation comes from geometry then it is well-known that the exponents of its singularities are all rational numbers (see for instance [12] and the references therein). This implies that the Gauss hypergeometric equation with parameters a, b, c comes from geometry if and only if the exponents of its singular set are rational numbers and hence if and only if a, b, c are rationals. The next non-trivial family of linear differential equations is the family of Heun equations:

$$y'' + \left(\frac{1-\theta_1}{z-t} + \frac{1-\theta_2}{z} + \frac{1-\theta_3}{z-1}\right)y' + \left(\frac{\theta_{41}\theta_{42}z - q}{z(z-1)(z-t)}\right)y = 0 \tag{1}$$

with

$$\theta_{41} = -\frac{1}{2}(\theta_1 + \theta_2 + \theta_3 - 2 + \theta_4),\ \theta_{42} = -\frac{1}{2}(\theta_1 + \theta_2 + \theta_3 - 2 - \theta_4).$$

As we mentioned, if (1) comes from geometry then the exponents $\theta_i,\ i = 1, 2, \ldots, 4$, are rational numbers. Now, our problem reduces to the following one: For which rational numbers $\theta_i,\ i = 1, \ldots, 4$, and complex numbers $t, q \in \mathbb{C}$ does the corresponding equation (1) come from geometry.

We have observed that in the classification of families of elliptic curves with exactly four singular fibers (see [10]) only 38 of the 50 examples give us Heun equations (five of them give us linear differential equations associated to Painlevé VI equations with algebraic solutions, see [6, 3], seven of them can be reduced to families with three singular fibers by

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Table 1: Heun equations coming from geometry, $a, b, c \in \mathbb{Q}$

*	q	t	θ_1	θ_2	θ_3	θ_4	θ_{42}	θ_{41}
1	$\frac{1}{3}(3a-2)(6a-1)t_1$	$\frac{t_1^2}{3},\ t_1^2+3t_1+3=0$	$a-\frac{1}{2}$	$a-\frac{1}{2}$	$a-\frac{1}{2}$	$9a-\frac{9}{2}$	$3a-\frac{1}{2}$	$-6a+4$
2	0	-1	$b-\frac{1}{2}$	$2b-1$	$b-\frac{1}{2}$	$4a+4b-4$	$2a$	$-2a-4b+4$
3	$-2(a+2b-2)(6b-5)$	-8	$b-\frac{1}{2}$	$3b-\frac{3}{2}$	$a+b-1$	$3a+3b-3$	$a-b+1$	$-2a-4b+4$
4	$-3(10a-7)(3a-2)t_1$	$-t_1^2,\ t_1^2-11t_1-1=0$	$a-\frac{1}{2}$	$5a-\frac{5}{2}$	$a-\frac{1}{2}$	$5a-\frac{5}{2}$	$-a+\frac{3}{2}$	$-6a+4$
5	0	-1	$a+c-1$	$2a+2b-2$	$a+c-1$	$2b+2c-2$	$-2a+2$	$-2a-2b-2c+4$
6	$\frac{-1}{3}(6a-5)(3a-2)t_1$	$\frac{t_1^2}{3}, t_1^2+3t_1+3=0$	$3a-\frac{3}{2}$	$3a-\frac{3}{2}$	$3a-\frac{3}{2}$	$3a-\frac{3}{2}$	$-3a+\frac{5}{2}$	$-6a+4$
7	$\frac{-2}{243}(96a-25)(3a-2)t_1$	$\frac{t_1^2}{9}, 3t_1^2-14t_1+27=0$	$a-\frac{1}{2}$	$\frac{1}{3}$	$a-\frac{1}{2}$	$8a-4$	$3a-\frac{2}{3}$	$-5a+\frac{10}{3}$
8	$\frac{-1}{288}(3a-2)(1029a-149)$	$\frac{81}{32}$	$a-\frac{1}{2}$	$\frac{1}{3}$	$2a-1$	$7a-\frac{7}{2}$	$2a-\frac{1}{6}$	$-5a+\frac{10}{3}$
9	$\frac{-125}{6}(4a-3)(3a-2)$	-80	$a-\frac{1}{2}$	$4a-2$	$\frac{1}{3}$	$5a-\frac{5}{2}$	$\frac{5}{6}$	$-5a+\frac{10}{3}$
10	$\frac{-25}{18}(3a-2)(6a-5)$	$-\frac{27}{5}$	$\frac{1}{3}$	$3a-\frac{3}{2}$	$2a-1$	$5a-\frac{5}{2}$	$\frac{5}{6}$	$-5a+\frac{10}{3}$
11	$\frac{1}{128}(49a-12)(3a-2)t_1$	$\frac{t_1^2}{8}, 4t_1^2+13t_1+32=0$	$a-\frac{1}{2}$	$\frac{1}{2}$	$a-\frac{1}{2}$	$7a-\frac{7}{2}$	$\frac{5}{2}a-\frac{1}{2}$	$-\frac{9}{2}a+3$
12	$\frac{-9}{16}a(a+2b-2)$	$\frac{1}{4}$	$2b-1$	$\frac{1}{2}$	$b-\frac{1}{2}$	$3a+3b-3$	$\frac{3}{2}a$	$-\frac{3}{2}a-3b+3$
13	$\frac{39}{500}(3a-2)(6a-5)$	$-\frac{3}{125}$	$\frac{1}{2}$	$3a-\frac{3}{2}$	$a-\frac{1}{2}$	$5a-\frac{5}{2}$	$\frac{1}{2}a+\frac{1}{2}$	$-\frac{9}{2}a+3$
14	$\frac{-3}{4}(a+2b-2)(6b-5)$	-3	$\frac{1}{2}$	$3b-\frac{3}{2}$	$a+b-1$	$2a+2b-2$	$\frac{1}{2}a-b+1$	$-\frac{3}{2}a-3b+3$
15	0	-1	$a-\frac{1}{2}$	$\frac{2}{3}$	$a-\frac{1}{2}$	$6a-3$	$2a-\frac{1}{3}$	$-4a+\frac{8}{3}$
16	$-\frac{14}{3}a+\frac{28}{9}$	$\frac{27}{2}$	$a-\frac{1}{2}$	$\frac{2}{3}$	$2a-1$	$5a-\frac{5}{2}$	$a+\frac{1}{6}$	$-4a+\frac{8}{3}$
17	$\frac{-2}{9}(3a-2)(6a-5)$	-1	$\frac{2}{3}$	$3a-\frac{3}{2}$	$2a-1$	$3a-\frac{3}{2}$	$-a+\frac{7}{6}$	$-4a+\frac{8}{3}$
18	$\frac{-1}{147}(58a-15)(3a-2)t_1$	$\frac{t_1^2}{49},\ t_1^2-13t_1+49=0$	$\frac{1}{3}$	$a-\frac{1}{2}$	$\frac{1}{3}$	$7a-\frac{7}{2}$	$3a-\frac{5}{6}$	$-4a+\frac{8}{3}$
19	0	-1	$\frac{1}{3}$	$2a-1$	$\frac{1}{3}$	$6a-3$	$2a-\frac{1}{3}$	$-4a+\frac{8}{3}$
20	$\frac{-4}{3}(4a-3)(3a-2)t_1$	$-\frac{t_1^2}{2}, t_1^2-10t_1-2$	$4a-2$	$\frac{1}{3}$	$4a-2$	$\frac{1}{3}$	$-4a+3$	$-4a+\frac{8}{3}$
21	$(\frac{-27}{2}\zeta-\frac{29}{4})(a-\frac{10}{9589}\zeta-\frac{7442}{28767})(a-\frac{2}{3})$	$-\frac{2}{7}(3\zeta+1),\ \zeta^2+3=0$	$a-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	$6a-3$	$\frac{5}{2}a-\frac{2}{3}$	$-\frac{7}{2}a+\frac{7}{3}$
22	$\frac{-14}{1125}(3a-2)(147a-22)$	$\frac{189}{125}$	$\frac{1}{2}$	$\frac{1}{3}$	$2a-1$	$5a-\frac{5}{2}$	$\frac{3}{2}a-\frac{1}{6}$	$-\frac{7}{2}a+\frac{7}{3}$
23	$\frac{77}{972}(3a-2)(6a-5)$	$-\frac{1}{27}$	$\frac{1}{2}$	$3a-\frac{3}{2}$	$\frac{1}{3}$	$4a-2$	$\frac{1}{2}a+\frac{1}{3}$	$-\frac{7}{2}a+\frac{7}{3}$
24	$-\frac{1}{6}a+\frac{1}{9}$	$-\frac{16}{9}$	$a-\frac{1}{2}$	$\frac{2}{3}$	$\frac{1}{3}$	$5a-\frac{5}{2}$	$2a-\frac{1}{2}$	$-3a+2$
25	$-3a+2$	9	$\frac{1}{3}$	$\frac{2}{3}$	$2a-1$	$4a-2$	a	$-3a+2$
26	$\frac{-1}{125}(3a-2)(38a-9)t_1$	$\frac{4t_1^2}{125},\ t_1^2-11t_1+125/4=0$	$\frac{1}{2}$	$a-\frac{1}{2}$	$\frac{1}{2}$	$5a-\frac{5}{2}$	$2a-\frac{1}{2}$	$-3a+2$
27	0	-1	$\frac{1}{2}$	$2b-1$	$\frac{1}{2}$	$2a+2b-2$	a	$-a-2b+2$
28	$\frac{-1}{6}(6a-5)(3a-2)t_1$	$-\frac{t_1^2}{3},\ t_1^2-6t_1-3=0$	$3a-\frac{3}{2}$	$\frac{1}{2}$	$3a-\frac{3}{2}$	$\frac{1}{2}$	$-3a+\frac{5}{2}$	$-3a+2$
29	$\frac{5}{162}a-\frac{5}{243}$	$-\frac{5}{27}$	$\frac{1}{2}$	$\frac{2}{3}$	$a-\frac{1}{2}$	$4a-2$	$\frac{3}{2}a-\frac{1}{3}$	$-\frac{5}{2}a+\frac{5}{3}$
30	$-\frac{5}{3}a+\frac{10}{9}$	5	$\frac{1}{2}$	$\frac{2}{3}$	$2a-1$	$3a-\frac{3}{2}$	$\frac{1}{2}a+\frac{1}{6}$	$-\frac{5}{2}a+\frac{5}{3}$
31	0	-1	$\frac{2}{3}$	$2a-1$	$\frac{2}{3}$	$2a-1$	$\frac{1}{3}$	$-2a+\frac{4}{3}$
32	0	-1	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{1}{2}$	$2a-1$	$a-\frac{1}{3}$	$-a+\frac{2}{3}$
33	$\frac{1}{12}(3a-1)(3a-2)t_1$	$\frac{t_1^2}{3},\ t_1^2+3t_1+3=0$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$3a-\frac{3}{2}$	$\frac{3}{2}a-\frac{1}{2}$	$-\frac{3}{2}a+1$
34	0	$-\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{1}{3}$	$3a-\frac{3}{2}$	$\frac{3}{2}a-\frac{1}{2}$	$-\frac{3}{2}a+1$
35	0	-1	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$4a-2$	$2a-\frac{2}{3}$	$-2a+\frac{4}{3}$
36	$\frac{-16}{243}(3a-1)(3a-2)t_1$	$\frac{t_1^2}{27},\ t_1^2-10t_1+27=0$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$	$4a-2$	$2a-\frac{2}{3}$	$-2a+\frac{4}{3}$
37	$\frac{25}{768}(3a-2)(3a-1)t_1$	$\frac{t_1^2}{64}, t_1^2+11t_1+64=0$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	$5a-\frac{5}{2}$	$\frac{5}{2}a-\frac{5}{6}$	$-\frac{5}{2}a+\frac{5}{3}$
38	$\frac{1}{3}(3a-2)(3a-1)t_1$	$\frac{t_1^2}{3},\ t_1^2+3t_1+3=0$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$6a-3$	$3a-1$	$-3a+2$

means of quadratic twists). Using this we have obtained a table of Heun equations coming from geometry, see Table 1 for $a, b, c \in \mathbb{Q}$. Table 1 contains the previously calculated list of Heun equations by R. S. Maier in [14].

One application of Table 1 can be found in [17], where the second author shows that one gets a list of Lamé equations with arithmetic Fuchsian monodromy group applying the inverse Halphen transform to some of the examples. In this way one obtains all those Lamé equations where the quaternion algebra associated to the arithmetic Fuchsian group is defined over $\mathbb{Q}$. Thus one can relate Krammer's example in [13], that was considered to be a counter example to a conjecture of Dwork, to a Gauss hypergeometric equation via geometric operations.

In §2 we explain how to compute Table 1 using the Weierstrass form of families of elliptic curves with four singular fibers. The corresponding algorithms are implemented in the library `painleve-heun.lib` in Singular, see [8]. Table 1 can be also computed using the j-function of the corresponding family of elliptic curves. We explain this in §3. In §4 we state Theorem 1 which characterizes pull-backs of hypergeometric functions by rational Belyi functions. In particular we get further Heun equations under restricted ramification data for the Belyi functions. Since the j-invariants of the mentioned 38 examples in [10] are Belyi functions, this method explains why we get Table 1. In §5 we have derived Table 2 of Lamé equations, i.e. $\theta_1 = \theta_2 = \theta_3 = \frac{1}{2}$, from Table 1. In §6 we compare Table 1 with examples we found in the literature.

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2 Calculating Table 1 using the Weierstrass form

In this section we explain how we have obtained Table 1 using the Weierstrass form of elliptic curves. Despite the fact that the j-invariants of the Herfurtner's list are special Belyi maps, the advantage of this method is that for each item in the table it gives an explicit family of Riemann surfaces with four singular fibers. This can be useful for arithmetic applications of Heun equations using the geometry of curves.

We take a family of elliptic curves

$$y^2 = f(x),\ f(x) := 4x^3 - g_2 x - g_3,\ g_2, g_3 \in \mathbb{C}(z)$$

with four singular fibers. There are 50 examples of such families which are listed by Herfurtner in [10]. In the next step we check whether the polynomial $f(x)$ factorizes over $\mathbb{C}(z)$. If $f(x)$ is a product of degree 2 and degree 1 polynomials then we redefine g_2 and g_3 in the following way

$$f(x) = (4x^2 - g_2 x + g_3)(x + \frac{g_2}{4}).$$

If $f(x)$ is a product of three degree 1 polynomials then we redefine g_2 and g_3 in the following way:

$$f(x) = (4x + g_2 + g_3)(x - \frac{g_2}{4})(x - \frac{g_3}{4})$$

Corresponding to the above three cases we consider the following family of transcendent curves:

$$y = (4x^3 - g_2 x - g_3)^a,$$

$$y = (x + \frac{g_2}{4})^a (4x^2 - g_2 x + g_3)^b,$$

$$y = (4x + g_2 + g_3)^a (x - \frac{g_2}{4})^b (x - \frac{g_3}{4})^c$$
$$a, b, c \in \mathbb{C}.$$

One can recover the family of elliptic curves by setting $a = b = c = \frac{1}{2}$. The corresponding systems in the variables g_2 and g_3 can be calculated from the system in three variables t_1, t_2, t_3

$$dY = AY$$

where

$$A = \tag{2}$$

$$\frac{1}{(t_1 - t_2)(t_1 - t_3)} \begin{pmatrix} \frac{1}{2}(b+c-2)t_1 + \frac{1}{2}(a+c-1)t_2 + \frac{1}{2}(a+b-1)t_3 & -a-b-c+2 \\ at_2t_3 + (b-1)t_1t_3 + (c-1)t_1t_2 & -\frac{1}{2}(b+c-2)t_1 - \frac{1}{2}(a+c-1)t_2 - \frac{1}{2}(a+b-1)t_3 \end{pmatrix} dt_1$$

$$+(\cdots)dt_2 + (\cdots)dt_3$$

and the matrix coefficient of dt_2 (resp dt_3) is obtained by permutation of t_1 with t_2 and a with b (resp. t_1 with t_3 and a with c) in the matrix coefficient of dt_1 written above. This system is associated to the family of transcendental curves

$$y = (t_1 - t_3)^{\frac{1}{2}(1-a-c)}(t_1 - t_2)^{\frac{1}{2}(1-a-b)}(t_2 - t_3)^{\frac{1}{2}(1-b-c)}(x - t_1)^a (x - t_2)^b (x - t_3)^c.$$

For further details and explicit formulas for the three cases above see [15]. In this way, we calculate the linear differential equation satisfied by integrals $\int \frac{dx}{y}$, namely

$$y'' + p_1(z)y' + p_2(z)y = 0$$

and then we write it in the SL-form. The SL-form of the above second order Fuchsian differential equation is by definition

$$y'' = p(z)y, \quad p(z) = -p_2(z) + \tfrac{1}{4}p_1(z)^2 + \tfrac{1}{2}p_1'(z).$$

In the 50 families of elliptic curves in [10] there are seven families of elliptic curves with I_0^* singularity. The corresponding singularity, namely ρ_4 which is an arbitrary parameter, does not appear as a singularity of the SL-form. Five other families depend on an extra parameter α and the corresponding SL-form has an apparent singularity. They give us algebraic solutions of the Painlevé VI equation and they are discussed in detail in [3, 6, 16]. Therefore, the first twelve families in [10] do not yield Heun differential equations. The next 38 families give us Heun equations in the SL-form:

$$y'' = p(z)y, \quad p(z) = \tfrac{a_1}{(z-t)^2} + \tfrac{a_2}{z^2} + \tfrac{a_3}{(z-1)^2} + \tfrac{a_4}{z(z-1)} + \tfrac{L}{z(z-t)(z-1)},$$

where

$$a_4 = -\frac{1}{4}(\sum_{i=1}^{3} \theta_i^2 - (\theta_4 + 1)^2) + \frac{1}{2},$$

$$L = q - t\theta_{41}\theta_{42} + \frac{(1-\theta_1)}{2}((1-\theta_2)(t-1) + (1-\theta_3)t).$$

Now it is just a matter of calculation to obtain the corresponding parameters from the SL-form. Our numbering row 1 till 38 in Table 1 corresponds to the 13th till 50th family in [10].

Among the 38 examples there are 13 examples with two Galois conjugate singularities and with $g_2, g_3 \in \mathbb{Q}(z)$. Since in Singular, see [8], we were not able to calculate in a ring with many transcendental and algebraic parameters, we have used the SL-form with singularities $t_1, t_2 = 0, t_3$ and ∞:

$$p(z) = \sum_{i=1}^{3} \frac{a_i}{(z-t_i)^2} + \frac{\tilde{a}_4}{z(z-t_3)} + \frac{t_1(t_1-t_3)/t_3 \cdot \tilde{L}}{(z-t_1)z(z-t_3)}.$$

We have to treat the 21th example in a especial way because it is the only example in which g_2 and g_3 are not defined over $\mathbb{Q}(z)$. The corresponding sequence of commands in Singular are implemented in the library `painleve-heun.lib`. This and the 38 families in [10] can be downloaded from the first author's webpage.

3 Calculating Table 1 using the j-invariant

In [10] Herfurtner has classified elliptic surfaces with four singular fibres in Weierstrass form. To each elliptic surface it corresponds a period, a complete elliptic integral of the first kind, depending on a parameter. Thus it satisfies a Picard-Fuchs equation with regular singular points. In 38 cases it is a Heun equation. All those equations are pull-backs of the Gauss hypergeometric equation L, where L is the uniformizing differential equation for $\mathrm{PSL}_2(\mathbb{Z})$, by the j-invariant of the elliptic curve, as already noted by Stiller studying classical uniformization problems in [19]. In 27 of the 38 cases Doran showed that the Picard-Fuchs equation is an orbifold uniformizing differential equation, see Chapter 4 in [7].

The idea is to replace the Picard-Fuchs equation L satisfied by elliptic integrals with suitable geometric Gauss hypergeometric equations satisfied by abelian integrals to obtain the one parameter families of geometric Heun equations in Table 1. In Herfurtner's list we find the following data: The family of elliptic curves

$$y^2 = 4x^3 - g_2(z)x - g_3(z),\ g_2(z), g_3(z) \in \mathbb{C}(z),$$

the discriminant $\Delta = g_2^3 - 27g_3^2$ and the j-invariant $j = \frac{g_2^3}{g_2^3 - 27g_3^2}$. It is easy to check that in the cases we consider, namely $I_1\ I_1\ I_1\ I_9 - I_6\ II\ II\ II$ in [10] (i.e. in Table 1, row 1-38) the j-invariant ramifies only at $0, 1$ and ∞. Such a function is also called a rational Belyi-function. In our cases the ramification indices at 0 are at most 3 and at 1 are at most 2. Therefore, we will consider the pull-back of the hypergeometric function

$${}_2F_1(\alpha, \beta, \frac{2}{3}, z), \quad \alpha = -\frac{a}{2} + \frac{1}{3}, \quad \alpha - \beta = -a + \frac{1}{2}$$

with $j(z)$. Since the Riemann scheme of the corresponding hypergeometric differential equation is

$$\begin{pmatrix} 0 & 1 & \infty \\ 0 & 0 & -\frac{a}{2} + \frac{1}{3} \\ \frac{1}{3} & \frac{1}{2} & -\frac{1}{6} + \frac{a}{2} \end{pmatrix}$$

this pull-back will satisfy (after a multiplication with an algebraic function) a Heun equation depending on the parameter a. We demonstrate this claim via the following example, Table 1, row 7, ($I_1 I_1 I_8 II$ in Herfurtner's list). First we recall two basic transformations of second order differential equations, which are readily to check:

Remark 1. Let $Y(z)$ be a solution of

$$y'' + p_1(z)y' + p_2(z)y = 0.$$

a) Then $Y(j(z))$ satisfies

$$y'' + (p_1(j(z))j'(z) - \frac{j''(z)}{j'(z)})y' + p_2(j(z))j'(z)^2 y = 0 \tag{3}$$

b) and $f(z)Y(z)$ satisfies

$$y'' + (p_1(z) - 2\frac{f'(z)}{f(z)})y' + (p_2(z) + \frac{2f'(z)^2}{f(z)^2} - \frac{p_1(z)f'(z)}{f(z)} - \frac{f''(z)}{f(z)})y = 0. \tag{4}$$

Example 1. Let $y^2 = 4x^3 - g_2(z)x - g_3(z)$, where

$$g_2(z) = 12z(z^3 - 6z^2 + 15z - 12), \qquad g_3(z) = 4z(2z^5 - 18z^4 + 72z^3 - 144z^2 + 135z - 27)$$

$$\begin{aligned} j(z) = \frac{g_2^3}{g_2^3 - 27g_3^2} &= -\frac{z(z^3 - 6z^2 + 15z - 12)^3}{(3z^2 - 14z + 27)} \\ j(z) - 1 &= -\frac{(2z^5 - 18z^4 + 72z^3 - 144z^2 + 135z - 27)^2}{(3z^2 - 14z + 27)}. \end{aligned}$$

Thus the ramification data is therefore given by the cycle decomposition

$$(3)(3)(3)(1), \quad (2)(2)(2)(2)(2), \quad (8)(1)(1).$$

The Hurwitz formula implies that the j-invariant is unramified outside $0, 1$ and ∞. Hence it is a Belyi-function. Since a hypergeometric function ${}_2F_1(\alpha, \beta, \gamma, z)$ satisfies

$$y'' + p_1(z)y' + p_2(z)y = 0, \qquad p_1(z) = \tfrac{\gamma - (\alpha+\beta+1)z}{z(1-z)}, \quad p_2(z) = \tfrac{\alpha\beta}{z(z-1)}$$

the pullback ${}_2F_1(\frac{a}{2} - \frac{1}{6}, -\frac{a}{2} + \frac{1}{3}, \frac{2}{3}, j(z))$, is a solution of (see Remark 1 a))

$$y'' + p_1(z)y' + p_2(z)y = 0,$$

$$p_1(z) = \frac{7z^2 - 21z + 18}{3z^3 - 14z^2 + 27z}, \qquad p_2(z) = \frac{-16(3a-1)(3a-2)(z^3 - 6z^2 + 15z - 12)}{(3z^2 - 14z + 27)^2 z}.$$

A solution multiplied by $f(z) = (3z^2 - 14z + 27)^{-1/3 + a/2}$ gives us a Heun equation (see Remark 1 b)):

$$y'' + (\frac{(\frac{3}{2} - a)(6z - 14)}{3z^2 - 14z + 27} + \frac{2}{3z})y' + \frac{3(9a-2)(-15a+10)z + 2(3a-2)(96a-25)}{9z(3z^2 - 14z + 27)}y = 0$$

Our entries in Table 1, row 7, are obtained via a Möbius transformation to get the singularities at $0, 1, t, \infty$.

The reason why this procedure always provides Heun equations will be clear in the next section.

4 Belyi functions

In order to derive further Heun-Picard-Fuchs equations which can be not necessarily obtained from Herfurtner's list we consider in this section pull-backs of hypergeometric functions by rational Belyi functions with restricted ramification data. These give rise to second order differential equations without apparent singularities and in particular Heun equations.

Proposition 1. *Let $j_1(z), j_2(z) \in \mathbb{C}[z]$ be polynomials such that $j(z) = \frac{j_1(z)}{j_2(z)} \in \mathbb{C}(z)$ is a rational Belyi function unramified outside $\{0,1,\infty\}$.*

a) We can assume that the factorization is of the form

$$j_1(z) = A\prod_{i\in I}(z-t_i)^{a_i},\ A \in \mathbb{C}^*, \quad j_2(z) = \prod_{k\in K}(z-u_k)^{c_k}, \quad j_1(z)-j_2(z) = A\prod_{j\in J}(z-s_j)^{b_j},$$

where $N := \deg(j_1) > M := \deg(j_2)$ and $(j_1(z), j_2(z)) = 1$.

b) Further for

$$\Lambda = \prod_{\{t\in\mathbb{C} | (j_1 j_2 (j_1-j_2))(t)=0\}} (z-t)$$

we have $\deg(\Lambda) = N+1$.

Proof. a) Via a Möbius-transformation and scaling we can assume that

$$j(z) = \frac{j_1(z)}{j_2(z)}, \quad \deg(j_1(z)) > \deg(j_2(z)).$$

b) Since $j(z)$ is only ramified at $0, 1,$ and ∞ the Riemann-Hurwitz formula implies that

$$2N - 2 \quad = \quad \sum_i (a_i - 1) + \sum_j (b_j - 1) + \sum_k (c_k - 1) + (N - \deg(j_2(z)) - 1)$$

Hence $deg(\Lambda) = N + 1$.

□

Theorem 1. *Let $j(z)$ be a rational Belyi function as in Proposition 1. Then*

$$j_2(z)^{-\alpha} \cdot {}_2F_1(\alpha, \beta, \gamma, j(z))$$

satisfies $y'' + q_1(z)y' + q_2(z)y = 0$, where

$$q_1(z) = \frac{\Lambda'}{\Lambda} + (\gamma-1)\frac{j_1'(z)}{j_1(z)} + (-\gamma+\alpha+\beta)\frac{(j_1(z)-j_2(z))'}{j_1(z)-j_2(z)} + (\alpha-\beta)\frac{j_2'(z)}{j_2(z)}$$

$$q_2(z) = \alpha\beta\frac{j_1'(z)}{j_1(z)}\frac{(j_1(z)-j_2(z))'}{j_1(z)-j_2(z)} - \alpha\frac{j_2'(z)}{j_2(z)} \cdot \left(\frac{j_2'(z)}{j_2(z)} - \frac{\Lambda'}{\Lambda} - (\gamma-\beta-1)\frac{j_1'(z)}{j_1(z)} + (\gamma-\alpha)\frac{(j_1(z)-j_2(z))'}{j_1(z)-j_2(z)} - \frac{j_2''(z)}{j_2'(z)}\right)$$

with the following Riemann scheme:

$$\left(\begin{array}{cccc} t_i & s_j & u_k & \infty \\ 0 & 0 & 0 & \alpha N \\ (1-\gamma)a_i & (\gamma-\alpha-\beta)b_j & (\beta-\alpha)c_k & \beta(N-M)+M\alpha \end{array}\right).$$

Proof. By Remark 1(a) the pull-back of the hypergeometric function ${}_2F_1(\alpha,\beta,\gamma,j(z))$ satisfies

$$y'' + (p_1(j(z))j'(z) - \frac{j''(z)}{j'(z)})y' + p_2(j(z))j'(z)^2 y = 0, \tag{5}$$

$$p_1(z) = \frac{\gamma - (\alpha+\beta+1)z}{z(1-z)} = \frac{\gamma}{z} + \frac{-\gamma + (\alpha+\beta+1)}{z-1}, \quad p_2(z) = \frac{\alpha\beta}{z(z-1)}.$$

By considering the exponents at the singularities locally one gets the following Riemann scheme:

$$\left(\begin{array}{cccc} t_i & s_j & u_k & \infty \\ 0 & 0 & \alpha c_k & \alpha(N-M) \\ (1-\gamma)a_i & (\gamma-\alpha-\beta)b_j & \beta c_k & \beta(N-M) \end{array}\right).$$

Let $a_1(z)$ be the coefficient of y'. Then sum of the exponents at a finite singularity t is given by $1 - \mathrm{Res}_t(a_1(z))$, cf. [11, Sec. 1.4]. Thus together with Remark 1 the pull-back (5) satisfies

$$\begin{aligned} y'' + (\frac{\Lambda'}{\Lambda} + (\gamma-1)\frac{j_1'(z)}{j_1(z)} + (-\gamma+\alpha+\beta)\frac{(j_1(z)-j_2(z))'}{j_1(z)-j_2(z)} + (-\alpha-\beta)\frac{j_2'(z)}{j_2(z)})y' + \\ \alpha\beta(\frac{j_1'(z)}{j_1(z)} - \frac{j_2'(z)}{j_2(z)}) \cdot (\frac{(j_1(z)-j_2(z))'}{j_1(z)-j_2(z)} - \frac{j_2'(z)}{j_2(z)})y &= 0 \end{aligned}$$

The solution multiplied by $f(z) = \prod(z-u_k)^{-\alpha c_k}$ satisfies $y'' + q_1(z)y' + q_2(z)y = 0$ with Riemann scheme

$$\left(\begin{array}{cccc} t_i & s_j & u_k & \infty \\ 0 & 0 & 0 & \alpha N \\ (1-\gamma)a_i & (\gamma-\alpha-\beta)b_j & (\beta-\alpha)c_k & \beta(N-M)+M\alpha \end{array}\right).$$

Again as above we can determine $q_1(z)$ and $q_2(z)$ is obtained by using Remark 1

$$\begin{aligned} q_1(z) = & \tfrac{\Lambda'}{\Lambda} + (\gamma-1)\tfrac{j_1'(z)}{j_1(z)} + (-\gamma+\alpha+\beta)\tfrac{(j_1(z)-j_2(z))'}{j_1(z)-j_2(z)} + (\alpha-\beta)\tfrac{j_2'(z)}{j_2(z)} \\ q_2(z) = & \alpha\beta(\tfrac{j_1'(z)}{j_1(z)} - \tfrac{j_2'(z)}{j_2(z)}) \cdot (\tfrac{(j_1(z)-j_2(z))'}{j_1(z)-j_2(z)} - \tfrac{j_2'(z)}{j_2(z)}) - \alpha\tfrac{j_2'(z)}{j_2(z)} \cdot (2(-\alpha)\tfrac{j_2'(z)}{j_2(z)} - (-(\alpha+1)\tfrac{j_2'(z)}{j_2(z)} + \tfrac{j_2''(z)}{j_2'(z)}) \\ & -(\tfrac{\Lambda'}{\Lambda} + (\gamma-1)\tfrac{j_1'(z)}{j_1(z)} + (-\gamma+\alpha+\beta)\tfrac{(j_1(z)-j_2(z))'}{j_1(z)-j_2(z)} + (-\alpha-\beta)\tfrac{j_2'(z)}{j_2(z)})). \end{aligned}$$

Simplifying the expression for $q_2(z)$ we get

$$q_2(z) = \alpha\beta\frac{j_1'(z)}{j_1(z)}\frac{(j_1(z)-j_2(z))'}{j_1(z)-j_2(z)} -$$

$$\alpha\frac{j_2'(z)}{j_2(z)} \cdot \left(\frac{j_2'(z)}{j_2(z)} - (\frac{\Lambda'}{\Lambda} + (\gamma-\beta-1)\frac{j_1'(z)}{j_1(z)} + (-\gamma+\alpha)\frac{(j_1(z)-j_2(z))'}{j_1(z)-j_2(z)}) - \frac{j_2''(z)}{j_2'(z)}\right).$$

□

Corollary 1. *Let*

$$1-\gamma = \frac{1}{A}, \quad -\gamma+\alpha+\beta = \frac{1}{B}, \quad \alpha-\beta = \frac{1}{C}, \quad A,B,C \in \mathbb{N}_\infty.$$

If for the ramification indices of a rational Belyi function $j(z)$ the following conditions hold

$$(*) \qquad A \mid a_i \Rightarrow a_i = A, \quad B \mid b_j \Rightarrow b_j = B, \quad C \mid c_k \Rightarrow c_k = C$$

then

$$j_2(z)^{-\alpha} \cdot {}_2F_1(\alpha, \beta, \gamma, j(z))$$

satisfies a second order differential equation $y'' + q_1(z)y' + q_2(z)y = 0$ without apparent singularities.

Corollary 2. *Let $A, B, C \in \mathbb{N}_\infty$ and $j(z)$ be a rational Belyi function satisfying the conditions $(*)$. Let also $4 = \#\{a_i \mid a_i \neq A\} + \#\{b_j \mid b_j \neq B\} + \#\{c_k \mid c_k \neq C\}$. Then we have*

a) The following function

$$j_2^{-\alpha}(z) \cdot {}_2F_1(\alpha, \beta, \gamma, j(z)), \quad 1 - \gamma = \frac{1}{A}, \quad -\gamma + \alpha + \beta = \frac{1}{B}, \quad \alpha - \beta = \frac{1}{C}$$

satisfies a Heun equation.

b) If $\{c_k \mid c_k = C\} = \emptyset$ we get a one parameter family of Heun equations corresponding to:

$$j_2^{-\alpha}(z) \cdot {}_2F_1(\alpha, \beta, \gamma, j(z)), \quad 1 - \gamma = \frac{1}{A}, \quad -\gamma + \alpha + \beta = \frac{1}{B}$$

c) If $\{c_k \mid c_k = C\} = \emptyset$ and $\{b_k \mid b_k = B\} = \emptyset$ we get a two parameter family of Heun equations corresponding to:

$$j_2^{-\alpha}(z) \cdot {}_2F_1(\alpha, \beta, \gamma, j(z)), \quad 1 - \gamma = \frac{1}{A}$$

d) If $\{c_k \mid c_k = C\} = \emptyset$, $\{b_k \mid b_k = B\} = \emptyset$ and $\{a_i \mid a_i = A\} = \emptyset$ we get a three parameter family of Heun equations corresponding to:

$$j_2^{-\alpha}(z) \cdot {}_2F_1(\alpha, \beta, \gamma, j(z)).$$

As noted by one of the anonymous referees there is an elegant way to derive Corollaries 1 and 2 by using simple local analysis and avoiding Theorem 1. However for future reference and in order to have an explicit formula for the differential equations we keep it despite the cumbersome calcuations appearing in its proof. Furthermore the computation of the Heun equation L follows from a two term local expansion of $L(f(z)$, where $f(z)$ is a known solution, e.g. at $z = 0$.

Remark 2. In order to prove Corollary 1 we at first determine the local exponents of the second order differential equation L satisfied by

$$j_2(z)^{-\alpha} \cdot {}_2F_1(\alpha, \beta, \gamma, j(z)).$$

We assume that $j(\infty) = \infty$ and concede that ∞ is going to be a singular point of L. For any finite point z_0 we can make the following observation. Suppose that $j(z_0) \notin \{0, 1, \infty\}$. Then $j'(z_0) \neq 0$ and L will have the same local exponents as the hypergeometric equation at the point $j(z_0)$, hence $0, 1$. So L does not have a singularity at z_0. Suppose $j(z_0) = 0$. Let a be the zero multiplicity of z_0. Then we have locally $j(z) \sim (z - z_0)^a$ and the local exponents of the hypergeometric equation are multiplied by a. Namely, if $f(x) = 1+O(x)$ and $x^{1-\gamma}g(x)$ with $g(x) = 1 + O(x)$ are local solutions of the hypergeometric equation, clearly $f(j(z)) = 1 + O(z - z_0)^a$ and $j(z)^{1-\gamma}g(j(z)) = (z - z_0)^{a(1-\gamma)}(1 + O(z - z_0)$ are the local solution of L around z_0. So the local exponents of L at z_0 read $0, (1 - \gamma)a$. When $(1 - \gamma)a = 1$ it is obvious that L does not have a singular point there. If at a point z_0 we have local exponents $0, 1$ and a basis of holomorphic solution the point z_0 is not singular. Of course, when $(1 - \gamma)a$ is an integer $\neq 1$ we have an apparent singularity we may not get rid of. But this is excluded by condition $(*)$ in Corollary 1. Similarly we proceed with the cases $j(z_0) = 1, \infty$. Corollary 1 is now also immediate.

The computation of the Heun equation is then straightforward. We know all local exponents, all that is needed is the value of the accessory parameter q. But this can be computed by applying the Heun operator L with the unknown parameter q to a known local solution $f(z)$ at $z = 0$, say. Then q can be solved by consideration of the first two terms of the local expansion of $Lf = 0$ in z (or some other point).

Remark 3. Herfurtner has classified all rational $j(z)$-functions such that $4 = \#\{a_i \neq 2\} + \#\{b_j \neq 3\} + \#\{c_k\}$. Thus we always obtain at least a 1 parameter family of Heun equations. Note that our a, b, c notation in Table 1 refers to the notations introduced in §2. If we are in the one parameter case in Table 1 ($a = b = c$) then the relation is $\alpha = \frac{a}{2} + \frac{1}{3},\ \gamma = \frac{2}{3},\ \beta = -\frac{1}{6} + \frac{a}{2}$.

Next we list also the 2 and 3-parameter families of Heun equation and the corresponding $j(z)$-functions satisfying the hypothesis of Corollary 2, part c,d. It just an easy consequence of the Hurwitz formula that we have computed all 2-and 3-parameter families of Heun equations in Table 1b:

Let G be a Gauss hypergeometric differential equation and $j(z)$ be a rational function such that the pullback of G with $j(z)$ of degree N gives rise to a Heun equation. Let a, b and c denote the orders of the local (projective) monodromy of G at the singularities $0, 1$ and ∞. We get the following conditions for the ramification: Over 0 we have $r + r_1$ points, where at the first r points have trivial monodromy and the last r_1 points have local monodromy of order dividing a. This can be written as

$$(ax_1, \ldots, ax_r, \alpha_1, \ldots, \alpha_{r_1}), \tag{6}$$

where $r, r_1 \in \mathbb{N}_0, x_i \in \mathbb{N}, a \nmid \alpha_i$. Note that the sum over all ramification orders is N. Similarly we get the corresponding ramification over 1 and ∞ :

$$(by_1, \ldots, by_s, \beta_1, \ldots, \beta_{s_1}), \quad (cz_1, \ldots, cz_t, \gamma_1, \ldots, \gamma_{t_1}), \tag{7}$$

where $s, t, s_1, t_1 \in \mathbb{N}_0, y_j, z_k \in \mathbb{N}, b \nmid \beta_j, c \nmid \gamma_k$. Since a Heun equation has only 4 singularities we get that $r_1 + s_1 + t_1 = 4$. The Riemann-Hurwitz formula implies that

$$-1 \geq (-N) + \frac{1}{2}((N - (r + r_1)) + (N - (s + s_1)) + (N - (t + t_1)) = \frac{1}{2}(N - (r + s + t + 4)).$$

Hence

$$N \leq r + s + t + 2. \tag{8}$$

To obtain a 2- or 3- parameter family we can assume $s = t = 0$. Thus $N \leq r + 2 \leq \frac{N}{2} + 2$ which implies $N \leq 4$. The following table follows from a classification of all triples

(g_1, g_2, g_3), $g_1 g_2 g_3 = id$, $g_i \in S_N$, where the elements g_i have the prescribed cycle decomposition and the construction of the corresponding Belyi-function.

Table 1b: 2-and 3-parameter families of geometric Heun equations, $\alpha, \beta, \gamma \in \mathbb{Q}$

$*$	q	t	θ_1	θ_2	θ_3	θ_{42}	θ_{41}
$I_3\ I_1\ II\ I_0^*$	$8\alpha(-3\gamma+2)$	-8	$1-\gamma$	$3(1-\gamma)$	$\gamma-2\alpha-\frac{1}{2}$	$-2\alpha+3\gamma-\frac{3}{2}$	4α
31	0	-1	$2(1-\gamma)$	$-4\alpha+2\gamma-1$	$2(1-\gamma)$	$2\gamma-1$	4α
32	0	-1	$-\alpha-\beta+\gamma$	$2(1-\gamma)$	$-\alpha-\beta+\gamma$	2β	2α
33	$-3\alpha\beta t_1$	$\frac{t_1^2}{3},\ t_1^2+3t_1+3=0$	$\frac{2}{3}-(\alpha+\beta)$	$\frac{2}{3}-(\alpha+\beta)$	$\frac{2}{3}-(\alpha+\beta)$	3β	3α
34	0	$-\frac{1}{3}$	$\frac{1}{2}$	$2(1-\gamma)$	$1-\gamma$	$-3\alpha+3\gamma$	$3\alpha-\frac{3}{2}$
$I_2\ I_1\ III\ I_0^*$	$6\alpha(2\beta-2\alpha-1)$	-8	$-2\alpha-2\beta+\frac{4}{3}$	$-2\alpha+2\beta$	$-\alpha-\beta+\frac{2}{3}$	$2\alpha+\beta$	3α
35	0	-1	$1-\gamma$	$2(1-\gamma)$	$1-\gamma$	$4(\gamma-\alpha)$	$4\alpha-2$

	$j(z)$	ramification data	(α, β, γ)
$I_3\ I_1\ II\ I_0^*$	$\frac{(z^4+8z^3)}{(64z-64)}$	(3)(1), (2)(2), (3)(1)	$\beta = \gamma - \alpha - \frac{1}{2}$
31	$\frac{(-z^4+2z^2-1)}{(4z^2)}$	(2)(2), (2)(2), (2)(2)	$\beta = \gamma - \alpha - \frac{1}{2}$
32	z^2	(2), (1)(1), (2)	
33	$(z+1)^3$	(3), (1)(1)(1), (3)	$\gamma = \frac{2}{3}$
34	$\frac{1}{4}z^2(z+3)$	(2)(1), (2)(1), (3)	$\beta = \gamma - \alpha - \frac{1}{2}$
$I_2\ I_1\ III\ I_0^*$	$-\frac{1}{27}\frac{(z-4)^3}{z^2}$	(3), (2)(1), (2)(1)	$\gamma = \frac{2}{3}$
35	$-4z^2(z-1)(z+1)$	(2)(1)(1), (2)(2), (4)	$\beta = \gamma - \alpha - \frac{1}{2}$

Note that all the 2- and 3-parameter Heun equations in Table 1 appear in Table 1b while the 2-parameter Heun equations in Table 1b, $I_3\ I_1\ II\ I_0^*$ and 33 extend Table 1. (We use the notation $I_3\ I_1\ II\ I_0^*$ and $I_2\ I_1\ III\ I_0^*$ to indicate that the j-function we use here is the same as in Herfurtner's list up to a Möbius-transformation. In Herfurtner's list the corresponding differential equations are Gauss hypergeometric ones.)

5 Lamé equations

The most studied Heun equations are the so called Lamé equations:

$$p(z)\frac{d^2y}{dz^2} + \frac{1}{2}p'(z)\frac{dy}{dz} - (n(n+1)z + B)y = 0,$$

where $p(z) = 4z^3 - g_2 z - g_3$. Hence we also list the cases, where the Heun equations in Table 1 specialize (after Möbius transformations) to Lamé equations. The relation between the above standard notation for a Lamé equation and our notation is given by the transformation $z \mapsto z + \frac{t+1}{3}$, that maps our singularities at $t, 0, 1$ to $t - \frac{t+1}{3}, -\frac{t+1}{3}, 1 - \frac{t+1}{3}$ and we have

$$n = \theta_4 - \frac{1}{2}, \quad B = 4q$$

Table 2: Lamé equations coming from geometry

*	conditions	q	t	θ_i, $i = 1,2,3$	θ_4	θ_{42}	θ_{41}
1	$a = 1$	$\frac{5}{3}t_1$	$\frac{t_1^2}{3},\ t_1^2 + 3t_1 + 3 = 0$	$\frac{1}{2}$	$\frac{9}{2}$	$\frac{5}{2}$	-2
5	$a + c = \frac{3}{2}, a + b = \frac{5}{4}$	0	-1	$\frac{1}{2}$	$\frac{7}{2} - 4a$	$2 - 2a$	$2a - \frac{3}{2}$
6	$a = \frac{2}{3}$	0	$\frac{t_1^2}{3}, t_1^2 + 3t_1 + 3 = 0$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0
11	$a = 1$	$\frac{37}{128}t_1$	$\frac{t_1^2}{8}, 4t_1^2 + 13t_1 + 32 = 0$	$\frac{1}{2}$	$\frac{7}{2}$	2	$-\frac{3}{2}$
12	$b = \frac{3}{4}, a = \frac{5}{12}$	$\frac{1}{32}$	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{8}$
14	$b = \frac{2}{3}, a = \frac{5}{6}$ $b = \frac{2}{3}, a = \frac{7}{12}$	$\frac{1}{8}$ $\frac{3}{64}$	-3 $\frac{3}{4}$	$\frac{1}{2}$ $\frac{1}{2}$	1 $\frac{1}{4}$	$\frac{3}{4}$ $\frac{3}{8}$	$-\frac{1}{4}$ $\frac{1}{8}$
26	$a = 1$ $a = \frac{3}{5}$	$-\frac{29}{125}t_1$ $\frac{1}{4t_1}$	$\frac{4t_1^2}{125},\ t_1^2 - 11t_1 + \frac{125}{4} = 0$ $\frac{125}{4t_1^2},\ t_1^2 - 11t_1 + \frac{125}{4} = 0$	$\frac{1}{2}$ $\frac{1}{2}$	$\frac{5}{2}$ $\frac{1}{10}$	$\frac{3}{2}$ $\frac{3}{10}$	-1 $\frac{2}{10}$
27	$b = \frac{3}{4}$	0	-1	$\frac{1}{2}$	$2a - \frac{1}{2}$	a	$-a + \frac{1}{2}$
28	$a = \frac{2}{3}$	0	$-\frac{t_1^2}{3},\ t_1^2 - 6t_1 - 3 = 0$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0
32	$a = \frac{3}{4}$	0	-1	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{7}{12}$	$-\frac{1}{12}$
33		$\frac{1}{12}(3a-1)(3a-2)t_1$	$\frac{t_1^2}{3},\ t_1^2 + 3t_1 + 3 = 0$	$\frac{1}{2}$	$3a - \frac{3}{2}$	$\frac{3}{2}a - \frac{1}{2}$	$1 - \frac{3}{2}a$
36	$a = \frac{5}{8}$	$\frac{1}{216}t_1$	$\frac{t_1^2}{27},\ t_1^2 - 10t_1 + 27 = 0$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{5}{12}$	$\frac{1}{12}$

6 Comparison with known results

The Heun equations computed by Maier, s. [14, Thm. 3.8], via polynomial pull-backs of hypergeometric differential equations appear with the exception of the equations (3.6.a) in our list:

Nr. in Maier's list	Nr. in List 1
(3.5a)	5
(3.5b)	12
(3.5c)	2
(3.6b)	36
(3.6c)	37
(3.6d)	38

The Heun equations (3.6.a),

$$y'' + \frac{2-(\alpha+\beta)}{3}\left(\frac{1}{z+\zeta_3} + \frac{1}{z} + \frac{1}{z-1}\right)y' + \frac{\alpha\beta z - \frac{\alpha\beta}{3}(1-\zeta_3)}{z(z-1)(z+\zeta_3)}y = 0, \quad \zeta_3^3 = 1$$

which depend on two free parameters α and β, appears in Table 1b), row 33. If the monodromy group is contained in $\mathrm{SL}_2(\mathbb{Z})$ (for instance for the parameters a, b, c in the Table 1 equal to $\frac{1}{2}$) then some of these differential equations appear also in literature, e.g. in the study of the Grothendieck p-curvature conjecture [4] and [5].

We list these examples, where we have used Möbius transformations to obtain coefficients in $\mathbb{Q}[z]$.

Table 1	Notation in [10]		$p(z)$
1	$I_1I_1I_1I_9$	$p(z)y'' + p'(z)y' + (z + \frac{1}{3})y = 0$	$z(z^2 + z + \frac{1}{3})$
2	$I_1I_1I_2I_8$	$p(z)y'' + p'(z)y' + zy = 0$	$z(z-1)(z+1)$
3	$I_1I_2I_3I_6$	$p(z)y'' + p'(z)y' + (z - \frac{1}{4})y = 0$	$z(z-1)(z + \frac{1}{8})$
4	$I_1I_1I_5I_5$	$p(z)y'' + p'(z)y' + (z+3)y = 0$	$z(z^2 + 11z - 1)$

Note that these examples also arise from Beauville's list, see [2].

Remark 4. At the time this paper was accepted further research was meanwhile carried out by other authors. All hypergeometric to Heun transformations with two or three continuous parameters, cf. Remark 3, up to Möbius transformations were independently studied by Filipuk and Vidunas in [9]. Also there is a *Table of all hyperbolic 4-to-3 rational Belyi maps and their dessins* available by van Hoeij and Vidunas [20], were all Belyi maps satisfying the conditions formulated in Corollaries 1 and 2 are listed together with further interesting properties. We also would like to point out that Sijsling has classified as a part of his PhD thesis all Lamé equations with arithmetic monodromy group of type $(1, e)$ that are pullbacks of hypergeometric differential equations, see [18].

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