FIBERED NEIGHBORHOODS OF CURVES IN SURFACES

C. CAMACHO, H. MOVASATI & P. SAD

This paper is devoted to the study of the simplest possible type of foliations on surfaces. We consider a compact, smooth, holomorphic curve inside a (holomorphic) surface and a holomorphic foliation by discs transverse to it; the general problem is to classify such objects. These foliations arise naturally when suspensions of groups of diffeomorphisms are constructed. In our specific case, we wish to know whether the foliation (or the fibration over the curve) is equivalent to the foliation by lines on the normal bundle to the curve. This is true when the curve has a sufficiently negative self-intersection number, but we are able to describe the obstructions that appear in the other cases.

Our study is related to the systematic study of neighborhoods of analytic varieties started by H. Grauert in his celebrated article ([5]). In this article negatively embedded submanifolds of codimension 1 are considered. There is a geometric aspect, where formal equivalences of neighborhoods are proven to be in fact holomorphic equivalences; the main point is the vanishing of some special cohomology groups due to the existence of holomorphically convex neighborhoods of the submanifold. As for the formal side, the notion of n-neighborhood , for $n \in \mathbb{N}$ is introduced and the obstruction to extend isomorphisms between n-neighborhoods to $(n+1)$-neighborhoods is described; it lies also in certain cohomology groups which depends on the normal bundle to the submanifold. The Kodaira Vanishing Theorem implies that for $n \in \mathbb{N}$ sufficiently large this group vanishes, so that the germ of a complex manifold along a negatively embedded submanifold depends only on a finite neighborhood. In ([7]) the special case of curves is considered; the use of Serre's duality allows simpler proofs. A careful analysis leads to the following linearisation result : a negatively embedded curve with self-intersection number smaller than $4-4g$ ($g \in \mathbb{N}$ is the genus of the curve) has always a neighborhood equivalent to a neighborhood of the zero section of the normal bundle.

Our main result states that, when the self-intersection number of the curve is negative and smaller than $2-2g$, the fibration is equivalent

Date: May 2001 .

to the linear one that exists in the normal bundle. We use the same cohomological property of a holomorphically convex neighborhood as in [5] to solve a Cousin problem. We remark that the linearisation result for neighborhoods, mentioned before, does not imply that a fibration is equivalent to a linear one, even if the self-intersection number is smaller than $4-4g$. As for the obstructions to linearisation, the existence of the fibration allows us to read them in the first cohomology group of the sheaf of 1-forms along the curve that take values in the normal bundle. We know then exactly when it is possible to linearise a fibered embedding; it remains to be developed the classification of all fibered embeddings with a given normal bundle. In the negative case, it is likely that $(2g-1)$-neighborhoods determine the embedding.

The paper is organized as follows. In Section 1 we state the results and give some examples. Sections 2 and 3 are devoted to the formal analysis of neighborhoods, and Sections 4 and 5 to the convergent aspects. Section 6 presents a simple technique to generate examples of fibered embeddings which are not linearisable. Finally, there is an Appendix with an application to holomorphic foliations.

1. Statement of Results and Examples

Let S be a compact, smooth, holomorphic curve embedded in some complex surface M in such a way that it has a neighborhood holomorphically fibered by a family $\mathcal{G}$ of transversal 1-dimensional discs (we will say that S is *fibered embedded* in M). This means that there exist a covering $\mathcal{U}$ of S by open sets $\{U_\alpha\} \subset M$ and charts

$$\psi_\alpha = (x_\alpha, y_\alpha) : U_\alpha \longrightarrow \mathbb{C}^2$$

such that

(1) $S \cap U_\alpha = y_\alpha^{-1}(0)$.
(2) whenever $U_\alpha \cap U_\beta \neq \emptyset$, the change of coordinates $F_{\alpha\beta} = \psi_\alpha \circ {\psi_\beta}^{-1}$ is given by

$$\begin{array}{rcl} x_\alpha &=& \phi_{\alpha\beta}(x_\beta) \\ y_\alpha &=& \eta_{\alpha\beta}(x_\beta, y_\beta), \end{array} \qquad (1)$$

where $\phi_{\alpha\beta}$ and $\eta_{\alpha\beta}$ are holomorphic functions. We remark that $F_{\beta\gamma} \circ F_{\gamma\alpha} \circ F_{\alpha\beta} = Id$ when $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$ and that $F_{\alpha\alpha} = Id$ for any α.
(3) in each set U_α the fibration $\mathcal{G}$ over S is given by $dx_\alpha = 0$.

The normal bundle N to S, defined by the transition maps $(x_\alpha, y_\alpha) = L_{\alpha\beta}(x_\beta, y_\beta)$ given by

$$\begin{array}{rcl} x_\alpha & = & \phi(x_\beta) \\ y_\alpha & = & A_{\alpha\beta}(x_\beta) y_\beta \end{array} \tag{2}$$

for $A_{\alpha\beta}(x_\beta) := \partial\eta_{\alpha\beta}/\partial y_\beta(x_\beta, 0)$, is in a natural way fibered by the lines $dx_\alpha = 0$. Let us call G_p the fiber of $\mathcal{G}$ and N_p the linear fiber in N both passing through $p \in S$.

The problem we study in this paper is the existence of a fibered equivalence between the embeddings of S in M and N, that is, *whether there exists a holomorphic diffeomorphism H, defined between neighborhoods of S in M and N, that sends S to S and G_p into N_p for all $p \in S$* (we say that the embedding is *linearisable*).

The following examples illustrate the situation.

Example 1. Take S as a rational curve with self-intersection number $S \cdot S = -1$ inside M. We may blow down a neighborhood of S and get near $(0,0) \in \mathbb{C}^2$ a foliation with a singularity of radial type (obtained from blowing down the fibration $\mathcal{G}$). A theorem of Poincaré (see [1]) allows us do define a diffeomorphism that sends the foliation to the one given by $ydx - xdy = 0$; such a diffeomorphism can be lifted to produce the equivalence with the linear fibration of N.

Example 2 ([1]). The curve S will be in this example a complex torus embedded with self-intersection number 0. Let

$$(z, w) \longrightarrow (\alpha z, \eta(z, w)) = (\alpha z, \lambda w + \sum_{k \geq 2} A_k(z) w^k)$$

define an embedding of the open set $\{1 - \epsilon < |z| < 1 + \epsilon\} \times \mathbb{D}$; here $0 < |\alpha| < 1, \lambda \in \mathbb{C}^*$ and $0 < \epsilon << |\alpha|$. The coordinates $(z, w), (z', w')$ for the surface M are settled as follows:

(1) $(1 + 2\epsilon)|\alpha| < |z| < 1 + \epsilon$, $|w| < 1$.
(2) $(1 - \epsilon)|\alpha| < |z'| < 1 - 2\epsilon$, $w' \in \mathbb{C}$.
(3) the change of coordinates is given by $z' = \alpha z$, $w' = \eta(z, w)$ for $1 - \epsilon < |z| < 1 + \epsilon$ and $z' = z$, $w' = w$ for $|\alpha|\,(1 + 2\epsilon) < |z| < 1 - 2\epsilon$.

As for the normal bundle, we have the coordinates (z, w) and (z', w'); z and z' are related as before, and $w' = \lambda w$.

Now we look for a diffeomorphism $H(z, w) = (z, h(z, w))$, defined for $(1 - \epsilon)|\alpha| < |z| < 1 + \epsilon$ and $|w|$ small, that satisfies $\partial h/\partial w(z, 0) = 1$

and $H(z', w') = (\alpha z, \lambda h(z, w))$, or $h(\alpha z, \eta(z, w)) = \lambda h(z, w)$. There are formal obstructions to finding such an h. Let us write $h(z, w) = w + \sum_{k\geq 2} h_k(z) w^k$ and try to find h_2; it has to satisfy the equation

$$A_2(z) + \lambda^2 h_2(\alpha z) = \lambda h_2(z).$$

Expanding as power series both the functions $h_2(z) = \sum_{-\infty}^{+\infty} l_n z^n$ in $(1-\epsilon)|\alpha| < |z| < 1 + \epsilon$ and $A_2(z) = \sum_{-\infty}^{+\infty} a_n z^n$ in a neighborhood of $\mathbb{S}^1$, we get

$$\lambda l_n (1 - \lambda \alpha^n) = a_n.$$

We demand $|\lambda \alpha^n - 1| \neq 0$ for all $n \in \mathbb{Z}$ (since $|\alpha| < 1$, this is a single condition; it is easy to see that once is satisfied, the series defining h_2 converges). Going further in the process of computing h_k for $k \geq 3$, we see that to get at least a formal map h we need $|\lambda^m \alpha^n - 1| \neq 0$ for all $n \in \mathbb{Z}$ and $m \in \mathbb{N}^*$.

In [1], it is also presented a small denominator condition in order to ensure the convergence of h; in the absence of this condition, we may have divergence (see [2]).

Example 3. There are embedded curves without fibration by discs in a neighborhood. For example, if the curve is plane of degree greater or equal to 2, any fibration would be the restriction of a globally defined foliation in $\mathbb{P}(2)$, by Levi's extension theorem; but such a foliation has necessarily tangency points with the curve.

Our main result in this paper is the following

Theorem 1. *Let S be a compact Riemann surface of genus $g \in \mathbb{N}$ fibered embedded in the surface M. Assume that $S \cdot S < 0$. Then, if $S \cdot S < 2 - 2g$, the embedding is fibered equivalent to the embedding of S in the normal bundle N.*

This statement can be seen as a generalization of Example 1. In order to put Example 2 in a general perspective, we have to introduce some notations and definitions. Let $\mathcal{O}^{(1,0)}(N^k)$ be the sheaf of holomorphic 1-forms of S with coefficients in the fiber bundle N^k, for $k \in \mathbb{N}$ (when $k = 0$, we use $\mathcal{O}^{(1,0)}$ for simplicity). We refer to the notation in the beginning of the section.

Definition 1. *Let $m \in \mathbb{N}^*$. A fibered embedding of S into M is m-linearisable when there are adapted charts with changes of coordinates $F_{\alpha\beta} = (\phi_{\alpha\beta}, \eta_{\alpha\beta})$ satisfying $\partial^j \eta_{\alpha\beta} / \partial y_\beta^j (x_\beta, 0) = 0$ for all $2 \leq j \leq m$ (whenever $U_\alpha \cap U_\beta \neq \emptyset$). The embedding is formally linearisable if it is m-linearisable for all $m \in \mathbb{N}^*$.*

We want to state a condition to guarantee that a fibered embedding is $(m+1)$-linearisable once it is already m-linearisable. So let us assume

$$\begin{array}{rcl} x_\alpha & = & \phi_{\alpha\beta}(x_\beta) \\ y_\alpha & = & y_\beta\{A_{\alpha\beta}(x_\beta) + C_{\alpha\beta}(x_\beta)y_\beta^m + \cdots\} \end{array} \tag{3}$$

We associate to $\omega \in H^0(S, \mathcal{O}^{(1,0)}(N^m))$, represented as the collection $\{\omega_\alpha\}$ in the covering $\mathcal{U}$, the 1-cochain

$$\theta_{\alpha\beta} := C_{\alpha\beta}A_{\alpha\beta}^{-1}\omega_\beta \tag{4}$$

The cocycle relations satisfied by the change of coordinates $\{F_{\alpha\beta}\}$ imply immediately that $\{C_{\alpha\beta}A_{\alpha\beta}^{-1}\} \in H^1(\mathcal{U}, N^{-m})$. It follows that

Lemma 1. $\{\theta_{\alpha\beta}\} \in H^1(\mathcal{U}, \mathcal{O}^{(1,0)})$.

As a consequence of Serre duality, as we will explain in Section 2, any element of $H^1(\mathcal{U}, \mathcal{O}^{(1,0)})$ is a 1-coboundary once we allow poles in the open sets U_α (in fact, the poles can be selected independently of the holomorphic 1-cocycle and outside all possible intersections of type $U_\alpha \cap U_\beta$). We have then a collection of meromorphic 1- forms $\{\theta_\alpha\}$ associated to ω such that $\theta_\alpha - \theta_\beta = \theta_{\alpha\beta}$ whenever $U_\alpha \cap U_\beta \neq \emptyset$.

Definition 2. *Let* $\omega \in H^0(S, \mathcal{O}^{(1,0)}(N^m))$. *Then* $r(\omega) = \sum_\alpha \mathcal{R}es(\theta_\alpha)$.

It can be proven that these numbers do not depend neither on the choice of the adapted charts nor on the choice of the meromorphic 1-forms involved. Also, the 1-cocycle $\{\theta_{\alpha\beta}\}$ defined in (4) represents an element of $H^1(S, \mathcal{O}^{(1,0)})$ in the covering $\mathcal{U}$.

Theorem 2. *Consider a fibered embedding of* S *into* M *which is* m-*linearisable. If* $r(\omega) = 0$ *for all* $\omega \in H^0(S, \mathcal{O}^{(1,0)}(N^m))$, *then the embedding is also* $(m+1)$-*linearisable.*

All vector spaces $H^0(S, \mathcal{O}^{(1,0)}(N^k))$ are finite dimensional, so that we have to check a finite number of conditions in order to apply the above theorem.

Since $H^0(S, \mathcal{O}^{(1,0)}(N^k)) = \{0\}$ if the Chern class of $\mathcal{O}^{(1,0)}(N^k)$ is negative, or equivalently $(S \cdot S)k < 2 - 2g$, it follows that when $S \cdot S < 0$ only a finite number of conditions have to be verified in order to ensure that the fibered embedding is formally linearisable. Furthermore, we are able to state: *if* $S \cdot S < 0$ *and the fibered embedding is formally linearisable, then it is also holomorphically linearisable.* is a consequence of this principle.

Let us go back to Example 2 and use the tools involved in the formulation of Theorem 2.

Example 2 bis. Let us take $\lambda = 1$ for simplicity; the normal bundle to the elliptic curve is therefore trivial. We wish to see whether the embedding is 2-linearisable. The vector space $H^0(S, \mathcal{O}^{(1,0)}(N))$ is simply $H^0(S, \mathcal{O}^{(1,0)})$, which is 1-dimensional; we choose the 1-form $z^{-1}dz$ as its generator. Take the covering $\mathcal{U}$ corresponding to the open sets $\{(1-\epsilon)|\alpha| < |z'| < 1-2\epsilon\}$ and $\{(1+2\epsilon)|\alpha| < |z| < 1+\epsilon\}$ and the holomorphic 1-cocycle θ_{12} defined as $\theta_{12} = A_2(z)z^{-1}dz$ in $V' = \{1-\epsilon < |z| < 1+\epsilon\}$ and $\theta_{12} = 0$ in $V = \{(1+2\epsilon)|\alpha| < |z| < 1-2\epsilon\}$. Now we look for meromorphic 1-forms θ_1 and θ_2 such that $\theta_1 - \theta_2 = \theta_{12}$.

Due to vanishing of θ_{12} in V, we see that $\theta_1 = \theta_2$ in that set. We conclude that there exists a global meromorphic 1- form $\theta = f(z)dz$ defined in a neighborhood of the annulus which satisfies the relation

$$f(\alpha z)\alpha dz - f(z)dz = A_2(z)z^{-1}dz$$

for $z \in V'$. Since

$$\mathcal{R}es(\theta) = \int_{|z|=1} \theta - \int_{|z|=|\alpha|} \theta$$

and taking into account the above relation, we arrive at

$$\mathcal{R}es(\theta) = -\int_{|z|=1} A_2(z)z^{-1}dz = -a_0.$$

Finally we observe that $S \cdot S = 0$, since N is trivial. Once again we arrive at the conclusion that $a_0 = 0$ is the condition for the embedding to be 2-linearisable.

We observe that $H^0(S, \mathcal{O}^{(1,0)}(N)) = \{0\}$ when N is not trivial.

2. The Residues

In this section we make two remarks about Definition 2 (notation as in Section 1).

First of all, we used that the 1-cocycle $\{\theta_{\alpha\beta}\}$ is a coboundary if we use meromorphic 1-forms. To see this, consider $\{\theta_{\alpha\beta}\} \in H^1(\mathcal{U}, \mathcal{O}^{(1,0)})$; clearly we have also $\{\theta_{\alpha\beta}\} \in H^1(\mathcal{U}, \mathcal{O}^{(1,0)}(D))$ for any effective divisor D. But $H^1(\mathcal{U}, \mathcal{O}^{(1,0)}(D))$ and $H^0(\mathcal{U}, \mathcal{O}(-D))$ have the same dimension, by Serre Duality ([6],pg. 76). It follows then that $H^1(\mathcal{U}, \mathcal{O}^{(1,0)}(D)) = \{0\}$, and $\{\theta_{\alpha\beta}\}$ is a coboundary for the cohomology with coefficients in $\mathcal{O}^{(1,0)}(D)$.

The second remark is a proof that the residues $r(\omega)$ of Definition 2 do not depend on the way we write the cocycle $\{\theta_{\alpha\beta}\}$ as a coboundary (see Remark 1). Let us take adapted coordinates $\{(x_\alpha, y_\alpha)\}$ and $\{(x_\alpha, z_\alpha)\}$ for a m-linearisable embedding, as in (3); we assume

$$\begin{aligned} y_\alpha &= y_\beta + C_{\alpha\beta}(x_\beta) y_\beta^{m+1} + \cdots \\ z_\alpha &= z_\beta + C'_{\alpha\beta}(x_\beta) z_\beta^{m+1} + \cdots \\ z_\alpha &= y_\alpha + (\cdots) + C_\alpha(x_\alpha) y_\alpha^{m+1} + \cdots \end{aligned}$$

It is easy to see that

$$A_{\alpha\beta}^{-1} C'_{\alpha\beta} = A_{\alpha\beta}^{-1} C_{\alpha\beta} + A_{\alpha\beta}^{m} C_\alpha - C_\beta. \tag{5}$$

Given $\{\omega_\beta\} \in H^0(\mathcal{U}, \mathcal{O}^{(0,1)}(N^m))$, we choose appropriate meromorphic 1-forms as to have

$$\begin{aligned} \theta_\alpha - \theta_\beta &= A_{\alpha\beta}^{-1} C_{\alpha\beta} \omega_\beta \\ \theta'_\alpha - \theta'_\beta &= A_{\alpha\beta}^{-1} C'_{\alpha\beta} \omega_\beta \end{aligned} \tag{6}$$

Combining (5) and (6) gives

$$\theta'_\alpha - \theta'_\beta = \theta_\alpha - \theta_\beta \ + \ C_\alpha \omega_\alpha - C_\beta \omega_\beta$$

Therefore

$$\theta'_\alpha - \theta_\alpha - C_\alpha \omega_\alpha = \theta'_\beta - \theta_\beta - C_\beta \omega_\beta$$

and the meromorphic 1-form ζ given as $\zeta_\alpha = \theta'_\alpha - \theta_\alpha - C_\alpha \omega_\alpha$ in each $\mathcal{U}_\alpha$ is globally defined. We apply the Residue Theorem to conclude that $\sum_\alpha \mathcal{R}es(\theta_\alpha) = \sum_\alpha \mathcal{R}es(\theta'_\alpha)$.

3. Proof of Theorem 2

Let us consider adapted coordinates as in (3) for the m-linearisable embedding of S into M, and replace the coordinates (x_α, y_α) by new coordinates (x_α, z_α):

$$z_\alpha = y_\alpha - c_\alpha(x_\alpha) y_\alpha^{m+1} \tag{7}$$

where c_α is a holomorphic function for each α. The new changes of coordinates are given by the maps

$$F'_{\alpha\beta}(x_\beta, z_\beta) = (x_\beta, \eta'(x_\beta, z_\beta))$$

where

$$\eta'(x_\beta, z_\beta) = A_{\alpha\beta}(x_\beta) z_\beta + C'_{\alpha\beta}(x_\beta) z_\beta^{m+1} + \cdots$$

and

$$A_{\alpha\beta}^{-1} C'_{\alpha\beta} = A_{\alpha\beta}^{-1} C_{\alpha\beta} + c_\beta - A_{\alpha\beta}^{m} c_\alpha$$

In order to prove that M is $(m+1)$-linearisable, it is enough then to verify that $\{A_{\alpha\beta}^{-1}C_{\alpha\beta}\} \in H^1(\mathcal{U}, N^{-m})$ is a co-boundary. Let us then start by writing

$$A_{\alpha\beta}^{-1}C_{\alpha\beta} = A_{\alpha\beta}^m t_\alpha - t_\beta$$

where $t_\alpha \in C^\infty(U_\alpha)$. The collection of 1-forms $\nu_\alpha = \bar\partial t_\alpha$ clearly defines an element $\nu \in H^0(S, \mathcal{E}^{(0,1)}(N^{-m}))$, a $C^\infty(0,1)$-form with coefficients in N^{-m}. By Serre duality (see [6], pg.76), if we are able to prove that

$$\int_S \nu \wedge \omega = 0$$

for all $\omega \in H^0(S, \mathcal{O}^{(1,0)}(N^m))$, then $\nu = \bar\partial h$ for $h \in H^0(S, \mathcal{E}^{(0,0)}(N^{-m}))$. Thus, each $t_\alpha - h_\alpha$ is a holomorphic function and

$$A_{\alpha\beta}^{-1}C_{\alpha\beta} = A_{\alpha\beta}^m(t_\alpha - h_\alpha) - (t_\beta - h_\beta)$$

Let us then fix $\omega = \{\omega_\alpha\} \in H^0(S, \mathcal{O}^{(1,0)}(N^m))$. From the existence of meromorphic 1-forms $\{\theta_\alpha\}$ such that

$$\theta_\alpha - \theta_\beta = A_{\alpha\beta}^{-1}C_{\alpha\beta}\omega_\beta$$

we get

$$\theta_\alpha - \theta_\beta = (A_{\alpha\beta}^m t_\alpha - t_\beta)\omega_\beta = t_\alpha\omega_\alpha - t_\beta\omega_\beta$$

Finally, an application of Stokes' theorem to the 1-form δ defined in each U_α as $\theta_\alpha - t_\alpha\omega_\alpha$ gives

$$r(\omega) = \sum_\alpha \mathcal{R}es(\theta_\alpha) = (2i\pi)^{-1}\int_S \nu \wedge \omega.$$

But $r(\omega) = 0$ by hypothesis. The proof is complete.

4. An Adapted Cousin Problem

Let us go back to the setting introduced in Section 1. *We assume hereafter that $S \cdot S < 0$.*

We define an (almost) global system of coordinates for the normal bundle N to S as follows. The first coordinate is the projection over S. As for the second one, consider the coordinates given by (2) and a meromorphic section $s = \{s_\alpha(x_\alpha)\}$; clearly $l_N = \{l_\alpha(x_\alpha) = y_\alpha s_\alpha^{-1}(x_\alpha)\}$ is a well defined meromorphic function of N. We may consider this function as giving the second coordinate for the points of N, with the exception of the fibers which pass through the zeroes and poles of S. If the divisor associated to s is $\sum_{i=1}^k n_i p_i$ (where $n_i \in \mathbb{Z}$ and $p_i \in S$, $i = 1, \ldots, k$), then l_N has divisor $S - \sum_{i=1}^k n_i N_{p_i}$. We want a similar structure for M (in fact, in some neighborhood of S in M; this will be our understanding from now on).

Theorem 3. *Assume the embedding of S into M is m-linearisable for some $m > 2g-2$. Then the divisor $S - \sum_{i=1}^{k} n_i G_{p_i}$ is a principal divisor of M.*

The proof of this theorem relies heavily on some results of Complex Analysis, which we now recall. First of all, the curve S has a fundamental system of strong Levi-pseudoconvex neighborhoods (see [7], Theorem 4.9). Let us consider:

(1) a neighborhood $U \subset M$ of S which is a strong Levi-pseudoconvex domain; $\mathcal{O}_U$ is its structural sheaf.
(2) the subsheaf $\mathcal{M} \subset \mathcal{O}_U$ of germs of holomorphic functions on U which vanish along S.

We may state the crucial property ([5], pg.357):

Theorem 4. *$H^1(U, \mathcal{M}^q) = \{0\}$ for all $q > max\{0, 2g-2\}$.*

We may now give the proof of Theorem 3. Let us write the divisors $S - \sum_{i=1}^{k} n_i G_{p_i}$ and $S - \sum_{i=1}^{k} n_i N_{p_i}$ in the coordinate systems given by (3) and (2) as $y_i x_i^{-n_i} = 0, 1 \leq i \leq k$. We define the multiplicative cocycle $\{y_i x_i^{-n_i}/(y_j x_j^{-n_j}) \circ F_{ij}^{-1}\}$ associated to the first divisor; our aim is to prove it is multiplicatively trivial. We have then:

$$\frac{y_i x_i^{-n_i}}{(y_j x_j^{-n_j}) \circ F_{ij}^{-1}} = \frac{y_i x_i^{-n_i}}{(y_j x_j^{-n_j}) \circ L_{ij}^{-1}} \cdot \frac{(y_j x_j^{-n_j}) \circ L_{ij}^{-1}}{(y_j x_j^{-n_j}) \circ F_{ij}^{-1}}$$

Since $S - \sum_{i=1}^{k} n_i N_{p_i}$ is principal (in the surface N), the multiplicative cocycle

$$\{y_i x_i^{-n_i}/(y_j x_j^{-n_j}) \circ L_{ij}^{-1}\}$$

is trivial: there exist nonvanishing functions $\{t_i\}$ such that

$$y_i x_i^{-n_i}/(y_j x_j^{-n_j}) \circ L_{ij}^{-1} = t_i/(t_j \circ L_{ij}^{-1}).$$

It follows that

$$\frac{y_i x_i^{-n_i}}{(y_j x_j^{-n_j}) \circ F_{ij}^{-1}} = \frac{t_i}{t_j \circ F_{ij}^{-1}} \cdot \frac{t_j \circ F_{ij}^{-1}}{t_j \circ L_{ij}^{-1}} \cdot \frac{(y_j x_j^{-n_j}) \circ L_{ij}^{-1}}{(y_j x_j^{-n_j}) \circ F_{ij}^{-1}}$$

As for the cocycle $\{(t_j \circ F_{ij}^{-1}/t_j \circ L_{ij}^{-1}) \cdot ((y_j x_j^{-n_j}) \circ L_{ij}^{-1}/(y_j x_j^{-n_j}) \circ F_{ij}^{-1})\}$, let us define u_{ij} in order to have

$$1 + u_{ij} = \frac{t_j \circ F_{ij}^{-1}}{t_j \circ L_{ij}^{-1}} \cdot \frac{(y_j x_j^{-n_j}) \circ L_{ij}^{-1}}{(y_j x_j^{-n_j}) \circ F_{ij}^{-1}}$$

and $v_{ij} := log(1+u_{ij})$ $(log 1 = 0)$; clearly $\{v_{ij}\}$ belongs (via the system of coordinates of M) to $H^1(\mathcal{U}, \mathcal{M}^m)$. Theorem 4 implies that this additive cocycle is trivial, and consequently $1+u_{ij} = e^{v_{ij}} = c_i/(c_j) \circ F_{ij}^{-1}$ for nonvanishing functions $\{c_i\}$. Finally we may write

$$\frac{y_i x_i^{-n_i}}{(y_j x_j^{-n_j}) \circ F_{ij}^{-1}} = \frac{t_i c_i}{(t_j c_j) \circ F_{ij}^{-1}},$$

so that $\{y_i x_i^{-n_i}/(y_j x_j^{-n_j}) \circ F_{ij}^{-1}\}$ is trivial.

5. Consequences

Let us present now some consequences of Theorem 3. We continue with the hypothesis $S \cdot S < 0$ and keep the notation of the last section.

Theorem 5. *Assume the embedding of S into M is m-linearisable for some $m > 2g-2$. Then the embedding is linearisable.*

Proof: Consider a neighborhood $U \subset M$ of S; a holomorphic diffeomorphism from U to a neighborhood of S in the normal bundle N will be now defined. Let $g : U \longrightarrow S$ and $g_N : M \longrightarrow S$ be the projections associated to the fibrations in U and N; the meromorphic functions introduced in the last section are denoted by l and l_N. Take $\Psi(p) \in N$ as the point such that $(g_N(\Psi(p)), l_N(\Psi(p)) = (g(p), l(p))$; in fact, this map is defined outside $\cup_{i=1}^k G_{p_i}$, the support of the divisor of the function l. In order to prove that it extends to a fiber G_{p_i}, we take local coordinates ψ and ψ_N around p_i such that $l \circ \psi = l_N \circ \psi_N = yx^{-n_i}$ and $x(\psi(q)) = x(\psi_N(\Psi(q))$ (this last condition comes from the fact that Ψ preserves the fibrations). It follows immediately that $y(\psi(q)) = y(\psi_N(\Psi(q))$. □

A short way of expressing this proof is that we have constructed a holomorphic diffeomorphism Ψ that carries the 1-form ldg into $l_N dg_N$: $\psi^*(l_N dg_N) = ldg$. In fact, the proof consists in finding local expressions for the map Ψ, which turn out to be unique; this property allows us to glue all the local diffeomorphisms.

Theorem 1 is a corollary to Theorem 5; it is enough to remark that if $S \cdot S < 2-2g$ then $H^0(S, \mathcal{O}^{(1,0)}(N^m)) = \{0\}$ for all $m \in \mathbb{N}^*$; consequently the embedding is m-linearisable for any $m \in \mathbb{N}^*$, and we just apply Theorem 2.

Finally we remark that if the embedding is formally linearisable, it is m-linearisable for any $m \in \mathbb{N}^*$(by Definition 1), and again we may apply Theorem 5 to get that the embedding is actually linearisable.

Remark 3. There is a nice fact about Theorem 4 when $S \cdot S < 2-2g$: $H^1(U, \mathcal{M}^{q_0}) = 0$ *for some* $q_0 > max\{0, 2g-2\}$ *implies* $H^1(U, \mathcal{M}^q) = 0$ *for all* $q > max\{0, 2g-2\}$.

The proof is standard. Let us consider the short exact sequence

$$0 \longrightarrow \mathcal{M}^{q+1} \longrightarrow \mathcal{M}^q \longrightarrow \mathcal{M}^q/\mathcal{M}^{q+1} \longrightarrow 0$$

It is easy to check that $\mathcal{M}^q/\mathcal{M}^{q+1}$ is isomorphic to $\mathcal{O}(N^{-q})$. By Serre's duality, we have that $H^1(S, \mathcal{O}(N^{-q})) \cong H^0(S, \mathcal{O}^{(1,0)} \otimes N^q)$. But $H^0(S, \mathcal{O}^{(1,0)} \otimes N^q) = 0$ if $S \cdot S < 2-2g$, so that $H^1(S, \mathcal{O}(N^{-q})) = 0$ as well. The long exact sequence associated to the above sequence gives

$$\cdots \longrightarrow H^1(U, \mathcal{M}^{q+1}) \longrightarrow H^1(S, \mathcal{M}^q) \longrightarrow 0$$

It follows that there is a surjective map $H^1(U, \mathcal{M}^{q_0}) \longrightarrow H^1(S, \mathcal{M}^q)$ for any $2-2g < q < q_0$; consequently, $H^1(S, \mathcal{M}^q) = 0$ in the case $q > max\{0, 2g-2\}$.

6. Examples

In this Section we construct examples of fibered embeddings that are not linearisable. We use the notations of Section 3.

Consider a fibered embedding of a curve S which is m-linearisable; it is easy to prove the converse to Theorem 2: if $r(\omega) \neq 0$ for some $\omega \in H^0(S, \mathcal{O}^{(1,0)}(N^m))$, then the embedding is not $(m+1)$-linearisable.

We start by selecting a non-trivial linear bundle N over a curve S such that $H^0(S, \mathcal{O}^{(1,0)}(N)) \neq \{0\}$, fix a nonzero $\omega \in H^0(S, \mathcal{O}^{(1,0)}(N))$ and take some meromorphic section s of N. The goal is to construct a fibered embedding of S into some surface which is not 2-linearisable and which has associated normal bundle N ($m = 1$ was chosen just for simplicity).

Step 0. Let $\mathcal{E}$ denote the sheaf over S of germs of biholomorphisms of the form $(x, y) \longrightarrow (x, g(x, y))$ where x belongs to an open subset of S, $y \in \mathbb{C}$ and $g(x, 0) = 0$. The elements of $H^1(S, \mathcal{E})$ correspond to the fibered embeddings of S into surfaces. In order to see this, an element of $H^1(S, \mathcal{E})$ is represented (in some covering of S by discs $\{V_\alpha\}$) as a collection $\{G_{\alpha\beta}\}$ of biholomorphisms of the form $G_{\alpha\beta}(x, y_\beta) = (x, g_{\alpha\beta}(x, y_\beta)), x \in V_\alpha \cap V_\beta$ which satisfy

$$G_{\alpha\alpha} = Id, G_{\gamma\beta} = G_{\gamma\alpha} \circ G_{\alpha\beta}$$

whenever $V_\alpha \cap V_\beta \cap V_\gamma \neq \emptyset$.

Once we get these maps, we may define a surface M as the quotient of $\cup_\alpha V_\alpha \times \mathbb{D}$ under the equivalence relation $(x_\alpha, y_\alpha) \sim (x_\beta, y_\beta)$ iff $(x_\alpha, y_\alpha) \in V_\alpha \times \mathbb{D}$, $(x_\beta, y_\beta) \in V_\beta \times \mathbb{D}$ and $(x_\alpha, y_\alpha) = (x_\beta, g_{\alpha\beta}(x_\beta, y_\beta))$; the curve S is fibered embedded since $G_{\alpha\beta}(x_\beta, 0) = 0$. The associated normal bundle is given by the cocycle $\{L_{\alpha\beta} = \partial g_{\alpha\beta}/\partial y_\beta(x_\beta, 0)\}$. The surface M has automatically a system of coordinate charts such that the changes of coordinates are given by the maps in $G_{\alpha\beta}$.

Given $\{G_{\alpha\beta}\} \in H^1(S, \mathcal{E})$, we may construct another element of $H^1(S, \mathcal{E})$ in the following way: 1) we take a section $s = \{s_\alpha\}$ of N, which can be supposed without zeroes or poles in the intersections $V_\alpha \cap V_\beta$; 2) we consider the collection of local biholomorphisms $\{\hat{G}_{\alpha\beta}\}$ given by

$$(x_\alpha, y_\alpha) = \hat{G}_{\alpha\beta}(x_\beta, y_\beta) = (x_\beta, \hat{g}_{\alpha\beta}(x_\beta, y_\beta)) = (x_\beta, s_\alpha^{-1} g_{\alpha\beta}(x_\beta, s_\beta y_\beta)).$$

It is easy to see that $\{\hat{G}_{\alpha\beta}\} \in H^1(S, \mathcal{E})$; but the associated normal bundle L to $\{G_{\alpha\beta}\}$ changes to $L \otimes N^*$. In particular, applying this construction to elements of $\mathcal{E}$ which have fixed associated normal bundle N produces embeddings with trivial normal bundles.

We have therefore a way of replacing an embedding of S into a surface M with associated normal bundle N by a different embedding of S into a surface $\hat{M}$ with associated trivial normal bundle, and vice-versa.

Step 1. Suppose now that we have a holomorphic surface $\hat{M}$ (where S is fibered embedded with trivial normal bundle) which is not 2-linearisable. There exists in S a 1-form $\sigma \in \mathcal{O}^{(1,0)}$ such that $\mathcal{R}es(\sigma) \neq 0$; we choose the section s as the quotient ω/σ. The covering $\{V_\alpha\}$ for $\hat{M}$ can be selected in order that no zeroes or poles of s appear in the intersections $V_\alpha \cap V_\beta$. Let the change of coordinates between coordinate charts of $\hat{M}$ be

$$\hat{F}_{\alpha\beta}(x, y_\beta) = (x, y_\beta + \sum_{m=2}^{\infty} \zeta_{\alpha\beta}^{(m)}(x) y_\beta^m).$$

We write as $\{A_{\alpha\beta}\}$ the transition functions for N. We put $\sigma = \{\sigma_\alpha\}$ and $\omega = \{\omega_\alpha\}$ and define M as in Step 0 to have change of coordinates

$$F_{\alpha\beta}(x, y_\beta) = A_{\alpha\beta}(x) y_\beta + \sum_{m=2}^{\infty} \eta_{\alpha\beta}^{(m)}(x) y_\beta^m$$

where

$$A_{\alpha\beta}^{-1} s_\beta^{m-1} \eta_{\alpha\beta}^{(m)} = \zeta_{\alpha\beta}^{(m)}.$$

In particular, $\mathcal{R}es(\omega) = \mathcal{R}es(\{A_{\alpha\beta}^{-1}\eta_{\alpha\beta}^{(2)}\omega_\beta\}) = \mathcal{R}es(\{A_{\alpha\beta}^{-1}\eta_{\alpha\beta}^{(2)} s_\beta \sigma_\beta\}) = \mathcal{R}es(\{\zeta_{\alpha\beta}^{(2)}\sigma_\beta\}) = \mathcal{R}es(\sigma) \neq 0$, and M is not 2-linearisable.

Step 2. At this point we have just to exhibit examples of fibered embeddings of S with associated trivial normal bundle which are not 2-linearisable. This is straightforward: we take a suspension of a special representation of the fundamental group of S into $\mathrm{Diff}_0(\mathbb{C})$, the group of germs of holomorphic diffeomorphisms of $\mathbb{C}$ which fix $0 \in \mathbb{C}$. Suppose the fundamental group is presented with the generators $a_1, b_1, \ldots, a_g, b_g$ (where $g \in \mathbb{N}$ is the genus of S) and the relation $a_1.b_1.a_1^{-1}.b_1^{-1} \ldots a_g.b_g.a_g^{-1}.b_g^{-1} = 1$. We represent a_1 by the local holomorphic diffeomorphism $l(t) = t + \sum_{m=2}^{\infty} l_n t^n$, a_2 by another local diffeomorphism which commutes with $l(t)$, and the remaining generators by the identity map. Let $\hat{M}$ be the suspension of such a representation; then $\hat{M}$ is not 2-linearisable if $l_2 \neq 0$, since this property persists under holomorphic changes of coordinates.

Remark 4. Examples of linear bundles over compact Riemann surfaces such that the space of holomorphic 1-forms (with coefficients in the bundle) has positive dimension can be found in ([6]), pg. 111. For example, if the Chern class c of the bundle is positive, the dimension is $c + (g-1)$, where $g \in \mathbb{N}$ is the genus of the curve. If $c = 2 - 2g$ then the dimension is 1 when the bundle is the anticanonical one.

7. Appendix: The Index Theorem for Foliations

Let us take a fibered embedding of S into M as before (the notation is taken from Sections 1 and 2). Consider in M a holomorphic line field I. In coordinates, one has a collection of meromorphic functions $\{I_\alpha(x_\alpha, y_\alpha)\}$ such that the line field is defined as $dy_\alpha - I(x_\alpha, y_\alpha)dx_\alpha = 0$; we assume that there are no poles in the intersections $U_\alpha \cap U_\beta$. The compatibility condition when $U_\alpha \cap U_\beta \neq \emptyset$ is

$$I_\alpha(x_\alpha, y_\alpha)dx_\alpha = U_{\alpha\beta}(x_\beta, y_\beta)I_\beta(x_\beta, y_\beta)dx_\beta + V(x_\beta, y_\beta)dx_\beta, \qquad (8)$$

where $U_{\alpha\beta} = \partial\eta_{\alpha\beta}/\partial y_\beta$ and $V_{\alpha\beta} = \partial\eta_{\alpha\beta}/\partial x_\beta$. The function I_α can be regarded as the slope of the line field in the coordinates (x_α, y_α). The restriction of the condition above to S gives

$$I_\alpha(x_\alpha, 0)dx_\alpha = A_{\alpha\beta}(x_\beta)I_\beta(x_\beta, 0)dx_\beta.$$

The line field along S defines a meromorphic *slope 1-form*

$$I_S = \{I_\alpha(x_\alpha, 0)dx_\alpha\}$$

with coefficients in N.

Let us apply the derivation $\frac{\partial}{\partial y_\beta}$ to both sides of (8) and restrict to S. We obtain:

$$\begin{gathered}\partial I_\alpha/\partial y_\alpha(x_\alpha, 0)dx_\alpha - \partial I_\beta/\partial y_\beta(x_\beta, 0)dx_\beta = \\ 2C_{\alpha\beta}(x_\beta)A_{\alpha\beta}^{-1}(x_\beta)I_\beta(x_\beta, 0)dx_\beta + A_{\alpha\beta}^{-1}(x_\beta)A_{\alpha\beta}'(x_\beta)dx_\beta\end{gathered} \qquad (9)$$

Theorem 6. Suppose the embedding of S is 2-linearisable. Then

$$\sum_\alpha \mathcal{R}es(\partial I_\alpha/\partial y_\alpha(x_\alpha,0)dx_\alpha) = S\cdot S.$$

in any of the following situations:

(1) the slope 1-form I_S is holomorphic.
(2) the coordinate system is adapted (see Definition 1).

Proof: 1) Let us suppose that the slope is holomorphic. From Lemma 1 we know that $\{\theta_{\alpha\beta} = C_{\alpha\beta}A_{\alpha\beta}^{-1}I_\beta(x_\beta,0)dx_\beta\}$ belongs to $H^1(\mathcal{U},\mathcal{O}^{(1,0)})$; we may write then (as in Definition 2) $\theta_\alpha - \theta_\beta = \theta_{\alpha\beta}$ and therefore $r(I_S) = \sum_\alpha \mathcal{R}es(\theta_\alpha)$. We claim that the forms θ_α may be chosen as holomorphic 1-forms.

In fact, let us replace the coordinates $\{y_\alpha\}$ by the coordinates $\{z_\alpha\}$ given as

$$z_\alpha = y_\alpha + B_\alpha y_\alpha^2$$

where $\{B_\alpha\}$ are holomorphic functions to be chosen. The changes of coordinates of the embedding are now given as

$$z_\alpha = A_{\alpha\beta}z_\beta + (C_{\alpha\beta} - A_{\alpha\beta}B_\beta + A_{\alpha\beta}^2B_\beta)z_\beta^2 + \cdots$$

Since the embedding is 2-linearisable, one can choose the functions $\{B_\alpha\}$ in order to have $A_{\alpha\beta}^{-1}C_{\alpha\beta} = B_\beta - A_{\alpha\beta}B_\alpha$, or

$$A_{\alpha\beta}^{-1}C_{\alpha\beta}I_\beta(x_\beta,0)dx_\beta = B_\beta I_\beta(x_\beta,0)dx_\beta - A_{\alpha\beta}B_\alpha I_\alpha(x_\alpha,0)dx_\alpha.$$

It follows that $r(I_S)=0$. From ([6]), pg. 102 and (9) we get that

$$S\cdot S = \sum_\alpha \mathcal{R}es(\partial I_\alpha/\partial y_\alpha(x_\alpha,0)dx_\alpha) - r(I_S).$$

2) In the case of an adapted coordinate system we have already that $C_{\alpha\beta}=0$. The proof follows immediately from (9). □

The numbers $\mathcal{R}es(\partial I_\alpha/\partial y_\alpha(x_\alpha,0)dx_\alpha)$ are the indices for the line field introduced in ([3]). We observe that Theorem 6 holds true without the hypothesis on the embedding if S is invariant for the line field. The case of foliations on line bundles was treated in ([4]).

References

[1] Arnold,V. *Chapitres Supplémentaires de la Théorie des Équations Différentielles Ordinaires*, Éditions Mir, 1980, Chapitre 5.
[2] Arnold,V. *Bifurcations of Invariant Manifolds of Differential Equations and Normal Forms in Neighborhoods of Elliptic Curves* , Funct. Anal. Appl. 10-4, 1977, pg. 249-259.

[3] Camacho,C. and Sad,P. *Invariant Varieties through Singularities of Vector Fields*, Annals of Math. 115, 1982, pg. 579-595.
[4] Camacho,C. *Dicritical Singularities of Holomorphic Vector Fields*, Contemporary Mathematics 269, 2001, pg. 39-46.
[5] Grauert, H. *Über Modifikationen und Exzeptionelle Analytische Mengen*, Math. Annalen. 146, 1962, pg. 331-368.
[6] Gunning, R. *Lectures on Riemann Surfaces*, Princeton University Press, 1966.
[7] Laufer, H. *Normal Two-Dimensional Singularities*, Princeton University Press, 1971.