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Fibered Neighborhoods of Curves in Surfaces

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ABSTRACT. The aim of this article is to study fibered neighborhoods of compact holomorphic curves embedded in surfaces. It is shown that when the self-intersection number of the curve is sufficiently negative the fibration is equivalent to the linear one defined in the normal bundle to the curve. The obstructions to equivalence in the general case are described.

This article is devoted to the study of the simplest possible type of foliations on surfaces. We consider a compact, smooth, holomorphic curve inside a (holomorphic) surface and a holomorphic foliation by discs transverse to it; the general problem is to classify such objects. These foliations arise naturally when suspensions of groups of diffeomorphisms are constructed. In our specific case, we wish to know whether the foliation (or the fibration over the curve) is equivalent to the foliation by lines on the normal bundle to the curve. This is true when the curve has a sufficiently negative self-intersection number, but we are able to describe the obstructions that appear in the other cases.

Our study is related to the systematic study of neighborhoods of analytic varieties started by H. Grauert in his celebrated article [5]. In this article, negatively embedded submanifolds of codimension 1 are considered. There is a geometric aspect, where formal equivalences of neighborhoods are proven to be in fact holomorphic equivalences; the main point is the vanishing of some special cohomology groups due to the existence of holomorphically convex neighborhoods of the submanifold. As for the formal side, the notion of n-neighborhood , for $n \in \mathbb{N}$ is introduced and the obstruction to extend isomorphisms between n-neighborhoods to $(n+1)$-neighborhoods is described; it lies also in certain cohomology groups which depends on the normal bundle to the submanifold. The Kodaira Vanishing Theorem implies that for $n \in \mathbb{N}$ sufficiently large this group vanishes, so that the germ of a complex manifold along a negatively embedded submanifold depends only on a finite neighborhood. In [7] the special case of curves is considered; the use of Serre's duality allows simpler proofs. A careful analysis leads to the following linearization result: a negatively embedded curve with self-intersection number smaller than $4 - 4g$ ($g \in \mathbb{N}$ is the genus of the curve) always has a neighborhood equivalent to a neighborhood of the zero section of the normal bundle.

Our main result states that, when the self-intersection number of the curve is negative and

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