Neighborhoods of Analytic Varieties

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To Maria Isabel and Sara

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Introduction

The systematic study of neighborhoods of analytic varieties was started by H.Grauert in his celebrated article [**Gr62**]. In that article he considers a manifold X and a negatively embedded submanifold $A \subset X$. He introduces the notion of n-neighborhood, $n \in \mathbb{N}$, of A, which is roughly the sheaf of holomorphic functions defined in neighborhoods of the points of A in X up to those functions which vanish on A of order n , and studies when an isomorphism of two n-neighborhoods can be extended to an isomorphism of $(n+1)$-neighborhoods. He observes that obstructions to this extension problem lie in the first cohomology group of certain sheaves involving the normal bundle of A in X. Using a version of Kodaira vanishing theorem (introduced by him in [**Gr62**]) he shows that for a large n these cohomology groups vanish and so he concludes that the germ of a negatively embedded manifold A depends only on a finite neighborhood of it. These methods are generalized to a germ of an arbitrary negatively embedded divisor A in [**HiRo64**] and [**La71**]. In the case where A is a Riemann surface embedded in a two dimensional manifold, by using Serre duality we can say exactly which finite neighborhood of A determines the embedding (see [**La71**]). P. Griffiths in [**Gri66**] studies the problem of extension of analytic objects (fiber bundles, analytic maps, cohomology elements etc.) in A to X. Again he introduces the finite extension of the object and he observes that obstructions lie in certain cohomology groups. We must remark that the above discussion leads to formal extensions and isomorphisms. Grauert solves the convergence problem by geometrical methods. Later, Artin's criterion (see [**Art68**]) on the existence of convergent solutions is used instead of Grauert's geometrical methods.

This is an expository text about negatively embedded varieties. The text is mainly based on Grauert's paper [**Gr62**] [1], but we have used also the contributions of subsequent authors. We have tried to state each theorem with a precise proof, except for some well-known theorems, for instance Grauert direct image theorem and Remmert proper mapping theorem, whose proofs can be found in classical books. Our principal aim is to extend this study to

[1] For a mathematical autobiography of Grauert and also a brief history of complex analysis, the reader is referred to the interesting text [**Re95**].

the germ of foliated neighborhoods and singularities. Since we did not find any book covering all the theorems and proofs related to Grauert's theorem on negatively embedded varieties, we decided to write this text and prepare it for a course in complex analysis.

In the first chapter we will review some well-known facts and definitions. The notion of reduced analytic variety, embedding dimension of singularities, formal neighborhood and obstruction to the existence of a formal isomorphism between two embeddings are discussed in this chapter. Cartan's theorem on the quotient of analytic varieties and Remmert reduction Theorem are presented. One of the main theorems in this chapter is Theorem 1.5. This theorem establishes the obstructions to the extension of a finite isomorphism of neighborhoods to a higher order isomorphism.

The second chapter is devoted to pseudoconvex domains. For some technical reasons, we have preferred to work with C^2 convex functions instead of C^2 plurisubharmonic functions. A convex function carries just the convexity information of its level varieties and is easy to handle, so we use convex functions rather than plurisubharmonic functions. Theorem 2.2 reveals an important cohomological property of pseudoconvex domains. It can be considered as Cartan's B theorem for Stein varieties. Using Remmert reduction theorem on pseudoconvex domains, one can see that pseudoconvex domains are the point modification of Stein varieties. This leads to the notion of exceptional or negatively embedded varieties.

One of the natural examples of an embedded manifold is the zero section of a vector bundle. We deal with these embeddings in chapter three. The zero section of a line bundle is an exceptional variety if and only the line bundle is negative in the sense of Kodaira, Theorem 3.1. Vanishing theorems for the germ of exceptional varieties are stated in this Chapter, Theorems 3.3 3.5.

Chapter four is devoted to the formal principle. Theorem 1.5 and Theorem 3.3 give us a formal isomorphism of two negatively embedded manifolds. Roughly speaking, the formal principle tells us when a formal isomorphism of two neighborhoods implies the existence of a biholomorphism. In this chapter we have stated Artin's Theorem 4.1. This theorem implies the formal principle for singularities, Theorem 4.2, and then the formal principle for exceptional varieties can be derived.

Chapter five is devoted to foliated neighborhoods. In the first steps we will consider the most simple foliations which are transversal foliations. The main theorem in this direction is Theorem 5.1. Next, foliations with tangencies and Poincaré type singularities is considered. We generalize Grauert's step by step extension of isomorphisms to the case where the germ of embedding is foliated. In this section we also introduce the notion of formal equivalence of two foliated neighborhoods and prove Theorem 5.3. Artin in [**Art68**] after stating his extension and lifting theorems poses the following question: Can one generalize these statements in various ways by requiring the map preserve extra structure, such as a stratification? We are interested in the case where this additional structure is a foliation.

Whenever it was possible, we have used figures to help to understand a definition, a theorem or its proof. Specially we hope that the figures will help on reconstructing the proofs in the mind. At the end of each chapter we have added some lines for the reader who wants to know more on the development of the material presented in the chapter. This will be useful also for classrooms activities.

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Chapter 1

Preliminaries

In this chapter we review some of the basic notions of complex analysis. We assume a basic knowledge of sheaf theory, coherent analytic sheaves and cohomology of sheaves. Good references for these are the books [**GuIII90**, **GrRe79**, **GrRe84**]. Throughout the text, for a given sheaf $\mathcal{S}$ over a topological space X, when we write $x \in \mathcal{S}$ we mean that x is a section of $\mathcal{S}$ in some open neighborhood in X or it is an element in a stalk of $\mathcal{S}$ over X, being clear from the text which we mean.

1.1 Varieties

For a topological space X and a point $x \in X$ we denote by (X, x) a neighborhood of x in X. This means that in our statements and arguments we fix a neighborhood of x in X but we can take it smaller if it is necessary. A $\mathbb{C}$-algebra is a commutative ring containing the field $\mathbb{C}$ as a subring, with $1 \in \mathbb{C}$ as the identity element of the ring. A homomorphism between two $\mathbb{C}$-algebras is a ring homomorphism that induces the identity mapping on the subfield $\mathbb{C}$. An example of $\mathbb{C}$-algebra we use in this text is:

- $\mathcal{O}_{\mathbb{C}^n,x}$, the ring of germs of holomorphic functions in a neighborhood of x in $\mathbb{C}^n$.

Its maximal ideal is

- $\mathcal{M}_{\mathbb{C}^n,x} := \{f \in \mathcal{O}_{\mathbb{C}^n,x} \mid f(x) = 0\}$.

One can consider $\mathcal{O}_{\mathbb{C}^n,x}$ as the ring of convergent power series $\sum_i^\infty a_i(y-x)^i$, where $i=(i_1,i_2,\ldots,i_n)$ runs through $(\mathbb{N}\cup\{0\})^n$ and $(y-x)^i=(y_1-x_1)^{i_1}(y_2-x_2)^{i_2}\cdots(y_n-x_n)^{i_n}$.

An analytic subvariety (X,x) of $(\mathbb{C}^n,x)$ is given by $f_1=0, f_2=0,\ldots,f_r=0$, where $f_1,f_2,\ldots,f_r\in\mathcal{M}_{\mathbb{C}^n,x}$.

- $\mathcal{I}_{X,x}:=\{f\in\mathcal{O}_{\mathbb{C}^n,x}\mid f\mid_X=0\}$;
- $\mathcal{O}_{X,x}:=\mathcal{O}_{\mathbb{C}^n,x}/\mathcal{I}_{X,x}$, the germs of holomorphic functions in a neighborhood of x on X;
- $\mathcal{M}_{X,x}:=\{f\in\mathcal{O}_{X,x}\mid f(x)=0\}$, the maximal ideal of $\mathcal{O}_{X,x}$;
- $\mathcal{M}^k_{X,x}$, the sub $\mathbb{C}$-algebra of $\mathcal{M}_{X,x}$ generated by $\Pi_{i=1}^k g_i,\ g_i\in\mathcal{M}_{X,x}$.

We collect all the necessary statements on $\mathbb{C}$-algebras which we need in the following proposition:

Proposition 1.1. *The following statements are true:*

1. $\mathcal{M}^m_{\mathbb{C}^n,0}$ *is exactly the set of holomorphic functions with the leading term (in the Taylor series) of degree greater than or to equal* m*;*
2. $\mathcal{O}_{\mathbb{C}^n,0}$ *is a Noetherian ring, i.e. every ideal in* $\mathcal{O}_{\mathbb{C}^n,0}$ *has a finite basis;*
3. $\cap_{k=1}^\infty\mathcal{M}^k_{\mathbb{C}^n,0}=\{0\}$*;*

Proof. We first prove the nontrivial part of the statement 1., i.e. if $f\in\mathcal{O}_{\mathbb{C}^n,0}$ with the leading term of degree $\geq m$ then $f\in\mathcal{M}^m_{\mathbb{C}^n,0}$. The proof is by induction on n. The case $n=1$ is trivial. By a linear change of coordinates we can assume that f is regular in the variable x_1, i.e. $f(x_1,0,\ldots,0)$ is not identically zero. By Weierstrass preparation theorem (see [**GuII90**] Theorem A4) we can write $f=u.(x_1^l+a_1x_1^{l-1}+\cdots+a_{l-1}x_1+a_l)$, where $a_1,a_2,\ldots,a_l$ are holomorphic functions in $x_2,x_3,\ldots,x_n$ and u is a holomorphic function in $x_1,x_2,\ldots,x_n$ with $u(0)\neq 0$. Since f and $\frac{f}{u}$ have the same leading term up to multiplication by a constant, it is enough to prove that $x_1^l+a_1x_1^{l-1}+\cdots+a_{l-1}x_1+a_l\in\mathcal{M}^m_{\mathbb{C}^n,0}$. This statement follows by our hypothesis and the hypothesis of induction for $n-1$.

The statement 2. can be found in [**GuII90**] Theorem A8. The statement 3. is a direct consequence of the first part (or the second part in the general context of local rings). □

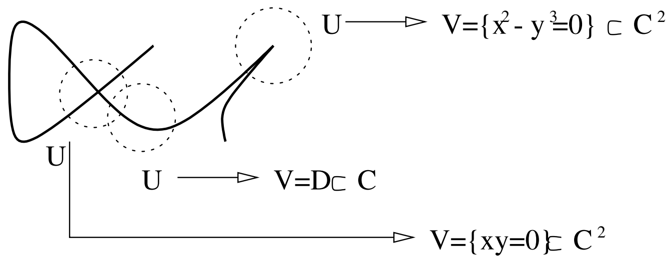

Figure 1.1: Analytic variety

A closed analytic subset X of an open domain in some $\mathbb{C}^n$ is locally given by the zero locus of some holomorphic functions and is called an affine (analytic) variety. We look at X as a topological space equipped with a sheaf $\mathcal{O}_X$ of $\mathbb{C}$-algebras $\mathcal{O}_{X,x}, x \in X$, that is called the structural sheaf of X. Let X and Y be two affine varieties. A continuous map $\tau : X \to Y$ is called holomorphic if the pull-back of functions, given by $\tau^*(f) = f \circ \tau$, defines a map τ^* from $\mathcal{O}_{Y,\tau(x)}$ into $\mathcal{O}_{X,x}$, which is a morphism of $\mathbb{C}$-algebras for all $x \in X$. The map τ is called a biholomorphism if there is a holomorphic map $\tau' : Y \to X$ such that $\tau' \circ \tau$ and $\tau \circ \tau'$ are identity maps respectively on X and Y.

Let X be a second-countable Hausdorff topological space and $\mathcal{C}_X$ be the sheaf of complex valued continuous functions on X. We say that X with a sheaf of $\mathbb{C}$-algebras $\mathcal{O}_X \subset \mathcal{C}_X$ is an analytic variety if every point of X has an open neighborhood U such that $(U, \mathcal{O}_U)$ is isomorphic to a $(V, \mathcal{O}_V)$, for some affine variety V, i.e. there is a homeomorphism $\psi : U \to V$ such that $\psi^* : \mathcal{O}_V \to \mathcal{O}_U, \psi^*(f) = f \circ \psi$, is an isomorphism of sheaves of $\mathbb{C}$-algebras.

Let X be a variety. For every point $x \in X$ there exist an open set U around x, V a closed analytic subset of an open domain D in some $\mathbb{C}^n$ and a homeomorphism $\psi : U \to V$ which induces an isomorphism between $\mathcal{O}_V$ and $\mathcal{O}_U$. A rough picture of this definition is depicted in Figure 1.1 This is called a chart around x and we denote it simply by

- $\psi : U \to V \subset D \subset \mathbb{C}^n$, a chart around x.

Given two such charts $\psi_\alpha : U_\alpha \to V_\alpha \subset D_\alpha \subset \mathbb{C}^{n_\alpha}$ and $\psi_\beta : U_\beta \to V_\beta \subset D_\beta \subset \mathbb{C}^{n_\beta}$ around x, the first is called a subchart of the second if there is an embedding $em : (D_\alpha, \psi_\alpha(x)) \hookrightarrow (D_\beta, \psi_\beta(x))$ such that $\psi_\beta = em \circ \psi_\alpha$. They are called equivalent if one is a subchart of the other and $n_\alpha =$

n_β. In this case the map em is a biholomorphism. This is an equivalence relation. Note that n, the dimension of D, differs chart by chart. For this reason it is better to define a variety using the language of $\mathbb{C}$-algebras rather than the formal definition by charts and transition functions, for instance see [**GuII90**], Definition B16.

The following proposition is Theorem B14 of [**GuII90**]. We give its proof because it is instructive.

Proposition 1.2. *Let $X \subset (\mathbb{C}^n, 0)$ and $Y \subset (\mathbb{C}^m, 0)$ be the germs of two affine varieties. Every holomorphic map $\tau : (X, 0) \to (Y, 0)$ is induced by a holomorphic map from $(\mathbb{C}^n, 0)$ to $(\mathbb{C}^m, 0)$.*

Proof. We have a morphism $\tau^* : \mathcal{O}_{Y,0} \to \mathcal{O}_{X,0}$ of $\mathbb{C}$-algebras. Since it sends the units to units, it sends the maximal ideal $\mathcal{M}_{Y,0}$ into the maximal ideal $\mathcal{M}_{X,0}$ and so $\tau^*(\mathcal{M}^k_{Y,0}) \subset \mathcal{M}^k_{X,0},\ k = 1, 2, \ldots$. Let us denote the coordinate functions of $(\mathbb{C}^m, 0)$ by $y_1, y_2, \ldots, y_m$ $(\in \mathcal{M}_{Y,0})$ and define $f_i := \tau^*(y_i)$. The map $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^m, 0)$ defined by $f = (f_1, f_2, \ldots, f_m)$ is the desired map. We consider the diagram

$$\begin{array}{ccc} \mathcal{O}_{\mathbb{C}^m,0} & \xrightarrow{f^*} & \mathcal{O}_{\mathbb{C}^n,0} \\ j \downarrow & & i \downarrow \\ \mathcal{O}_{Y,0} & \xrightarrow{\tau^*} & \mathcal{O}_{X,0} \end{array} \qquad (1.1)$$

where $\imath, j$ are the canonical maps. We observe that the maps

$$\imath \circ f^*, \tau^* \circ j : \mathcal{O}_{\mathbb{C}^m,0} \to \mathcal{O}_{X,0}$$

coincide on polynomials in y_i's. For an arbitrary $k \in \mathbb{N}$, every $g \in \mathcal{O}_{\mathbb{C}^m,0}$ can be written as $g_1 + g_2$, where g_1 is a polynomial in y_i's and $g_2 \in \mathcal{M}^k_{\mathbb{C}^m,0}$ (here we have used Proposition 1.1,1). Therefore $(\imath \circ f^* - \tau^* \circ j)(g) \in \mathcal{M}^k_{X,0}$ for all $k = 1, 2, \ldots$. Now Proposition 1.1, 3 implies that $\imath \circ f^* = \tau^* \circ j$.

$\mathcal{I}_{Y,0}$ is a subset of the kernel of $\tau^* \circ j$ and so of the kernel of $\imath \circ f^*$. This implies that whenever a $g \in \mathcal{O}_{\mathbb{C}^m,0}$ is zero on $(Y, 0)$ then it is zero on $f(X, 0)$ and so $f(X, 0) \subset (Y, 0)$. Since $\tau, f : X \to Y$ induce the same map $\tau^* = f^*$, the proof is finished. □

For a germ of an analytic variety (X, x) we set

- $T^*_x X := \mathcal{M}_{X,x}/\mathcal{M}^2_{X,x}$, the cotangent space of X at x;
- $T_x X :=$ the dual of $T^*_x X$. $T_x X$ is called the tangent space of X at x.

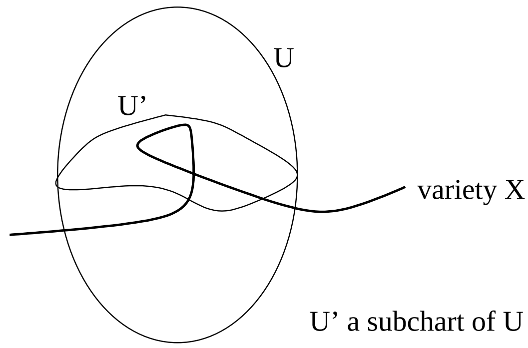

Figure 1.2: Embedding dimension

A holomorphic map $f : (X, x) \to (Y, y)$ induces the map

$$T_x^* f : T_{f(x)}^* Y \to T_x^* X$$

It would be instructive to check that the definition of the tangent space in the case where X is smooth coincides with the usual definition of tangent space with differential of transition maps of X. In the singular case the bundle of tangent spaces $\{T_x X, x \in X\}$ has a natural structure of an analytic variety (see [**GuII90**] J) and so we can define in a natural way the notion of a vector field in a variety.

Let X be a variety and $x \in X$. Using some chart around x we can identify the germ of the singularity (X, x) as an analytic subspace of $\mathbb{C}^n$, for some n. The smallest integer n with this property is called the embedding dimension of X at x and is denoted by $emb_x X$.

The following proposition can also be found in [**GrRe84**] p. 115.

Proposition 1.3. *We have $emb_x X = dim_{\mathbb{C}} T_x^* X$. More precisely, for a point $x \in X$ if $x_1, x_2, \ldots, x_m \in \mathcal{M}_{X,x}$ form a basis for $T_x^* X$ then $\psi : (X, x) \to \mathbb{C}^m$ given by $\psi = (x_1, x_2, \ldots, x_m)$ is a chart map around x whose associated affine space is of dimension $dim_{\mathbb{C}} T_x^* X$. Every two charts with the dimension of the affine spaces equal to $dim_{\mathbb{C}} T_x^* X$ are equivalent and every chart has a subchart whose associated affine space is of dimension $m = dim_{\mathbb{C}} T_x^* X$.*

Proof. Since our statement is local, we can assume that $X \subset (\mathbb{C}^n, 0)$ and $x = 0$. Let $\lambda : \mathcal{I}_{X,0} \to \mathcal{M}_{\mathbb{C}^n,0}/\mathcal{M}^2_{\mathbb{C}^n,0}$ be the canonical map. Its coimage $(\mathcal{M}_{\mathbb{C}^n,0}/\mathcal{M}^2_{\mathbb{C}^n,0})/Im(\lambda)$ is isomorphic to $\mathcal{M}_{X,0}/\mathcal{M}^2_{X,0}$. Therefore if $r :=$

$dimIm(\lambda)$, $m := dim_{\mathbb{C}}\mathcal{M}_{X,0}/\mathcal{M}_{X,0}^2$ then $r+m=n$. Let $f_1, f_2, \ldots, f_r \in \mathcal{I}_{X,0}$ such that their image by λ form a $\mathbb{C}$-basis for $Im(\lambda)$. This means that the linear part of the map $f = (f_1, f_2, \ldots, f_r)$ has the maximum rank r. Therefore f is a regular map and $N = \{x \in (\mathbb{C}^n, 0) \mid f(x) = 0\}$ is a smooth complex submanifold of $(\mathbb{C}^n, 0)$ and $dim_{\mathbb{C}}N = m$. But we have also $X \subset N$. We have proved that each chart has a subchart whose associated affine space is of dimension $m = dimT_0^*X$.

Let us be given two charts for $(X, 0)$ whose associated affine spaces are of dimension T_0^*X. This means that $(X, 0)$ is embedded in two different ways in $(\mathbb{C}^m, 0)$, say X_1, X_2. By Proposition 1.2 the map induced by the identity $\imath : (X_1, 0) \to (X_2, 0)$ can be extended to a holomorphic map $f : (\mathbb{C}^m, 0) \to (\mathbb{C}^m, 0)$. Using the argument of the previous paragraph and the dimension condition we have

$$T_0^*\mathbb{C}^m = T_0^*X_i,\ i = 1, 2 \tag{1.2}$$

But we know that $\imath^* : \mathcal{O}_{X_2,0} \to \mathcal{O}_{X_1,0}$ is an isomorphism of $\mathbb{C}$-algebras and so it induces an isomorphism $\mathcal{M}_{X_2,0}/\mathcal{M}_{X_2,0}^2 \to \mathcal{M}_{X_1,0}/\mathcal{M}_{X_1,0}^2$. The equality (1.2) and the inverse mapping theorem imply that f is a biholomorphism.

Let $x_1, x_2, \ldots, x_m \in \mathcal{M}_{X,0}$ form a basis for T_x^*X. The map $\psi : (X, x) \to \mathbb{C}^m$ given by $\psi = (x_1, x_2, \ldots, x_m)$ is a holomorphic map. Take an arbitrary embedding of $(X, 0)$ in $(\mathbb{C}^m, 0)$. According to Proposition 1.2 ψ is obtained by restriction of a holomorphic map $f : (\mathbb{C}^m, 0) \to (\mathbb{C}^m, 0)$. Since $T_0^*X = T_0^*\mathbb{C}^m$, the map $T_0^*f : T_0^*\mathbb{C}^m \to T_0^*\mathbb{C}^m$ is an isomorphism and so f is a biholomorphism. This proves that ψ is an embedding. □

Proposition 1.4. *For a holomorphic map $f : (X, x) \to (Y, y)$ if T_x^*f is surjective then f is an embedding.*

Proof. Let σ be the canonical map $\mathcal{M}_{Y,y} \to T_y^*Y$ and $n = dim_{\mathbb{C}}T_x^*X$. Choose $f_1, f_2, \ldots, f_n \in \mathcal{M}_{Y,y}$ such that their image by $T_x^*f \circ \sigma$ form a basis of T_x^*X. The map

$$g = (f_1, f_2, \ldots, f_n) : (Y, 0) \to (\mathbb{C}^n, 0)$$

has the following property: $g \circ f$ is an embedding of X in $(\mathbb{C}^n, 0)$, for this see the first part of Proposition 1.3. We identify X with its image by $g \circ f$ in $(\mathbb{C}^n, 0)$. The set $X_1 := f(X)$ is an analytic variety because it is $g^{-1}(X)$. The inverse of $f : X \to X_1$ is given by g. □

1.2 Stein varieties

Stein varieties share many properties with germs of varieties. In this section we list the definition and some theorems about Stein varieties. For more detailed study the reader is referred to [**GrRe79**, **GuIII90**].

Let K be a subset of a variety X, $\hat{K}_X := \{x \in X \mid |f(x)| \leq sup_{y \in K}|f(y)| \; \forall f \in \mathcal{O}_X(X)\}$ is called the (holomorphic) convex hull of K in X. Then, X is called holomorphically convex if for any compact set $K \subset X$ the convex hull $\hat{K}_X$ is also compact.

Theorem 1. *(***Definition***) Let X be a holomorphically convex variety. X is called Stein if one of the following equivalent condition is satisfied:*

1. *For any point $x \in X$ there exist holomorphic functions $f_1, f_2, \dots, f_m$ on X such that x is an isolated point of the set $\{x \in X \mid f_1(x) = f_2(x) = \dots = f_m(x) = 0\}$;*

2. *Holomorphic functions on X separate the points of X, i.e. for any pair of points x and y in X there exists a holomorphic function on X such that $f(x) \neq f(y)$;*

3. *X does not contain nowhere discrete compact analytic subsets;*

The reader is referred to [**GuIII90**], Theorems 4M,5M,11M for the proof of the equivalences.

Proposition 1.5. *If U_1 and U_2 are two Stein open subsets of a variety X then $U_1 \cap U_2$ is Stein.*

Proof. Since holomorphic functions separate points in U_1, this is the case also in every open subset of U_1. Therefore it is enough to prove that $U_1 \cap U_2$ is holomorphically convex. For a compact set $K \subset U_1 \cap U_2$ we have $\hat{K}_{U_1 \cap U_2} \subset \hat{K}_{U_1} \cap \hat{K}_{U_2}$. Since $\hat{K}_{U_1 \cap U_2}$ is closed and is a subset of a compact set, it is compact. □

Let X be a variety, $\mathcal{S}$ an analytic sheaf on X and $\mathcal{U}$ a covering of X by open sets. The covering $\mathcal{U}$ is called acyclic with respect to $\mathcal{S}$ if $\mathcal{U}$ is locally finite, i.e. each point of X lies in a finite number of open sets in $\mathcal{U}$, and $H^\mu(U_{i_1} \cap \dots \cap U_{i_k}, \mathcal{S}) = 0$ for all $U_{i_1}, \dots, U_{i_k} \in \mathcal{U}$ and $\mu \geq 1$. Now let us state two well-known facts

Theorem 2. *(Leray lemma) Let $\mathcal{U}$ be an acyclic covering of a variety X. There is a natural isomorphism $H^\mu(\mathcal{U}, \mathcal{S}) \cong H^\mu(X, \mathcal{S})$.*

Theorem 3. *(Cartan's Theorem B) For a Stein variety X and a coherent analytic sheaf $\mathcal{S}$ on X we have $H^\mu(X, \mathcal{S}) = 0$ for $\mu \geq 1$.*

A covering $\mathcal{U}$ of a variety X is called Stein if it is locally finite and each open set in $\mathcal{U}$ is Stein.

Combining Proposition 1.5, Theorem B of Cartan we conclude that a Stein covering is acyclic and so by Leray lemma $H^\mu(\mathcal{U}, \mathcal{S}) \cong H^\mu(X, \mathcal{S})$ for any coherent analytic sheaf $\mathcal{S}$ on X and $\mu \geq 1$.

1.3 Equivalence relations in varieties and Remmert reduction theorem

Given a topological space T. We denote by $\mathcal{C}_T$ the sheaf of continuous complex valued functions on T. Let us be given a variety X and an equivalence relation R on X. Let $\phi : X \to X/R$ be the canonical map. We can define the sheaf $\mathcal{O}_{X/R}$ of $\mathbb{C}$-algebras on X/R as follows: The data

$$U \to \{f \in \mathcal{C}_{X/R}(U) \mid f \circ \phi \in \mathcal{O}_X(\phi^{-1}(U))\},\ U \text{ an open subset of } X/R$$

form the sheaf $\mathcal{O}_{X/R}$. In this section we want to answer the following question: When $(X/R, \mathcal{O}_{X/R})$ is an analytic variety? By definition of $\mathcal{O}_{X/R}$ if $(X/R, \mathcal{O}_{X/R})$ is an analytic variety then $\phi : X \to X/R$ is a holomorphic mapping. Cartan's article [**Ca60**] is the main source for this section.

Theorem 4. *(Remmert proper mapping theorem [**Re57**]) If f is a proper holomorphic mapping of a variety X into a variety Y then the image $f(X)$ is a subvariety of Y.*

The direct image $f_*\mathcal{S}$ is defined as follows: $f_*\mathcal{S}$ is the sheaf associated to the presheaf $U \to \mathcal{O}_X(f^{-1}(U))$, for open sets U in Y. One can define higher order direct images $R^\mu f_*\mathcal{S}, \mu \geq 0$ as the sheaf associated to the presheaf

$$U \to H^\mu(f^{-1}(U), \mathcal{S})$$

Theorem 5. *(Grauert direct image theorem [**Gr60**]) Let f be a proper holomorphic mapping of a variety X into a variety Y. If $\mathcal{S}$ is a coherent sheaf on X then $R^\mu f_*\mathcal{S}, \mu \geq 0$ is a coherent analytic sheaf on Y.*

The reader is referred to [**GrRe84**] for the proof of the above classical theorems and their applications.

Let X, Y be two varieties and $f : X \to Y$ be a holomorphic map. We can define the equivalence relation R_f in X as follows:

$$\forall x, y \in X,\ xR_f y \text{ if and only if } f(x) = f(y)$$

Theorem 1.1. *If $f : X \to Y$ is a proper holomorphic map then $(X/R_f, \mathcal{O}_{X/R_f})$ is an analytic variety.*

Proof. By Remmert proper mapping theorem we can assume that f is a surjective map and then we can identify Y with X/R_f pointwise. By this identification we denote $\mathcal{O}_{X/R_f}$ by $\mathcal{S}$. The structural sheaf $\mathcal{O}_Y$ of Y is a subsheaf of $\mathcal{S}$. For a moment suppose that $\mathcal{S}$ is a coherent ($\mathcal{O}_Y$-module) sheaf. A part of the definition of a coherent sheaf is the following: For every point $y' \in Y$ there is an open neighborhood U of y' in Y and sections $s_1, s_2, \ldots, s_k$ of $\mathcal{S}(U)$ such that $s_{1y}, s_{2y}, \ldots, s_{ky}$ generate $\mathcal{S}_y$ as a $\mathcal{O}_{Y,y}$-module for all $y \in U$. Now by definition $f_i := s_i \circ f$'s are holomorphic functions on $V := f^{-1}(U)$. Define the map

$$g : V \to U \times \mathbb{C}^k,\ g(x) = (f(x), f_1(x), f_2(x), \ldots, f_k(x))$$

g is a proper holomorphic mapping and so we can apply Remmert proper mapping theorem and obtain a subvariety $Z := g(V)$ of $U \times \mathbb{C}^k$. Now the map $f : V \to U$ decomposes into

$$V \xrightarrow{g} Z \xrightarrow{h} U$$

where h is the projection on the first coordinate and so it is a holomorphic map. Since the f_i's are constant along the fibers of f, h is a one to one map. Therefore we can identify Z with U through h. By this identification, one can easily see that $\mathcal{S}$ on Z is nothing but the structural sheaf $\mathcal{O}_Z$ of Z. We have proved that $(U, \mathcal{O}_{X/R_f} \mid_U)$ is isomorphic to the variety Z.

Now it remains to prove that $\mathcal{S}$ is a coherent sheaf on Y. Let T be the analytic variety in $X \times X$ given by the inverse image of the diagonal of $Y \times Y$ by the map $f \times f : X \times X \to Y \times Y$ and $\pi_i : T \to X,\ i = 1, 2$ be the projections on the first and second coordinates. We have a diagram

$$\begin{array}{ccc} T & \xrightarrow{\pi_i} & X \\ g & \searrow & \downarrow & f \\ & & Y \end{array} \tag{1.3}$$

where $g = f \circ \pi_1 = f \circ \pi_2$. Now the maps $\pi_i, i = 1, 2$ induce the maps $\pi_i^* : \mathcal{O}_X \to \mathcal{O}_T$ and so the maps

$$\alpha_{i*} : f_*\mathcal{O}_X \to g_*\mathcal{O}_T, \ i = 1, 2$$

and we have

$$\mathcal{S} = ker(\alpha_{1*} - \alpha_{2*})$$

To prove this equality, take an open set U in Y and r a holomorphic function on $f^{-1}(U)$. If r is constant on the fibers of f (in the case where f has disconnected fibers this statement cannot be derived from the fact that r is holomorphic and f is proper) then $\alpha_{1*}(r) = \alpha_{2*}(r)$. If $\alpha_{1*}(r) = \alpha_{2*}(r)$ then the definition of T implies that r is constant on the fibers of f and so it is a section of $\mathcal{S}$ on U.

By Grauert direct image theorem $g_*\mathcal{O}_T$ and $f_*\mathcal{O}_X$ are coherent sheaves and so $\mathcal{S}$ is a coherent sheaf. □

Now let us consider a family of proper holomorphic mappings $f_i : X \to Y_i, i \in I$, where I is an index set. One can define the equivalence relation R_I on X as follows:

$$\forall x, y \in X, \ xR_Iy \text{ if and only if } f_i(x) = f_i(y) \ \forall i \in I$$

In the case where I is finite the pair $(X/R_I, \mathcal{O}_{X/R_I})$ is an analytic variety because $R_I = R_{f_I}$, where

$$f_I := \Pi_{i\in I} f_i : X \to Y_I, \ Y_I := \Pi_{i\in I} Y_i$$

For an infinite family of holomorphic functions we have the following proposition:

Proposition 1.6. *Let X and $Y_i, i \in I$ be varieties and $f_i : X \to Y_i, \ i \in I$ holomorphic functions. For any compact subset K of X there is a finite subset $J \subset I$ such that R_I and R_J induce the same relation on K.*

Proof. For a finite set $J \subset I$ let Δ_J be the subset of $X \times X$ given by the inverse image of the diagonal of $Y_J \times Y_J$ by $f_J \times f_J : X \times X \to Y_J \times Y_J$. Each Δ_J is a subvariety of $X \times X$ and if $J \subset J'$ be finite subsets of I then $\Delta_{J'} \subset \Delta_J$. Such a family of varieties becomes constant on a given compact subset $\tilde{K}$ of $X \times X$. Take a point $p \in \tilde{K}$. Since in the family Δ_J the dimension of Δ_J around p cannot drop infinitely many times, our claim is true locally. One can cover $\tilde{K}$ by finitely many small open sets and get the assertion for $\tilde{K}$. □

The equivalence relation R on a variety is called a proper equivalence relation if for any compact set $K \subset X$ the K-saturated set, i.e. the union of R-equivalence classes cutting K, is compact. For a proper equivalence relation the set X/R is locally compact, X/R is Hausdorff and the continuous map $X \to X/R$ is proper.

Theorem 1.2. *(H. Cartan [**Ca60**]) Let R be a proper equivalence relation on a variety X with the following property: Each point of $x \in X/R$ has an open neighborhood U such that $\mathcal{O}_{X/R}(U)$ separates the points of U, i.e. for any two points $x_1, x_2 \in U$ there is $f \in \mathcal{O}_{X/R}(U)$ such that $f(x_1) \neq f(x_2)$. Then $(X/R, \mathcal{O}_{X/R})$ is an analytic variety.*

Proof. Let $U \subset X/R$ be the open set introduced in the theorem. Since $\mathcal{O}_{X/R}(U)$ separates the points of U, the equivalence relation R_I defined by the family $I = \phi^* \mathcal{O}_{X/R}(U)$ in $\phi^{-1}(U)$ is R. Therefore if U' is a relatively compact open subset of U containing y, then by Proposition 1.6 and Theorem 1.1 $(U', \mathcal{O}_{X/R}\mid_{U'})$ is a variety. □

Now as an application of Theorem 1.2 we state and prove Stein factorization and Remmert reduction theorems.

Theorem 1.3. *(Stein factorization) Let $f : X \to Y$ be a proper holomorphic map of varieties. Then there exist a variety Z and holomorphic maps*

$$X \xrightarrow{g} Z \xrightarrow{h} Y$$

such that 1. $f = h \circ g$, 2. h is a finite map, 3. $g_* \mathcal{O}_X = \mathcal{O}_Z$. The triple (h, g, Z) with properties 1,2 and 3 satisfies: 4. g has connected fibers 5. It is unique up to biholomorphism, i.e. for any other triple (h', g', Z') with the properties 1,2,3 of the theorem there is a biholomorphic map $a : Z \to Z'$ such that $g' = a \circ g$ and $h' = h \circ a^{-1}$.

Proof. We define the equivalence relation R in X as follows: For all $x, y \in X$ we have xRy if and only if $f(x) = f(y)$ and x and y are in the same connected component of $f^{-1}(f(x))$. A simple topological argument shows that R is a proper equivalence relation. The map f decomposes into $X \xrightarrow{g} X/R \xrightarrow{h} Y$, where g and h are continuous maps. For a Stein small open set U in Y, $\mathcal{O}_Y(U)$ separates the point of U and $h^* \mathcal{O}_Y(U) \subset \mathcal{O}_{X/R}(h^{-1}(U))$. Therefore R satisfies the condition of Theorem 1.2 and so $Z := (X/R, \mathcal{O}_{X/R})$ is a variety and g is a holomorphic map. The map h is also holomorphic because

$h^*\mathcal{O}_Y \subset \mathcal{O}_Z$. Since f is proper, a fiber of f has finitely many connected components and so h is a finite map. The condition 3 is true by definition of $\mathcal{O}_{X/R}$.

Assume that $g^{-1}(x)$ is not connected and has two connected components A and B. In an open neighborhood of $g^{-1}(x)$ we can define a two valued function which takes 1 in a neighborhood of A and 0 in a neighborhood of B. This function is not a pullback of any holomorphic function in a neighborhood of x in Y, which is a contradiction with 3. The property 1,2 and 3 imply that the points of Z' are in one to one correspondence with connected components of the fibers of f. Therefore we have a one to one map $a : Z \to Z'$. It can be easily seen that a is the desired map for 5. □

Remark: Let $f : X \to Y$ be a surjective proper holomorphic map of varieties with $f_*\mathcal{O}_X = \mathcal{O}_Y$. The argument which we used for 4. of Theorem 1.3 implies that f has connected fibers. For any open set $U \subset Y$ and a holomorphic function r in $\phi^{-1}(U)$ there exists a holomorphic function s in U such that $r = s \circ f$.

Theorem 1.4. *(Remmert reduction [**Re56**]) Let X be a holomorphically convex space. Then there exist a Stein space Y and a proper surjective holomorphic map $\phi : X \to Y$ such that 1. $\phi_*\mathcal{O}_X = \mathcal{O}_Y$. Moreover the fact that Y is Stein and 1 imply 2. ϕ has connected fibers 3. The map $\phi^* : \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$ is an isomorphism 4. The pair (ϕ, Y) is unique up to biholomorphism, i.e. for any other pair (ϕ', Y') with Y' Stein and property 1, there is a biholomorphism $a : Y \to Y'$ such that $\phi' = a \circ \phi$.*

Proof. Let $R = R_I$ be the relation in X defined by the family $I = \mathcal{O}_X(X)$. For a compact set K in X the set $\cup_{x\in K} R_x$ is closed and contained in the convex hull of K in X. Since X is holomorphically convex, this means that R is a proper equivalence relation. It satisfies also the condition of Theorem 1.2. Therefore $(X/R, \mathcal{O}_{X/R})$ is a variety. By definition X/R is holomorphically convex and holomorphic functions on X/R separate the points of X/R. This means that $Y := X/R$ is a Stein variety. The canonical map $\phi : X \to Y$ is the desired map. It is enough to prove that ϕ has connected fibers. If a fiber of ϕ has two connected components A and B then we can use Stein factorization and obtain a holomorphic function f on X such that $f(A) \neq f(B)$. But this means that A and B are two distinct equivalence classes of R which is a contradiction.

For any other pair (ϕ', Y') the existence of a bijective map $a : Y \to Y'$ follows from the fact that ϕ and ϕ' have the same fibers. The property 1. and remark after Theorem 1.3 proves that a is a biholomorphism. □

1.4 Neighborhood notations

Let A be a subvariety of an analytic variety X. We define:

- $\mathcal{M} := \mathcal{M}_A$, the subsheaf of $\mathcal{O}_X$ consisting of elements that vanish at A;
- $A_{(*)} := \mathcal{O}_X \mid_A$, $A_{(*)}$ is called the neighborhood sheaf of A;
- $A_{(\nu)} := \mathcal{O}_X/\mathcal{M}^\nu \mid_A$, $A_{(\nu)}$ is called the ν-neighborhood of A. $A_{(1)}$ is the structural sheaf of A;
- $\mathcal{Q}_\nu := \mathcal{M}^\nu/\mathcal{M}^{\nu+1} \mid_A$, $\mathcal{Q}_\nu$ is a $\mathcal{O}_A$-module sheaf;
- $\mathcal{M}_{(\nu)} := \mathcal{M}/\mathcal{M}^\nu \mid_A$.
- For any analytic sheaf $\mathcal{S}$ on X

  $$res(\mathcal{S}) := \mathcal{S}/\mathcal{S}.\mathcal{M}$$

  is called the structural restriction of $\mathcal{S}$ on A. Note that the sheaf theory restriction $\mid$ has nothing to do with the complex structure of A but this restriction has. For instance the structural restriction of $\mathcal{O}_X$ to A is $\mathcal{O}_A$. When there is no danger of confusion we will write the same symbol $\mathcal{S}$ instead of $res(\mathcal{S})$. The sheaf $res(\mathcal{S})$ has a natural structure of $\mathcal{O}_A$-module. Moreover if $\mathcal{S}$ is a coherent $\mathcal{O}_X$-module sheaf then $res(\mathcal{S})$ is a coherent $\mathcal{O}_A$-module sheaf;
- $\mathcal{S}(\nu) := res(\mathcal{S}) \otimes_{\mathcal{O}_A} \mathcal{Q}_\nu$, for an analytic sheaf $\mathcal{S}$ on X. There is a natural homomorphism $\mathcal{S}\mathcal{M}^\nu \to \mathcal{S}(\nu)$ for which we have the short exact sequence

  $$0 \to \mathcal{S}\mathcal{M}^{\nu+1} \to \mathcal{S}\mathcal{M}^\nu \to \mathcal{S}(\nu) \to 0$$
- For any vector bundle (linear space) $F \to X$, F^* denotes its dual and $\underline{F}$ the sheaf of holomorphic sections of F;

- $\mathcal{T} := \underline{TX}$, the sheaf of holomorphic vector fields in X (sections of the tangent bundle TX);
- $\mathcal{T}_A$, the subsheaf of $\mathcal{T}$ consisting of vector fields tangent to A;
- $N := TX|_A/TA$ the normal bundle of A in X.

The reader is referred to [**GuII90**] I,J for the notion of tangent space of a variety. Specially it is proved there that the bundle of tangent spaces of a variety has a canonical structure of an analytic variety.

We have

$$nil(A_{(\nu)}) := \{x \in A_{(\nu)} \mid \exists n \in \mathbb{N}, x^n = 0\} = \mathcal{M}/\mathcal{M}^\nu \tag{1.4}$$

$$\mathcal{Q}_{\nu-1} = \{x \in A_{(\nu)} \mid x.nil(A_{(\nu)}) = 0\} \tag{1.5}$$

and a canonical short exact sequence

$$0 \to \mathcal{Q}_{\nu-1} \to A_{(\nu)} \to A_{(\nu-1)} \to 0$$

There are natural isomorphisms

$$\mathcal{Q}_1 \cong \underline{N}^*,\ \mathcal{T}/\mathcal{T}_A \cong (\mathcal{Q}_1)^*$$

$$\mathcal{Q}_\nu \cong \mathcal{Q}_1 \otimes \mathcal{Q}_1 \otimes \cdots \otimes \mathcal{Q}_1(\nu \text{ times}),\ \mathcal{Q}_\nu \cong (\underline{N}^*)^\nu$$

Let us be given two embeddings $A \hookrightarrow X,\ A \hookrightarrow X'$. If we denote the image of the first embedding by A and the second by A' we have a natural biholomorphism

$$\phi : A' \to A$$

which gives us an isomorphism

$$\phi_{(1)} : A_{(1)} \to A'_{(1)} \tag{1.6}$$

This isomorphism is fixed from now on. We always assume that the pairs (X, A) and (X', A') have the same local structure, i.e. for any $a' \in A'$ and its corresponding $a = \phi(a') \in A$ there is a local biholomorphism

$$(X', A', a') \to (X, A, a)$$

Notations related to A' will be written by adding $'$ to the notations of A.

1.5 Formal and finite neighborhoods

The natural inclusions

$$\cdots \subset \mathcal{M}^{\nu+1} \subset \mathcal{M}^{\nu} \subset \mathcal{M}^{\nu-1} \subset \cdots \subset \mathcal{M}$$

give us the natural chain of canonical functions:

$$\cdots \xrightarrow{\pi} A_{(\nu+1)} \xrightarrow{\pi} A_{(\nu)} \xrightarrow{\pi} A_{(\nu-1)} \xrightarrow{\pi} \cdots \xrightarrow{\pi} A_{(1)}$$

We define

$$A_{(\infty)} := lim_{\infty \leftarrow \nu} A_{(\nu)}$$

In other words, every element of $A_{(\infty)}$ is given by a sequence

$$\ldots, f_{\nu+1}, f_{\nu}, f_{\nu-1}, \ldots, f_1 \quad f_v \in A_{(\nu)}$$

$$\pi(f_{\nu+1}) = f_{\nu}$$

The $\mathbb{C}$-algebra structure of $A_{(\infty)}$ is defined naturally. $A_{(\infty)}$ is called the formal neighborhood of A or the formal completion of X along A. There exists a natural canonical homomorphism

$$A_{(*)} \rightarrow A_{(\nu)}$$

which extends to the inclusion

$$A_{(*)} \hookrightarrow A_{(\infty)}$$

Define in the set

$$\tilde{\mathbb{N}} = \{1, 2, 3, \cdots, \infty, *\}$$

the order

$$1 < 2 < 3 < \cdots < \infty < *$$

we conclude that for any pair $\mu, \nu \in \tilde{\mathbb{N}}, \mu \leq \nu$ there exists a natural homomorphism

$$\pi : A_{(\nu)} \rightarrow A_{(\mu)}$$

If no confusion is possible, we will not use any symbol for the homomorphisms considered above. Let us analyze the global sections of the above sheaves. Every global section of $A_{(*)}$ is a holomorphic function in a neighborhood of A. Let g be a global section of $A_{(\nu)}, \nu < \infty$. We can choose a collection of local charts $\{U_\alpha\}_{\alpha\in I}$ in X covering A and holomorphic functions g_α in U_α such that $g = g_\alpha$ in the sheaf $A_{(\nu)}$. This means that

$$g_\alpha - g_\beta \in \mathcal{M}^\nu \mid_{U_\alpha \cap U_\beta}, \ \alpha, \beta \in I$$

Conversely, every collection of $\{g_\alpha\}_{\alpha\in I}$ satisfying the above conditions defines a global section of $A_{(\nu)}$.

Let $\mu, \nu \in \tilde{\mathbb{N}}, \mu \leq \nu$. We say that the homomorphism $\phi_{(\nu)} : A_{(\nu)} \to A'_{(\nu)}$ induces the homomorphism $\phi_{(\mu)} : A_{(\mu)} \to A'_{(\mu)}$, if the following diagram is commutative:

$$\begin{array}{ccc} A_{(\nu)} & \stackrel{\phi_{(\nu)}}{\to} & A'_{(\nu)} \\ \downarrow & & \downarrow \\ A_{(\mu)} & \stackrel{\phi_{(\mu)}}{\to} & A'_{(\mu)} \end{array} \tag{1.7}$$

We also say that $A_{(\nu)} \to A'_{(\nu)}$ extends $A_{(\mu)} \to A'_{(\mu)}$.

$\mathcal{Q}_1$ is the set of nilpotent elements of $A_{(2)}$ and so every homomorphism (isomorphism) $\phi_{(2)} : A_{(2)} \to A'_{(2)}$ induces a homomorphism (isomorphism) $\phi_{()} : \mathcal{Q}_1 \to \mathcal{Q}_1$. We also say that $\phi_{(2)}$ extends $\phi_{()}$.

Proposition 1.7. *Every homomorphism (isomorphism) $\phi_{(\nu)} : A_{(\nu)} \to A'_{(\nu)}, 2 \leq \nu < \infty$ induces natural homomorphisms (isomorphisms)*

$$A_{(\mu)} \to A'_{(\mu)}, \mu \leq \nu$$

Proof. It is enough to prove our claim for $\mu = \nu - 1$. For an arbitrary μ one can repeat the argument for the pair $\nu - 1, \nu - 2$ and so on. The kernel of $\pi : A_{(\nu)} \to A_{(\nu-1)}$ is $\mathcal{Q}_{\nu-1}$ and $\mathcal{Q}_{\nu-1}$ is has the property (1.5). Therefore $\phi_{(\nu)}$ sends $\mathcal{Q}_{\nu-1}$ to $\mathcal{Q}_{\nu'-1}$. This implies that $\phi_{(\nu)}$ induces the desired map $A_{(\nu)}/\mathcal{Q}_{\nu-1} \to A'_{(\nu)}/\mathcal{Q}_{\nu'-1}$, because $A_{(\nu-1)} = A_{(\nu)}/\mathcal{Q}_{\nu-1}$ and $A'_{(\nu-1)} = A'_{(\nu)}/\mathcal{Q}_{\nu'-1}$. □

The homomorphism $\phi_{(\infty)} : A_{(\infty)} \to A'_{(\infty)}$ is called convergent if it takes $A_{(*)}$ into $A'_{(*)}$.

The following proposition gives us the local information for analyzing a homomorphism $\phi_{(\nu)} : A_{(\nu)} \to A'_{(\nu)}, \ \nu \in \tilde{\mathbb{N}}$.

Proposition 1.8. *Let $a \in A$ and U be a small neighborhood of a in A. Let also $a' = \phi^{-1}(a)$ and $U' = \phi^{-1}(U)$. The following statements are true:*

1. *Every homomorphism (isomorphism) $\phi_{(*)} : A_{(*)} \mid_U \to A'_{(*)} \mid_{U'}$ which induces an isomorphism $A_{(1)} \mid_U \to A'_{(1)} \mid_{U'}$ is induced by a unique holomorphic (biholomorphic) map $(X', A', a') \to (X, A, a)$;*

2. *Every homomorphism* $\phi_{(*)} : A_{(*)} |_U \to A'_{(*)} |_{U'}$ *which induces isomorphisms* $A_{(1)} |_U \to A'_{(1)} |_{U'}$ *and* $A_{(2)} |_U \to A'_{(2)} |_{U'}$ *is an isomorphism also;*

3. *Every homomorphism (isomorphism)* $\phi_{(\nu)} : A_{(\nu)} |_U \to A'_{(\nu)} |_{U'}$, $2 \leq \nu < \infty$ *is induced by a homomorphism (isomorphism)* $A_{(*)} |_U \to A'_{(*)} |_{U'}$.

In the case where a is a regular point of both A and X, the proof of this proposition is easy. The proof in general uses simple properties of local rings and their homomorphisms. The reader is referred to [**Nag62**] for more informations about local ring theory.

Proof. By Proposition 1.2 the homomorphism $\phi_{(*)} : A_{(*)_a} \to A'_{(*)_{a'}}$ is induced by a unique map $(X', a') \to (X, a)$. We must prove that this map takes A' to A. Since $\phi_{(*)}$ induces an isomorphism $A_{(1)} |_U \to A'_{(1)} |_{U'}$, it takes the ideal of A in X to the ideal of A' in X'. This implies that $(X', A', a') \to (X, A, a)$.

The second and third statements have a completely algebraic nature. To prove them we use the following notations

$$R := A_{(*)_a} \cong A'_{(*)_{a'}},\ I := \mathcal{M}_{A,a}, \cong \mathcal{M}_{A',a'},\ \tau := \phi_{(*)},\ \tau_\nu := \phi_{(\nu)},\ \nu \in \mathbb{N}$$

(Note that (X, A) and (X', A') have the same local structure). Let us prove the second statement. Since $\tau_2 : R/I^2 \to R/I^2$ is an isomorphism and the nilpotent set of R/I^2 is the set I/I^2, we have $I = \tau(I) + I^2$. Let us prove that $\tau(I) = I$. Put

$$R' := I/\tau(I)$$

We have $IR' = R'$. Let $a_1, a_2, \ldots, a_r$ be a minimal set of generators for R'. We have $a_r \in R' = IR'$ and so

$$a_r = \sum_{i=1}^{r} s_i a_i,\ s_i \in I$$

or $(1 - s_r)a_r$ lies in the ideal generated by $a_1, a_2, \ldots, a_{r-1}$. Since $1 - s_r$ is a holomorphic function in (X, a) and its value in a is 1 it is invertible and so we get a contradiction with this fact that no proper subset of $a_1, a_2, \ldots, a_r$ generates R' (The used argument is similar to the proof of Nakayama's lemma (see [**GuII90**] A, Lemma 9)).

We have proved that $\tau(I) = I$. Since $\tau_1 : R/I \to R/I$ is an isomorphism and $\tau(I) = I$, τ is surjective. Now let us prove that τ is injective. Define

$$R_n := \{x \in R \mid \tau^n(x) = 0\}$$

τ induces a map from R_n to itself and the image of this map contains R_{n-1} Since R is a Noetherian ring and we have an increasing sequence of ideals $\cdots \subset R_n \subset R_{n+1} \subset \cdots$, there is a natural number n_0 such that $R_{n_0} = R_{n_0+1} = \cdots = R_*$. Now $\tau_* = \tau \mid_{R_*}$ is a surjective map from R_* to R_*. But by definition of R_*, τ_* must be zero. Therefore $R_* = 0$ and so $R_1 = 0$. This means that τ is injective.

Now let us prove the third statement. Let $x_1, x_2, \ldots, x_n$ form a basis for the vector space $\frac{\mathcal{M}_R}{\mathcal{M}_R^2}$, where $\mathcal{M}_R$ denotes the maximal ideal of R. We have seen in Proposition 1.3 that $(x_1, x_2, \ldots, x_n)$ form an embedding of (X, a) in $(\mathbb{C}^n, 0)$. We can choose elements $f_1, f_2, \ldots, f_n$ in R such that $\tau_\nu([x_i]) = [f_i],\ i = 1, 2, \ldots, n$, where $[.]$ denotes the equivalence class. Now it is easy to verify that the homomorphism

$$\tau : R \to R$$

$$f(x_1, x_2, \ldots, x_n) \to f(f_1, f_2, \ldots, f_n)$$

induces the desired map. If τ_ν is an isomorphism then by the second part of the proposition τ is also an isomorphism. $\square$

Now using Proposition 1.8 we can find geometrical interpretations of homomorphisms $A_{(\mu)} \to A'_{(\mu)},\ \mu \in \tilde{\mathbb{N}}$ as follows

1. There exists an isomorphism $\phi_{(*)} : A_{(*)} \to A'_{(*)}$ if and only if there exists a biholomorphism of some neighborhood of A into some neighborhood of A' in X' extending $\phi : A \to A'$;

2. Any isomorphism $\phi_{(\nu)} : A_{(\nu)} \to A'_{(\nu)}, 1 < \nu \in \mathbb{N}$ is given by a collection of biholomorphisms $(U_\alpha, A) \to (U'_\alpha, A')$, where $\{U_\alpha\}_{\alpha\in I}$ (resp. $\{U'_\alpha\}_{\alpha\in I}$) is an open covering of A (resp. A') in X (resp. X'), and such that $\phi_\alpha \circ \phi_\beta^{-1}$ is the identity up to holomorphic functions vanishing on A of order ν;

The first statement justifies the name neighborhood sheaf adopted for $A_{(*)}$.

Unfortunately an isomorphism

$$\phi_{(\infty)} : A_{(\infty)} \to A'_{(\infty)} \tag{1.8}$$

may not be given by a collection of isomorphisms $\phi_{(\nu)} : A_{(\nu)} \to A'_{(\nu)}, \nu \in \mathbb{N}$ such that for $\nu \geq \mu$, $\phi_{(\nu)}$ extends $\phi_{(\mu)}$. However, the $\phi_{(\infty)}$ which we will construct in the next section will have this property. For this reason when we talk about an isomorphism (1.8) we assume that it induces isomorphisms in finite neighborhoods.

1.6 Obstructions to formal isomorphism

In this section we will identify the obstructions for the existence of an isomorphism between formal neighborhoods of A and A'. We formulate our main problem in this section as follows: Let A' be the image of another embedding of A in a manifold X'.

1. Given an isomorphism $\phi : \mathcal{Q}_1 \to \mathcal{Q}'_1$. Under which conditions is it induced by an isomorphism $\phi_{(2)} : A_{(2)} \to A'_{(2)}$?
2. Given an isomorphism $\phi_{(\nu)} : A_{(\nu)} \to A'_{(\nu)}, \nu \geq 2$. Under which conditions does it extend to $\phi_{(\nu+1)} : A_{(\nu+1)} \to A'_{(\nu+1)}$?

In other words we want to describe the germ of an embedding $A \hookrightarrow X$ with minimal data. The first elementary data of an embedding is its normal bundle (when A is not smooth the sheaf $\mathcal{Q}_1 = \mathcal{M}/\mathcal{M}^2$ plays the role of the normal bundle). The other data of an embedding are its finite neighborhoods.

Note that if all such conditions in the above questions are satisfied for A and A', we get only an isomorphism of formal neighborhoods of A and A'. The applied methods are quite formal and can be found in [**Gr62**, **HiRo64**, **La71**]. In what follows, every homomorphism $A_{(\nu)} \to A'_{(\nu)}, \nu \in \tilde{\mathbb{N}}$ which we consider will be an extension of the fixed isomorphism (1.6) (Note that $A_{(1)}$ is the structural sheaf of A).

Let $a \in A$ and $a' = \phi^{-1}(a)$ be its corresponding point in A'. The stalk of the sheaf $A_{(\nu)}, \nu \in \tilde{N}$ at a is denoted by ${A_{(\nu)}}_a$. Any isomorphism

$$\phi_{(\nu)_a} : {A_{(\nu)}}_a \to {A'_{(\nu)}}_{a'} \tag{1.9}$$

determines an isomorphism between $A_{(\nu)} \mid_{U_a}$ and $A'_{(\nu)} \mid_{U_{a'}}$, where U_a and $U_{a'}$ are two open neighborhood of a and a' in A and A', respectively (see Proposition 1.8).

The following proposition gives us the local solutions of our problem:

Proposition 1.9. *Any isomorphism* $\phi_{(\nu)_a} : {A_{(\nu)}}_a \to {A'_{(\nu)}}_{a'}$ *is induced by an isomorphism*

$$\phi_{(*)_a} : {A_{(*)}}_a \to {A_{(*)}}_{a'} \tag{1.10}$$

and hence extends to

$$\phi_{(\nu+1)_a} : {A_{(\nu+1)}}_a \to {A_{(\nu+1)}}_{a'} \tag{1.11}$$

Proof. The above proposition is the third part of Proposition 1.8 in another form. Note that the isomorphism $\phi_{(*)} : A_{*,a} \to A'_{*,a'}$ is not unique. $\square$

In the introduction of [**GrRe84**] we find the following statement of H. Cartan: *la notion de faisceau s'introduit parce qu'il s'agit de passer de données locales à l'etude de propriétés globales.* Like many other examples in complex analysis, the obstructions to glue the local solutions lie in a first cohomology group of a sheaf over A. The precise identification of that sheaf and its first cohomology group is our main objective in this section.

Now, let us be given an isomorphism $\phi_{(\nu)} : A_{(\nu)} \to A'_{(\nu)}$. We want to extend $\phi_{(\nu)}$ to $\phi_{(\nu+1)} : A_{(\nu+1)} \to A'_{(\nu+1)}$, i.e. to find an isomorphism $\phi_{(\nu+1)} : A_{(\nu+1)} \to A'_{(\nu+1)}$ such that the following diagram is commutative:

$$\begin{array}{ccc} A_{(\nu+1)} & \overset{\phi_{(\nu+1)}}{\to} & A'_{(\nu+1)} \\ \downarrow & & \downarrow \\ A_{(\nu)} & \overset{\phi_{(\nu)}}{\to} & A_{(\nu)} \end{array} \tag{1.12}$$

Proposition 1.9 gives us the local solutions

$$\begin{array}{ccc} A_{(\nu+1)_a} & \overset{\phi_{(\nu+1)_a}}{\to} & A'_{(\nu+1)_{a'}} \\ \downarrow & & \downarrow \\ A_{(\nu)_a} & \overset{\phi_{(\nu)_a}}{\to} & A'_{(\nu)_{a'}} \end{array} \tag{1.13}$$

where $A_{(\nu)_a}$ is the stalk of the sheaf $A_{(\nu)}$ over the point a. Now, cover A with small open sets for which we have the diagrams of the type (1.13). Combining two diagrams in the intersection of neighborhoods of the points a and b we get:

$$\begin{array}{ccc} A_{(\nu+1)_{a,b}} & \overset{\phi_{(\nu+1)_{a,b}}}{\to} & A_{(\nu+1)_{a,b}} \\ \downarrow & & \downarrow \\ A_{(\nu)_{a,b}} & \overset{id}{\to} & A_{(\nu)_{a,b}} \end{array} \tag{1.14}$$

where

$$\phi_{(\nu+1)_{a,b}} = \phi_{(\nu+1)_a}^{-1} \circ \phi_{(\nu+1)_b} \tag{1.15}$$

Note that we have used the notation $\phi_{(\nu+1)_{a,b}}$ instead of $\phi_{(\nu+1)} |_{U_a \cap U_b}$, $\phi_{(\nu+1)_a}$ instead of $\phi_{(\nu+1)} |_{U_a}$ and so on. The above transition elements are obstruction to our extension problem. Now it is natural to define the following sheaf:

$Aut(\nu)$ is the sheaf of isomorphisms $\phi_{(\nu+1)} : A_{(\nu+1)} \to A_{(\nu+1)}$ inducing the identity in $A_{(\nu)}$, i.e. the following diagram is commutative

$$\begin{array}{ccc} A_{(\nu+1)} & \stackrel{\phi_{(\nu+1)}}{\rightarrow} & A_{(\nu+1)} \\ \downarrow & & \downarrow \\ A_{(\nu)} & \stackrel{id}{\rightarrow} & A_{(\nu)} \end{array} \tag{1.16}$$

Later in Proposition 1.11 we will see that $Aut(\nu)$ is a sheaf of Abelian groups. Now the data in (1.15) form an element of

$$H^1(A, Aut(\nu))$$

The elements of $H^1(A, Aut(\nu))$ are obstructions to the extension problem.

It is clear that the case $\nu = 1$ needs an special treatment. $A_{(1)}$ is the structural sheaf of A and the condition $H^1(A, Aut(1)) = 0$ means that any two embeddings of A have the same 2-neighborhood and in particular have isomorphic $\mathcal{M}/\mathcal{M}^2$'s. This implies that the normal bundles of A and A' are isomorphic! Therefore, the definition of $Aut(1)$ is not useful. We modify this definition as follows:

$Aut(1)$ is the sheaf of isomorphisms $\phi_{(2)} : A_{(2)} \to A_{(2)}$ inducing the identity on $\mathcal{M}/\mathcal{M}^2$ and for which the following diagram is commutative

$$\begin{array}{ccc} A_{(2)} & \stackrel{\phi_{(2)}}{\rightarrow} & A_{(2)} \\ \downarrow & & \downarrow \\ A_{(1)} & \stackrel{id}{\rightarrow} & A_{(1)} \end{array} \tag{1.17}$$

Proposition 1.10. *If $H^1(A, Aut(\nu)) = 0$ then any isomorphism*

1. $\phi_{(\nu)} : A_{(\nu)} \to A'_{(\nu)}$ *if* $\nu > 1$
2. $\phi_{()} : \mathcal{Q}_1 \to \mathcal{Q}'_1$ *if* $\nu = 1$

extends to an isomorphism $\phi_{(\nu+1)} : A_{(\nu+1)} \to A'_{(\nu+1)}$.

Proof. The obstruction to the above extension is obtained by diagram (1.14) and so is an element of $H^1(A, Aut(\nu))$. □

Now we have to identify $Aut(\nu)$ and especially we have to verify when $H^1(A, Aut(\nu)) = 0$ is satisfied.

Proposition 1.11. *Suppose that X is a smooth variety. For $\nu \geq 2$ we have*

$$Aut(\nu) \cong \mathcal{T}(\nu)(:= \mathcal{T} \otimes_{\mathcal{O}_A} \mathcal{Q}_\nu)$$

where $\mathcal{T}$ is the sheaf of holomorphic vector fields in X (sections of the tangent bundle of X); for the case $\nu = 1$ we have

$$Aut(1) \cong \mathcal{T}_A(1)(:= \mathcal{T}_A \otimes_{\mathcal{O}_A} \mathcal{Q}_1)$$

where $\mathcal{T}_A$ is the sheaf of holomorphic vector fields in X tangent to A.

Proof. Let us introduce the function which will be our candidate for the desired isomorphisms. First consider the case $\nu \geq 2$.

$$* : \mathcal{T}(\nu) \to Aut(\nu)$$

For any $\psi \in \mathcal{T}(\nu)$ define

$$\beta, \beta' : A_{(\nu+1)} \to A_{(\nu+1)}$$

$$\beta(f) = f + \psi.df$$

$$\beta'(f) = f - \psi.df$$

we have

$$\beta \circ \beta'(f) = f - \psi.df + \psi d(f - \psi.df) = f - \psi.d(\psi.df) = f \text{ mod } \mathcal{M}^{2\nu-1}$$

We have $2\nu - 1 \geq \nu + 1$ and so

$$\beta \circ \beta'(f) = f \text{ mod } \mathcal{M}^{\nu+1} \tag{1.18}$$

In other words β' is the inverse function of β. We define

$$*(\psi) = \beta$$

Now it is enough to prove that $*$ is the desired isomorphism. Since X is nonsingular $*$ is injective. Let $\beta \in Aut(\nu)$. We write

$$\beta(f) - f = \psi'(f)$$

$\psi'(f) = 0 \text{ mod } \mathcal{M}^\nu$ and so $\psi' \in Hom(A_{(\nu+1)}, \mathcal{M}^\nu/\mathcal{M}^{\nu+1})$. Composing with $A_{(*)} \to A_{(\nu+1)}$ and without change in notations we can assume

$$\psi' \in Hom(A_{(*)}, \mathcal{M}^\nu/\mathcal{M}^{\nu+1})$$

Let $z_1, z_2, \ldots, z_n$ be local coordinates. Define

$$\psi(dz_i) = \psi'(z_i)$$

Then $\psi \in \mathcal{T}(\nu)$ and the mapping $\beta \to \psi$ is the inverse of $*$.

The case $\nu = 1$ is the same as previous one. We need to substitute $\mathcal{T}_A$ for $\mathcal{T}$ to get the congruency (1.18). □

How can we calculate the cohomology groups $H^1(A, \mathcal{T}(\nu))$? To do this, we break $\mathcal{T}(\nu)$ into two other simple sheaves as follows:
There is a natural short exact sequence

$$0 \to \mathcal{T}_A \to \mathcal{T} \to \mathcal{Q}_1^* \to 0$$

By tensorial multiplication over $\mathcal{O}_A$ with $\mathcal{Q}_\nu$, we have

$$0 \to \mathcal{T}_A(\nu) \to \mathcal{T}(\nu) \to \mathcal{Q}_{\nu-1} \to 0$$

This gives us the long exact sequence

$$\ldots \to H^1(A, \mathcal{T}_A(\nu)) \to H^1(A, \mathcal{T}(\nu)) \to H^1(A, \mathcal{Q}_{\nu-1}) \to \ldots$$

We summarize the above arguments in the following proposition:

Theorem 1.5. *If $H^1(A, \mathcal{T}_A(\nu)) = 0$ and $H^1(A, \mathcal{Q}_{\nu-1}) = 0$ then $H^1(A, \mathcal{T}(\nu)) = 0$ and so any isomorphism*

$$\phi_{(\nu)} : A_{(\nu)} \to A'_{(\nu)} \text{ if } \nu > 1$$

$$\phi : \mathcal{Q}_1 \to \mathcal{Q}'_1 \text{ if } \nu = 1$$

extends to an isomorphism $\phi_{(\nu+1)} : A_{(\nu+1)} \to A'_{(\nu+1)}$.

In the case of A a Riemann surface embedded in a two dimensional manifold we can substitute the conditions $H^1(A, \mathcal{T}_A(\nu)) = 0$ and $H^1(A, \mathcal{Q}_{\nu-1}) = 0$ by some numerical ones. The Serre duality will be used for this purpose.

Theorem 6. *(Serre Duality) Let A be a complex manifold of complex dimension n and V a holomorphic vector bundle over A. Then there exists a natural $\mathbb{C}$-isomorphism*

$$H^q(A, \underline{\Omega^p \otimes V}) \cong (H^{n-q}(A, \underline{\Omega^{n-p} \otimes V^*}))^*$$

where $\Omega^p = T^*A \wedge T^*A \wedge \cdots \wedge T^*A$ p times.

For a proof of this theorem the reader is referred to [**Ra65**]. From now on we do not use the line under bundles (it denotes the sheaf of sections), for instance instead of $H^1(A, \underline{\Omega^1})$ we write $H^1(A, \Omega^1)$. Let A be a Riemann surface. Putting $p = 0,\ q = 1$ we have

$$H^1(A, V) \cong (H^0(A, \Omega^1 \otimes V^*))^*$$

Now

$$H^1(A, \mathcal{Q}_{\nu-1}) = H^1(A, (N^*)^{\nu-1})) = (H^0(A, \Omega^1 \otimes N^{v-1}))^*$$

$\Omega^1 \otimes N^{v-1}$ has no global holomorphic section if

$$c(\Omega^1 \otimes N^{\nu-1}) = 2g - 2 + (\nu - 1)A.A < 0 \tag{1.19}$$

In the same way

$$H^1(A, \mathcal{T}_A(\nu)) = (H^0(A, \Omega^1 \otimes (TA)^* \otimes N^\nu))^* = (H^0(A, \Omega^1 \otimes \Omega^1 \otimes N^\nu))^* = 0$$

if

$$c(\Omega^1 \otimes \Omega^1 \otimes N^\nu) = 2(2g - 2) + \nu A.A < 0 \tag{1.20}$$

Finally we conclude that

Theorem 1.6. *Let A be a Riemann surface of genus g embedded in a two dimensional manifold X. Suppose that*

- $A.A < 0$ *if* $g = 0$*;*
- $A.A < 2(2 - 2g)$ *if* $g \geq 1$

Then the embedding $A \hookrightarrow X$ is formally equivalent with $A' \hookrightarrow X'$, where the normal bundle of A' in X' equals the normal bundle of A in X.

Proof. Since the normal bundle of A' in X' equals the normal bundle of A in X, there exists an isomorphism $\phi_{()} : \mathcal{Q}_1 \to \mathcal{Q}'_1$. To extend this isomorphism to a formal isomorphism of the neighborhoods of A and A' in X and X', respectively, we must have the inequalities (1.20) for all $\nu \geq 1$ and (1.19) for all $\nu > 1$ satisfied. This implies exactly $A.A < 0$ if $g = 0$ and $A.A < 2(2-2g)$ if $g \geq 1$. □

1.7 Construction of embedded Riemann surfaces

In this section we discuss various ways for constructing an embedding of a Riemann surface A in a two dimensional manifold. The positive embeddings are abundant. They can be obtained by hyperplane sections of two dimensional algebraic manifolds. The first natural way to get a negative embedding is the following:

Let A be a Riemann surface and $A \hookrightarrow X$ a positive embedding of A in a two dimensional manifold, i.e. $A.A \geq 0$. Performing a blow up in a point x of A gives us another embedding of A in a two dimensional manifold with self-intersection $A.A-1$. In fact, the new normal bundle of A is $N.L_{-x}$, where N is the normal bundle of A in X and L_{-x} is the line bundle associated to the divisor $-x$. Performing more blow ups in the points of A gives us negative embeddings of A with arbitrary self-intersection.

We have learned another way of changing the normal bundle of an embedding from P. Sad which goes as follows. The basic idea comes from [**CMS02**].

Fix a germ of an embedding (X, A) (for instance we can suppose $X = A \times \mathbb{C}$). Let $\mathcal{S}$ be the sheaf of local biholomorphisms $(X, A, x) \to (X, A, x)$, $x \in A$ sending A to A identically. $\mathcal{S}$ is clearly a non-Abelian sheaf. We define an equivalence relation in $H^1(A, \mathcal{S})$ as follows: For $F = \{F_{ij}\}, F' = \{F'_{ij}\} \in H^1(A, \mathcal{S})$, $F \sim F'$ if and only if there exists a collection of biholomorphisms $\{g_i\}$ such that

$$F'_{ij} = g_i \circ F_{ij} \circ g_j^{-1}$$

We define

$$\mathcal{I}(X) = H^1(A, \mathcal{S})/\sim$$

To each $F \in \mathcal{I}(X)$ we can associate the line bundle $L_F = \{det(DF_{ij}\mid_A)\}$.

Let $\{\psi_i\}$ be a collection of chart maps for the germ (X, A) and $F = \{F_{ij}\} \in H^1(A, \mathcal{S})$. The new collection of transition functions

$$\psi_i \circ F_{ij} \circ \psi_j^{-1}$$

defines an embedding of A with the normal bundle $L_F N$. We can see easily that two $F, F' \in H^1(A, \mathcal{S})$ give us the same embedding if and only if $F \sim F'$. Therefore we have

Proposition 1.12. $\mathcal{I}(X)$ *is the moduli space of germs of all embeddings of A in two dimensional manifolds. Moreover the line bundle of the embedding associated to $F \in \mathcal{I}(X)$ is $L_F.N$.*

Now let $X = N$ be a linear bundle. Consider another line bundle M over A with a meromorphic section s of M. There is defined the biholomorphism

$$\delta : N \to N.M$$

$$v \to v.s$$

which is well-defined out of the fibers passing through the zeros and poles of s. Now we can define

$$\Delta : \mathcal{I}(N) \to \mathcal{I}(NM)$$

$$\{F_{ij}\} \to \{\delta \circ F_{ij} \circ \delta^{-1}\}$$

The line bundles associated to F and $\Delta(F)$ are equal but the normal bundle of the embedding associated to F is $L_F N$ and to $\Delta(F)$ is $L_F NM$.

Another interesting method which can give us embedded Riemann surfaces is the action of groups. Consider a subgroup G of $Diff(\mathbb{C}^2, 0)$ and denote by G_0 its linear group. After a blow up in $0 \in \mathbb{C}^2$ we can consider G as a group which acts in a neighborhood of $\mathbb{P}^1$, the projective line of the blow up. Now we assume that G_0 is a Kleinian group which acts on $\mathbb{P}^1$. If U_0 is a region in $\mathbb{P}^1$ such that $A = U_0/G_0$ is a compact Riemann surface then it would be interesting to find a region U in a neighborhood of U_0 in the blow up space such that $(U/G, A)$ is an embedding of A. For more information about Kleinian groups the reader is referred to [**Ma88**] and [**Le66**].

Complementary notes

1. The complementary material to section 1.1 can be Weierstrass preparation and division theorems, Theorem 4A,5A of [**GuII90**]. One can include also sections A,B of [**GuIII90**] for the notion of sheaf and cohomology of a sheaf.
2. Section I,J of [**GuII90**] are devoted to the tangent space of an analytic variety and can be included in section 1.1. Particularly Theorem 16I claims that the both notions of tangent space there and here are the same. This will be useful for section 1.6 if one wants to follow the arguments in a general case of an embedded A in a variety X. One can also include the notion of linear spaces over varieties from the survey in [**GPR94**] chapter 2 section 3.
3. One can use [**Gri66**] in section 1.6 for more extension problems such as the extension of fiber bundles, holomorphic maps and cohomology elements.
4. Section M of [**GuIII90**] covers various equivalent definitions of Stein spaces and fill the proof of the equivalent definitions of a Stein variety stated in the beginning of section 1.2. Cartan's B theorem and preliminary materials on Cech cohomology can be found there.

5. There are many contributions to complex analysis which are concerned with the following problem: When the quotient space of an equivalence relation in a complex space is again a complex space. Grauert's direct image theorem plays an important role in these works. For a more detailed study in this direction we recommend the article [**Gr83**] and its references.

6. It would be nice if the proofs of Remmert proper mapping theorem and Grauert direct image theorem would be discussed along the study of this text. These proofs and more applications of these classical theorems can be found in [**GrRe84**]

Chapter 2

Pseudoconvex Domains

The notions of plurisubharmonic functions and pseudoconvex domains appeared in complex analysis after E.E. Levi discovered around 1910 that the boundary of a domain of holomorphy in $\mathbb{C}^n$ satisfies certain conditions of pseudoconvexity. The question of whether conditions on the boundary might determine a domain of holomorphy became known as the Levi problem. The first definitions were made by K. Oka [**Ok42**] and P. Lelong [**Le45**]. The reader is referred to T. Peternell survey in [**GPR94**] Chapter V and [**GuI90**] K-R for more history and developments not treated here. In this text we will consider only the C^2 category of plurisubharmonic functions. We start this chapter by introducing the notion of strongly convex functions. They just carry the convexity information of their fibers and contain the class of strongly plurisubharmonic functions. Strongly convex functions are easy to handle and this is the main reason we have chosen them instead of strongly plurisubharmonic functions. We also define the notion of convex function parallel to plurisubharmonic functions. But this seems to be useless, since they do not satisfy the maximum principle!

In this chapter for topological spaces A and B we write $A \subset\subset B$ to denote that A is relatively compact in B, i.e. the closure of A in B is compact in B.

2.1 Strongly convex functions

Let $\psi : (\mathbb{C}^n, p) \to \mathbb{R}$ be a C^2-function. Recall that

$$\frac{\partial\psi}{\partial z_j} = \frac{1}{2}(\frac{\partial\psi}{\partial x_j} - i\frac{\partial\psi}{\partial y_j}),\ \frac{\partial\psi}{\partial \bar{z}_j} = \frac{1}{2}(\frac{\partial\psi}{\partial x_j} + i\frac{\partial\psi}{\partial y_j})$$

The Levi form of ψ at $p \in \mathbb{C}^n$ is defined by

$$L_p(\psi)(v) := \sum_{i,j=1}^{n} \frac{\partial^2 \psi}{\partial z_i \partial \bar{z}_j}(p) v_i \bar{v_j},\ v = (v_1, v_2, \dots, v_n) \in \mathbb{C}^n$$

The following simple equalities will be useful in forthcoming arguments.

$$(2.1)\ L_p(h\psi)(v) = h(p)L_p(\psi)(v) + \psi(p)L_p(h)(v) + 2Re(D_p\psi(v).\bar{D}_p h(v))$$

$$L_p(\phi \circ \psi) = \phi''(\psi(p))|D\psi|^2 + \phi'(\psi(p))L_p(\psi)$$

$$L_p(-\log \psi)(v) = \frac{1}{\psi(p)}(\frac{|D_p\psi(v)|^2}{\psi(p)} - L_p(\psi)(v))$$

where $\psi, h : (\mathbb{C}^n, p) \to \mathbb{R}$ are two C^2 functions and ϕ is a $\mathbb{R}$-valued C^2 function defined in a neighborhood of the image of ψ.

A C^2 function $\psi : (\mathbb{C}^n, p) \to \mathbb{R}$ is called convex (resp. strongly convex) at the point p in the sense of Levi if

$$(2.2) \qquad D_p\psi(v) = 0 \Rightarrow L_p(\psi)(v) \geq 0 \text{ (resp. } > 0),\ \forall v \in \mathbb{C}^n, v \neq 0$$

Practically in the above definition we will assume the additional condition $v \in S := \{v \in \mathbb{C}^n \mid |v| = 1\}$ to obtain a compact space for the parameter v. This does not change the definition. Let $G \subset \mathbb{C}^n$ be an open domain and $\psi : G \to \mathbb{R}$ a C^2 function. We say that ψ is convex (resp. strongly convex) in G if it is convex (resp. strongly convex) at each point $p \in G$.

Note that in the one dimensional case a C^2 function $G \to \mathbb{R},\ G \subset \mathbb{C}$ is (strongly) convex if

$$\frac{\partial \psi}{\partial z}(p) = 0 \Rightarrow \frac{\partial^2 \psi}{\partial z \partial \bar{z}}(p) \geq 0 \text{ (resp. } > 0),\ \forall p \in G$$

ψ is strongly convex if and only if it has no local maximum in G. Unfortunately we cannot say a similar statement for ψ convex. For example $\psi(x + iy) = -(x^4 + y^4)$ has a local maximum at 0 and is a convex function. From now on we work only with strongly convex functions.

Let $\psi : (\mathbb{C}^n, p) \to \mathbb{R},\ \psi(p) = 0$ be strongly convex at p and $h : (\mathbb{C}^n, p) \to \mathbb{R}^+$ be a C^2 function. Using (2.1) one can easily check that $h\psi$ is also strongly convex at p. The following propositions reveal some important properties of strongly convex functions.

Proposition 2.1. *If $\psi : (\mathbb{C}^n, p) \to \mathbb{R}$ is strongly convex at p then it is strongly convex in a neighborhood of p in $\mathbb{C}^n$.*

Proof. The projection on the second coordinate $\pi : Y \to (\mathbb{C}^n, p)$, where $Y := \{(v, x) \in S \times (\mathbb{C}^n, p) \mid D_x\psi(v) = 0\}$ is a continuous proper map. Now $L_.(.) : Y \to \mathbb{R}$ is continuous and strictly positive on the fiber $\pi^{-1}(p)$. Therefore it must be strictly positive on the fibers $\pi^{-1}(x)$ for x in a neighborhood of p in $\mathbb{C}^n$. □

Proposition 2.2. *Let ψ be a strongly convex function in a neighborhood of a compact set K in $\mathbb{C}^n$. There exists an $\epsilon > 0$ such that if h is a real-valued C^2 function on a neighborhood of K in $\mathbb{C}^n$ and the absolute values of its first and second derivatives on this neighborhood are less than ϵ, then $\psi + h$ is strongly convex in a neighborhood of K in $\mathbb{C}^n$.*

Proof. $L_.(.)$ is strictly positive on $Y = \{(v, x) \in S \times K \mid D_x\psi(v) = 0\}$. Therefore it is positive in a compact neighborhood U of Y in $S \times \mathbb{C}^n$. Since the projection on the second coordinate is a continuous proper map, we can choose a neighborhood U' of K in $\mathbb{C}^n$ such that for all $(v, x) \in S \times U'$ if $D\psi_x(v) = 0$ then $(v, x) \in U$. We take ϵ_1 such that if for $(v, x) \in S \times U'$ we have $|D_x\psi(v)| < \epsilon_1$ then $(v, x) \in U$ for all $x \in U'$. We take also $\epsilon_2 = min_{(v,x)\in U} \frac{L_x(\psi)(v)}{(\sum |v_i|)^2}$. Now U' and $\epsilon := min\{\epsilon_1, \epsilon_2\}$ are the desired objects. If $D_x(\psi + h)(v) = 0$ then $|D_x\psi(v)| = |D_x h(v)| < \epsilon|v| = \epsilon$ and so $(v, x) \in U$. Now

$$L_x(\psi + h)(v) \geq L_x(\psi) - |L_x(h)| \geq L_x(\psi) - \epsilon \sum |v_i||\bar{v}_j| =$$

$$L_x(\psi) - \epsilon(\sum |v_i|)^2 > 0, \ (v, x) \in U$$

□

Now it is an easy exercise to verify that: 1. The pull-back of a strongly convex function by a biholomorphic map is a strongly convex function. This statement is not true when we replace biholomorphic with holomorphic, for instance take a constant function which is of course a holomorphic function 2. The restriction of a strongly convex function $\psi : (\mathbb{C}^n, 0) \to \mathbb{R}$ to $\mathbb{C}^m = \{(z_1, z_2, \ldots, z_m, 0, \ldots, 0) \in (\mathbb{C}^n, 0)\}$ is a strongly convex function.

We are in a position to extend the notion of strongly convex functions to varieties. Let X be an analytic variety and $\psi : X \to \mathbb{R}$ a continuous function. Then ψ is called strongly convex if for every local chart $\phi : U \to V \subset D \subset \mathbb{C}^n$, U an open subset of X and V a closed analytic subset of the open subset

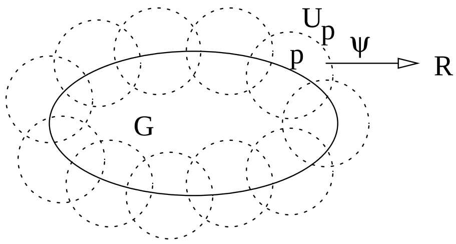

Figure 2.1: Strongly pseudoconvex domain

D of some $\mathbb{C}^n$, there exists a strongly convex function $\check{\psi}$ on D such that $\psi = \check{\psi} \circ \phi$. Now the mentioned facts and Proposition 1.3 imply that the above definition is independent of the choice of a local chart.

Proposition 2.3. *(maximum principle) Let (X, p) be a germ of a variety and $X \neq p$. There does not exist a strongly convex function $\psi : (X, p) \to \mathbb{R}$ such that $\psi(y) \leq \psi(p),\ \forall y \in (X, p)$*

Proof. We take a holomorphic function $\gamma : (\mathbb{C}, 0) \to (X, p)$.The pullback $\psi \circ \gamma$ is a strongly convex function and hence does not take maximum at 0. □

The above statement can be reformulated as follows: If $\psi : (X, p) \to \mathbb{R}, \psi(p) = 0$, is a strongly convex function then there do not exist non discrete analytic varieties Y such that $Y \subset \{\psi(x) \leq 0\}$.

2.2 Strongly pseudoconvex domains

Let X be an analytic variety and G a relatively compact open subset of X. We say that G is strongly pseudoconvex if for every point p in the boundary of G there exist a neighborhood U_p of p and a real valued strongly convex C^2-function ψ defined in U_p such that

$$G \cap U_p = \{x \in U_p \mid \psi(x) < 0\}$$

(see Figure 2.1). The next proposition shows that instead of local C^2-functions ψ, we can choose a global one.

Proposition 2.4. *Let $G \subset X$ be a relatively compact strongly pseudoconvex domain. There exists a neighborhood U of ∂G and a strongly convex C^2-function ψ in U such that*

$$U \cap G = \{x \in U \mid \psi(x) < 0\}$$

Proof. Let $p \in \partial G$. We have a strongly convex function $\psi : U_p \to \mathbb{R}$ defined in a neighborhood U_p of p such that $U_p \cap G = \{x \in U \mid \psi(x) < 0\}$. Let $h : U_p \to \mathbb{R}^+$ be a C^2-function on U_p and V an open subset of U_p such that $p \in V \subset\subset supp(h) \subset\subset U_p$. Since ∂G is compact, we can cover it by a finite number of such V's, say $\partial G \subset \cup_{i=1}^r V_i$. Let h_i be the associated function to V_i as above. We claim that the function

$$\psi = \sum_{i=1}^{r} h_i \psi_i$$

is the desired function. In fact ψ restricted to ∂G is zero and is strictly negative in $U \cap G$ (because ψ_i are negative and at least one of them is strictly positive at each point). At each point $p \in \partial G$ one of the $h_i \psi_i$ is strongly convex and all the others are convex functions. This implies that the above sum is strongly convex at $p \in \partial G$. By Proposition 2.1 ψ is strongly convex in a neighborhood of p in X. Since ∂G is compact, a finite union of these open sets gives us the desired neighborhood. □

Proposition 2.5. *Let $G \subset X$ be a relatively compact strongly pseudoconvex domain and ψ be the function defined in a neighborhood U of ∂G as in Proposition 2.4. There exists an ϵ such that if the values of a C^2-function h defined in U and its first and second derivatives are less than ϵ then $\{x \in U \mid \psi(x) < h(x)\}$ is strongly pseudoconvex.*

Proof. This is a direct consequence of Proposition 2.2 and 2.4. □

Theorem 2.1. *Let $G \subset X$ be a relatively compact strongly pseudoconvex domain. Then there exists a compact set $K \subset G$ containing all nowhere discrete analytic compact subsets of G.*

Proof. Let ψ be as in Proposition 2.4 and

$$U_1 = \{x \in U \mid -\epsilon < \psi(x) < 0\}$$

for a small ϵ. We claim that $K = G - U_1$ is the desired compact set. Let A be an analytic nowhere discrete compact subset of G and $A \not\subset K$ or equivalently

$A \cap U_1$ is not empty. Then ψ has a maximum greater than $-\epsilon$ in $A \cap U_1$. By Proposition 2.3 this is a contradiction with the fact that ψ is strongly convex. □

If K is analytic, compact and nowhere discrete we say that K is maximal.

2.3 Plurisubharmonic functions

A C^2-function $\psi : (\mathbb{C}^n, p) \to \mathbb{R}$ is called plurisubharmonic (resp. strongly plurisubharmonic) at p if its Levi form at p is positive semidefinite (resp. positive definite), i.e.

$$L_p(\psi)(v) := \sum_{i,j=1}^{n} \frac{\partial^2 \psi}{\partial z_i \partial \bar{z}_j}(p) v_i \bar{v}_j \geq 0 \, (resp. > 0) \, \forall v = (v_1, v_2, \cdots, v_n) \neq 0 \in \mathbb{C}^n,$$

Let $G \subset \mathbb{C}^n$ be an open domain and $\psi : G \to \mathbb{R}$ a C^2 function. We say that ψ is plurisubharmonic (resp. strongly plurisubharmonic) in G if it is plurisubharmonic (resp. strongly plurisubharmonic) at each point $p \in G$. A strongly plurisubharmonic function at p satisfies (2.2) and so it is a strongly convex function.

The most simple strongly plurisubharmonic function is $\psi(z) = |z|^2 = \sum_{i=1}^{n} z_i \bar{z}_j$, $\psi : \mathbb{C}^n \to \mathbb{R}^+$. For a holomorphic function f on an open domain $D \subset \mathbb{C}^n$ the function $log|f|$ is plurisubharmonic. In fact

$$2\frac{\partial^2 log|f|}{\partial \bar{z}_i \partial z_j} = \frac{\partial \frac{\frac{\partial f}{\partial z_j}}{f}}{\partial \bar{z}_i} = 0$$

Propositions 2.1, 2.2 are valid when we replace strongly convex with strongly plurisubharmonic. The proofs go as follows: Since $S := \{v \in \mathbb{C}^n \mid |v| = 1\}$ is compact, the map $L_p(\psi) : S \to \mathbb{R}$ has a minimum > 0. The functions $\frac{\partial^2 \psi}{\partial z_i \partial \bar{z}_j}(x)$ are continuous in x and so in a small neighborhood of p $L_x(\psi)$ reaches its minimum at a positive real number. This proves Proposition 2.1.

We take $\epsilon = min_{p \in U, v \in S} \frac{L_p(\psi)(v)}{(\sum |v_i|)^2}$, where U is a compact neighborhood of K in which ψ is strongly plurisubharmonic. In U we have

$$L_p(\psi + h)(v) \geq L_p(\psi) - \mid L_p(h) \mid \geq L_p(\psi) - \epsilon \sum \mid v_i \mid\mid \bar{v}_j \mid =$$

$$L_p(\psi) - \epsilon(\sum \mid v_i \mid)^2 > 0, \; p \in U, \; v \in S$$

Proposition 2.6. *Let $\psi : (\mathbb{C}^n, p) \to \mathbb{R}$ with $\psi(p) = 0$ be a strongly convex function. There exists a C^2 function $h : (\mathbb{C}^n, p) \to \mathbb{R}^+$ such that $h\psi$ is strongly plurisubharmonic.*

Proof. Let h be a C^∞ function in $(\mathbb{C}^n, p)$ such that $D_p h = D_p \psi,\ h(p) = \epsilon > 0$. We claim that for an ϵ enough small $h\psi$ is strongly plurisubharmonic at p. Using the formula (2.1) we have: $L_p(h\psi)(v) = \epsilon L_p(\psi)(v) + 2 \mid D_p\psi(v) \mid^2$. Let $S := \{v \mid\mid v \mid= 1\}$ and $H = S \cap \{v \in \mathbb{C}^n \mid D_p\psi(v) = 0\}$. By hypothesis $L_p(h\psi)(.)$ is strictly positive in H and hence in a compact neighborhood K_1 of H in S. Let K_2 be a compact subset of S such that $S = K_1 \cup K_2$ and $K_2 \cap H$ is empty. On K_2, $A = \frac{L_p(\psi)(.)}{2|D_p\psi(.)|^2}$ is a well-defined function with a minimum c. If c is positive then $L_p(h\psi)$ is already positive definite. If c is negative we can take $0 < \epsilon < \frac{-1}{c}$ and conclude that $h\psi$ is strongly plurisubharmonic at p and so by the discussion before Proposition 2.6 it is strongly plurisubharmonic in a neighborhood of p in $\mathbb{C}^n$. □

Note that the above argument in dimension $n = 1$ implies that an smooth point of the boundary of an open domain in $\mathbb{C}$ is given by the zero locus of a plurisubharmonic function.

Proposition 2.7. *Let $\psi : (\mathbb{C}^n, p) \to \mathbb{R}$ with $\psi(p) = 0$ be a strongly plurisubharmonic function and $h : (\mathbb{C}^n, p) \to \mathbb{R}^+$ be a C^2 function such that $D_p(h) = D_p(\psi)$. Then $h\psi$ is a strongly plurisubharmonic function in a neighborhood of p in $\mathbb{C}^n$.*

Proof. Using the hypothesis and 2.1 we get $L_p(h\psi)(v) = h(p)L_p(\psi)(v) + 2 \mid D_p\psi(v) \mid^2$. Therefore $h\psi$ is strongly plurisubharmonic at p and so by the discussion before Proposition 2.6 it is strongly plurisubharmonic in a neighborhood of p in $\mathbb{C}^n$. □

Proposition 2.4 is also true when we replace strongly convex with strongly plurisubharmonic. This proposition is stated in [**Gr62**] p. 338 Satz 2 and [**GPR94**] p. 228. In the proof by Grauert one reads: *Wie man leicht nachrechnet, ist the Levi form $L(\phi^*)$ in z positiv definit.* This easy calculation in Narasimahn's paper [**Na62**] p. 204 and Laufer's book [**La71**] Lemma 4.12 takes form as a complicated argument. This was one of the main reasons for us to prefer strongly convex functions instead of plurisubharmonic functions.

Let X be an analytic variety and $\psi : X \to \mathbb{R}$ a continuous function. We say that ψ is plurisubharmonic (resp. strongly plurisubharmonic) if for every local chart $\phi : U \to V \subset D \subset \mathbb{C}^n$, U an open subset of X and V a closed analytic subset of the open subset D of some $\mathbb{C}^n$, there exists a plurisubharmonic (resp. strongly plurisubharmonic) function $\check{\psi}$ on D such that $\psi = \check{\psi} \circ \phi$.

The fact that the pull-back by a holomorphic (biholomorphic) function sends (strongly) plurisubharmonic functions to (strongly) plurisubharmonic functions and Proposition 1.3 imply that the above definition is independent of the choice of a local chart. When X is a an open domain in the complex plane $\mathbb{C}$ then plurisubharmonic functions on X are precisely C^2 subharmonic functions on X(see [**GuI90**], J Theorem 8). For the following proposition see Figure 2.3.

Proposition 2.8. *Let $\psi : (X, p) \to \mathbb{R}, \psi(p) = 0$ be a strongly convex function. Then there exists a holomorphic function f defined in a neighborhood of p in X such that*

$$\{f = 0\} \cap \overline{\{x \in X \mid \psi(x) < 0\}} = \{p\}$$

Proof. The theorem for $(\mathbb{C}^n, p)$ implies easily the general case (X, p). So we assume that $X = \mathbb{C}^n$. By Proposition 2.6 we can assume that ψ is strongly plurisubharmonic. The Taylor series of ψ at p reads

$$\psi(z) = 2Re(\sum_i \frac{\partial \psi}{\partial z_i}(p)z_i + \sum_{ij} \frac{\partial^2 \psi}{\partial z_i \partial z_j}(p)z_i z_j) + L_p(\psi)(z) + o(|z - \psi(p)|^2)$$

Now $f(z) := \sum_i \frac{\partial \psi}{\partial z_i}(p)z_i + \sum_{ij} \frac{\partial^2 \psi}{\partial z_i \partial z_j}(p)z_i z_j$ is the desired function. □

The proof of the above proposition tells us something more: we can choose a neighborhood U of p in X such that the function f associated to the point $x \in \psi^{-1}(0) \cap U$ is defined in U. We are going to use this fact in the proof of the following proposition.

Proposition 2.9. *Let $\psi : (X, p) \to \mathbb{R}$ with $\psi(p) = 0$ be strongly convex at p. There is a Stein neighborhood X' of p in X such that $U := \{x \in X' \mid \psi(x) < 0\}$ is Stein.*

Proof. Let X' be a Stein neighborhood of p such that for all $p \in \psi^{-1}(0) \cap X'$ there is a holomorphic function f defined on X' with the property mentioned in Proposition 2.8. Then X' is the desired Stein open set. Since X' can be embedded in some affine space $\mathbb{C}^n$, it is enough to prove that U is holomorphically convex. Let K be a compact subset of U. We have $\hat{K}_U \subset \hat{K}_{X'}$ and

$\hat{K}_{X'}$ is compact in X'. Therefore if $\hat{K}_U$ is not compact in U, its closure in X' must have a point $p \in \psi^{-1}(0)$. Let f be the holomorphic function in X' associated to the point p as in Proposition 2.8. The function $\frac{1}{f}$ is a holomorphic function on U such that $lim_{x \to p} |\frac{1}{f}| = +\infty$. But $|\frac{1}{f}(y)| \leq max_{x \in K} |\frac{1}{f}(x)|$ for all $y \in \hat{K}_U$. This leads to a contradiction. □

Since the intersection of two Stein open sets is Stein again (see Proposition 1.5) the assertion of the above proposition is true for Stein open sets smaller than X'.

Proposition 2.10. *Let* (z, z_{n+1}) *be the coordinate system of* $(\mathbb{C}^n \times \mathbb{C}, (p, p_{n+1}))$, $\psi : (\mathbb{C}^n, p) \to (0, 1)$ *be a* C^2 *function and* $0 < \epsilon \leq 1$. *The function* $|z_{n+1}|^2 - \epsilon\psi(z)$ *is strongly convex at* (p, p_{n+1}), $|p_{n+1}|^2 = \epsilon\psi(p) \neq 0$ *if and only if* $-\log\psi$ *is strongly plurisubharmonic at* p.

We have stated this proposition with ϵ in order to have also the following statement: If $-\log\psi$ is strongly plurisubharmonic then $\frac{|z_{n+1}|^2}{\psi}$ is a strongly convex function at any point with $z_{n+1} \neq 0$. We can replace $|z_{n+1}|^2 - \epsilon\psi(z)$ by $|z_{n+1}| - \epsilon\psi(z)$ in the above proposition.

Proof. First let us suppose that $-\log\psi$ is strongly plurisubharmonic. Let $(v, v_{n+1}) \in \mathbb{C}^{n+1}$ such that

$$D_{(p,p_{n+1})}(|z_{n+1}|^2 - \epsilon\psi)(v, v_{n+1}) = \bar{p}_{n+1}v_{n+1} - \epsilon D_p\psi(v) = 0$$

Then

$$L_p(|z_{n+1}|^2 - \epsilon\psi)(v, v_{n+1}) = |v_{n+1}|^2 - \epsilon L_p(\psi)(v) = \frac{\epsilon^2|D_p\psi(v)|^2}{|p_{n+1}|^2} - \epsilon L_p(\psi)(v)$$

$$= \epsilon(\frac{|D_p\psi(v)|^2}{\psi(p)} - L_p(\psi)(v)) = \epsilon\psi(p)L_p(-\log\psi)(v) > 0$$

Now let us prove the inverse. Let $|z_{n+1}|^2 - \epsilon\psi(z)$ be strongly convex at (p, p_{n+1}) with $\epsilon\psi(p) = |p_{n+1}|^2 \neq 0$. Fix $v \in \mathbb{C}^n$ and take $v_{n+1} = \frac{\epsilon D_p(\psi)(v)}{\bar{p}_{n+1}}$. We have

$$L_p(-\log\psi)(v) = \frac{1}{\epsilon\psi(p)}L_p(|z_{n+1}|^2 - \psi(z))(v, v_{n+1}) > 0$$

□

2.4 Cohomological properties of pseudoconvex domains

Let us state an important theorem concerning the cohomology of strongly pseudoconvex domains with values in a coherent sheaf.

Theorem 2.2. *([**Gr58**]) Let G be a relatively compact strongly pseudoconvex domain in a complex variety X and $\mathcal{S}$ a coherent analytic sheaf on G. Then the cohomology groups $H^\mu(G, \mathcal{S})$ are finite dimensional vector spaces for $\mu > 0$.*

This section is devoted to the proof of the above theorem. We will use the C^2-function ψ defined in a neighborhood U of ∂G such that $G \cap U = \{x \in U \mid \psi(x) < 0\}$ and we assume that $X = U \cup G$. If U' is a small Stein open set in U then according to Proposition 2.9, the intersection $U' \cap G$ is again Stein. Let us state two lemmas whose proofs are just topological manipulations.

Lemma 2.1. *Consider the situation of Theorem 2.2. If U' is a small Stein open set in X then the restriction map $r : H^\mu(G \cup U', \mathcal{S}) \to H^\mu(G, \mathcal{S})$ is surjective for $\mu > 0$.*

Proof. Consider an arbitrary Stein covering $\mathcal{U}$ of G containing the Stein open set $G \cap U'$. We have $Z^\mu(G,\mathcal{U}) = Z^\mu(G \cup U',\mathcal{U} \cup \{U'\})$. This is due to the fact that the intersection of at least two open sets in $\mathcal{U} \cup \{U'\}$ is a subset of G. Since $U' \cap G$ is Stein, $\mathcal{U} \cup \{U\}$ is a Stein covering of $G \cup U'$. Leray Lemma finishes the proof. □

Lemma 2.2. *Consider the situation of Theorem 2.2. To each Stein covering $\mathcal{U} = \{U_i \mid i = 1, 2, \ldots, r\}$ of $\bar{G}$ in X one can find a strongly pseudoconvex domain G' such that 1. $G \subset\subset G' \subset\subset \cup_{i=1}^r U_i$ 2. The restriction map $r : H^\mu(G', \mathcal{S}) \to H^\mu(G, \mathcal{S})$ is surjective for $\mu > 0$.*

Proof. Take $K_i \subset\subset \partial G \cap U_i$ such that $\partial G = \cup_{i=1}^r K_i$. According to Proposition 2.2 in each U_i there is $\epsilon_i > 0$ such that if h_i is a C^2 function on U_i and the absolute value of h_i and its first and second derivatives are less than ϵ_i then $\{x \in U_i \mid \psi(x) < h_i(x)\}$ is strongly pseudoconvex at each point of $(\psi - h_i)^{-1}(0)$ in a neighborhood of K_i in U_i. We take $\epsilon = \frac{min \epsilon_i}{r}$ and in each chart U_i we take a C^2 function h_i such that

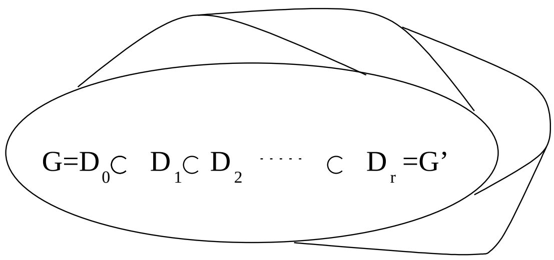

Figure 2.2: The idea of the Proof of Lemma 2.2

1. h_i has a compact support in U_i;
2. h_i restricted to K_i is strictly positive;
3. the absolute value of h_i and its first and second derivatives are less than ϵ.

Define

$$D_j := \{x \in U \mid \psi(x) < \sum_{i=1}^{j} h_i(x)\} \cup G, \; j = 1, 2, \ldots, r, \; D_0 := G$$

We have $D_0 \subset D_1 \subset \cdots \subset D_r$. By the choice of ϵ and by Proposition 2.2 we can conclude that D_i is a strongly pseudoconvex domain. Since h_i is strictly positive on K_i and K_i's cover ∂G, we have $G \subset\subset D_r$. We claim that $G' := D_r$ satisfies our lemma. We must check that the restriction map $r : H^\mu(D_r, \mathcal{S}) \to H^\mu(G, \mathcal{S})$ is surjective. It is enough to check that $r : H^\mu(D_i, \mathcal{S}) \to H^\mu(D_{i-1}, \mathcal{S})$ is surjective for all $i = 1, 2, \ldots, r$. Since the support of h_i is in U_i, we have $D_i = D_{i-1} \cup (D_i \cap U_i)$. By Proposition 2.9 $D_i \cap U_i$ is Stein. Lemma 2.1 finishes the proof. □

Let F_1 and F_2 be two Fréchet spaces (see [**GuI90**] F). Recall that a linear mapping between two topological vector spaces is called compact (or completely continuous) if some open neighborhood of the origin in the domain is mapped to a relatively compact set in the range. A theorem of L. Schwarz says

Theorem 7. *(L. Schwarz) Let $u, v : F_1 \to F_2$ be two continuous linear maps. If u is compact and v is surjective then the $\mathbb{C}$-vector space $F_2/Im(u+v)$ is finite dimensional.*

A proof of this statement can be found in [**GuRo**] App. B 12.

Proof of Theorem 2.2. Let $\mathcal{U} = \{U_i, i = 1, 2, \ldots, r\}$ and $\mathcal{U}' = \{U'_i, i = 1, 2, \ldots, r\}$ be two Stein coverings of $\bar{G}$ such that $U_i \subset\subset U'_i$ and $\mathcal{U} \cap G$ is a Stein covering of G. By Lemma 2.2, we have a strongly pseudoconvex domain G' such that $G \subset\subset G' \subset\subset \cup_{i=1}^{r} U_i$ and $\mathcal{U}' \cap G', \mathcal{U} \cap G'$ are Stein coverings of G'. We consider the maps

$$u, v : Z^\mu(\mathcal{U}', \mathcal{S}) \oplus C^{\mu-1}(\mathcal{U}, \mathcal{S}) \to Z^\mu(\mathcal{U}, \mathcal{S})$$

$$u(a, b) = r(a) + \delta(b),\ v(a, b) = -r(a)$$

where r is the restriction and δ is the coboundary map. Since $H^\mu(G', \mathcal{S}) = H^\mu(\mathcal{U}' \cap G', \mathcal{S})$ and $H^\mu(G, \mathcal{S}) = H^\mu(\mathcal{U} \cap G, \mathcal{S})$, the map v is surjective. The following theorem finishes the proof.

Theorem 8. *Let $U \subset\subset U'$ be two open domains in a variety X and $\mathcal{S}$ be a coherent analytic sheaf on X. Then one can endow $\mathcal{S}(U')$ and $\mathcal{S}(U)$ with Fréchet space structures such that the restriction $r : \mathcal{S}(U') \to \mathcal{S}(U)$ is a compact mapping.*

In the case $\mathcal{S} = \mathcal{O}_X$ this is Montel's Theorem (see [**GuI90**]). For an arbitrary coherent sheaf we refer to [**KK83**] Lemma 62.6. □

The tools used in the proof of Theorem 2.2 provide us with a proof of the following theorem due to Serre and Cartan (see [**Ma68**]).

Theorem 2.3. *Let A be a compact variety and $\mathcal{S}$ be a coherent sheaf on A. Then $H^\mu(A, \mathcal{S}), \mu > 0$, are finite dimensional $\mathbb{C}$-vector spaces.*

Now we are in a position to prove that a strongly convex domain is holomorphically convex.

Theorem 2.4. *(R. Narasimhan [**Na60**]). Let $G \subset X$ be a relatively compact strongly pseudoconvex domain. Then G is holomorphically convex.*

Proof. We prove that for every boundary point $p \in \partial G$ one can find a holomorphic function g on G such that $lim_{x \to p} |g| = +\infty$. This implies that G is holomorphically convex. Let K be a compact subset of G and $p \in \partial G$ be a boundary point . For all $y \in \hat{K}$ we have $|g(y)| \leq max_{x \in K} |g(x)| < +\infty$. This means that $\hat{K}$ cannot have p in its closure. Since G itself is relatively compact in X, we conclude that $\hat{K}$ is compact.

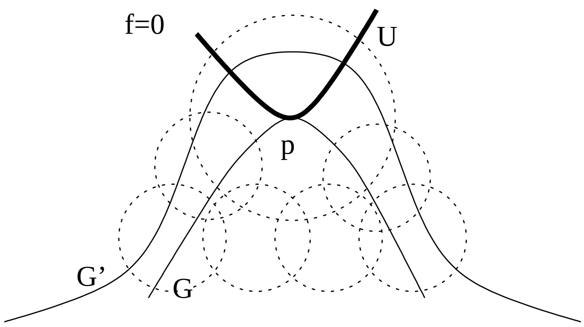

Figure 2.3: The idea of the Proof of Theorem 2.4

According to Proposition 2.8 for every boundary point $p \in \partial G$ one can find a holomorphic function f defined in a neighborhood of p in X such that $\{f = 0\} \cap \bar{G} = \{p\}$. Let U be a Stein open set around p. One can choose a strongly pseudoconvex domain G' enough near G such that $D := \{f = 0\} \cap G'$ is closed in G' and is relatively compact in U. Now $U \cap G'$ is a Stein open set in G' and one can choose a Stein covering $\mathcal{U} = \{U_i, i = 1, 2, \ldots, r\}$ of G' such that $U_1 := U \cap G'$ and $U_i, i = 2, 3, \ldots, r$ do not intersect D. Put $f_i = 0$ if $i = 2, \ldots, r$ and $f_1 = \frac{1}{f^m}$. We have the cocycle $\delta_m := \{f_i - f_j\} \in H^1(\mathcal{U}, \mathcal{S}) = H^1(G', \mathcal{S})$. But by Theorem 2.2 this vector space is finite dimensional. Therefore there exist $m_i \in \mathbb{N}, c_i \in \mathbb{C}, i = 1, 2, \ldots, s$ such that $\sum c_i \delta_{m_i} = 0$. This means that there is a meromorphic function g on G' with poles along D and such that in a neighborhood of p $g - \sum_{i=1}^{s} \frac{c_i}{f^{m_i}}$ is holomorphic. Therefore g is not holomorphic at p. Thus $g \mid_G$ is the desired holomorphic function in G. □

2.5 Exceptional varieties

Let $G \subset X$ be a relatively compact strongly pseudoconvex domain. By Theorem 2.4 G is holomorphically convex, and so, we can apply Remmert reduction theorem to G and obtain a Stein space Y and a holomorphic map $\phi : G \to Y$.

Theorem 2.5. *Let $G \subset X$ be a relatively compact strongly pseudoconvex domain and $\phi : G \to Y$ its Remmert reduction. Then the degeneracy set*

$$A = \{x \in G \mid x \textit{ is not an isolated point of } \phi^{-1}(\phi(x))\}$$

is the maximal compact analytic nowhere discrete subset of G.

Proof. The subsets $\phi^{-1} \circ \phi(x), x \in G$, are connected, and so by the definition, A is nowhere discrete. We prove that A is a closed analytic set. The set $R = \{(x_1, x_2) \in X \times X \mid \phi(x_1) = \phi(x_2)\}$ is an analytic set and the projection on the first coordinate $\pi : R \to X$ is analytic. By [**Gr83**] Proposition 1 p.138 we know that

$$\tilde{A} = \{x \in R \mid dim(\pi^{-1}\pi(x)) > 0\}$$

is a closed analytic set. Since $A = \pi(\tilde{A})$ and π is proper, A is also an analytic closed set. By Theorem 2.4, there exists a compact set K which contains all compact analytic nowhere discrete subsets of G. For any $x \in A$, $\phi^{-1}\phi(x)$ is connected, and so by definition, is compact nowhere discrete subset of A. This implies that $\phi^{-1}\phi(x) \subset A$ and hence $A \subset K$. Since A is a closed set in the compact set K, A is compact. □

The Remmert reduction $\phi : G \to Y$ is proper and A is compact so $\phi(A)$ is a compact analytic subset of Y. But Y is Stein, and so, $\phi(A)$ is discrete set and A is a union of compact connected analytic subsets $A_1, A_2, \ldots, A_r$ of G. In this case Remmert reduction substitute each A_i with a point. This leads us to the definition of exceptional sets.

Let X be an analytic variety and A be a compact connected subvariety of X. A is exceptional in X if there exists an analytic variety X' and a proper surjective holomorphic map $f : X \to X'$ such that

- $\phi(A) = \{p\}$ is a single point;
- $\phi : X - A \to X' - \{p\}$ is an analytic isomorphism;
- For small neighborhoods U' and U of p and A, respectively, $\mathcal{O}_{X'}(U') \to \mathcal{O}_X(U)$ is an isomorphism.

We also say that A can be blown down to a point or is contractible or negatively embedded.

Theorem 2.6. *(Grauert,[***Gr62***] Satz 5 p. 340) Let A be a compact connected analytic subset of X. Then A is an exceptional variety if and only if it has a strongly pseudoconvex neighborhood G in X such that A is the maximal compact analytic subset of G.*

Proof. Let us first suppose that A is exceptional. The analytic variety X' obtained by definition can be embedded in a $(\mathbb{C}^n, 0)$ (definition of analytic sets). The neighborhood of p in X' given by

$$U = \{x \in X' \mid z_1(x)\overline{z_1(x)} + \cdots + z_n(x)\overline{z_n(x)} < \epsilon\}, \ \epsilon \text{ a small positive number}$$

is a pseudoconvex domain. Now it is easy to see that $G = \phi^{-1}(U)$ is the desired open neighborhood of A.

Now let us suppose that A has a strongly pseudoconvex neighborhood G in X such that A is the maximal compact analytic subset of G. Let $\phi : G \to X'$ be the Remmert reduction of G. We can see easily that A is the degeneracy set of ϕ and $\phi(A)$ is a single point p. Since the fibers $\phi^{-1}\phi(x)$ are connected, ϕ is one to one map between $G - A$ and $X' - \{p\}$. Combining this and the property of ϕ in Remmert reduction theorem we can conclude that ϕ induces an isomorphism of stalks and so it is a biholomorphism between $G - A$ and $X' - \{p\}$. The third condition of an exceptional variety can be read directly from Remmert reduction theorem. □

Complementary notes

1. Theorem 2.9 can be generalized as follows: Let X be a Stein variety and ψ a real valued C^2 function such that $U := \{x \in X \mid \psi(x) < 0\}$ is convex at each point x with $\psi(x) = 0$ then U is a Stein variety. The proof can be found in [**Na60**] section 4 corollary 1.
2. Let $\psi : (\mathbb{C}^n, 0) \to \mathbb{R}$ be a strongly plurisubharmonic function. One may try to show that being strongly convex is an intrinsic property of the boundary point $p \in \partial U$, where $U = \{x \in (\mathbb{C}^n, p) \mid \psi(x) < \psi(p)\}$ and then say that U is strongly convex at $p \in \partial U$ without mentioning ψ. The argument is the following: Let ψ' be a C^2-function in $(\mathbb{C}^n, p)$ such that ${\psi'}^{-1}(\psi'(p)) = \psi^{-1}(\psi(p))$ and $D\psi'$ is not identically zero on $\psi^{-1}(\psi(p))$ and is positive on U. If for a C^2 function h

$$\psi - \psi(p) = h.(\psi' - \psi'(p)) \tag{2.3}$$

then ψ' is also strongly convex at p. We have $D\psi_p(v) = h(p)D\psi'_p(v)$ and so if $D\psi'_p(v) = 0$ then $D\psi_p(v) = 0$. By (2.1) and the fact that ψ is strongly convex we have $L_p(\psi)(v) = h(p)L_p(\psi')(v) > 0$ if $D\psi'_p(v) = 0$. This implies that $h(p) > 0$ (because ψ' is positive in U) and ψ' is strongly convex at p. If $\psi^{-1}(p)$ is a smooth real submanifold of $(\mathbb{C}^n, p)$ and of codimension one then ψ' is regular at p and one can obtain the condition (2.3) using the Taylor series of ψ in the variable $\psi' - \psi'(p)$ (see [**GuI90**] p. 189). But it is not clear whether (2.3) is true always or not.
3. The reader who is interested to know the proof of Proposition 2.4 with a global plurisubharmonic ψ can look at the articles [**Ri68**, **Wa72**]. Note that we have an alternative proof for exceptional varieties using the notion of strongly convex functions and Remmert reduction, see Theorem 2.6.

4. The Riemann extension theorems are valid for upper semi-continuous strongly plurisubharmonic functions (see [**GuII90**], K for definition). The precise statement and proof can be recovered from [**Gr56**].

5. In the proof of Theorem 2.2 we have used: For an small open relatively compact set U of a variety and a coherent sheaf $\mathcal{S}$ on X, $H^0(U, \mathcal{S})$ has a canonical structure of a Fréchet space. The construction of such a canonical structure is done [**GrRe79**] Chapter VI, Section 3, [**Ma68**] Chapter 4 and [**KK83**] chapter 6.

6. When A is a union of curves in a two dimensional manifold we have a numerical criterion for contractablity of A.

   Theorem: Let A be a compact connected one dimensional subvariety of a manifold X. Suppose that A contains only normal crossing singularities. Then A is exceptional in X if and only if the intersection matrix $S = [A_i.A_j]$ of A in X is negative definite, where $A = \cup A_i$ is the decomposition of A into irreducible components.

   This is Theorem 4.9 of [**La71**].

Chapter 3

Vanishing theorems

The aim of this section is to introduce vanishing theorems in complex analysis and algebraic geometry. Theorem 3.5 is the main theorem in this chapter. We will follow Grauert's article [**Gr62**], but our proof for Theorem 3.5 works for a general exceptional variety while Grauert's argument works for codimension one exceptional varieties in manifolds. We use the same letter i for the complex number $\sqrt{-1}$ and for indexing; being clear in the text the distinction between them.

3.1 Positive and negative bundles

Let us start the section with the definition of a negative vector bundle. The vector bundle $V \to A$ over a complex manifold A is called negative (in the sense of Grauert) if its zero section is an exceptional variety in V. Naturally $V \to A$ is called positive if its dual is negative.

There is another definition in algebraic geometry for a positive line bundle as follows: The line bundle $L \to A$ over a complex manifold is called positive (in the sense of Kodaira) if its Chern class $c(L)$ in the de Rham cohomology $H^2(A, \mathbb{C})$ is represented by a positive real $(1,1)$-form ω, i.e. $\omega_p(v,v) > 0$ for any point $p \in A$ and non-zero vector v in the real tangent space at p. We can write the form ω in a local chart as follows:

$$\omega = i(\sum_{i,j=1}^{n} g_{ij} dz_i \wedge d\overline{z_j})$$

where g_{ij}'s are real functions and $g_{ij} = g_{ji}$. For more information about this definition of positive line bundles the reader is referred to [**GrHa78**].

Theorem 3.1. *A line bundle L over A is positive in the sense of Kodaira if and only if there exist a covering $\{U_i, i \in I\}$ of A by open sets and a collection of C^2 functions $p_i : U_i \to \mathbb{R}^+,\ i \in I$ such that*

1. *$-\log p_i$ is strongly plurisubharmonic for any $i \in I$;*
2. *$p_i = |h_{ij}|p_j$, where L is given by $\{h_{ij}\} \in H^1(A, \mathcal{O}^*)$ in the covering.*

Proof. In Cech cohomology the Chern class of L is obtained by $\delta\{f_{ij}\} \in H^2(A, \mathbb{Z})$, where $f_{ij} := \frac{1}{2\pi i} \log h_{ij}$ (write the long exact sequence associated to $0 \to \mathbb{Z} \to \mathcal{O} \stackrel{e^{2\pi i.}}{\to} \mathcal{O}^* \to 0$ and recall the construction of the coboundary map $\delta : H^1(A, \mathcal{O}^*) \to H^2(A, \mathbb{Z})$). Now let us look at the diagram which produces an isomorphism between Cech cohomology and de Rham cohomology (see [**BT82**] Chapter 2).

$$\begin{array}{ccccccccc}
0 & \to & \Omega^2(A) & \to & \Pi_i\Omega^2(U_i) & \to & \Pi_{ij}\Omega^2(U_{ij}) & \to & \Pi_{ijk}\Omega^2(U_{ijk}) \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \to & \Omega^1(A) & \to & \Pi_i\Omega^1(U_i) & \to & \Pi_{ij}\Omega^1(U_{ij}) & \to & \Pi_{ijk}\Omega^1(U_{ijk}) \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \to & \Omega^0(A) & \to & \Pi_i\Omega^0(U_i) & \to & \Pi_{ij}\Omega^0(U_{ij}) & \to & \Pi_{ijk}\Omega^0(U_{ijk}) \\
 & & & & \uparrow & & \uparrow & & \uparrow \\
 & & & & C^0(\mathcal{U}, \mathbb{C}) & \to & C^1(\mathcal{U}, \mathbb{C}) & \to & C^2(\mathcal{U}, \mathbb{C}) \\
 & & & & \uparrow & & \uparrow & & \uparrow \\
 & & & & 0 & & 0 & & 0
\end{array}$$

where the right arrow maps are δ's and the up arrow maps are d's. We start with $\{f_{ij}\} \in \Pi_{ij}\Omega^0(U_{ij})$. We have $\delta\{f_{ij}\} \in C^2(\mathcal{U}, \mathbb{C})$. The equality $d\delta\{f_{ij}\} = 0$ implies $\delta(d\{f_{ij}\}) = 0$ and so there is a collection $\{\omega_i\}$ of 1-forms such that

$$\delta\{\omega_i\} = \partial\{f_{ij}\} \tag{3.1}$$

The collection $\{d\omega_i\}$ defines a global closed form ω which represents the Chern class $c(L)$ in the de Rham cohomology. Now if $\omega_i = \omega_i^{10} + \omega_i^{01}$ is the decomposition of ω_i into $(1,0)$ and $(0,1)$ forms then $\delta\omega_i^{01} = 0$ and so $\{\omega_i^{01}\}$ form a global form and so it does not contribute to the cohomology class of ω and we can assume that ω_i's are $(1,0)$-forms. Now $\omega = \partial\omega_i + \bar{\partial}\omega_i$ is the decomposition of ω into $(2,0)$ and $(1,1)$ forms. The form ω represents a real class in $H^2(M, \mathbb{C})$ and it has not $(0,2)$ part, therefore the $(2,0)$-part of ω must be $d\alpha$, where α is a global 1-form on A, and there exists a global 1-form β such that $\omega - (d\beta + d\alpha)$ is a real form. We replace $\{\omega_i\}$ with $\{\omega_i - \alpha - \beta\}$

and so we can assume that ω is a $(1,1)$ real form. Since $\partial\omega_i$ is a $(2,0)$ form, this means that $\partial\omega = 0$ and so $\omega_i = \partial q_i$. Therefore we have $\omega = \bar{\partial}\partial q_i$. Now $\omega = \bar{\omega} = -\bar{\partial}\partial\bar{q}_i$. This implies $\bar{\partial}\partial Re q_i = 0$ and so we replace q_i with $iIm(q_i)$ and we assume that q_i is pure imaginary. Now by (3.1) and $\omega_i = \partial q_i$ we have:

$$\partial(\{\delta q_i - f_{ij}\}) = 0 \Rightarrow \delta q_i - f_{ij} = \bar{g}_{ij}$$

where g_{ij} are holomorphic functions. Since δq_i is pure imaginary and f_{ij}, g_{ij} are holomorphic, we have $g_{ij} = -f_{ij}$ and $\delta q_i = 2iIm(f_{ij})$ and so

$$\frac{e^{-\pi i q_i}}{e^{-\pi i q_j}} = e^{-2\pi Im(f_{ij})} = |h_{ij}|$$

Now define $p_i = e^{-\pi i q_i}$. Of course when ω is positive definite then the p_i's are the desired functions. If we have p_i's with the properties 1 and 2 then we define $q_i = \frac{\log p_i}{-\pi i}$ and $\{\bar{\partial}\partial q_i\}$ form a global form which is the Chern class of L in the de Rham cohomology and is positive definite. □

Theorem 3.2. *The line bundle $L \to A$ is positive in the sense of Kodaira if and only if it is positive in the sense of Grauert.*

Proof. Let L be positive in the sense of Kodaira. We have the p_i's given by the Theorem 3.1. Let $z_i : L^{-1} |_{U_i} \to \mathbb{C}$ be a coordinate system along the fibers of L^{-1} in U_i. We have $z_i = h_{ij} z_j$ and so $\frac{|z_i|}{p_i}$ form a global function in a neighborhood of the zero section of L^{-1}. By Proposition 2.10 this function is strongly convex and so we have a strongly pseudoconvex neighborhood of the zero section.

Now suppose that the zero section has a strongly pseudoconvex neighborhood. By Theorem 2.6 one can find a C^∞ function ψ defined in a neighborhood U of the zero section in L such that

1. ψ is strongly plurisubharmonic in $U - A$;
2. $\psi \geq 0$ and $\psi^{-1}(0) = A$.

Take $V = \cap_{0 \leq \theta < 2\pi} e^{i\theta} U$. V is an open neighborhood of the zero section and is invariant under multiplication by $e^{i\theta}$, $0 \leq \theta < 2\pi$. Define

$$\psi'(z) = \int_0^{2\pi} \psi(e^{i\theta} z) d\theta, \ z \in V$$

Since $L_z(\psi')(v) = \int_0^{2\pi} e^{2i\theta} L_{ze^{i\theta}}(\psi)(v)d\theta$, ψ' is also a strongly plurisubharmonic function. The intersection of $V_\epsilon := \{z \in V \mid \psi'(z) < \epsilon\}$ with each fiber is a disk with the center in the zero section. Therefore if z_i is a non zero local section of L defined in an open neighborhood U_i in the zero section then V_ϵ in U_i is given by $\{z \in L \mid \frac{|z|}{|z_i|} < p_i(\pi(z))\}$, where p_i is the radius of $V_\epsilon \cap L_p$ and $\pi : L \to A$ is the bundle map. Now the functions p_i are the desired functions. (By Theorem 2.6 p_i is strongly plurisubharmonic). □

Let A be an exceptional variety in the variety X with normal bundle N. Recall that N has a natural structure of a linear space. Furthermore we assume that N is negative, i.e. the zero section of N is an exceptional variety in N.

A holomorphic function defined in an open set in N is called homogeneous of degree ν along the fibers of N if in a trivialization chart $(x, z) \in U \times \mathbb{C}^n$ it is a homogeneous polynomial of degree ν in the variable z. Since the transition functions are linear in z, this definition does not depend on the chart we choose. Let $\mathcal{H}_\nu$ be the sheaf of homogeneous functions of degree ν along the fibers of N. The sheaf $\mathcal{H}_\nu$ has a natural structure of a $\pi^*\mathcal{O}_A$-module, where $\pi : N \to A$ is the bundle map. The sheaf $\pi^*\mathcal{O}_A$ is the sheaf of holomorphic functions in N which are constant along the fibers of N. We have a natural isomorphism

$$\mathcal{H}_\nu \mid_A \tilde{\to} \mathcal{M}^\nu / \mathcal{M}^{\nu+1}$$

obtained by the inclusion. Let $\mathcal{S}$ be a coherent sheaf defined in a strongly pseudoconvex neighborhood of A in N. (If we have a coherent sheaf $\check{\mathcal{S}}$ on A then the pre image of $\check{\mathcal{S}}$ by the bundle map $\pi : N \to A$ is a coherent sheaf $\mathcal{S}$ on N).

One can define the homogeneous subsheaf of degree ν of $\mathcal{S}$ as

$$\mathcal{S}_\nu := \mathcal{S} \otimes_{\pi^*\mathcal{O}_A} \mathcal{H}_\nu$$

The structural restriction of $\mathcal{S}_\nu$ to A is isomorphic to $\mathcal{S}(\nu) := res(\mathcal{S}) \otimes_{\mathcal{O}_A} \mathcal{M}^\nu / \mathcal{M}^{\nu+1}$, where $res(\mathcal{S}) = \mathcal{S} / \mathcal{M}\mathcal{S}$.

Theorem 3.3. *(Grauert, [**Gr62**], Hilfssatz 1, p. 344) Let $\mathcal{S}$ be a coherent analytic sheaf on a neighborhood of the zero section of the normal bundle N. There exists a positive integer ν_0 such that*

$$H^\mu(A, \mathcal{S}(\nu)) = 0, \quad \mu \geq 1, \ \nu \geq \nu_0$$

Proof. Theorem 2.2 is the key of the proof of this theorem. We have the maps

$$a : \mathcal{S} \to \mathcal{S}_1 \oplus \mathcal{S}_2 \oplus \cdots \oplus \mathcal{S}_\nu$$

$$b : \mathcal{S}_1 \oplus \mathcal{S}_2 \oplus \cdots \oplus \mathcal{S}_\nu \to \mathcal{S}$$

where a is the canonical map and b is the inclusion, with $a \circ b$ equal to the identity. Taking the μ-th cohomology from the above data we conclude that $b^* : H^\mu(A, \mathcal{S}_1) \oplus H^\mu(A, \mathcal{S}_2) \oplus \cdots \oplus H^\mu(A, \mathcal{S}_\nu) \to H^\mu(U, \mathcal{S})$ is an injection, because $a^* \circ b^*$ is the identity. Since by Theorem 2.2 $H^\mu(U, \mathcal{S})$ is finite dimensional, we get the desired number in the theorem. $\square$

When N is a negative line bundle over a manifold A, this theorem is exactly Kodaira's vanishing theorem. We have $\mathcal{M}^\nu/\mathcal{M}^{\nu+1} \simeq \underline{L}^\nu$ and $\mathcal{S}(\nu) \simeq \mathcal{S} \otimes_{\mathcal{O}_A} L^\nu$, where $L = N^*$, and L is a positive line bundle.

In the case where A is a Riemann surface and N is a line bundle this theorem was already proved in Chapter 1 Section 1.6 using the Serre Duality. In this case we can explicitly state the minimum number ν_0 with the property of Theorem 3.3.

3.2 A vanishing theorem

Let A be an exceptional subvariety of a variety X. By Theorem 2.6 one can find a C^∞ function ψ defined in a neighborhood of A in X such that ψ is strongly plurisubharmonic outside A, $\psi \geq 0$ and $\psi^{-1}(0) = A$. Therefore we have a fundamental system $U_\epsilon := \{\psi(x) < \epsilon\}, 0 < \epsilon << 1$ of relatively compact strongly pseudoconvex neighborhoods around A. Fix a U_ϵ. Let $\mathcal{S}$ be an analytic sheaf on U_ϵ and f be a holomorphic function on U_ϵ. Since A is compact connected, f restricted to A is constant. We denote this constant by $f(A)$. Take a Stein covering $\mathcal{U}$ of U_ϵ. We have $H^\mu(\mathcal{U}, \mathcal{S}) = H^\mu(U_\epsilon, \mathcal{S})$, where $\mu > 0$. The usual multiplication of f by cocycles in $Z^\mu(\mathcal{U}_\epsilon, \mathcal{S})$ yields a well-defined map from $H^\mu(U_\epsilon, \mathcal{S})$ to itself.

Lemma 3.1. *Let f be a holomorphic function and $\mathcal{S}$ be a coherent sheaf defined on a neighborhood of A in X. There exist a natural number n_1 and a positive number ϵ_1 such that*

$$(f - f(A))^n H^\mu(U_\epsilon, \mathcal{S}) = 0, \ \forall n \geq n_1, n \in \mathbb{N}, \ 0 < \epsilon \leq \epsilon_1, \ \mu \geq 1$$

Proof. Without loosing the generality suppose that $f(A) = 0$, i.e. f vanishes on A. Let $\phi : (X, A) \to (Y, p)$ be the Remmert reduction mapping (blow down). By Grauert direct image theorem $R^\mu \phi_* \mathcal{S}$ is a coherent sheaf. Since $\phi |_{X-A}$ is a biholomorphism, Cartan's theorem B implies that the support of $R^\mu \phi_* \mathcal{S}$ lies in $p \in Y$ and so the stalk $(R^\mu \phi_* \mathcal{S})_p$ is a finite dimensional $\mathbb{C}$-vector space. Now by the property 1 listed in Remmert reduction theorem there is a holomorphic function g in (Y, p) such that $f = g \circ \phi$. Multiplication by g with the stalk $(R^\mu \phi_* \mathcal{S})_p$ has not eigenvalue different from zero. Therefore it is unipotent and so there is n such that $g^n R^\mu \phi_* \mathcal{S}$ is the zero sheaf. □

Theorem 3.4. *There exist a natural number* ν_0 *and a positive number* ϵ' *such that for all* $\nu \geq \nu_0, \nu \in \mathbb{N}$ *and* $0 < \epsilon < \epsilon'$ *the map induced by inclusion*

$$H^\mu(U_\epsilon, \mathcal{S}\mathcal{M}^\nu) \to H^\mu(U_\epsilon, \mathcal{S})$$

is the zero map.

Our proof for this theorem is similar to Grauert's proof. Grauert after proving this theorem for pure codimension one A in a manifold X ([**Gr62**] Satz 1 p. 355) tells us that for an arbitrary exceptional variety A in X this theorem follows from Hauptsatz II of [**Gr60**]. This theorem is also proved for pure codimension one A in a manifold X in [**La71**] Theorem 5.4.

Proof. One can blow down A to a point and obtain a singularity (Y, p). Let $z_1, z_2, \ldots, z_n$ be the coordinate functions of (Y, p) and $f_1, f_2, \ldots, f_n$ be the pullback of z_i's by the blow down map. According to Lemma 3.1 there is a natural number n_i and a positive number ϵ_i such that $f_i^n H^\mu(U_\epsilon, \mathcal{S}) = 0, \ n \geq n_i, 0 < \epsilon < \epsilon_i$. Let n be the maximum of n_i's, ϵ' be the minimum of ϵ_i's. From now on we write $U = U_\epsilon$ for a fixed $0 < \epsilon < \epsilon'$.

Let $\tilde{\mathcal{M}}$ be the subideal of $\mathcal{O}_U$ generated by f_i^n's. The zero locus of f_i^n's is A and so by Hilbert Nullstellensatz theorem (see [**GuII90**]) there exists a natural number ν_1 such that

$$\mathcal{M}^{\nu_1} \subset \tilde{\mathcal{M}}$$

The proof of the theorem is by inverse induction on μ. If $\mathcal{U}$ is a finite Stein covering of U with r open sets, then by Cech cohomology $H^r(U, \mathcal{S}\mathcal{M}^\nu) = 0$ for all natural numbers ν and for all sheaves $\mathcal{S}$. Therefore our theorem is trivial for $\mu = r$. Now suppose that it is true for $\mu + 1$. We want to prove that it is true for μ also.

Let

$$\pi : \mathcal{O}_U^n \to \tilde{\mathcal{M}}$$

$$\pi(a_1, a_2, \dots, a_n) = \sum_{i=1}^{n} a_i f_i$$

and $\mathcal{R} := Ker\pi$. We write the short exact sequence

$$0 \to \mathcal{R} \to \mathcal{O}_U^n \to \tilde{\mathcal{M}} \to 0$$

and we make a tensor product of this short exact sequence with $\mathcal{S}$ (resp. $\mathcal{S}\mathcal{M}^{\nu_2}$, where ν_2 is an unknown natural number) over $\mathcal{O}_U$ and then we write the associated long exact sequence. Since $\mathcal{S}^n = \mathcal{S} \otimes \mathcal{O}_U^n$ and $H^\mu(U, \mathcal{S}^n) \to H^\mu(U, \mathcal{S}\tilde{\mathcal{M}})$ is the zero map, we get the commutative diagram

$$\begin{array}{ccccccc} 0 & \to & H^\mu(U, \mathcal{S}\tilde{\mathcal{M}}) & \to & H^{\mu+1}(U, \mathcal{S} \otimes \mathcal{R}) & \to \\ & & \uparrow & & \uparrow & \\ \cdots & \to & H^\mu(U, \mathcal{S}\tilde{\mathcal{M}}\mathcal{M}^{\nu_2}) & \to & H^{\mu+1}(U, \mathcal{S} \otimes \mathcal{R} \otimes \mathcal{M}^{\nu_2}) & \to \end{array} \tag{3.2}$$

By induction for a big ν_2 the second up arrow map is zero and so by the above diagram the first is zero also. The map $H^\mu(U, \mathcal{S}\mathcal{M}^\nu) \to H^\mu(U, \mathcal{S}), \nu \geq \nu_1 + \nu_2$ splits into

$$H^\mu(U, \mathcal{S}\mathcal{M}^\nu) \to H^\mu(U, \mathcal{S}\tilde{\mathcal{M}}\mathcal{M}^{\nu_2}) \to H^\mu(U, \mathcal{S}\tilde{\mathcal{M}}) \to H^\mu(U, \mathcal{S})$$

and so it is the zero map. □

Let us be given a subvariety of a variety X. We say that A is strongly exceptional in X if A is exceptional and the normal bundle of A in X is negative.

Theorem 3.5. *(Grauert [**Gr62**],Satz 2, p. 357) Let us be given a strongly exceptional subvariety A of a variety X. There exists a positive integer ν_0 such that*

$$H^\mu(U, \mathcal{S}\mathcal{M}^\nu) = 0, \quad \mu \geq 1, \ \nu \geq \nu_0$$

where U is a small strongly pseudoconvex neighborhood of A in X.

Proof. Let ν_0 be the number such that $H^\mu(A, \mathcal{S}(\nu)) = 0, \ \nu \geq \nu_0, \ \mu \geq 1$. Consider the short exact sequence

$$0 \to \mathcal{S}\mathcal{M}^{\nu+1} \to \mathcal{S}\mathcal{M}^\nu \to \mathcal{S}(\nu) \to 0$$

For $\nu \geq \nu_0$ the map $H^\mu(U, \mathcal{S}\mathcal{M}^{\nu+1}) \to H^\mu(U, \mathcal{S}\mathcal{M}^\nu)$ is surjective and so for any $k \geq \nu$ the map $H^\mu(U, \mathcal{S}\mathcal{M}^k) \to H^\mu(U, \mathcal{S}\mathcal{M}^\nu)$ is surjective. According to Theorem 3.4 for a large k this map is zero and so $H^\mu(U, \mathcal{S}\mathcal{M}^\nu) = 0, \ \nu \geq \nu_0$. □

Note that ν_0 in the above theorem is the same number ν_0 in $H^\mu(A, \mathcal{S}(\nu)) = 0, \ \nu \geq \nu_0, \ \mu \geq 1$.

The concept of being exceptional is contained in which neighborhood of A? Let A' be the image of another embedding $A \hookrightarrow X'$ of A. The following theorem gives us an answer.

Theorem 3.6. *If A is exceptional and there exists an isomorphism $\phi_{(2)} : A_{(2)} \to A'_{(2)}$ of 2-neighborhoods then A' is also exceptional.*

This is Theorem 4.9 (see also Theorem 6.12) of [**La71**], Satz 8 p.353 of [**Gr62**] and Lemma 11 of [**HiRo64**]. The maim core of the proof is a geometric construction due to Grauert (see [**La71**] p. 70-71). In the case where A is an exceptional curve in a smooth surface, $\mathcal{M}/\mathcal{M}^2$ is the nilpotent subsheaf of $A_{(2)}$ and so every isomorphism of 2-neighborhoods induce an isomorphism of $\mathcal{M}/\mathcal{M}^2$'s. Therefore A and A' have the same intersection matrix.

3.3 Blow down and blow up

The classical definition of blow up at $0 \in \mathbb{C}^n$ goes as follows: The projective space $\mathbb{P}^n$ is the set of one dimensional sub vector spaces of $\mathbb{C}^{n+1}$ and its canonical line bundle

$$L := \{(x, y) \in \mathbb{P}^n \times \mathbb{C}^{n+1} \mid y \in x\}$$

is a negative line bundle, because the projection on the second coordinate $\pi : L \to \mathbb{C}^{n+1}$ exhibits the zero section of L as an exceptional variety. It is usual to write $L = \tilde{\mathbb{C}^{n+1}}$ and say that $\pi : \tilde{\mathbb{C}^{n+1}} \to \mathbb{C}^{n+1}$ is the blow up map of $\mathbb{C}^{n+1}$ at 0. If no confusion is possible we identify $\pi^{-1}(0)$ with $\mathbb{P}^n$. The manifold $\tilde{\mathbb{C}^{n+1}}$ is covered by affine charts

$$([x_0 : x_1 : \cdots : x_n], (x_0, x_1, \ldots, x_n)) \to (\frac{x_0}{x_i}, \ldots \frac{x_{i-1}}{x_i}, x_i, \frac{x_{i-1}}{x_i}, \ldots, \frac{x_n}{x_i})$$

The coordinate system in this affine chart is denoted by $(t_0, \ldots, t_{i-1}, x_i, t_{i+1}, \ldots, t_n)$

Let $(X,0) \subset (\mathbb{C}^{n+1},0)$ be a germ of a variety and $\mathcal{I} \subset \mathcal{O}_{\mathbb{C}^{n+1},0}$ be the ideal of holomorphic functions vanishing on X. For an element $f \in \mathcal{I}$, let f^* be the leading term of f and $\mathcal{I}^*$ be the ideal generated by $\{f^*, f \in \mathcal{I}\}$. The variety $Zero(\mathcal{I}^*)$ is called the tangent cone of X at 0. It is a homogeneous variety, i.e. for all $x \in Zero(\mathcal{I}^*)$ we have $\mathbb{C}.x \subset Zero(\mathcal{I}^*)$. Therefore we can projectivize the tangent cone and obtain the projectivized tangent cone $\mathbb{TC}_0 X \subset \mathbb{P}^n$. Note that if X is given by $f_1 = 0, f_2 = 0, \ldots, f_k = 0$ then not necessarily $f_1^* = 0, f_2^* = 0, \ldots, f_k^* = 0$ defines the tangent cone of X at 0. We may need more leading terms of elements in $\mathcal{I}$. Let $\pi : \tilde{\mathbb{C}^{n+1}} \to \mathbb{C}^{n+1}$ be the blow-up map.

Proposition 3.1. *The closure $\tilde{X}$ of $\pi^{-1}(X - \{0\})$ in $\tilde{\mathbb{C}^{n+1}}$ is an analytic variety and $\tilde{X} \cap \mathbb{P}^n \cong \mathbb{TC}_0 X$. In particular $\mathbb{TC}_0 X$ is of pure codimension one in $\tilde{X}$, i.e. each irreducible component of $\mathbb{TC}_0 X$ is of codimension one in $\tilde{X}$*

Proof. In an affine chart $(x_0, t_1, t_2, \ldots, t_n)$ $\tilde{X}$ is given by

$$f_m(1, t_1, \ldots, t_n) + x_0 f_{m+1}(1, t_1, \ldots, t_m) + \cdots, \; f = f_m + f_{m+1} + \cdots \in \mathcal{I}$$

and so it is a variety. Intersection of $\tilde{X}$ with $\mathbb{P}^n$ in this coordinate system is $f_m(1, t_1, t_2, \ldots, t_m) = 0, f \in \mathcal{I}$ which is $\mathbb{TC}_0 X$ in the coordinate system $(t_1, t_2, \ldots, t_n)$ of $\mathbb{P}^n$.

The dimension m of each irreducible component of $\tilde{X} \cap \mathbb{P}^n$ satisfies $dim\tilde{X} \geq m \geq dim(\tilde{X}) + dim\mathbb{P}^n - (n+1) = dim\tilde{X} - 1$ (see [**Ke**] Theorem 3.6.1). Since $\tilde{X}$ has no irreducible component in $\mathbb{P}^n$, we conclude that $m = dim\tilde{X} - 1$. □

By definition the blow up variety $\tilde{X}$ is embedded in $\mathbb{P}^n \times \mathbb{C}^{n+1}$ and so we have the projection on the second coordinate $\pi : \tilde{X} \to \mathbb{C}^{n+1}$, called blow up map at $0 \in X$, and the projection on the first coordinate $\pi_1 : \tilde{X} \to \mathbb{P}^n$. Put $A = \mathbb{TC}_0 X$ and U a small neighborhood of A in $\tilde{X}$. We have

1. π induces a biholomorphism between $U - A$ and $\pi(U) - \{0\}$;
2. $\pi_1 \mid_A$ is an embedding of A in $\mathbb{P}^n$.

Theorem 3.7. *Suppose the that all irreducible components of an exceptional variety A in a manifold X are of codimension one and the normal bundle of A in X is negative. There is a positive integer ν_1 such that for all $k \geq \nu_1$ if $s_0, s_1, \ldots, s_n$ form a basis for the vector space $H^0(U, \mathcal{M}^k)/H^0(U, \mathcal{M}^{k+2})$ then*

$$F_k : U \to \mathbb{P}^n \times \mathbb{C}^{n+1}$$

$$F_k(x) = ([s_0(x) : s_1(x) : \cdots : s_n(x)], (s_0(x), s_1(x), \ldots, s_n(x)))$$

is a well-defined map and is an embedding of a small neighborhood of A in U with the properties 1,2 listed above.

Of course the number n depends on k. Without loosing the generality we can assume that $s_0, s_1, \ldots, s_m,\ 0 \leq m \leq n$ form a basis for $H^0(U, \mathcal{M}^k)/H^0(U, \mathcal{M}^{k+1})$.

Proof. We prove that there exists $\nu_1 \in \mathbb{N}$ such that for $k \geq \nu_1$ the statements 1,2 and 3 listed below are true:

1. F_k is well-defined. Let $Zero(s_i), 0 \leq i \leq m$ be the zero divisor of s_i. One can write $Zero(s_i) = D_i + k.A$, where D_i is a divisor in U and it does not contain A. If $\cap_{i=0}^n |D_i|$ is empty then for a point $x \in U$ there is some D_i such that $x \notin D_i$ and so $\frac{s_j}{s_i},\ j = 1, 2, \ldots, n, j \neq i$ is a holomorphic function near x. This means that $[s_0(x) : s_1(x) : \ldots : s_n(x)] = [\frac{s_0(x)}{s_i(x)} : \frac{s_1(x)}{s_i(x)} : \ldots : \frac{s_n(x)}{s_i(x)}]$ is well-defined in a neighborhood of x. Recall that for a coherent sheaf $\mathcal{S}$ on X and a subvariety$Y \subset X$ we set $Res_Y(\mathcal{S}) = \mathcal{S}/\mathcal{S}\mathcal{M}_Y$, where $\mathcal{M}_Y$ is the zero ideal of Y. For k big enough we have $H^1(U, \mathcal{M}_{x_1}\mathcal{M}^k) = 0$ and so

$$H^0(U, \mathcal{M}^k) \to Res_x(\mathcal{M}^k) \to 0 \tag{3.3}$$

Now (3.3) is true for all points in a neighborhood of x. Since $\bar{U}$ is compact, we can cover U by a finite number of such open sets. Therefore there exists a positive integer k_1 such that (3.3) is true for all $x \in U$. If $x \in \cap_{i=0}^n |D_i|$ then $H^0(U, \mathcal{M}^k) \subset H^0(U, \mathcal{M}_x\mathcal{M}^k)$. By the above sequence we conclude that $Res_x(\mathcal{M}^k)$ is empty which is a contradiction.

2. F_k is one to one. Let $x, y \in U$. We take k big enough such that $H^1(U, \mathcal{M}_{x,y}\mathcal{M}^k) = 0$, where by x, y we mean the set $\{x, y\}$. We have

$$H^0(U, \mathcal{M}^k) \to Res_{x,y}(\mathcal{M}^k) \to 0 \tag{3.4}$$

The above sequence is true in a neighborhood of (x, y) in $\bar{U} \times \bar{U}$. Since $\bar{U}$ is compact, we can cover $\bar{U} \times \bar{U}$ by a finite number of such open sets. Therefore there exists a positive integer k_2 such that (3.4) is true for all $x, y \in U,\ k \geq k_2$.

3. F_k is a locally embedding map. In the above argument we can take $\mathcal{M}_{x,x} = \mathcal{M}_x^2$ and so for $k \geq k_2$ we have

$$H^0(U, \mathcal{M}^k) \xrightarrow{\alpha} H^0(U, \mathcal{M}^k/\mathcal{M}^k\mathcal{M}_x^2) \to 0,\ \forall x \in U$$

Since $H^0(U, \mathcal{M}^{k+2}) \subset ker\alpha$ we have

$$(3.5) \quad H^0(U, \mathcal{M}^k)/H^0(U, \mathcal{M}^{k+2}) \xrightarrow{\beta} H^0(U, \mathcal{M}^k/\mathcal{M}^k\mathcal{M}_x^2) \to 0, \ \forall x \in U$$

Fix a point $x \in A$. We can suppose that $s_0(x) \neq 0$. The support of $\mathcal{M}^k/\mathcal{M}^k\mathcal{M}_x^2$ is the point x and at this point $(\mathcal{M}^k/\mathcal{M}^k\mathcal{M}_x^2)x \cong s_0\frac{\mathcal{O}_{X,x}}{\mathcal{M}_x^2}$. The image of s_i by β is $s_0.\frac{s_i}{s_0}$ and so by (3.5) the pullback of the coordinates functions $\frac{x_i}{x_0}$ of $\mathbb{P}^n$ by F_k span $\mathcal{O}_{X,x}/\mathcal{M}_x^2$. This implies that the map $T^*_{F_k(x)}\mathbb{P}^n \times \mathbb{C}^{n+1} \to T^*_xU$ is surjective and so by Proposition (1.4) F_k is an embedding in a neighborhood of x.

We set $\nu_1 = max\{k_1, k_2\}$ and get the global embeddings $F_k, k \geq \nu_1$. $\square$

Complementary notes

1. The various definitions of positive line bundles coincide. However, for vector bundles whose fibers have dimension greater than one these definitions are not equivalent (see [**Gri69**], [**Gri65**] and [**Um73**]).
2. A linear space L over a variety A is a natural generalization of a vector bundle over a manifold, for this see the survey [**GPR94**] chapter 1 section 3 and also [**Gr62**] Definition 5 p. 351. L has a zero section biholomorphic to A and we say that a linear space is negative if its zero section is exceptional. In Theorem 3.3 we have strongly used the fact that the normal bundle of a an embedded variety has a structure of a linear space.

Chapter 4

Formal principle and Artin's theorem

The formal principle says: Every isomorphism from the formal neighborhood $A_{(\infty)}$ of A in X to the formal neighborhood $A'_{(\infty)}$ of a subvariety A' of a variety X' implies the existence of a biholomorphism from an open neighborhood of A in X onto an open neighborhood of A' in X' (Note that we do not say that the formal isomorphism of neighborhoods is convergent). It is known that the formal principle does not hold in every case. V.I. Arnold in [**Ar76**] has introduced a torus embedded in a complex manifold of dimension two with trivial normal bundle. The formal neighborhood of this torus is isomorphic with the formal neighborhood of the zero section of the normal fiber bundle, but, there does not exist a biholomorphism between a neighborhood of the torus and of the zero section of the normal fiber bundle. However, the formal principle holds when the embedding of A in X has suitable properties of negativity [**Gr62**], or positivity [**Hi81**]. In [**Art68**] M. Artin proves the formal principle for singularities, i.e. $A = \{a\}$ is a single point of a variety X. The next section is devoted to Artin's theorem. We use the notations

- For any local ring R, we denote by $\mathcal{M}_R$ its maximal ideal;
- $\mathbb{C}[[x]]$, the ring of formal series in x;
- $\mathbb{C}\{x\}$, the ring of convergent series in x.

4.1 Artin's theorem

Consider an arbitrary system of analytic equations

$$f_1(x,y)=0, f_2(x,y)=0,\dots,f_k(x,y)=0 \tag{4.1}$$

where $f_1, f_2, \dots, f_k$ are germs of holomorphic functions in $(\mathbb{C}^n \times \mathbb{C}^m, 0)$.

Theorem 4.1. *(M. Artin [**Art68**]) Suppose that*

$$\hat{y}(x) = (\hat{y}_1(x), \hat{y}_2(x), \dots, \hat{y}_m(x))$$

are formal power series without constant term which solve (4.1), i.e.

$$f(x,\hat{y}(x)) = 0, \ f = (f_1, f_2, \dots, f_k)$$

Let c be a positive integer. There exists a convergent series solution

$$y(x) = (y_1(x), y_2(x), \dots, y_m(x))$$

of (4.1) such that

$$y(x) \equiv \hat{y}(x) \ \textit{ modulo } \mathcal{M}^c_{\mathbb{C}[[x]]}$$

Another way of stating the result is to say that the analytic solutions are dense in the space of formal solutions with its $\mathcal{M}_{\mathbb{C}[[x]]}$-adic metric (see [**Nag62**] for definitions).

Proposition 4.1. *Theorem 4.1 with $c = 1$ implies Theorem 4.1 with an arbitrary positive integer c.*

Proof. In order to prove this we need:

1. For a given positive integer c, $\mathcal{M}^c_{\mathbb{C}[[x]]}$ is the set of formal power series with the leading term of degree $\geq c$.

The above statement is the formal version of Proposition 1.1 part 1. The proof is essentially the same. We must use the formal Weierstrass preparation theorem (see [**Nag62**] p. 191). Now let us suppose that Theorem 4.1 is true for $c = 1$ and we have a formal solution $\hat{y}(x)$ for (4.1). Let $y_c(x)$ be a vector of polynomials of degree $\leq c$ such that the components of $\hat{y}(x) - y_c(x)$ have leading term of degree greater than c. By 1. we can write each component of $\hat{y}(x) - y_c(x)$ as a finite sum of terms of the form $s_1.s_2 \dots s_{c+1}$, where

$s_i \in \mathcal{M}_{\mathbb{C}[[x]]}, i = 1, 2, \ldots, c+1$. Now let $\{a_1, a_2, \ldots, a_p \in \mathcal{M}_{\mathbb{C}[[x]]}\}$ be the set of all such s_i's. We have $\hat{y}(x) - y_c(x) = F(a_1, a_2, \ldots, a_p)$, where the components of F are polynomials in $a_1, a_2, \ldots, a_p$. We introduce the new variables $A_1, A_2, \ldots, A_p$ and the equations

$$f(x, y_c(x) + F(A_1, A_2, \ldots, A_p)) = 0$$

with variables $x_1, x_2, \ldots, x_n, A_1, A_2, \ldots, A_p$. These equations have a formal solution $A_i = a_i \in \mathcal{M}_{\mathbb{C}[[x]]}$, $i = 1, 2 \ldots, p$ and so by Theorem 4.1 for $c = 1$ we have a convergent solution. This gives us the Theorem 4.1 for c. $\square$

Fix the formal solution $\hat{y}(x)$ of (4.1) and put

$$\mathcal{I} = \{f \in \mathcal{O}_{\mathbb{C}^n \times \mathbb{C}^m, 0} \mid f(x, \hat{y}(x)) = 0\}$$

$f_1, f_2, \ldots, f_k \in \mathcal{I}$ and $\mathcal{I}$ is a prime ideal. Therefore the zero locus of the ideal $\mathcal{I}$, namely V, is irreducible. Knowing Proposition 4.1, Theorem 4.1 is equivalent to: There is a submanifold $(N, 0)$ of $(\mathbb{C}^{n+m}, 0)$ of dimension n such that 1. $N \subset V$, 2. the projection on the first n coordinates $N \to (\mathbb{C}^n, 0)$ is a biholomorphism.

4.2 Formal principle for singularities

This section is devoted to the proof of the formal principle for singularities following Artin's article [**Art68**]. When $A = \{a\}$ is a single point, following the literature we adopt the notations:

- $\mathcal{O}_{X,a} = A_{(*)}$, the local ring of X at a;
- $\hat{\mathcal{O}}_{X,a} = A_{(\infty)}$, the completion of the local ring $\mathcal{O}_{X,a}$;
- $\mathcal{O}^{\nu}_{X,a} = A_{(\nu)}$;

Theorem 4.2. *The formal principle holds for singularities, i.e. let X and X' be germs of holomorphic varieties at a and a' respectively. The isomorphism*

$$\hat{\tau} : \hat{\mathcal{O}}_{X,a} \cong \hat{\mathcal{O}}_{X',a'}$$

of the formal completions implies the isomorphism $X \cong X'$ of the germs of the varieties.

The preceding theorem is the corollary 1.6 of [**Art68**]. It will be instructive to see how the corollary can be obtained from Theorem 4.1.

Proof. Without losing generality, we can assume that X and X' are germs of holomorphic varieties in $(\mathbb{C}^n, 0)$ and $(\mathbb{C}^m, 0)$, respectively. We use the following notations:

$$\mathcal{O}_{X,0} = \mathbb{C}\{x\}/(f_1, \ldots, f_r)$$
$$\mathcal{O}_{X',0} = \mathbb{C}\{y\}/(g_1, \ldots, g_s)$$

where $\mathbb{C}\{x\}$ is the ring of convergent series in $(\mathbb{C}^n, 0)$. Let $\hat{p}_i(y) \in \mathbb{C}[[y]]$ represents the image $\hat{\tau}(x_i)$ of x_i in $\mathbb{C}[[y]]$. The fact that $\hat{\tau}$ is a homomorphism implies that

$$f_i(\hat{p}_1(y), \hat{p}_2(y), \ldots, \hat{p}_n(y)) \equiv 0 \quad \text{modulo } (g_1, \ldots, g_s),\ i = 1, 2, \ldots, r$$

i.e. there are formal series $\hat{\beta}_{ij}(y) \in \mathbb{C}[[y]]$ with

$$f_i(\hat{p}_1(y), \hat{p}_2(y), \ldots, \hat{p}_n(y)) = \sum_{j=1}^{s} \hat{\beta}_{ij}(y) g_j(y)$$

Now consider the system of holomorphic equations

$$f_i(p_1, p_2, \ldots, p_n) - \sum_{j=1}^{s} \beta_{ij} g_j(y) = 0,\ i = 1, 2, \ldots, r$$

with unknown variables y, p_i, β_{ij}. This system has the formal solution $\hat{p}_i(y), \hat{\beta}_{ij}(y)$. Applying Theorem 4.1 with $c = 2$, we obtain a homomorphism

$$\tau_1 : \mathcal{O}_{X,0} \to \mathcal{O}_{X',0}$$

which is congruent to $\hat{\tau}$ modulo $\mathcal{M}_{\mathcal{O}_{X',0}}$. Let us prove that τ_1 is an isomorphism. With the same argument for $\hat{\tau}^{-1}$ we obtain τ_2 congruent $\hat{\tau}^{-1}$ modulo $\mathcal{M}_{\mathcal{O}_{X,0}}$. Now $\tau_1 \circ \tau_2 : \mathcal{O}_{X,0} \to \mathcal{O}_{X,0}$ is congruent to the identity modulo $\mathcal{M}_{\mathcal{O}_{X',0}}$. The second part of Proposition 1.8 finishes the proof of our corollary. □

4.3 Formal principle for exceptional varieties

The formal principle was proved for the first time by Grauert in [**Gr62**] for codimension one compact strongly exceptional manifolds. Extending his method, Hironaka and Rossi in [**HiRo64**] proved the formal principle for pure codimension one strongly exceptional varieties in manifolds. Their proof is a direct generalization of Grauert's proof. In this section we present a more algebraic proof of the formal principle for strongly exceptional varieties of pure codimension one in manifolds. The basic tools for this proof are the vanishing Theorem 3.5 and the Embedding theorem 3.7.

Theorem 4.3. *([**Gr62**],[**HiRo64**]) The formal principle holds for strongly exceptional varieties of pure codimension one in manifolds.*

In what follows we will write $H^\mu(A, \mathcal{SM}^k)$ instead of $H^\mu(U, \mathcal{SM}^k)$. Here we consider sheaf theory restriction of $\mathcal{SM}^k$ to A.

Proof. Let us be given an isomorphism $\phi_{(\infty)} : A_{(\infty)} \to A'_{(\infty)}$. Recall that by definition this is a collection of isomorphisms $\phi_{(\nu)} : A_{(\nu)} \to A'_{(\nu)}, \nu \in \mathbb{N}$, such that the diagram

$$\begin{array}{ccc} A_{(\nu)} & \overset{\phi_{(\nu)}}{\to} & A'_{(\nu)} \\ \downarrow & & \downarrow \\ A_{(\mu)} & \overset{\phi_{(\mu)}}{\to} & A'_{(\mu)} \end{array} \tag{4.2}$$

is commutative for all $\mu < \nu,\ \mu, \nu \in \mathbb{N}$. In particular By (1.4) $\phi_{(\mu)}$ induces an isomorphism $\mathcal{Q}_\mu \to \mathcal{Q}'_\mu$. Now consider the following (not complete) subdiagram of (4.2)

$$\begin{array}{ccc} \mathcal{M}^{\mu-1}/\mathcal{M}^\nu & \to & ? \\ \downarrow & & \downarrow \\ \mathcal{Q}_\mu & \to & \mathcal{Q}'_\mu \end{array} \tag{4.3}$$

A simple argument shows that instead ? we have $\mathcal{M}'^{\mu-1}/\mathcal{M}'^\nu$ and the map $\mathcal{M}^{\mu-1}/\mathcal{M}^\nu \to \mathcal{M}'^{\mu-1}/\mathcal{M}'^\nu$ is an isomorphism. Replacing $\mu - 1$ with ν and ν with $\nu + 2$ we get an isomorphism

$$\mathcal{M}^\nu/\mathcal{M}^{\nu+2} \to \mathcal{M}'^\nu/\mathcal{M}'^{\nu+2} \tag{4.4}$$

Now suppose that A is an exceptional variety with a negative normal bundle. The isomorphism $\mathcal{Q}_1 \to \mathcal{Q}'_1$ implies that A and A' have biholomorphic normal bundles and so A' is also exceptional. According to Theorem 3.5 if we take ν big enough then $H^1(A, \mathcal{M}^\nu) = 0$. We write the long exact sequence of

$$0 \to \mathcal{M}^{\nu+2} \to \mathcal{M}^\nu \to \mathcal{M}^\nu/\mathcal{M}^{\nu+2} \to 0$$

and we conclude that $H^0(A, \mathcal{M}^\nu/\mathcal{M}^{\nu+2}) = H^0(A, \mathcal{M}^\nu)/H^0(A, \mathcal{M}^{\nu+2})$. In the same way $H^0(A', \mathcal{M}'^\nu/\mathcal{M}'^{\nu+2}) = H^0(A', \mathcal{M}'^\nu)/H^0(A', \mathcal{M}'^{\nu+2})$. We apply the functor H^0 on the map (4.4) and get an isomorphism

$$\alpha : H^0(A, \mathcal{M}^\nu)/H^0(A, \mathcal{M}^{\nu+2}) \to H^0(A, \mathcal{M}'^\nu)/H^0(A, \mathcal{M}'^{\nu+2}) \tag{4.5}$$

We take $s_1, s_2, \ldots, s_n \in H^0(A, \mathcal{M}^\nu)$ such that they form a basis of the $\mathbb{C}$-vector space $H^0(A, \mathcal{M}^\nu)/H^0(A, \mathcal{M}^{\nu+2})$. According to theorem 3.7 if ν is big enough then

$$(X, A) \to \mathbb{P}^n \times \mathbb{C}^{n+1} \tag{4.6}$$

$$x \to [s_0(x) : s_1(x) : \cdots : s_n(x)] \times (s_0(x), s_1(x), \cdots, s_n(x))$$

is an embedding. Let ${s'}_i \in H^0(A, \mathcal{M}'^\nu)$ be the image of s_i by the map α. Since the map α is an isomorphism ${s'}_i$'s give also an embedding of (X', A') in a similar way (We may take ν bigger). We will need the following lemma:

Lemma 4.1. *Let f be a holomorphic function in a neighborhood of A in X with $Zero(f) = \nu A + D$, where D is a divisor without A in a neighborhood of A in X. If $Zero(\alpha(f)) = \nu' A' + D'$, D' a divisor without A', then $\nu = \nu'$ and $D \cap A$ is mapped to $D' \cap A'$ by $\phi : A \to A'$.*

Fix a point $a \in A$ and $a' = \phi(a) \in A'$ and assume that $\frac{s_1}{s_0}, \frac{s_2}{s_0}, \ldots, \frac{s_n}{s_0}, s_0, s_1, \ldots, s_n$ form a coordinate system in a neighborhood of a in X. Applying the above lemma on s_0 implies that $\frac{s'_1}{s'_0}, \frac{s'_2}{s'_0}, \ldots, \frac{s'_n}{s'_0}, s'_0, s'_1, \ldots, s'_n$ form also a coordinate system in a neighborhood of a' in A'.

Lemma 4.2. *We have $b_i := \frac{s_i}{s_0}(a) = \frac{s'_1}{s'_0}(a')$ for all $i = 1, 2, \ldots, n$.*

For holomorphic functions $h_0, h_1, \ldots, h_n$ with values in $(\mathbb{C}, 0)$ and h'_i, $1 \leq i \leq n$ with values in $(\mathbb{C}, b_i)$ define

$$\mathcal{A}(h_0, h_1, \ldots, h_n) = \{f(h_0, h_1, \ldots, h_n) \mid f \in \mathcal{O}_{\mathbb{C}^{n+1},0}\}$$

$$\mathcal{A}(h_0, h_1, \ldots, h_n, h'_1, h'_2, \ldots, h'_n) = \{f(h_0, h_1, \ldots, h_n, h'_1, h'_2, \ldots, h'_n) \mid f \in \mathcal{O}_{\mathbb{C}^{n+1}\times\mathbb{C}^n,(0,b)}\}$$

where $b = (b_1, b_2, \ldots, b_n)$. Consider the map

$$\beta : \mathcal{A}(s_0, s_1, \ldots, s_n) \to \mathcal{A}(s'_0, s'_1, \ldots, s'_n),$$

$$f(s_0, s_1, \dots, s_n) \to f(s'_0, s'_1, \dots, s'_n)$$

We have $\mathcal{A}(s_0, s_1, \dots, s_n) \subset A_{(*)_a}$ and $\mathcal{A}(s'_0, s'_1, \dots, s'_n) \subset A'_{(*)_{a'}}$ and we claim that β extends to a unique isomorphism $\phi_{(*)_a} : A_{(*)_a} \to A'_{(*)_{a'}}$. Since β does not depend on the point a we get an isomorphism $\phi_{(*)} : A_{(*)} \to A'_{(*)}$ and so by the first part of Proposition 1.7 a biholomorphism $(X, A) \to (X', A')$.

We extend β to a map

$$\beta' : \mathcal{A}(s_0, s_1, \dots, s_n, \frac{s_1}{s_0}, \frac{s_2}{s_0}, \dots, \frac{s_n}{s_0}) \to \mathcal{A}(s'_0, s'_1, \dots, s'_n, \frac{s'_1}{s'_0}, \frac{s'_2}{s'_0}, \dots, \frac{s'_n}{s'_0})$$

$$f(s_0, s_1, \dots, s_n, \frac{s_1}{s_0}, \frac{s_2}{s_0}, \dots, \frac{s_n}{s_0}) \to f(s'_0, s'_1, \dots, s'_n, \frac{s'_1}{s'_0}, \frac{s'_2}{s'_0}, \dots, \frac{s'_n}{s'_0}),$$

for $f \in \mathcal{O}_{\mathbb{C}^{n+1} \times \mathbb{C}^n, (0,b)}$. Now by the fact that (4.1) is an embedding we have

$$A_{(*)_a} = \mathcal{A}(s_0, s_1, \dots, s_n, \frac{s_1}{s_0}, \frac{s_2}{s_0}, \dots, \frac{s_n}{s_0})$$

and the same statement for A'. Theorem 4.3 is proved. □

Proof of Lemma 4.1 and 4.2: The statement $\nu = \nu'$ of Lemma 4.1 is a consequence of the isomorphism $\mathcal{Q}_\nu \to \mathcal{Q}'_\nu$ for all $v \in \mathbb{N}$. The second part of Lemma 4.1 and also Lemma 4.2 is a consequence of the fact that all the isomorphisms $\phi_{(\nu)}$ induces a fixed isomorphism $\phi_{(1)}$. This implies that if $f_1 \in A_{(\nu)_a}$ then $f_1(a) = \phi_{(\nu)}(f_1)(a')$. In particular $f_1(a) = 0$ if and only if $\phi_{(\nu)}(f_1)(a') = 0$. □

4.4 Grauert's theorem

Now we are in a position to state Grauert theorem about rigidity of strongly exceptional varieties. Let $A \hookrightarrow X$ be a strongly exceptional variety, A' be the zero section of the normal bundle N of A in X and T_A the tangent bundle of A.

Theorem 4.4. *(Grauert [***Gr62***] Satz 7 p. 363) Let $\phi_{(l)} : A_{(l)} \to A'_{(l)}$ be an isomorphism and $H^1(A, T_A \otimes N^\nu) = 0, H^1(A, N^{\nu-1}) = 0$ $\nu \geq l$. Then ϕ extends to a biholomorphism of neighborhoods $A_{(*)} \to A'_{(*)}$. In particular if $H^1(A, T_A \otimes N^\nu) = 0, H^1(A, N^\nu) = 0$, $\nu \geq 1$ then there exists a biholomorphism between a neighborhood of A in X and a neighborhood of A' in N.*

Proof. By the hypotheses and Theorem 1.5 we can get a formal isomorphism of (X, A) and (N, A'). Now by formal principle for strongly exceptional varieties Theorem 4.3 we can find the desired biholomorphism.

□

Notice that by Kodaira vanishing theorem (Theorem 3.3) there exists a ν_0 such that $H^1(A, T_A \otimes N^\nu) = 0, H^1(A, N^{\nu-1}) = 0$ $\nu \geq \nu_0$. Roughly speaking, the germ of a strongly exceptional variety of pure codimension one is determined by a ν-neighborhood for ν big enough.

Now consider the case in which A is a Riemann surface embedded in a two dimensional manifold. A line bundle on A is negative if and only if it has a negative Chern class. Therefore A is strongly exceptional if and only if the self intersection of A is negative. Now in Theorem 1.6 instead of a formal equivalence we have a biholomorphism.

Complementary notes

1. It would be nice if the proof of Theorem 4.1 to be discussed from [**Art62**]. We have not given the proof, because we were not able to simplify Artin's argument. To the authors knowledge, generalizations of this theorem do not give a simpler proof, because these generalizations use Theorem 4.1 (see [**BDLD79**],[**Wa75**]).

2. To state a generalization of Theorem 4.3 we introduce the concept of modification. A proper surjective holomorphic map $\phi : X \to Y$ of analytic varieties X and Y is called a modification if there are closed analytic sets $A \subset X$ and $Y \subset Y$ with codimension at least one such that 1. $\phi(A) = B$ 2. $\phi : X - A \to Y - B$ is biholomorphic 3. A and B are minimal with the properties 1 and 2. Note that for us an analytic variety is always assumed to be reduced. A more general theorem about formal principle is the following:

   Theorem: ([**Kos81**],[**An80**]) If $\phi : (X, A) \to (Y, B)$ is a modification with A and B compact then the formal principle holds for (X, A) if and only if it holds for (Y, B).

   Formal principle is true for singularities and so Theorem 4.3 is a consequence of Theorem 2.

Chapter 5

Foliated neighborhoods

Let A be a Riemann surface embedded in a two dimensional manifold X. In what follows we use both X and (X, A) to denote the germ of X in A. A (holomorphic) foliation in X with isolated singularities is given by a collection of holomorphic 1-forms ω_α defined on $U_\alpha, \alpha \in I$, where $\{U_\alpha\}_{\alpha \in I}$ is an open covering of X, and such that

$$\omega_\alpha = g_{\alpha\beta}\omega_\beta,\ \alpha, \beta \in I, g_{\alpha\beta} \in \mathcal{O}_X^*(U_\alpha \cap U_\beta) \tag{5.1}$$

where $\mathcal{O}_X^*$ is the sheaf of holomorphic without zero functions in X. Furthermore we assume that the set of points in which ω_α is zero has codimension greater than one (discrete set). In other words ω_α has not a zero divisor. Therefore for any foliation $\mathcal{F}$ there is associated a line bundle L given by the transition functions

$$L = \{g_{\alpha\beta}\}_{\alpha,\beta \in I} \in Pic(X) := H^1(X, \mathcal{O}_X^*)$$

The data (5.1) can be considered as a holomorphic section $\omega \in H^0(X, \Omega^1 \otimes L)$ without zero divisor, where Ω^1 is the cotangent bundle of X.

Fix a line bundle L in X. Any section $\omega \in H^0(X, \Omega^1 \otimes L)$ gives us a foliation, say $\mathcal{F}(\omega)$. If ω has a zero divisor we use the following trick: Let $\omega \in \Omega^1 \otimes L$ be a holomorphic section with the zero divisor Z. Let L_Z be the line bundle associated to Z and $s \in H^0(M, L_Z)$ be the holomorphic section with the zero divisor Z. Now $\frac{\omega}{s}$ is a holomorphic without zero divisor section of $H^0(X, \Omega \otimes L \otimes L_Z^{-1})$, and so, we can substitute L by $L \otimes L_Z^{-1}$.

Let L be a line bundle over X. We denote by $\mathcal{F}(X, A, L)$ the space of holomorphic foliations in X given by the forms $\omega \in \Omega^1 \otimes L$. If two

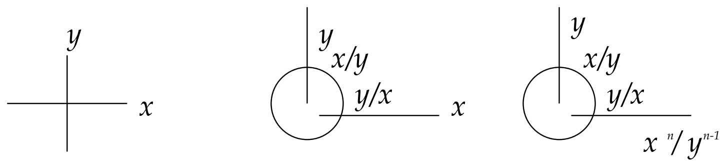

Figure 5.1: A projective line with self intersection $-n$

holomorphic without zero divisor sections $\omega, \omega' \in \Omega^1 \otimes L$ induce the same foliation then $\omega = f.\omega'$, where f is a holomorphic function on X and $f(A)$ (which is a constant) is not zero. Therefore we have the map

$$\mathcal{F}(X, A, L) \to \mathbb{P}(H^0(A, (\Omega^1 \otimes L) \mid_A))$$

Two foliations $\mathcal{F}, \mathcal{F}' \in \mathcal{F}(X, A, L)$ in (X, A) are called equivalent, say $\mathcal{F}R\mathcal{F}'$, if there exists a biholomorphism

$$\psi : (X, A) \to (X, A)$$

such that 1. $\psi \mid_A$ is identity, 2. $\psi^{-1}(\mathcal{F}') = \mathcal{F}$. We are interested in the space $\mathcal{F}(X, A, L)/R$ of equivalence classes. Natural questions in this direction arise: Is $\mathcal{F}(X, A, L)/R$ finite dimensional? Does $\mathcal{F}(X, A, L)/R$ has a natural structure of complex space? When $\mathcal{F}(X, A, L)/R$ is a discrete set? To answer these questions we start with the most simple foliations, namely, foliations without singularity and transverse to A. The reader is referred to [**La71**] for more information about a one dimensional exceptional variety embedded in a manifold.

5.1 Transversal foliations

Let A be a exceptional Riemann surface in a two dimensional manifold X. According to [**La71**] Theorem 4.9 A has a negative self-intersection. In this section we are concerned with germs of transverse holomorphic foliations in (X, A), i.e. the foliations with no singularity and with leaves transverse to A. Let us introduce some examples in the case $A = \mathbb{P}_1$.

By successive blow-ups at the origin of $\mathbb{C}^2$, we can get a $A \cong \mathbb{P}_1$ embedded in a two dimensional manifold and with $A.A = -n$. A neighborhood of A is covered by coordinate systems $(u, y) = (\frac{X}{Y}, Y)$ and $(x, t) = (\frac{X^n}{Y^{n-1}}, \frac{Y}{X})$, where

X and Y are the pullback of a coordinates system at the origin of $\mathbb{C}^2$. The change of coordinates is given by

$$(x,t) \to (\frac{1}{t}, xt^n) = (u,y)$$

In this example we have a germ of transverse holomorphic foliation $\mathcal{F}$ given by the 1-form

$$\omega = XdY - YdX = (xt^{n-1})^2 dt = -y^2 du$$

It is easy to check that

$$zer(\omega) = 2.A + 2(n-1)L$$

$$zer(Y) = 1.A + n.L, \ zer(X) = 1.A + (n-1)L + L'$$

where $zer()$ means the zero divisor and L (resp. L') is the leaf of $\mathcal{F}$ given by $t = 0$ (resp. $u = 0$) in the coordinates (x,t) (resp. (u,y)); it is the pullback of X-axis (resp. Y-axis). The mentioned example contains the basic idea of the proof of the following theorem.

Theorem 5.1. *([**CMS02**]) Let A be a Riemann surface of genus g embedded in a manifold X of dimension two with $A.A < min\{2-2g, 0\}$. The germs of any two holomorphic transverse foliations are equivalent.*

Let us first state the main lemma we need in the proof of the above theorem:

Lemma 5.1. *Let A be a complex manifold of dimension n negatively embedded in a manifold X of dimension $n+1$. Moreover suppose that $H^1(A, N^*) = 0$, where N is the normal bundle of the embedding and N^* is the dual bundle. The restriction map $r : Pic(X) \to Pic(A)$ is injective. The negativity condition and $H^1(U, N^*) = 0$ in the case $n = 1$ translates into $A.A < min\{2-2g, 0\}$ using the Serre duality.*

Note that the negativity condition does not imply $H^1(U, N^*) = 0$. If X has a transversal foliation to A by curves then we have a holomorphic map $\sigma : X \to A$ which is constant along the leaves of the foliation. The pull-back of line bundles on A by the map σ shows that r is surjective.

Proof. In the case $n = 1$ we use the Serre duality and we have: If $A.A < 2-2g$ then $H^1(A, N^*) = H^0(A, \Omega^1 \otimes N)^* = 0$. Therefore we have proved the last statement of our theorem.

Now let us prove the first part of the lemma. The sheaf of holomorphic sections of N^* is isomorphic to $\mathcal{M}/\mathcal{M}^2$ and so we have

$$H^1(A, \mathcal{M}/\mathcal{M}^2) = 0,$$

By Theorem 3.5 and the remark after we have

$$H^1(U, \mathcal{M}) = 0$$

where U is a strongly pseudoconvex neighborhood of A in X. The diagram

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \downarrow & & & & \\ & & & & \mathcal{M} & & & & \\ & & & & \downarrow & & & & \\ 0 & \to & \mathbb{Z} & \to & \mathcal{O}_X & \to & \mathcal{O}_X^* & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathbb{Z} & \to & \mathcal{O}_A & \to & \mathcal{O}_A^* & \to & 0 \\ & & & & \downarrow & & & & \\ & & & & 0 & & & & \end{array} \tag{5.2}$$

gives us

$$\begin{array}{ccccccc} & & H^1(U, \mathcal{M}) = 0 & & & & \\ & & \downarrow & & & & \\ H^1(U, \mathbb{Z}) & \to & H^1(U, \mathcal{O}_X) & \to & H^1(U, \mathcal{O}_X^*) & \to & H^2(U, \mathbb{Z}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^1(A, \mathbb{Z}) & \to & H^1(A, \mathcal{O}_A) & \to & H^1(A, \mathcal{O}_A^*) & \to & H^2(A, \mathbb{Z}) \end{array} \tag{5.3}$$

By considering a small neighborhood U, if necessary, we can assume that A and U have the same topology and so the first and forth column functions are isomorphisms. In the argument which we are going to consider now we do not mention the name of mappings, being clear from the above diagram which mapping we mean.

Let $x_1 \in H^1(U, \mathcal{O}_X^*)$ maps to zero (the trivial bundle) in $H^1(A, \mathcal{O}_A^*)$. Since the fourth column is an isomorphism, x_1 maps to zero in $H^2(U, \mathbb{Z})$. This means that there is a $x_2 \in H^1(U, \mathcal{O}_X)$ which maps to x_1. Let x_3 be the image of x_2 in $H^1(A, \mathcal{O}_A)$. Since the above diagram is commutative, x_3 maps to the trivial bundle in $H^1(A, \mathcal{O}_A^*)$. Therefore there exists a x_4 in $H^1(A, \mathbb{Z})$ which maps to x_3. Since the first column is an isomorphism and the second is injective, we conclude that $x_4 \in H^1(U, \mathbb{Z}) \cong H^1(A, \mathbb{Z})$ maps to x_2 and so x_2 maps to $x_1 = 0$ in $H^1(U, \mathcal{O}_X^*)$. □

Proof of Theorem 5.1: Let $\mathcal{F}$ be the germ of a transverse foliation in (X, A) and N the normal bundle of A in X. The normal bundle N of A in X has a meromorphic global section namely s. Let

$$div(s) = \sum n_i p_i, \ p_i \in A, \ n_i \in \mathbb{Z}$$

We define the divisor D in X as follows:

$$D = A - \sum n_i \mathcal{L}_{p_i}$$

where $\mathcal{L}_{p_i}$ is the leaf of $\mathcal{F}$ through p_i. The line bundle L_D associated to D restricted to A is the trivial line bundle, and so by Lemma 5.1, L_D is trivial or equivalently there exists a meromorphic function g on (X, A) with $div(g) = D$.

Let $\tilde{f}$ be an arbitrary meromorphic function on A and f its extension along the foliation . Define the 1-form

$$\omega = g df$$

The 1-form ω has the following properties

1. ω induces the foliation $\mathcal{F}$;
2. The divisor of ω is $A + K$, where K is $\mathcal{F}$-invariant and its restriction to A depends only on $\tilde{f}$ and the meromorphic section s.

Let $\mathcal{F}'$ be another transverse foliation in (X, A). In the same way we can construct the 1-form ω' for $\mathcal{F}'$. We claim that at each point $a \in A$ there exists a unique biholomorphism

$$\psi_a : (X, A, a) \to (X, A, a)$$

inducing identity on A and with the property $\psi^{-1}(\omega') = \omega$. The uniqueness property implies that these local biholomorphisms are parts of a global biholomorphism $\psi : (X, A) \to (X', A')$ which send ω to ω'.

Now we prove our claim. Fix a coordinate system x in a neighborhood of a in A. Let $k_1 = (x_1, y_1)$ be a coordinates system in a neighborhood of a in X such that A and $\mathcal{F}$ in this coordinates system are give respectively by $y_1 = 0$ and $dx_1 = 0$ and $x_1 \mid_A = x$. We can write $\omega = px_1y_1^m dx$, where $m \in \mathbb{Z}$ depends only on $\tilde{f}$, s and $p \in \mathcal{O}^*_{X,a}$. By changing the coordinates $(x_1, y_1) \to (x_1, p^{\frac{1}{m}}y_1)$, we can assume that $p = 1$. It is easy to check that the coordinate system (x_1, y_1) with the mentioned properties is unique. In the same way we can find a coordinates system $k_2 = (x_2, y_2)$ in a neighborhood of a in X such that in this coordinates system $\omega' = x_2y_2^m dx_2$ and $x_2 \mid_A = x$. We identify the images of k_1 and k_2. The map $k_2^{-1} \circ k_1$ is the desired biholomorphism. □

Now we give another application of Lemma 5.1.

Proposition 5.1. *Consider the situation of Lemma 5.1. Let $\mathcal{F}$ be a foliation by curves transversal to A in X. Then there exists a holomorphic vector field V defined a neighborhood of A in X with the following properties: 1. V is tangent to $\mathcal{F}$ 2. The zero divisor of V is $1.A$.*

Proof. Let $U_\alpha, \alpha \in I$ be an open covering of A in U such that in each U_α there is defined a vector field V_α without zero locus and tangent to $\mathcal{F}$. Then $g_{\alpha\beta} := \frac{X_\beta}{X_\alpha} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$ is a cocycle and hence

$$L := \{g_{\alpha\beta}\} \in H^1(U, \mathcal{O}^*)$$

Since the X_α's are tangent to $\mathcal{F}$, we can think of L as the tangent line bundle to $\mathcal{F}$ and consequently as the normal bundle of A in X when we restrict it to A. Now let us consider A as a divisor with coefficient $+1$ in X and let f_α be a holomorphic function on U_α vanishing on V of order one (If it is necessary we can take a finer covering of A). The line bundle associated to A is given by

$$L' := \{g'_{\alpha\beta}\},\ g'_{\alpha\beta} = \frac{f_\alpha}{f_\beta}$$

L' restricted to A is again the normal bundle N. Therefore LL'^{-1} restricted to A is the trivial bundle and so by lemma 5.1 LL'^{-1} is the trivial bundle or equivalently there are holomorphic functions $s_\alpha \in \mathcal{O}^*(U_\alpha)$ such that

$$g_{\alpha\beta} = \frac{s_\alpha}{s_\beta} g_{\alpha\beta} \Rightarrow \frac{X_\beta}{X_\alpha} = \frac{s_\alpha}{s_\beta}\frac{f_\alpha}{f_\beta}$$

Now the desired global vector field U_α is defined by $V \mid_{U_\alpha} := s_\alpha f_\alpha X_\alpha.$: □

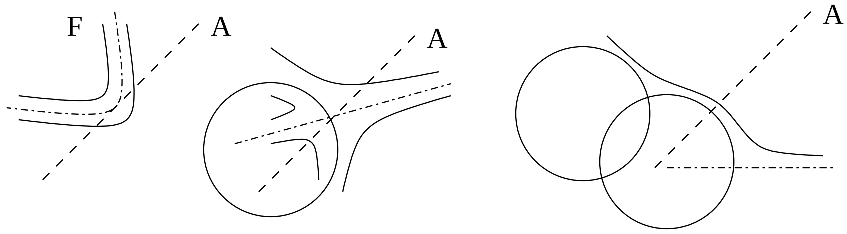

Figure 5.2: Blowing up a tangency

5.2 Foliations with tangencies and singularities

Let A be a Riemann surface of genus g embedded in a manifold X of dimension two with $A.A < min\{2-2g, 0\}$. In Theorem 5.1 we proved that the germs of any two holomorphic transverse foliations are equivalent. Now consider a foliation $\mathcal{F}$ in (X, A) which is transverse to A except in a finite number of points $S \subset A$. A point in S can be a tangency point of $\mathcal{F}$ with A or a singularity of $\mathcal{F}$ (see Figure 5.3).

Proposition 5.2. *If $A.A < min\{2-2g, 0\}$ then there exists a meromorphic 1-form ω in X with the following properties: 1. ω induces the foliation $\mathcal{F}$ 2. $div(\omega) = A - \sum_{i=1}^{k} \mathcal{L}_{p_i}$, where $\{p_1, p_2, \ldots, p_k\} \subset A - S$ and $\mathcal{L}_{p_i}$ is the leaf of $\mathcal{F}$ through p_i.*

Proof. Let s be a meromorphic section of the normal bundle of A in X such that if $div(s) = \sum_{i=1}^{k_1} n_i p_i,\ n_i \in \mathbb{Z}$ then $\{p_1, p_2, \ldots p_{k_1}\} \subset A - S$. In the proof of Theorem 5.1 we have constructed a meromorphic function in X such that $div(g) = A - \sum_{i=1}^{k_1} n_i \mathcal{L}_{p_i}$.

Let the foliation $\mathcal{F}$ be given by $\omega \in \Omega^1 \otimes L$, where L is a line bundle on X (see the beginning of this chapter). Moreover, suppose that ω has not zero divisor. We take a meromorphic section r of $L\mid_A$ such that $div(r) = \sum_{i=k_1+1}^{k} n_i p_i,\ n_i \in \mathbb{Z}$ and $\{p_{k_1+1}, \ldots, p_k\} \subset A - S$. Now by Lemma 5.1 the line bundle associated to $\sum_{i=k_1+1}^{k} n_i \mathcal{L}_{p_i}$ in X is L and so r extends to a meromorphic section of L, say again r, and $div(r) = \sum_{i=k_1+1}^{k} n_i \mathcal{L}_{p_i}$. The form $\frac{\omega}{r}$ is meromorphic in X and $g.\frac{\omega}{r}$ is the desired meromorphic form. □

Let $\mathcal{F}$ and $\mathcal{F}'$ be the germs of two foliations in (X, A) which are locally

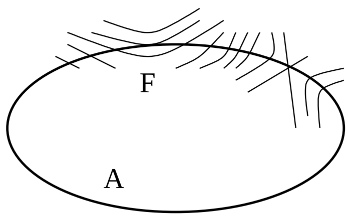

Figure 5.3: Foliations with tangencies and singularities

biholomorphic, i.e. for every point $a \in A$ there exists a biholomorphism

$$\phi_a : (X, A, a) \to (X, A, a)$$

sending the foliation $\mathcal{F}$ to $\mathcal{F}'$. Roughly speaking, $\mathcal{F}$ and $\mathcal{F}'$ have the same local analytic structure around A.

Assume that $A.A < min\{2-2g, 0\}$. By proposition 5.2 we can find a meromorphic 1-form ω (resp. ω') such that $\mathcal{F}$ (resp. $\mathcal{F}'$) is given by $\omega = 0$ (resp. $\omega' = 0$). If $\mathcal{F}$ and $\mathcal{F}'$ have the same line bundle L then we can assume that $div(\omega) \mid_A = div(\omega') \mid_A$. By the argument we used in the proof of Theorem 5.1 we can find a biholomorphism from a neighborhood of $A - S$ to itself, sending $\mathcal{F}$ to $\mathcal{F}'$. But there is no reason to claim that this biholomorphism extends to a full neighborhood of A, as we will see in the next example.

The example which we are going to explain it is due to M. Suzuki(see [**Su74**] and [**Su78**]). Consider the germ of holomorphic foliations given by

$$\mathcal{F}(\omega) : \omega = (y^3 + y^2 - xy)dx - (2xy^2 + xy - x^2)dy = 0$$

$$\mathcal{F}'(\omega') : \omega' = (2y^2 + x^3)dx - 2xydy$$

The foliation $\mathcal{F}$ (resp. $\mathcal{F}'$) has a meromorphic (resp. Liouvillian) first integral $f = \frac{y^2 - x^3}{x^2}$ (resp. $\frac{x}{y} e^{\frac{y(y+1)}{x}}$). In both cases after blowing up at 0 we get two non singular foliations around $A := \mathbb{P}_1$, the divisor of blow up, and with the following property: Both $\mathcal{F}'$ and $\mathcal{F}'$ are transverse to A in all points except one point and in this point they have a tangency of order two with A. The foliation $\mathcal{F}$ (resp. $\mathcal{F}'$) has a the tangency point in the affine chart (x, t) with the coordinates $p := (0, 0)$ (resp. $p' := (0, 1)$). Now by the change of coordinates $(x, y) \to (x, y - x)$ ($(x, t) \to (x, t - 1)$ in the affine chart) for the foliation $\mathcal{F}'$ we can assume that p' coincides with p.

It is shown in [**CeMa82**] that these two foliations are topologically equivalent, i.e. there is a homeomorphism from a neighborhood of A in X to itself which sends $\mathcal{F}$ to $\mathcal{F}'$. In the next section we will define the formal equivalence of two foliations and we will prove that $\mathcal{F}$ and $\mathcal{F}'$ are formally equivalent.

The intersection of a leaf of $\mathcal{F}$ (resp. $\mathcal{F}'$) with A around p (resp. p') is given by $t^2 = c$ (resp. $\frac{e^{t-1}}{t-1} = c'$). It is easy to see that there is no biholomorphism of A which sends the intersection structure of $\mathcal{F}$ with A to the intersection structure of $\mathcal{F}'$ with A. Therefore there is no biholomorphic map between $\mathcal{F}$ and $\mathcal{F}'$. Note that $\mathcal{F}$ and $\mathcal{F}'$ are not even locally biholomorphic if we fix a biholomorphism $\phi : A \to A,\ \phi(p) = p$ and require that local biholomorphisms induce ϕ on A.

5.3 Formal isomorphism of foliations

The aim of this section is to extend the methods used in the section 1.6 to the case where we have foliated neighborhoods. We define the formal isomorphism between two foliated neighborhoods and then we identify the obstructions for the existence of such formal isomorphism. We prove that if the foliation $\mathcal{F}$ has not singularities on A and $A.A < min\{0, 2 - 2g - tang(\mathcal{F}, A)\}$ then any other holomorphic foliation in a neighborhood of A having the same local analytic structure of $\mathcal{F}$, is formally isomorphic with $\mathcal{F}$. Here $tang(\mathcal{F}, A)$ is the number of tangency points between A and $\mathcal{F}$ counting with multiplicity. In another words the formal moduli space of foliations with the local structure of $\mathcal{F}$ contains only one point.

Let $\mathcal{F}$ and $\mathcal{F}'$ be the germs of two foliations in (X, A) which are locally biholomorphic, i.e. for any point $a \in A$ there exists a biholomorphism

$$\phi_a : (X, A, a) \to (X, A, a)$$

sending the foliation $\mathcal{F}$ to $\mathcal{F}'$. Roughly speaking, $\mathcal{F}$ and $\mathcal{F}'$ has the same local analytic structure around A.

Let ν be a natural number. We say that the isomorphism

$$\phi_{(\nu)} : A_{(\nu)} \to A_{(\nu)}$$

is ν-isomorphism between $\mathcal{F}$ and $\mathcal{F}'$ if for every point $a \in A$ there exists a local biholomorphism

$$\phi_a : (X, A, a) \to (X, A, a)$$

which induces $\phi_{(\nu)}$ and sends $\mathcal{F}$ to $\mathcal{F}'$. We also say that $\phi_{(\nu)}$ sends $\mathcal{F}$ to $\mathcal{F}'$. The isomorphism of formal neighborhoods

$$\phi_{(\infty)} : A_{(\infty)} \to A_{(\infty)}$$

is a formal biholomorphism between $\mathcal{F}$ and $\mathcal{F}'$ if for every natural number ν the ν-isomorphism $A_{(\nu)}$ induced by $\phi_{(\infty)}$ sends $\mathcal{F}$ to $\mathcal{F}'$.

Now we are going to identify the obstructions for the existence of formal biholomorphism between two foliations.

Let us be given an ν-isomorphism $\phi_{(\nu)} : A_{(\nu)} \to A_{(\nu)}$ between the foliations $\mathcal{F}$ and $\mathcal{F}'$. We want to extend $\phi_{(\nu)}$ to $\phi_{(\nu+1)} : A_{(\nu+1)} \to A_{(\nu+1)}$, i.e. to find a $(\nu+1)$-isomorphism $\phi_{(\nu+1)} : A_{(\nu+1)} \to A_{(\nu+1)}$ between $\mathcal{F}$ and $\mathcal{F}'$ such that the following diagram is commutative:

(5.4)
$$\begin{array}{ccc} A_{(\nu+1)} & \overset{\phi_{(\nu+1)}}{\to} & A_{(\nu+1)} \\ \downarrow & & \downarrow \\ A_{(\nu)} & \overset{\phi_{(\nu)}}{\to} & A_{(\nu)} \end{array}$$

$\mathcal{F}$ and $\mathcal{F}'$ have the same local analytic structure. Therefore we have the local solutions of our problem.

(5.5)
$$\begin{array}{ccc} A_{(\nu+1)_a} & \overset{\phi_{(\nu+1)_a}}{\to} & A_{(\nu+1)_a} \\ \downarrow & & \downarrow \\ A_{(\nu)_a} & \overset{\phi_{(\nu)_a}}{\to} & A_{(\nu)_a} \end{array}$$

where $A_{(\nu)_a}$ is the stalk of the sheaf $A_{(\nu)}$ over the point a. If $\nu = 1$ we can furthermore assume that $\phi_{(\nu+1)_a}$ is the identity on $\mathcal{M}/\mathcal{M}^2$. We cover A with small open sets for which we have the diagrams of the type (5.5). Combining two diagrams in the intersection of neighborhoods of the points a and b we get:

(5.6)
$$\begin{array}{ccc} A_{(\nu+1)_{a,b}} & \overset{\phi_{(\nu+1)_{a,b}}}{\to} & A_{(\nu+1)_{a,b}} \\ \downarrow & & \downarrow \\ A_{(\nu)_{a,b}} & \overset{id}{\to} & A_{(\nu)_{a,b}} \end{array}$$

where

(5.7)
$$\phi_{(\nu+1)_{a,b}} = \phi_{(\nu+1)_a} \circ \phi_{(\nu+1)_b}^{-1}$$

sends the foliation $\mathcal{F}$ to itself.

Remark: Notice that we have used the notation $\phi_{(\nu+1)_{a,b}}$ instead of $\phi_{(\nu+1)} |_{U_a \cap U_b}$, $\phi_{(\nu+1)_a}$ instead of $\phi_{(\nu+1)} |_{U_a}$ and so on.

The above transition elements are obstructions to our extension problem. Now it is natural to define the following sheaf: $Aut(\nu, \mathcal{F})$ is the sheaf of $(\nu+1)$-isomorphisms $\phi_{(\nu+1)} : A_{(\nu+1)} \to A_{(\nu+1)}$ which sends $\mathcal{F}$ to itself and induces the identity in $A_{(\nu)}$, i.e. the following diagram is commutative

$$\begin{array}{ccc} A_{(\nu+1)} & \overset{\phi_{(\nu+1)}}{\rightarrow} & A_{(\nu+1)} \\ \downarrow & & \downarrow \\ A_{(\nu)} & \overset{id}{\rightarrow} & A_{(\nu)} \end{array} \tag{5.8}$$

in the case $\nu = 1$ we assume furthermore that $\phi_{(\nu+1)}$ is the identity on $\mathcal{M}/\mathcal{M}^2$.

Now it is easy to see that the data in (5.7) form an element of

$$H^1(A, Aut(\nu, \mathcal{F}))$$

The elements of $H^1(A, Aut(\nu, \mathcal{F}))$ are obstructions to the extension problem. More precisely we have proved the following proposition:

Proposition 5.3. *If $H^1(A, Aut(\nu, \mathcal{F})) = 0$ then any ν-isomorphism between the foliation $\mathcal{F}$ and $\mathcal{F}'$ extends to a $(\nu+1)$-isomorphism between them.*

Now we have to identify $Aut(\nu, \mathcal{F})$ and especially we have to verify when $H^1(A, Aut(\nu, \mathcal{F})) = 0$ is satisfied.

Proposition 5.4. *If A is not $\mathcal{F}$-invariant then $Aut(1, \mathcal{F})_a = 0$ for all points a in which $\mathcal{F}$ is transverse to A and so $H^1(A, Aut(1, \mathcal{F})) = 0$.*

Proof. Let $\mathcal{F}$ be transverse to A at a. Choose a coordinate system (x, y) around a such that $\mathcal{F}$ in this coordinate system is given by x =constant. Now it is easy to see that every biholomorphism $(\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ which sends $\mathcal{F}$ to $\mathcal{F}$ and induces the identity on $\mathcal{M}/\mathcal{M}^2$ has the form

$$(x, y) \to (x, y + y^2 s_2(x) + h.o.t.)$$

and hence induces the identity in $A_{(2)}$. □

The above proposition says that when A is not $\mathcal{F}$-invariant we can always find a 2-isomorphism between the foliations $\mathcal{F}$ and $\mathcal{F}'$.

Theorem 5.2. *Assume that A is not $\mathcal{F}$-invariant and $\mathcal{F}$ does not have singularities on A. For $\nu \geq 2$ we have*

$$Aut(\nu, \mathcal{F}) \cong \mathcal{T}_{\mathcal{F}}(\nu)$$

where $\mathcal{T}_{\mathcal{F}}$ is the sheaf of holomorphic vector fields in X inducing the foliation $\mathcal{F}$.

Proof. Recall that $\mathcal{T}_{\mathcal{F}}(\nu) = \mathcal{T}_{\mathcal{F}} \otimes_{\mathcal{O}_A} \mathcal{Q}_\nu = \mathcal{T}_{\mathcal{F}}.\mathcal{M}^\nu / \mathcal{T}_{\mathcal{F}}.\mathcal{M}^{\nu+1}$. Let us introduce our candidate for the isomorphism:

$$* : \mathcal{T}_{\mathcal{F}}(\nu) \to Aut(\nu, \mathcal{F})$$

The operator $*$ associate to every holomorphic vector field $X \in \mathcal{T}_{\mathcal{F}}(\nu)$ the $(\nu+1)$-isomorphism

$$*(X) : A_{(\nu+1)} \to A_{(\nu+1)}$$
$$f \to f + df.X$$

Since X has zero of order ν in A, $*(X)$ induces identity in $A_{(\nu)}$. We must prove that $*(X)$ sends $\mathcal{F}$ to $\mathcal{F}$.

Let $X_t(x)$ be the solution of the vector field X passing through x in the time t. Since X is zero in A, $X_1 = X_t \mid_{t=1}$ is well-defined in a smaller neighborhood around A. X is tangent to the foliation and so X_1 sends $\mathcal{F}$ to $\mathcal{F}$. It is enough to prove that X_1 induces the map $*(X)$ in $A_{(\nu+1)}$. We have

$$X_t^* f = f \circ X_t = f + t df(X) + \sum_{i \geq 2} \frac{\partial^i (f \circ X_t)}{\partial t^i} \mid_{t=0} t^i$$

Since

$$\frac{\partial^2 (f \circ X_t)}{\partial t^2} = ((d^2 f \circ X_t).(X \circ X_t)).(X \circ X_t) + (df \circ X_t).((dX \circ X_t).(X \circ X_t))$$

X has zero of order ν along A and $v \geq 2$, we conclude that

$$\frac{\partial^i (f \circ X_t)}{\partial t^i} \mid_{t=0} = 0 \text{ mod } \mathcal{M}^{\nu+1}$$

or equivalently

$$X_1^* f = f + df(X) \text{ mod } \mathcal{M}^{\nu+1}$$

$*$ is trivially injective. Let us now prove that $*$ is surjective.
Let $\beta \in Aut(\mathcal{F}, \nu)$ and

$$h : (x, y) \to (x, y) + (f, g)$$

be an isomorphism in a coordinate system (x, y) around a point $a \in A$ which extends β and sends $\mathcal{F}$ to $\mathcal{F}$.We have $f, g \in \mathcal{M}^\nu$. Suppose that in this coordinate system $\mathcal{F}$ is given by the 1-form $\omega = Pdy - Qdx = 0$, where P and Q are relatively prime. Since $h^*(\omega) \wedge \omega = 0$ we have

$$P\tilde{Q}f_x + Q\tilde{Q}f_y - P\tilde{P}g_x - \tilde{P}Qg_y = 0 \tag{5.9}$$

where

$$\tilde{P} = P(x + f, y + g),\ \tilde{Q} = Q(x + f, y + g)$$

Since A is not $\mathcal{F}$-invariant, y does not divide Q. Therefore considering the equality (5.9) modulo $\mathcal{M}^\nu$ we see that

$$Qf_y - Pg_y \text{ mod } \mathcal{M}^\nu$$

This implies that

$$Qf - Pg = 0 \text{ mod } \mathcal{M}^{\nu+1}$$

The foliation $\mathcal{F}$ has not singularity at a and so $PQ(a) \neq 0$. Using this fact we can fined new holomorphic functions $\tilde{f}$ and $\tilde{g}$ such that

$$\tilde{f} = f,\ \tilde{g} = g \text{ mod } \mathcal{M}^\nu$$

$$\omega(X) = Q\tilde{f} - P\tilde{g} = 0$$

where $X = (\tilde{f}, \tilde{g})$. The vector field X is the desired. □

Now suppose that A is not $\mathcal{F}$-invariant. $\mathcal{F}$ is transverse to A except in a finite number of points. These points may be tangency points of $\mathcal{F}$ with A or singularities of $\mathcal{F}$. Suppose that there does not exists a singularity of $\mathcal{F}$ on A.

Using Serre duality, we have

$$H^1(A, \mathcal{T}_{\mathcal{F}}(\nu)) = (\Gamma(A, \Omega \otimes T^*_{\mathcal{F}} \otimes N^\nu))^* = 0$$

if

$$c(\Omega \otimes T^*_{\mathcal{F}} \otimes N^\nu) = (2g - 2) - c(T_{\mathcal{F}}) + \nu A.A < 0 \tag{5.10}$$

We have

(5.11) $$c(T_{\mathcal{F}}) = A.A - tang(\mathcal{F}, A)$$

where $tang(\mathcal{F}, A)$ is the number of tangency points of $\mathcal{F}$ and A, counting with multiplicity (see [**Br02**]). Now substituting (5.11) in (5.10), we conclude that:

Theorem 5.3. *Let A be a Riemann surface of genus g embedded in a two dimensional manifold X and $\mathcal{F}$ and $\mathcal{F}'$ be two locally biholomorphic and without singularity foliations around A. If $A.A < min\{0, 2-2g-tang(\mathcal{F}, A)\}$ then there exists a formal isomorphism between $\mathcal{F}$ and $\mathcal{F}'$.*

Complementary notes

1. Let $\mathcal{F}$ and $\mathcal{F}$ be two locally biholomorphic foliations around a Riemann surface $A \subset X$. Suppose that $\mathcal{F}$ and $\mathcal{F}'$ are transverse to A except at a finite set $S \subset A$. One can use the methods of [**CeMa82**] for Suzuki's example and find a homeomorphism between $\mathcal{F}$ and $\mathcal{F}'$.
2. It would be interesting if Theorem 5.1 is true in the following case: Let A be a manifold of dimension n negatively embedded in a manifold X of dimension $n+m$. Let $\mathcal{F}$ and $\mathcal{F}'$ two non singular transversal foliations to A with leaves of dimension m. If the normal bundle of A in X is negative "enough" then $\mathcal{F}$ and $\mathcal{F}'$ are equivalent.

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